Proving the Integrity of the Weighted Sum Squared Error (WSSE) Loran Cycle Confidence Algorithm

Sherman C. Lo, *Stanford University*
Benjamin B. Peterson, *Peterson Integrated Geopositioning*
Per K. Enge, *Stanford University*

BIOGRAPHY

Sherman C. Lo is currently a research associate at the Stanford University Global Positioning System (GPS) Laboratory. He is the Associate Investigator for the Stanford University efforts on the Department of Transportation's technical evaluation of Loran.

Per Enge is a professor in the Department of Aeronautics and Astronautics at Stanford University. He is the director of the Stanford GPS Laboratory and the Center for Position, Navigation and Time.

Benjamin Peterson served head of the Engineering Department at the U. S. Coast Guard Academy, in New London, CT. After his retirement from the Coast Guard, he founded Peterson Integrated Geopositioning, LLC

ABSTRACT

For Loran to provide redundancy to GPS for aviation, Loran must meet aviation integrity requirements. The integrity under nominal conditions derives from being able to bound the horizontal position error via the horizontal protection level (HPL). This is accomplished by guaranteeing that the correct Loran cycles are being tracked, adequate and complete error bound models are used for the HPL, and HPL calculations are performed correctly.

The means of providing a guarantee on cycle selection is through a calculation of its confidence ("cycle confidence"). The cycle confidence algorithm needs to be conservative since it ensures that the measurements used by the navigation solution and the HPL is free of cycle error. When available, the algorithm uses redundant measurements in a manner similar to GPS receiver autonomous integrity measurement (RAIM) algorithms where testing using χ^2 distributions are conducted. This paper will examine the use of redundant measurements in the form of the weighted sum squared error (WSSE) and determine when the χ^2 assumption is valid. It will apply those results to the Loran cycle confidence algorithm and use them to help develop the demonstration of integrity.

Keywords: Loran, cycle selection, WSSE, RAIM

INTRODUCTION

The demonstration of Loran integrity is essential to showing that it can support aviation applications such as Required Navigation Performance (RNP) 0.3 non precision approach (NPA) [1] and enroute RNP 1.0. A key part of the demonstration is the development of a Loran cycle selection algorithm that has integrity.

The ability to determine if the correct cycle is selected is necessary for meeting integrity. An incorrect cycle selection, should it be used for positioning, results in an undetected range error of three kilometer or more. While Loran cycle selection algorithms have been around for some time, an algorithm that will serve aviation requires an indication of the confidence of the cycle selection. Hence, an aviation receiver will have a cycle confidence algorithm to quantify the certainty of the cycle selection. The selected cycles can only be used provided the confidence calculated is of an adequate level. The current level, as determined by the Federal Aviation Administration (FAA) Loran evaluation team, is for the probability of having an incorrect cycle selection amongst all the signals used to be at most 7×10^{-8} if the cycle selection is to be used [1][2]. As the cycle selection integrity depends on the cycle confidence calculated, the cycle confidence algorithm itself must have integrity. In other words, it must provide a conservative estimate of the probability. One goal of the paper is to provide an implementation that can guarantee integrity.

Peterson et. al. developed a method for determining Loran cycle confidence in [3] That methodology represents the starting basis for this paper. The algorithm of particular interest is the determination of cycle confidence when redundant measurements are available. The algorithm is based on Receiver Autonomous Integrity Monitoring (RAIM) developed for using redundant measurements from Global Positioning System (GPS) satellites to determine if any unusual biases exist. It uses the weighted sum squared error (WSSE) statistic for the determination. Hence in assessing and developing the integrity of the Loran cycle confidence algorithm, we may gain more insight on GPS RAIM.

OUTLINE

A means of implementing of the WSSE cycle confidence algorithm with integrity will be provided by the paper. First, background on residual testing for navigation signals and the use of the chi squared (χ^2) distribution for these test is provided. This section will also provide a derivation of when the WSSE distribution can be considered χ^2. The next section examines the Loran cycle confidence algorithm and the basic concept of providing integrity. Loran differs from GPS in that nominal biases are significant and must be accounted for. It will use the results from the first section to provide an implementation that can have demonstratable integrity. The integrity is demonstrated using outlines of proofs (provided in the appendix) and arguments supporting the conservatism of each algorithm. The final section will provide some simulation results illustrating key points of the derivation and algorithm.

Note that this paper will use the term cycle integrity or confidence algorithm to denote the method for determining the certainty of cycle selection.

BACKGROUND

There has been considerable work done on using residuals test for navigation, particularly for use with Global Positioning System (GPS) satellite measurements [4][5][6]. These methods leverage redundant measurement such that the signals are essentially cross checked with an aggregate measurement. Since the likelihood of incorrect cycle selection on multiple signals is much lower than having one, this check increases our confidence of detecting outliers. Many of these methods, such as the sum squared error (SSE) and WSSE, utilize the chi square (χ^2) distribution to describe the decision statistic derived from the residuals. This section will provide some background on the use of residuals for navigation error detection, particular for the WSSE. The Loran cycle confidence algorithm is based on the WSSE. It will also describe the χ^2 distribution. Finally, it will derive the relationship between WSSE and the χ^2 distribution.

CALCULATION THE WSSE FROM RESIDUALS

Statistics such as the SSE and WSSE provide a simple scalar metric of the variation of the residual errors. Start with the basic measurement equation where y is our pseudorange measurement vector, x is the true position, N is the number of measurements, G is the geometry matrix and ε is the error vector on the pseudorange measurements. And so we have

$$y = Gx + \varepsilon$$
$$\varepsilon = \left[\varepsilon_1 \cdots \varepsilon_N\right]^T$$

Solving for the weighted least squares with weighting matrix W_1, the estimated position ($\hat{x}$), estimated pseudoranges ($\hat{y}$) and residual error (‘residuals or $\hat{\varepsilon}$) can be derived.

$$\hat{x} = \left(G^T W_1 G\right)^{-1} G^T W_1 y$$
$$P = G\left(G^T W_1 G\right)^{-1} G^T W_1$$
$$\hat{y} = G\hat{x} = Py$$
$$\hat{\varepsilon} = (I - P)\varepsilon$$

The WSSE is generated by multiplying the estimated residuals by a weighting matrix, W_2, as seen in Equation (1). Typically, W_2 is equaled to W_1 but this need not be the case. From this formulation it can be seen that the WSSE is the weighted sum squared of the residual errors on each signal is checked against an aggregate of the position solution using the entire signal set.

$$WSSE = \hat{\varepsilon}^T W_2 \hat{\varepsilon} = \varepsilon^T W_2 \left(I - P\right)\varepsilon \qquad (1)$$

The distribution of WSSE depends on the underlying errors, ε_i. One common, general assumption is that each ε_i is normal with $\varepsilon_i \sim N\left(b_i, \sigma_i\right)$ and $b = \left[b_1 \cdots b_N\right]^T$ $\sigma = \left[\sigma_1 \cdots \sigma_N\right]^T$

For GPS, it is typically assumed that ε_i are zero mean normal random variables. Under such conditions, Walter et. al. stated that the resulting WSSE statistic is a central χ^2 with N-3 degrees of freedom [6]. Furthermore, if ε are normal but not zero mean, the WSSE statistic is often assumed to follow a noncentral χ^2 distribution with N-3 degrees of freedom. However, the result not necessarily true. They are only true under specific conditions

CHI SQUARED DISTRIBUTION

The χ^2 distribution is important as it provides a tractable description of WSSE distribution. The χ^2 distribution is formed from the sum of independent identically distributed (iid) normal random variables (rv) with variance of one. This is termed a standard normal distribution. If the random variables are zero mean, the resulting distribution is central χ^2 otherwise the distribution is noncentral χ^2. All χ^2 distributions are characterized by the degree of freedom (*dof*) which represents the number of iid distributions used to form the sum. A central χ^2 is characterized only by its *dof*. Because of the nonzero mean, noncentral χ^2 distribution needs an additional parameter for characterization. The parameter is termed the noncentrality parameter which we will designate as *ncp* in this paper. One can consider the central χ^2 as having a *ncp* of zero. Figure 1 presents a flowchart on how central and noncentral χ^2 distributions are formed. Provided that the WSSE is χ^2 and the biases b are known, the noncentrality parameter can be calculated as $ncp = b^T W_2 (I - P) b$.

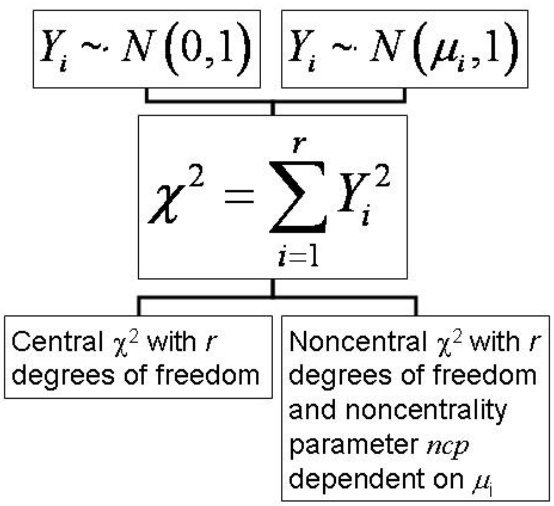

Figure 1. Central and Noncentral χ^2 Distribution

Figure 2 shows the probability density function (pdf) of a central and noncentral χ^2 distribution. As seen in the figure, as the pdf of the distribution shifts to the right (to higher values of the statistic) as the *ncp* increases. It is shown in Appendix B that the cumulative distribution function (cdf) of a χ^2 distribution is overbounded by the cdf of another χ^2 distribution provided that 1) the *ncp* of the latter distribution is smaller and 2) they have the same degree of freedom. Mathematically, this means that for any given *dof*, υ, if $ncp_1 \le ncp_2$, the following is true:

$$CDF_{\chi^2(\upsilon,ncp_1)}(x) \ge CDF_{\chi^2(\upsilon,ncp_2)}(x)$$

where χ^2(a,b) is a χ^2 distribution with *dof = a* & *ncp = b*. Figure 3 illustrates the result. It presents two χ^2 distributions with the same *dof* but different *ncp*. The cdf of the distribution with the smaller *ncp* is greater than cdf of the other distribution. This result is used later to demonstrate the integrity of the Loran cycle confidence algorithm.

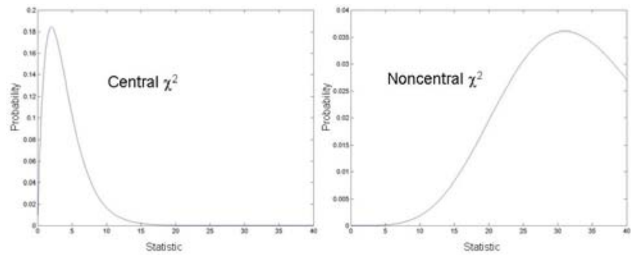

Figure 2. PDF of Central (L) and Noncentral (R) χ^2 Distribution

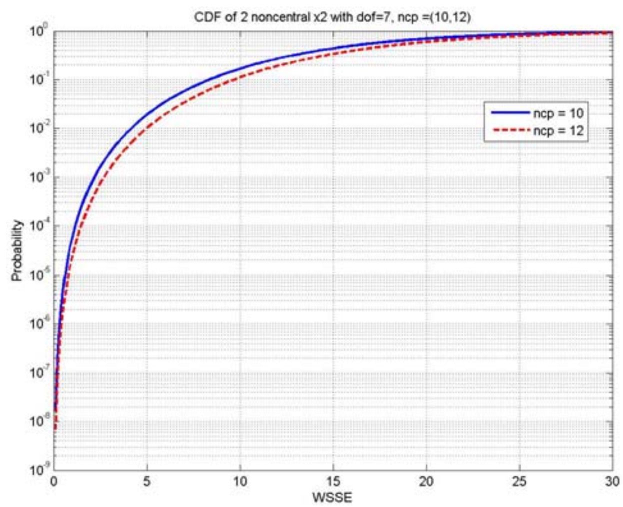

Figure 3. Comparison of the CDF of two noncentral χ^2 with the same degree of freedom

RELATIONSHIP BETWEEN THE WSSE AND CHI SQUARED DISTRIBUTION

While it is generally assumed that the WSSE is χ^2 distributed, this is generally not true and only holds under specific conditions. Appendix A provides a proof showing that the statement holds if $W_1=W_2=R^{-1}$ where R is the true covariance matrix for random errors. In the paper, we shall refer to this true covariance matrix simply as the covariance matrix while estimates of the truth will be preceded by the identifier "estimate". There are instances where, for $W_1 \neq W_2$, χ^2 holds. But in those cases W_2 still equals R^{-1}. For this remainder of the paper, we will assume that $W=W_1=W_2$.

It is enlightening to examine part of the proof in detail. If ε are distributed as normal random variables and we denote s as being a vector of iid standard normal rv, then the WSSE can be written as:

$$WSSE = s^T As$$
$$A = \left(R^{\frac{1}{2}}\right)^T W\left(I - G\left(G^T WG\right)^{-1} G^T W\right)\left(R^{\frac{1}{2}}\right) \qquad (2)$$

If W is symmetric (it generally is chosen that way), it turns out that A can be written as

$$s^T As = s^T C^T \Lambda Cs$$

where C is an orthogonal matrix and Λ is a diagonal matrix. Define dim(x) as the number of dimensions of the position vector x (i.e. 3 if x = [lat lon time]). Then Λ has at most N-dim(x) non zero eigenvalues, λ_i. Since C is orthogonal, $s^T C^T \Lambda Cs$ and $s^T \Lambda s$ have the same distribution. This implies:

$$WSSE = \sum_{i=1}^{N-\dim(x)} z_i^2 \text{ where } z_i \sim N\left(\beta_i, \sqrt{\lambda_i}\right) \qquad (3)$$

and β_i depends on A and b.

If $W = R^{-1}$, then there are exactly N-dim(x) eigenvalues identically equaled to 1. Equation (3) shows that this is the case where we have the sum of the squares of N-dim(x) standard normal distributions, hence the WSSE is χ^2. The eigenvalues should not change significantly should W be close, but not equaled to R^{-1}. Deviation from this only occurs if we are close to a singularity. However, a difference between R^{-1} and W also changes the WSSE distribution by affecting how the biases interact with the random errors. The insight that we get from this derivation is that χ^2 is a good approximation for the WSSE distribution provided that W is close to R^{-1} and the bias effects are small.

In summary, for WSSE statistic used in cycle confidence to be χ^2, two conditions must hold. First, the true residual errors must be normally distributed. Second, the weighting matrix used must be the inverse of the true error covariance matrix.

APPLICATION OF RESIDUALS TEST TO LORAN CYCLE CONFIDENCE

A WSSE based residuals tests was used as the basis of our Loran cycle confidence algorithm. In this section, we provide background on the cycle selection process and discuss why cycle confidence is necessary. Then we discuss how cycle confidence can be estimated using the χ^2 distribution. However, since we do not know all the desired parameters, we can only bound the distribution. The last section discusses how this bound can be created in a way that integrity is guaranteed using only known information.

Cycle selection is the process of choosing the same cycle on the Loran pulse to track for all signals. This ensures consistency between measurements. The tracked cycle is typically the sixth zero crossing which is the standard Loran phase tracking point. An incorrect cycle selection can result in a range error of three kilometers (one Loran wavelength or λ) or more and hence it is important that one has confidence in our cycle selection. Figure 4 shows a standard Loran pulse and the phase tracking point.

The cycle selection process in Loran typically involves examining the envelope of the signal and choosing the desired cycle based on the envelope slope or ratio. The determination complicated by the presence of noise on the signal and group delay between the envelope and the carrier of the signal due to propagation. The group delay results in an effect typically referred to as the envelope to cycle difference (ECD). These measurement uncertainties can result in incorrectly cycle identification. As such a cycle confidence algorithm is necessary in aviation for determining the certainty of the cycles selected and whether the selection should be used.

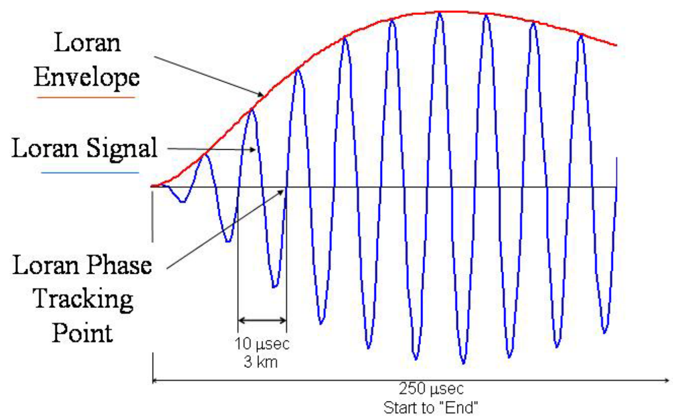

Figure 4. Loran Pulse, Envelope, and Tracking Point

Once cycle selection has been conducted, a process is necessary to determine if that cycle selection should be used. This can be done in junction with a threshold test as discussed in the next section. However, using a threshold is not necessary in Loran cycle confidence. The necessary feature is that the overall probability of having cycle error on any signal (P_{WC}) is known. If this probability is lower than the maximum allowable probability of 7×10^{-8}, then we can use the cycles selected. Otherwise, the cycle selection is rejected. The probability is calculated by summing over all possible cases of having an incorrect cycle and not detecting that incorrect cycle (i.e. choosing to use the selection even though it is in error). The probability for a given case is given by the probability of having the incorrect on the given station(s) i (P_{ICi}) and the probability of missed detection given that there is an incorrect cycle on i (P_{MDi}). This is expressed in Equation (5). Since the probability of having and not detecting three or more cycle errors is low relative to our $7\text{x}10^{-8}$ threshold, those cases typically do not need to be considered.

$$P_{WC} = \sum P_{MD_i} P_{IC_i} + \sum P_{MD_{ij}} P_{IC_i} P_{IC\,j} + \sum P_{MD_{jki}} P_{IC_i} P_{ICj} P_{IC\,k} + \ldots \quad (4)$$

The probability of having a cycle error can be well estimated given our signal to noise ratio (SNR) and ECD uncertainty. This value has been theoretically and empirically calculated [7]. So the primary unknown for calculating the probability of having an incorrect cycle in our cycle selection is the conditional probability of missed detection. If we can generate conservative estimates of P_{MD} for all cases, then our estimate of P_{WC} will be conservative.

DETERMINING PROBABILITY OF MISSED DETECTION

The determination of the probability of missed detection is complicated by the nominal biases that exist on Loran. The assumption that biases are insignificant, often used in GPS, is not valid for Loran. Loran has propagation biases, known as additional secondary factors (ASF), that cannot be completely estimated. The unknown portion of the bias can be significant – up to a couple hundred meters. Since these biases are not known, the true distribution cannot be exactly known. Additionally, there is still random noise and transmitter jitter that corrupt the signal. These factors make the calculation challenging.

Let's start by examining the ideal case where we have knowledge of all bias and random errors. Assume the random errors are normally distributed. If these are known and we use the covariance matrix for weighting, we should have χ^2 distribution for the WSSE, as discussed previously. The distribution will differ for the case of having no fault (no incorrect cycle) and the faulted case (incorrect cycle). Having a different distribution for the no fault (H_0) and faulted case (H_1) leads to the test for determining the probability of missed detection of an incorrect cycle selection. The comparison of the two distribution is shown in Figure 5 with the no fault distribution always lying to the right of the faulted distribution.

A threshold is set to decide if an incorrect cycle selection exists. If the WSSE is less than the threshold, the user will assume that all cycles are correct. The threshold value is a design parameter and trades off between the false alarm and missed detection rate. This can be seen in Figure 5. The overlapping regions to the left and right of the threshold, shown in red and yellow on the figure, depict the probability of miss detection and false alarms, respectively. The probability of false alarm (P_{FA}) is the complementary cumulative distribution function (ccdf) of the no fault distribution at the threshold. The probability of missed detection, P_{MD}, for the given incorrect cycle case is the value of the cdf of the faulted distribution at the threshold and is given by Equation (5).

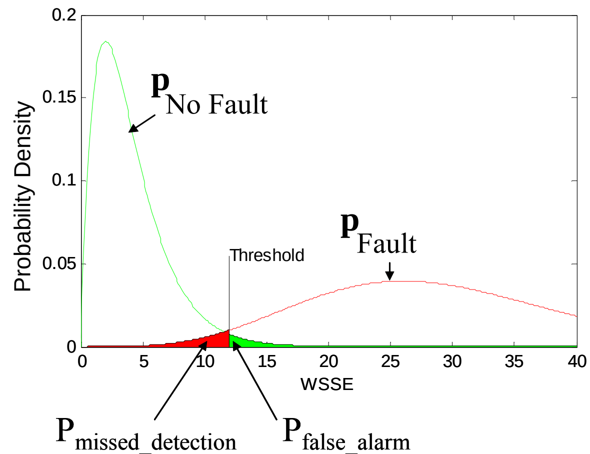

Figure 5. Probability Density Function for No Fault (Right) and Fault (Left) Distribution

$$P_{MD} = \int_{-\infty}^{threshold} p_{Fault}(x)dx = \int_{0}^{threshold} p_{Fault}(x)dx \quad (5)$$

However to determine P_{FA} and P_{MD}, the no fault and faulted distributions must be known, respectively. For the no fault case, it is assumed that there are only the nominal biases due to ASF. The WSSE, because of the biases, has a noncentral χ^2 distribution. In the faulted case, there is at least one incorrect cycle selection. This means there is at least one signal with a bias consisting of the nominal biases and a bias of one λ. The WSSE is also noncentral χ^2 distributed but with a different noncentrality parameter than the nominal case.

While there is only one no fault case, there are multiple possibilities for a fault to exist. Hence, if we have ten stations and consider all possible cases where there is a cycle error on one or two signals, there are 100 possible faulted cases to consider. Hence, it is generally easier to set the threshold based on false alarm rate since there is only one case to consider. Since false alarm represents a loss of availability, it is set at 99 to 99.9% since that represents the minimum availability requirement for Loran RNP.

The problem with the scenario is that we do not know the true biases. If we did, they could be eliminated and there would be no biases. Hence, we do not know the true fault and no fault distributions. However, we know the nominal extent of the biases (bounds on biases) for Loran. The next section will show how it is possible to use this information and *dof* to generate noncentrality parameters that will appropriately bound to have a conservative estimate of P_{FA} and $P_{MD.}$

DEMONSTRATING INTEGRITY

The methodology used for demonstrating that the cycle selection has integrity is to show that the calculated P_{MD} is larger than the true P_{MD} for each possible missed cycle situation. As the true distributions are not known, bounding distributions have to be used. It is assumed that the variance of the distribution can be well modeled and that only the biases are not well known. This is a reasonable assumption as there are reasonable models for the random errors and their distributions. These errors depend primarily on transmitter jitter and SNR.

To get a conservative estimate of P_{MD}, an overbound concept similar to that presented in [8] is useful. For integrity, it is only necessary to overbound the lower (left) tail of H_1 distribution to get a conservative estimate for P_{MD}. If the lower tail of the faulted distribution is overbounded, the estimated P_{MD} will be greater than the true P_{MD}. This conservatism helps insure that the integrity requirement is met.

As a side note, the estimated H_1 distribution only needs to overbound the true distribution up to a cumulative distribution function (cdf) of $7x10^{-8}$ divided by the probability of the fault occurring. This translates to a contribution to P_{WC} of $7x10^{-8}$. Since any probability of undetected wrong cycle (P_{WC}) greater than $7x10^{-8}$ will be regarded as unacceptable, regardless of how much it exceeds that level, the overbounding of these higher probabilities is unnecessary. This because any contribution that exceeds $7x10^{-8}$ that does not affect the cycle decision.

Overbounding the upper (right) tail of the no fault distribution is useful as it results in a higher threshold than the one that would be derived from the true no fault distribution. The significance of this is that the higher threshold results a true $P_{FA} < P_{FA}$ calculated.

Hence, it is desirable to overbound both the upper (right) tail and lower (left) tail of the no fault and faulted distribution, respectively. This is seen in Figure 6. This provides both availability at or higher than specified by the P_{FA} and integrity to the estimate of cycle confidence. The next two sections discuss how the overbounding can be accomplished. In both these demonstrations, we will use the result from Appendix B which states that, given a *dof*, the cdf of a χ^2 distribution with a given *ncp* at any value *w* is always larger than the cdf at *w* of a χ^2 distribution with a larger *ncp*.

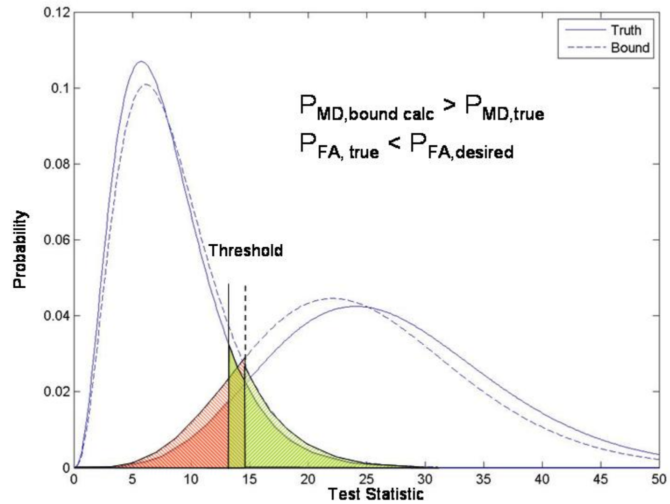

Figure 6. Overbounding to Achieve Conservative Estimate of False Alarm & Missed Detection

BOUNDING NO FAULT DISTRIBUTION

Providing a conservative estimate of the probability of false alarm means bounding the upper tail of the no fault distribution. One way to achieve this is to have the cdf of the estimate be less than or equal to the true cdf. Using Appendix B, this means using an estimated *ncp* that is greater than the true *ncp*. The question is how to do this without knowledge of the true biases.

While we do not have knowledge of the true biases, we have knowledge of the limits of these biases. Denote th bias bound on station *i* as B_i, where $B_i > 0$. So we know that the true bias, b_i, is in $[-B_i, + B_i]$. It can be proven that there is a choice of signs, s_i, for each bias bound such that a bias bound vector $B = [s_1B_1 \dots s_NB_N]$ will result in a $ncp_{Ho,estimate} = B^TW(I-P)B$ that is greater than or equal to $ncp_{Ho,true}$. The proof is given in Appendix C, lemma 1.

BOUNDING FAULTED DISTRIBUTION

We have a similar issue in estimating a conservative value of the probability of missed detection. A conservative estimate means bounding the lower tail of the faulted distribution. This can be achieved using an estimated cdf that is greater than or equal to the true cdf. This means using an estimated *ncp* that is less than the true *ncp*.

Again, the bias limits can be employed to provide the *ncp*. One way is to do a search over the space of all possible bias vectors (since we know the limits of each bias) for the minimum *ncp*. Then that minimum is used as our estimate. This provides the best solution but is computationally intensive and receiver manufacturers may not want or be able to implement such an algorithm.

Implementation must be an important consideration. We developed a second method that also uses an appropriate choice of signs for the bias bounds. In lemma 2 from Appendix C, it is shown, with an appropriate choice of signs for the bias bounds, that we can determine a value $\Delta ncp < ncp_{H1,true} - ncp_{Ho,true}$. Let $\Delta ncp = ncp_{H1,estimate}$. Δncp is guaranteed to be less than $ncp_{H1,true}$ as $ncp_{H0,true} > 0$. While the estimate is more conservative estimate than that calculated by the first method, it is computationally less intensive.

The bottomline conclusion is that integrity is provable provided the WSSE is χ^2 distributed.

SIMULATION RESULTS

Simulations were conducted to confirm some of our insight gained from the WSSE χ^2 proof (Appendix A) and visualize the effects of deviations from the assumptions of the proof. Test scenarios were created to test the distribution of the WSSE statistics under different weighting assumptions. Three different weighting matrix cases will be presented in this section. The first case is one where the weighting is the inverse of the covariance matrix. The other two cases use weighting matrices that deviate from that weighting. These cases are important as we will generally not know the covariance matrix but only have an approximation to it. The geometry of the example scenario used is shown in Figure 7.

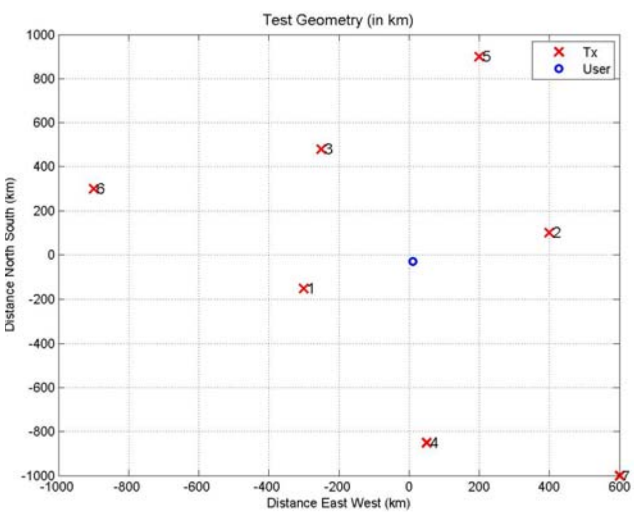

Figure 7. An Example Simulation Geometry

The first case simulated is the nominal case where the weighting matrix is the inverse of the covariance matrix. Figure 8 shows comparison of the WSSE distribution for the simulation and ideal χ^2 distribution with knowledge of the true biases. The top plot shows the no fault case while the bottom plot shows the faulted case with the incorrect cycle selection occurring on each signal i with a probability proportional to the $P_{IC,i}$. As expected and anticipated by the proof, the simulation and χ^2 distribution are nearly identical.

In the faulted case, the bias is the nominal bias plus (or minus) a three kilometer error due to incorrect cycle selection on the faulted signal(s). The signal on which the error occurs depends on the probability of incorrect cycle, P_{IC}, on that signal normalized to the total probability of incorrect cycle. The sign on the cycle bias has equal probability of being positive or negative. The bottom plot on Figure 8 shows comparison of the faulted WSSE from simulation with the weighted sum of the various faulted χ^2 distributions. The weighting is based on the probability of occurrence. Again the match between simulation and theory is nearly perfect.

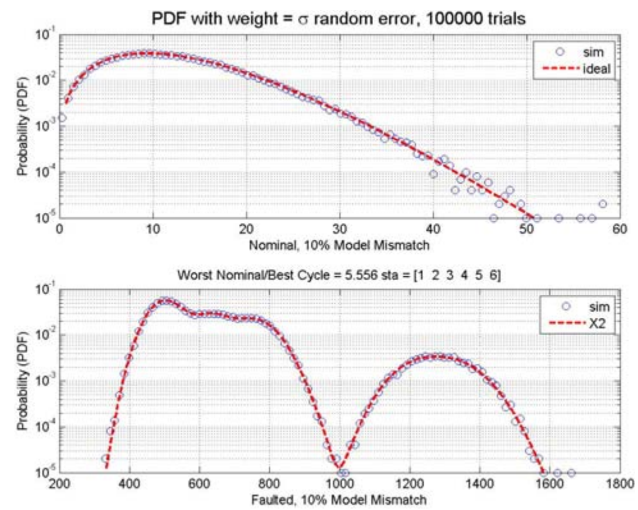

Figure 8. PDF of WSSE Distribution with σ Weighting: No Fault (Top), Faulted (Bottom)

In the second case examined, we deviated from the weighting matrix used in the first case. The matrix used has diagonal elements based on the standard deviation and bias (σ+b) of the measurement errors rather than just the standard deviation as is the case with the true covariance matrix. The off diagonal elements are based on the covariance of bias term since it is assumed that the random terms are independent and there is no cross correlation. This case is examined since it was the original implementation used in the original proposal [3] and 2004 FAA Technical Evaluation of Loran [1]. It also serves as an example of what a large deviation from the true covariance matrix may cause. Figure 9 shows the simulation results for the no fault (top) and aggregate faulted (bottom) case. The no fault case is compared with a central χ^2 and an estimated noncentral χ^2 using the true biases. The faulted case is compared to a χ^2 from assuming the most likely bias occurs and a χ^2 using the true biases. As seen from the plots, the WSSE differs significantly from the estimates of the resulting distribution

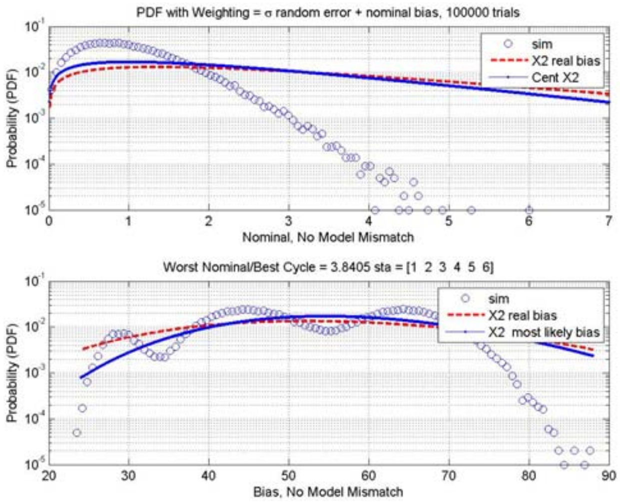

Figure 9. PDF of WSSE Distribution with σ+b Weighting: No Fault (Top), Faulted (Bottom)

The proof also claims that if we use a weighting that deviates from the inverse of the true covariance of the random errors only, the WSSE cannot be guaranteed to have a χ^2 distribution. This can be seen in the second case as well. Figure 10 shows an attempt to fit of the WSSE distribution from the σ+b weighting to a χ^2 distribution. It is important to note that the χ^2 distribution seen in the figure cannot be generated even with the known true bias and variances. It was generated by fitting to the actual WSSE distribution. The result is that no matter which χ^2 distribution is chosen, it will not perfectly model when the weighting deviates from the inverse of the covariance matrix.

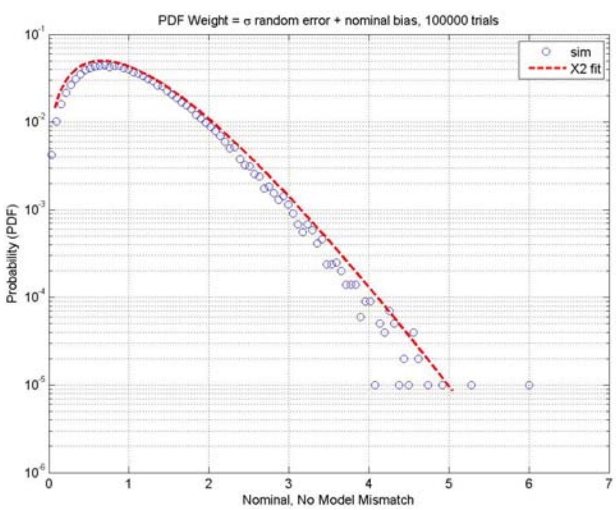

Figure 10. PDF σ+b weighting, Nominal, Best Fit

The simulations of the previous section represent the ideal case where the random error statistics and bias errors are known. This is generally not the case. However, exact knowledge is not necessary as the goal is to bound the distributions such that the true integrity level is greater than calculated integrity.

Uncertainty in the knowledge of the true random error statistics means that only an estimate for R is available. As a result $W_\sigma \neq R^{-1}$ and, thus, the WSSE generally will not be χ^2. Generally, the Loran statistics are well known, being based on years of measurements. As argued previously, if the estimated covariance is reasonably close to truth, the WSSE distribution is well approximated by a χ^2 distribution provided the biases are too large. Figure 11 shows an example of a case where each estimated covariance term is in error from the corresponding true term by 10%.

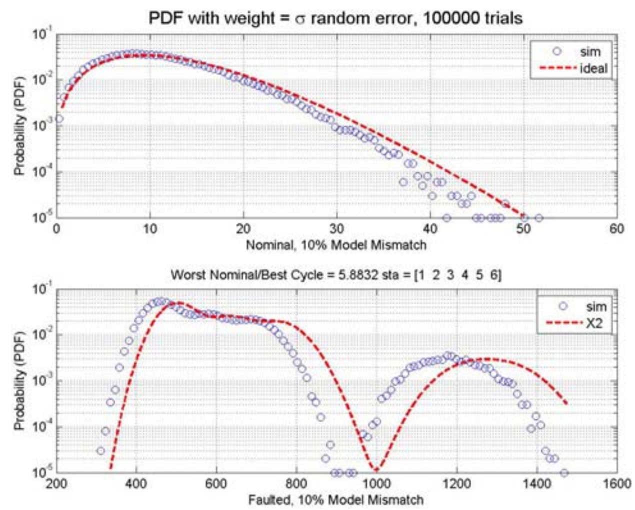

Figure 11. Simulation of σ weighting with 10% error on σ

The figure shows an interesting effect. The top plot shows the case where there is only the nominal bias. In this instance, the distribution of the WSSE does not vary from the χ^2 distribution generated using the true biases. The result is not unexpected as it was anticipated by the discussion in a previous section. In that section, we hypothesized that, if our estimated covariance matrix is not far from the truth, the eigenvalues of the matrix A (see Equation (3)) would be close to one. If the effects of the biases are ignored, the WSSE is the sum of the squares on normal random variables with variance equaled to the eigenvalues. Since these are close to one, the result should be close the χ^2 (where the variances are one).

However, the weighting matrix also affects how the bias component interacts with the random components of the underlying distributions in generating the WSSE. The bottom plot shows the fault case where there are one or two biases that are significantly greater than the others. In this case, the difference between the used and actual weighting matrix causes a difference that is very noticeable. If all the biases are roughly the same magnitude (as is the case in the no fault case), the effects would average out and the difference would be less noticeable. So while the random component may be close, the bias effect is not, especially if there are large differences in bias values and they are significant compared to the random errors. This is the case of when there is a cycle error.

SIMULATING THE BOUND

The previous section examined the distribution given that the biases are known. It essentially tested our results concerning the relationship between the WSSE and χ^2 distribution. In this section, we examine the bound performance using bias bounds.

Figure 12 shows the true and estimated no fault distributions for both weightings. The plots show the estimated distribution calculated using the median *ncp*. As mentioned previously, for the weighting based on the true covariance matrix, it can be shown that the worst case (maximum) *ncp* for the no fault distribution is guaranteed to overbound the true no fault distribution. In this case, as in many cases, the estimated no fault distribution using the median *ncp* overbounds the true no fault distribution.

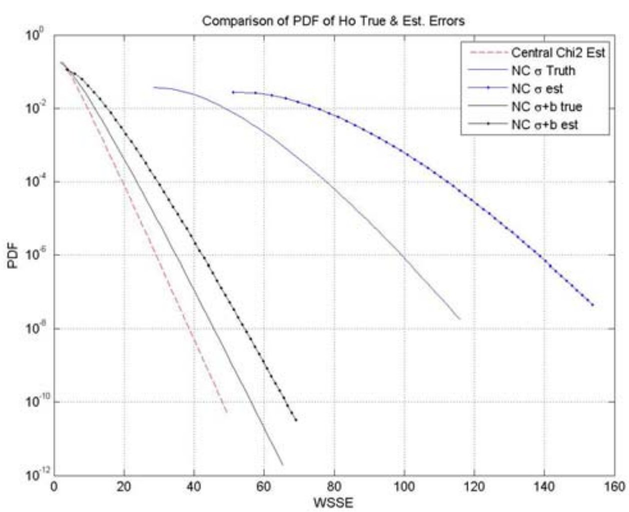

Figure 12. Comparison of the PDF of the No Fault True & Estimated WSSE Distribution (Estimated distrib. chooses median *ncp* combination of bias), NC = Noncentral

Figure 13 shows the true and estimated fault distributions for both weightings. Again, the plots show the estimated distributions calculated using the median *ncp*. As there are multiple faulted cases, the plot is that of the average distribution from all faulted cases weighted by fault probability of each case. It can be seen that even with the use of a median *ncp*, the inverse covariance weighting, the lower tail is overbounded in this case. In the case of the inverse σ+b weighting, it is difficult to tell whether the tail is overbounded. However, since the faulted distribution is reasonably close to that of the no fault distribution, the resulting estimated P_{WC} will indicate inadequate cycle confidence.

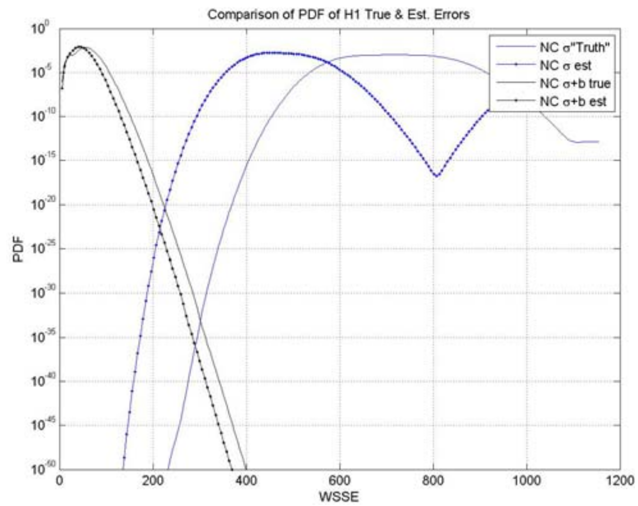

Figure 13. Comparison of the PDF of the Faulted True & Estimated WSSE Distribution (Estimated distributions use *ncp* from no fault case)

CONCLUSIONS

For Loran to serve as a primary navigation aid for aviation, it must demonstrate that it can meet the integrity requirements of that application. Cycle determination and confidence is critical for integrity as an undetected error leads to an undetected range error of three kilometers, resulting in a loss of integrity. This paper demonstrated a cycle confidence algorithm that has integrity. It also demonstrated that the algorithm can be implemented. The demonstration of the applicability of the χ^2 distribution allowed the algorithm to be both mathematically tractable (hence implementable) and help with the integrity proof. Additionally, it was shown that the bounding distribution on the no fault and faulted cases can be generated using a priori knowledge (nominal biases limits) and simple calculations (assessment over all set of signs on the limits).

In solving this issue, we examined the use of the WSSE to better understand when χ^2 assumption holds. The paper showed that WSSE is χ^2 distributed provided weighting is the inverse of the true covariance matrix. The result was used to develop a conservative estimate of missed detection. It was also used to provide integrity to the cycle confidence calculation in the form of a conservative estimate of the probability of having an incorrect cycle.

FUTURE WORK

Additional work still needs to be conducted both for availability and implementation. For example, the availability using the algorithm still needs to be explored though other work using the algorithm has shown it can provide reasonable availability [9].

ACKNOWLEDGMENTS

The authors would like to thank Mitch Narins of the FAA, Loran Program Office for supporting this work. We would also like to acknowledge the Thomas Gunther and the rest of the Loran Evaluation Team for their inputs.

REFERENCES

[1] FAA report to FAA Vice President for Technical Operations Navigation Services Directorate, "Loran's Capability to Mitigate the Impact of a GPS Outage on GPS Position, Navigation, and Time Applications," March 2004.

[2] Lo, Sherman, et al., "Loran Integrity Analysis for Required Navigation Performance 0.3", Proceedings of the 5th International Symposium on Integration of LORAN-C/Eurofix and EGNOS/Galileo, Munich, Germany, June 2004.

[3] Peterson, B. et. al., "Hazardously Misleading Information Analysis for Loran LNAV," Proceedings of the International Symposium on Integration of LORAN-C/Eurofix and EGNOS/Galileo, DGON, June 2002.

[4] Brown, R. Grover and Chin, Gerald Y., "GPS RAIM: Calculation of Threshold and Protection Radius Using Chi-Square Methods – A Geometric Approach," *Global Positioning System: Papers Published in Navigation*, Volume V, Institute of Navigation, Alexandria, VA, 1993

[5] Parkinson, B. W., and Axelrad, P., "Autonomous GPS Integrity Monitoring Using Pseudorange Residual," *Navigation: The Journal of the Institute of Navigation*, Vol. 35 No. 2, 1988

[6] Walter, Todd, and Enge, Per, "Weighted RAIM for Precision Approach," Proceedings of the Institute of Navigation GPS Conference, September 12-15, 1995, Palm Springs, California

[7] Boyce, Lee, Lo, Sherman, Powell, J. D., and Enge, Per, "Atmospheric Noise: Data Collection and Analysis", Proceedings of the International Loran Association 35th Annual Meeting, Groton, CT, October 2006

[8] DeCleene, Bruce, "Defining Pseudorange Integrity – Overbounding", Proceedings of the Institute of Navigation GPS Conference, Salt Lake City, UT September 2000

[9] Lo, Sherman, Peterson, Benjamin, and Enge, Per, "Assessing the Loran Cycle Confidence Algorithm", Proceedings of the International Loran Association 35th Annual Meeting, Groton, CT, October 2006

APPENDIX A: PROOF OF CONDITIONS WHEN WSSE IS χ^2 DISTRIBUTED

Lemma 1: The WSSE is χ^2 distributed provided that $W = R^{-1}$. R is the true covariance matrix of the random (non bias) component of the measurement errors, ε.

Proof: Assume $\varepsilon = \left[\varepsilon_1 \cdots \varepsilon_N\right]^T$ where $\varepsilon_i \sim N\left(b_i, \sigma_i\right)$ are independent. Let vector s of N independent normal random variables with unit variance can be found such that ε and $R^{1/2}s$ are identically distributed. This means that the two are equivalent with equivalence between two statistics meaning they have the same distribution

Note that since R is a covariance matrix, it is symmetric and positive definite, it has a square root $R^{1/2}$ which is also symmetric[1]. If ε are zero mean, then s is a vector of N independent identically distributed (iid) standard normal random variables. So the WSSE can be rewritten from Equation (6) to Equation (7).

$$WSSE = \hat{\varepsilon}^T W \hat{\varepsilon} = \varepsilon^T W \left(I - P\right)\varepsilon = \varepsilon^T W \left(I - G\left(G^T W G\right)^{-1} G^T W\right)\varepsilon = \varepsilon^T M \varepsilon \quad (6)$$

$$WSSE = \varepsilon^T M \varepsilon \sim s^T \left(R^{1/2}\right)^T M R^{1/2} s = s^T A s \quad (7)$$

where $P = G\left(G^T W G\right)^{-1} G^T W$ and $M = W\left(I - P\right)$

It can be shown that A is symmetric if W is symmetric.

$$A = R^{\frac{1}{2}} W \left(I - G\left(G^T W G\right)^{-1} G^T W\right) R^{\frac{1}{2}}$$

$$A^T = R^{\frac{1}{2}} \left(I - G\left(G^T W G\right)^{-1} G^T W\right)^T W^T R^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}} \left(I - \left(G\left(G^T W G\right)^{-1} G^T W\right)^T\right) W R^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}} \left(I - \left(W G\left(G^T W G\right)^{-1}\right)\right) W R^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}} W \left(I - G\left(G^T W G\right)^{-1} G^T W\right) R^{\frac{1}{2}} = A$$

M is symmetric by a similar argument.

---

[1] It is assume that all measurement errors ε have non zero variance.

From lemma, A can be written in terms of an orthogonal projection matrix, C, and a NxN diagonal matrix, Λ, as seen in Equation (8). Since the rank of A is at most $N\text{-}dim(x)$, Λ has at most $N\text{-}dim(x)$ non zero eigenvalues. In the case of Loran, $dim(x) = 3$.

$$s^T As = s^T C^T \Lambda Cs \tag{8}$$

Since C is an orthogonal matrix, $Cs \sim s$ (i.e. they have equivalent distributions). The result is that, if W is symmetric, the WSSE is distributed as the sum of the squares of $N\text{-}3$ normal random variables with variance equaled to the non zero eigenvalues of Λ.

$$WSSE \sim s^T C^T \Lambda Cs \sim s^T \Lambda s \tag{9}$$

However, if A is idempotent ($A=A*A$), then, from lemma, the $N\text{-}3$ eigenvalues are all ones. Define this matrix as Λ_I. It can be shown that if $W = R^{-1}$, then A is idempotent. This definition also satisfies W being symmetric. First show A is idempotent.

$$A = R^{\frac{1}{2}} R^{-1} \left(I - G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} \right) R^{\frac{1}{2}}$$

$$A * A = R^{\frac{1}{2}} R^{-1} \left(I - G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} \right)\left(I - G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} \right) R^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}} R^{-1} \left(I - 2G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} + G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} \right) R^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}} R^{-1} \left(I - G\left(G^T R^{-1} G\right)^{-1} G^T R^{-1} \right) R^{\frac{1}{2}} = A$$

If A idempotent and symmetric, then

$$\Lambda = \Lambda_I \equiv \begin{bmatrix} I_{(N-3)x(N-3)} & 0_{(N-3)x3} \\ 0_{3x(N-3)} & 0_{3x3} \end{bmatrix}$$

The result is that WSSE is distributed as the sum of the squares of $N\text{-}3$ standard normal random variables. This is the definition of a χ^2 distribution with $N\text{-}3$ degrees of freedom. If ε is zero mean, then the WSSE is a central χ^2 distribution with $N\text{-}3$ *dof*. If ε has a bias, then the WSSE is a noncentral χ^2 distribution with N-3 *dof* and noncentrality parameter, *ncp*, given by Equation (10)

$$ncp = b^T Mb \tag{10}$$

Thus it is demonstrated that WSSE is χ^2 if $W = R^{-1}$. Let σ weighting denote the case where $W = W_\sigma =$ inverse of covariance matrix based on σ. If the random error statistics are exactly known, then $W_\sigma = R^{-1}$. If the random error statistics used are reasonably close to the true statistics, then, for σ weighting, $W_\sigma \sim R^{-1}$. And so the eigenvalues of A should be close to 1. This implies that the WSSE should be well modeled by a χ^2 distribution if the error statistics are close.

APPENDIX B: USING THE NONCENTRALITY PARAMETER FOR BOUNDING THE CDF

Lemma 1: Assume any two χ^2 distribution with the same degree of freedom, υ, and noncentrality parameter, ncp_1 and ncp_2, respectively. Denote the distribution as $\chi^2(\upsilon, ncp_1)$ and $\chi^2(\upsilon, ncp_2)$, respectively. Without loss of generality, assume $ncp_1 \le ncp_2$. It can be shown that:

$$CDF_{\chi^2(\upsilon,ncp_1)}(x) \ge CDF_{\chi^2(\upsilon,ncp_2)}(x) \tag{11}$$

We can construct the distributions as follows:

$$\chi^2(\upsilon, ncp_1) = \sum_{n=1}^{\upsilon-1} \tilde{y}_n^2 + \tilde{x}_1^2$$

$$\chi^2(\upsilon, ncp_2) = \sum_{n=1}^{\upsilon-1} \tilde{z}_n^2 + \tilde{x}_2^2$$

$$\tilde{y}_n \sim N(0,1), \tilde{z}_n \sim N(0,1), \forall n \in (1,...,\upsilon-1),$$

independent

$$\tilde{x}_1 \sim N\left(\sqrt{ncp_1}, 1\right), \tilde{x}_2 \sim N\left(\sqrt{ncp_2}, 1\right)$$

where $\sqrt{ncp_2}, \sqrt{ncp_1} \ge 0$

Since the part of the each of the χ^2 distributions derived from the zero mean normal random variables are identical, we only need to consider the contribution of the non zero mean component (and the convolution of that portion with the zero mean contribution).

$$CDF_{\chi^2(\upsilon,ncp)}(x) = \int_0^x PDF_{\chi^2(\upsilon-1,0)}(z) * CDF_{\chi^2(1,ncp)}(x-z)dz$$

Hence we can write

$$CDF_{\chi^2(\upsilon,ncp_1)}(x) - CDF_{\chi^2(\upsilon,ncp_2)}(x)$$

$$= \int_0^x PDF_{\chi^2(\upsilon-1,0)}(z) * \left[CDF_{\chi^2(1,ncp_1)}(x-z) - CDF_{\chi^2(1,ncp_2)}(x-z) \right] dz$$

So showing Equation (11) means showing

$$CDF_{\chi^2(1,ncp_1)}(x) \ge CDF_{\chi^2(1,ncp_2)}(x), \forall x \tag{12}$$

First start with the definition of

$$CDF_{\chi^2(1,ncp)}(y) = \int_{-\sqrt{y}}^{\sqrt{y}} N\left(\sqrt{ncp}, 1\right) dx = CDF_{N\left(\sqrt{ncp},1\right)}\left(\sqrt{y}\right) - CDF_{N\left(\sqrt{ncp},1\right)}\left(-\sqrt{y}\right)$$

Note that since we are dealing with a normal distribution, $CDF_{N(\mu,\sigma)}(x) = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$ where $erf(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$

We will be using a couple properties of the erf function:

- it is monotonically increasing in x.
- it is an odd function

Rewriting Equation (12) using the erf function, we get

$$CDF_{\chi^2(1,ncp_1)}(y) - CDF_{\chi^2(1,ncp_2)}(y) = CDF_{N(\sqrt{ncp_1},1)}(\sqrt{y}) - CDF_{N(\sqrt{ncp_2},1)}(\sqrt{y})$$
$$+CDF_{N(\sqrt{ncp_2},1)}(-\sqrt{y}) - CDF_{N(\sqrt{ncp_1},1)}(-\sqrt{y})$$
$$CDF_{N(\sqrt{ncp_1},1)}(\sqrt{y}) - CDF_{N(\sqrt{ncp_2},1)}(\sqrt{y}) = \frac{1}{2}\left[erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right)\right]$$
$$CDF_{N(\sqrt{ncp_2},1)}(-\sqrt{y}) - CDF_{N(\sqrt{ncp_1},1)}(-\sqrt{y}) = \frac{1}{2}\left[erf\left(\frac{-\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right) - erf\left(\frac{-\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right)\right]$$
$$= \frac{1}{2}\left[erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right)\right]$$

The last equality comes from erf being an odd function

$$CDF_{\chi^2(1,ncp_1)}(y) - CDF_{\chi^2(1,ncp_2)}(y) = \tag{13}$$
$$\frac{1}{2}\left[erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) + erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right) - \left(erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right) + erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right)\right)\right]$$

Now let's define

$$\left(\sqrt{y} - \sqrt{ncp_1}\right) - \left(\sqrt{y} - \sqrt{ncp_2}\right) = \Delta rncp = \sqrt{ncp_2} - \sqrt{ncp_1} \geq 0$$
$$\left(\sqrt{y} + \sqrt{ncp_2}\right) - \left(\sqrt{y} + \sqrt{ncp_1}\right) = \Delta rncp = \sqrt{ncp_2} - \sqrt{ncp_1} \geq 0$$
$$\mu_{rncp} = \frac{\sqrt{ncp_2}+\sqrt{ncp_1}}{2} \geq 0$$

If we take the derivative of the *erf* function, it is noted that it is monotonically decreasing in *x* for *x* >0. This can be seen if we take its derivative

$$\frac{d}{dx}erf(x) = \frac{2}{\pi}e^{-x^2}$$

So demonstrating (12) means showing the following is true.

$$erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right) \geq \left(erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right)\right)$$

We can show this is true by using the derivative

$$\frac{erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right)}{\Delta rncp} \approx \frac{2}{\pi}e^{-\left(\sqrt{y}-\mu_{rncp}\right)^2}$$
$$erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right) \approx \frac{2\Delta rncp}{\pi}e^{-\left(\sqrt{y}-\mu_{rncp}\right)^2}$$
$$\frac{erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right)}{\Delta rncp} \approx \frac{2}{\pi}e^{-\left(\sqrt{y}+\mu_{rncp}\right)^2}$$
$$erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right) \approx \frac{2\Delta rncp}{\pi}e^{-\left(\sqrt{y}+\mu_{rncp}\right)^2}$$

Note that $\frac{2\Delta rncp}{\pi}e^{-\left(\sqrt{y}-\mu_{rncp}\right)^2} \geq \frac{2\Delta rncp}{\pi}e^{-\left(\sqrt{y}+\mu_{rncp}\right)^2}$ since $\sqrt{y} - \mu_{rncp} \geq \sqrt{y} + \mu_{rncp}$

Hence we get $erf\left(\frac{\sqrt{y}-\sqrt{ncp_1}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}-\sqrt{ncp_2}}{\sqrt{2}}\right) \geq \left(erf\left(\frac{\sqrt{y}+\sqrt{ncp_2}}{\sqrt{2}}\right) - erf\left(\frac{\sqrt{y}+\sqrt{ncp_1}}{\sqrt{2}}\right)\right)$.

Another way of deriving the result is starting from Equation (13), we get

$$CDF_{\chi^2_{(1,ncp1)}}(y) - CDF_{\chi^2_{(1,ncp2)}}(y)$$
$$= \frac{1}{\sqrt{\pi}}\left[\int_{\left(\frac{\sqrt{y}-\sqrt{ncp2}}{\sqrt{2}}\right)}^{\left(\frac{\sqrt{y}-\sqrt{ncp1}}{\sqrt{2}}\right)} e^{-t^2}dt - \int_{\left(\frac{\sqrt{y}+\sqrt{ncp1}}{\sqrt{2}}\right)}^{\left(\frac{\sqrt{y}+\sqrt{ncp2}}{\sqrt{2}}\right)} e^{-t^2}dt\right]$$
$$= \frac{1}{\sqrt{\pi}}\left[\int_0^{\left(\frac{\sqrt{ncp2}-\sqrt{ncp1}}{\sqrt{2}}\right)} \left(e^{-\left(\frac{\sqrt{y}-\sqrt{ncp2}}{\sqrt{2}}+t\right)^2} - e^{-\left(\frac{\sqrt{y}+\sqrt{ncp1}}{\sqrt{2}}+t\right)^2}\right)dt\right] \geq 0$$

since

$$\int_0^{\frac{\left(\sqrt{ncp2}-\sqrt{ncp1}\right)}{\sqrt{2}}} \left(e^{-\left(\frac{\sqrt{x}-\sqrt{ncp2}}{\sqrt{2}}+t\right)^2} - e^{-\left(\frac{\sqrt{x}+\sqrt{ncp1}}{\sqrt{2}}+t\right)^2}\right)dt \geq 0, \sqrt{ncp_2} \geq \sqrt{ncp_1} \geq 0, x > 0$$

and $\left(\frac{\sqrt{x}-\sqrt{ncp2}}{\sqrt{2}}+t\right)^2 \leq \left(\frac{\sqrt{x}+\sqrt{ncp1}}{\sqrt{2}}+t\right)^2$ so $e^{-\left(\frac{\sqrt{x}-\sqrt{ncp2}}{\sqrt{2}}+t\right)^2} \geq e^{-\left(\frac{\sqrt{x}+\sqrt{ncp1}}{\sqrt{2}}+t\right)^2}$

Therefore, $CDF_{\chi^2(1,ncp_1)}(x) - CDF_{\chi^2(1,ncp_2)}(x) \geq 0, \forall x$.

As a result, $CDF_{\chi^2(\upsilon,ncp_1)}(x) \geq CDF_{\chi^2(\upsilon,ncp_2)}(x), \forall x$

APPENDIX C: BIAS BOUNDS FOR CALCULATING BOUNDING NONCENTRALITY PARAMETER

Lemma 1: If there exist bounds for the true biases, i.e., we have $B_i > 0$ such that $B_i \geq |b_i|$ for all signals *i*, then a noncentrality parameter $ncp_{0,E}$ that is larger than the true noncentrality parameter, $ncp_{0,T}$ exists and can be computed using the correct choice of signs, s_i, for the bounding biases.

Proof: Denote

$$\bar{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$$

$$\bar{B} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}^T$$

Where $\beta_i = s_i B_i$

$$ncp_{0,E} = \bar{B}^T M \bar{B}, ncp_{0,T} = \bar{b}^T M \bar{b}$$

$$ncp_{0,E} = \bar{B}^T M \bar{B} \geq ncp_{0,T} = \bar{b}^T M \bar{b}$$

$$\bar{B}^T M \bar{B} - \bar{b}^T M \bar{b} \geq 0$$

where $P = G\left(G^T W G\right)^{-1} G^T W$ and $M = W\left(I - P\right)$

Note: $ncp_{0,E}$ represents the estimated *ncp* on the no fault (H_0) hypothesis while $ncp_{0,T}$ represents the true *ncp* on H_0.

$$\bar{B} - \bar{b} = \bar{\Delta}$$

$$\bar{B}^T M \bar{B} = \left(\bar{b} + \bar{\Delta}\right)^T M \left(\bar{b} + \bar{\Delta}\right)$$

$$= \bar{b}^T M \bar{b} + \bar{b}^T M \bar{\Delta} + \bar{\Delta}^T M \bar{b} + \bar{\Delta}^T M \bar{\Delta}$$

$$\bar{B}^T M \bar{B} - \bar{b}^T M \bar{b} = \bar{b}^T M \bar{\Delta} + \bar{\Delta}^T M \bar{b} + \bar{\Delta}^T M \bar{\Delta}$$

$\bar{\Delta}^T M \bar{\Delta} \geq 0$ since *M* is positive semi definite

$R^{\frac{1}{2}} M R^{\frac{1}{2}} = A = C^T \Lambda C$ see Appendix A

$M = R^{-\frac{1}{2}} C^T \Lambda C R^{-\frac{1}{2}}$, $R^{1/2}$ is invertible

Let $d = C R^{-\frac{1}{2}} b$, so $b^T M b = d^T \Lambda d \geq 0$

So *M* is positive semi-definite since Λ is positive semi-definite.

$\bar{b}^T M \bar{\Delta} = \bar{\Delta}^T M \bar{b}$ since *M* is symmetric

So we only to show that $\bar{b}^T M \bar{\Delta} \geq 0$ to show that $ncp_{0,E} = \bar{B}^T M \bar{B} \geq \bar{b}^T M \bar{b} = ncp_{0,T}$

Expanding $\bar{b}^T M \bar{\Delta}$, we get

$$\bar{b}^T M \bar{\Delta} = \Delta_1 \left(m_{11} b_1 + \cdots + m_{n1} b_n\right)$$
$$+ \Delta_2 \left(m_{12} b_1 + \cdots + m_{n2} b_n\right) + \cdots + \Delta_n \left(m_{1n} b_1 + \cdots + m_{nn} b_n\right)$$

since $B_i \geq |b_i|$, we can choose the sign of Δ_i. Ideally, if we knew b_i, we would choose the sign of Δ_i such that

$$\Delta_i \begin{cases} \geq 0 \text{ if } \left(m_{1i} b_1 + \cdots + m_{ni} b_n\right) \geq 0 \\ \leq 0 \text{ if } \left(m_{1i} b_1 + \cdots + m_{ni} b_n\right) < 0 \end{cases}$$

Such a selection would insure that all terms $\Delta_i \left(m_{1i} b_1 + \cdots + m_{ni} b_n\right)$ are non negative and hence $\bar{b}^T M \bar{\Delta} \geq 0$. This is possible by selecting sign s_i.

$$\Delta_i = \left(s_i \beta_i - b_i\right) \begin{cases} \geq 0 \text{ if } s_i = +1 \\ \leq 0 \text{ if } s_i = -1 \end{cases}$$

Thus there exist at least one set of signs that yields the desired result. It turns out that we do not need to know b_i. Since the *ncp* for that case must be less than or equaled to max($ncp_{0,E}$) over all possible sign combinations s_i. The results means that the max($ncp_{0,E}$) over all signs is greater than or equal to the $ncp_{0,T}$.

Lemma 2: If there exist bounds for the true biases, then there exists a choice of signs for bounding bias such that the calculated difference between the faulted and no fault noncentrality parameter, $\Delta ncp_E = ncp_{1,E} - ncp_{0,E}$, is smaller than true difference $\Delta ncp_T = ncp_{1,T} - ncp_{0,T}$. This can be shown for the one and two fault case.

Proof:

For H_0,

$$ncp_{0,T} = \sum_{i=1}^{m} m_{ii} b_i^2 + 2\sum_{i=1}^{m} \sum_{j \geq i}^{m} m_{ij} b_i b_j$$

$$ncp_{0,E} = \sum_{i=1}^{m} m_{ii} B_i^2 + 2\sum_{i=1}^{m} \sum_{j \geq i}^{m} m_{ij} B_i B_j$$

Examine the one fault case and assume the cycle error (λ) is on signal *k*.

$$ncp_{1,T} = \sum_{i=1,i\neq k}^{m} m_{ii}b_i^2 + m_{kk}\left(b_k+\lambda\right)^2$$

$$+2\sum_{i=1,i\neq k}^{m}\sum_{j\geq i,j\neq k}^{m} m_{ij}b_ib_j + 2\sum_{j\neq k}^{m} m_{kj}\left(b_k+\lambda\right)b_j$$

$$ncp_{1,E} = \sum_{i=1,i\neq k}^{m} m_{ii}B_i^2 + m_{kk}\left(B_k+\lambda\right)^2$$

$$+2\sum_{i=1,i\neq k}^{m}\sum_{j\geq i,j\neq k}^{m} m_{ij}B_iB_j + 2\sum_{j\neq k}^{m} m_{kj}\left(B_k+\lambda\right)B_j$$

For the case of up to two faults, we have:

$$ncp_{1,T} = \sum_{i=1,i\neq k,l}^{m} m_{ii}b_i^2 + m_{kk}\left(b_k+\lambda\right)^2 + m_{ll}\left(b_l+\lambda\right)^2$$

$$+2\sum_{i=1,i\neq k,l}^{m}\sum_{j\geq i,j\neq k,l}^{m} m_{ij}b_ib_j + 2\sum_{j\neq k,l}^{m}\left[m_{kj}\left(b_k+\lambda\right)b_j + m_{lj}\left(b_l+\lambda\right)b_j\right]$$

$$+2m_{kl}\left(b_k+\lambda\right)\left(b_l+\lambda\right)$$

$$ncp_{1,E} = \sum_{i=1,i\neq k,l}^{m} m_{ii}B_i^2 + m_{kk}\left(B_k+\lambda\right)^2 + m_{ll}\left(B_l+\lambda\right)^2$$

$$+2\sum_{i=1,i\neq k,l}^{m}\sum_{j\geq i,j\neq k,l}^{m} m_{ij}B_iB_j + 2\sum_{j\neq k,l}^{m}\left[m_{kj}\left(B_k+\lambda\right)B_j + m_{lj}\left(B_l+\lambda\right)B_j\right]$$

$$+2m_{kl}\left(B_k+\lambda\right)\left(B_l+\lambda\right)$$

1) Examine the difference in *ncp* when there is only one fault.

$$\Delta ncp_T = m_{kk}\left(2b_k\lambda+\lambda^2\right) + 2\sum_{j=1,j\neq k}^{m} m_{kj}\lambda b_j$$

$$\Delta ncp_E = m_{kk}\left(2B_k\lambda+\lambda^2\right) + 2\sum_{j=1,j\neq k}^{m} m_{kj}\lambda B_j$$

$$\Delta\lambda_T - \Delta\lambda_E = m_{kk}\left(2b_k-2B_k\right)\lambda + 2\lambda\sum_{j=1,j\neq k}^{m} m_{kj}\left(b_j-B_j\right) = 2\lambda\sum_{j=1}^{m} m_{kj}\left(b_j-B_j\right)$$

Choose signs for each element such that

$$B_i = \begin{cases} +\beta_i & \text{if } m_{ki} < 0 \\ -\beta_i & \text{if } m_{ki} \geq 0 \end{cases}$$

since $\left|B_i\right| \geq \left|b_i\right|$, that results in $\Delta ncp_T - \Delta ncp_E > 0$

If cycle error is –λ, then

$$\Delta ncp_T - \Delta ncp_E = -2\lambda\sum_{j=1}^{m} m_{kj}\left(b_j-B_j\right)$$

In this case, choose signs for each element such that

$$B_i = \begin{cases} +\beta_i & \text{if } m_{ki} > 0 \\ -\beta_i & \text{if } m_{ki} \leq 0 \end{cases}$$

2) Examine the difference in *ncp* when there is one or two faults.

$$\Delta ncp_T = m_{kk}\left(2b_k\lambda+\lambda^2\right) + m_{ll}\left(2b_l\lambda+\lambda^2\right)$$

$$+2\sum_{j=1,j\neq k,l}^{m}\left(m_{kj}+m_{lj}\right)\lambda b_j + 2m_{kl}\left(b_kb_l+\left(b_k+b_l\right)\lambda+\lambda^2\right)$$

$$\Delta ncp_E = m_{kk}\left(2B_k\lambda+\lambda^2\right) + m_{ll}\left(2B_l\lambda+\lambda^2\right)$$

$$+2\sum_{j=1,j\neq k,l}^{m}\left(m_{kj}+m_{lj}\right)\lambda B_j + 2m_{kl}\left(B_kB_l+\left(B_k+B_l\right)\lambda+\lambda^2\right)$$

$$\Delta ncp_T - \Delta ncp_E = 2\lambda m_{kk}\left(2b_k-2B_k\right) + 2\lambda m_{ll}\left(2b_l-2B_l\right)$$

$$+2\lambda\sum_{j=1,j\neq k,l}^{m} m_{kj}\left(b_j-B_j\right) + 2\lambda\sum_{j=1,j\neq k,l}^{m} m_{lj}\left(b_j-B_j\right)$$

$$+2\lambda m_{kl}\left(b_k-B_k\right) + 2\lambda m_{kl}\left(b_l-B_l\right)$$

$$\Delta ncp_T - \Delta ncp_E = 2\lambda\sum_{j=1}^{m}\left(m_{kj}+m_{lj}\right)\left(b_j-B_j\right)$$

Apply the rule signs for each element such that

$$B_i = \begin{cases} +\beta_i & \text{if } \left(m_{ki}+m_{li}\right) < 0 \\ -\beta_i & \text{if } \left(m_{ki}+m_{li}\right) \geq 0 \end{cases}$$

If both cycle errors are –λ, then

$$\Delta ncp_T - \Delta ncp_E = -2\lambda\sum_{j=1}^{m}\left(m_{kj}+m_{lj}\right)\left(b_j-B_j\right)$$

Without loss of generality, assume different signs for errors. Signal *k* has +λ and l has –λ cycle error.

$$\Delta ncp_T - \Delta ncp_E = 2\lambda\sum_{j=1}^{m}\left(m_{kj}-m_{lj}\right)\left(b_j-B_j\right)$$

$$B_i = \begin{cases} +\beta_i & \text{if } \left(m_{ki}-m_{li}\right) < 0 \\ -\beta_i & \text{if } \left(m_{ki}-m_{li}\right) \geq 0 \end{cases}$$

The result is that there does exist a sign choice where $\Delta ncp_T - \Delta ncp_E > 0$. The same argument can be used on cases where more faults exist.