

BOUNDING TEMPORALLY CORRELATED MEASUREMENT NOISE WITH AN APPLICATION TO GNSS INTEGRITY

Juan Blanch, Eric Phelts, Kaz Gunning, Todd Walter

Stanford University

Lance de Groot, Laura Norman

Hexagon Autonomy & Positioning

ABSTRACT

In GNSS techniques like Precise Point Positioning (PPP), many of the errors affecting the position solution are modelled and estimated over time. In order to use PPP or related techniques for applications that require integrity, we need models of the errors that will lead to upper bounds of the estimated error covariance. In this work, we use the properties of the power spectral density to develop error models that bound autoregressive models with uncertain parameters. As an example, we show how this approach can be applied to the determination of a first order model that bounds the clock and ephemeris errors of GPS for multiple satellites.

INTRODUCTION

Most safety-of-life systems based GNSS like SBAS, GBAS, or RAIM, and even the future Advanced RAIM use simple snapshot estimators. These estimators can be analyzed assuming that they only use measurements from the current epoch. As a consequence, the integrity of the instantaneous position solution is not affected by the temporal correlation of the measurement errors, or more precisely, when it is, it follows a simple monotonic relationship. This simplifies considerably the integrity analysis of these systems, but it also means that the temporal structure of the errors cannot be exploited to improve performance. The situation changes when we consider filtered solutions, and Kalman filters in particular.

In filtered solutions of the type used in Precise Point Positioning (PPP) [1], [2], [3], [4], [5], [6], [7], the errors affecting the position solution are modelled and estimated over time, so the temporal structure of these errors is key. These techniques are very attractive because they have the potential of improving performance very significantly for airborne applications [4], [5], [6], [7], and of providing meter level protection levels for automotive applications [2], [3]. These improvements rely mostly on the temporal structure of the errors, it is therefore very important to develop temporal error models that will result in integrity error bounds (protection levels (PL)) that bound the actual errors to the required probability.

Estimation under model uncertainty has been studied for more than three decades. However, it has proven challenging to find results that are practical and adapted to the integrity problem [8]. In particular, the proposed solutions are often reliant on techniques that are not yet standard, at least not for real time applications (this includes linear programming, quadratic programming, etc). For our purpose, one of the goals is to provide measurement noise characterizations that allow the use of standard Kalman filtering techniques, or at least with minimum modifications. An example of the kind of methods that are well adapted to our objective is given by [8], [9], because it provides a simple and practical bound for uncertain

first order Gauss-Markov processes with unknown, but bounded time constants. In this work, we will make use the properties of the power spectral density as it appears to be a very practical representation for stationary processes. In GNSS applications, this method has been used at least in [10], [18], [19], [20], [21]. It is based on the fact that a function is semidefinite positive (in the sense that the associated quadratic form is semidefinite positive) if and only if its Fourier transform is positive [11], [12]. This result, which is a very narrow form of Bochner's Theorem [13], has been extensively used in the analysis of stationary time series, and in spatial statistics [14], because it provides a relatively simple rule to ensure that a given form of the autocovariance function of the process is valid (in the sense that it defines a positive covariance matrix). Here we use this result to ensure that a certain temporal process characterization bounds one that is either not entirely known (but with bounded parameters), or described by an empirical autocovariance function that is difficult to parameterize.

In the first part we will review the approach. In the second part, we will derive simple conditions under which the autocovariance of one process bounds another one for some common processes (like a Gauss-Markov process). In the third part, and to evaluate the viability of this method we apply it to the (very preliminary) determination of a first order model that bounds the GPS clock and ephemeris errors.

USING THE PSD TO DERIVE BOUNDING CONDITIONS

This method is based on the fact that a function is semidefinite positive (in the sense that the associated quadratic form is semidefinite positive) if and only if its Fourier transform is positive [11], [12]. This result, which is a very narrow form of Bochner's Theorem [13], has been extensively used in the analysis of stationary time series, and in spatial statistics [14], because it provides a relatively simple rule to ensure that a given form of the autocovariance function of the process is valid (in the sense that it defines a positive covariance matrix). Here we use this result to ensure that a certain temporal process characterization bounds one that is either not entirely known (but with bounded parameters), or described by an empirical autocovariance function that is difficult to parameterize.

Propagation and observation equations for Kalman filter estimation

We consider the following propagation and observation equations:

$$\begin{aligned}x_{k+1} &= F_k x_k + w_k \\ y_k &= H_k x_k + v_k\end{aligned}\tag{14}$$

where:

x_k is the state

y_k is the observation

w_k is the process noise

v_k is the measurement noise

To determine the Kalman filter equations, we need to assign a covariance to w_k and v_k . Let us assume that:

$$w_k \sim N(0, Q_k)$$

$$v_k \sim N(0, R_k)$$

To ensure that the covariance estimate is conservative, it is necessary to use bounds on the two matrices Q_k and R_k . We will assume that we can determine two matrices such that:

$$Q_k \preceq \bar{Q}_k$$

$$R_k \preceq \bar{R}_k$$

This is however not sufficient. For the Kalman covariance estimate to be conservative, we also need:

$$\text{cov}(w_k, w_{k'}) = 0$$

$$\text{cov}(v_k, v_{k'}) = 0 \text{ for } k < k'$$

This is in general not true. The question therefore is how to model the error to account for a possible temporal correlation.

State augmentation

If the errors are correlated over time, but the structure is known and can be described as a sum of autoregressive models, we can account for the correlation using state augmentation (which consists of writing the recursive equation defining the correlation and augmenting the original state with the measurement state). This will be exploited later on in the report.

As pointed out in [23], it is also possible to account for the (known) correlated noise using a measurement-differencing filter.

Normalization

We now assume that the temporal correlation is not exactly known. To go further, we change the notations as follows:

$$w_k = A_k e_k$$

$$v_k = B_k e_k$$

where A and B are designed such that the covariance of e_k is the identity. This can be easily achieved when we assume that there is no correlation across measurements from different types (code and carrier, for example) and from different satellites.

Equivalence between Kalman filter and batch approach

This step is only necessary to develop the method. It does not imply that we will need to use a batch approach. It is well known that the Kalman filter estimate at the most recent time step is strictly equivalent to a batch estimate. That is, we can write the system of equations (14) from time 1 to n as follows:

$$\begin{bmatrix} y_0 \\ \vdots \\ y_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Gamma \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} + \Phi \begin{bmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_n \end{bmatrix} \quad (15)$$

where the matrices Γ and Φ are formed with H_k , F_k , A_k , and B_k . We label C the covariance of ε (C is unknown). We have $\varepsilon \sim N(0, C)$.

Bounding condition

The condition to have bounded noise is that our model covariance Σ is an upper bound of C :

$$C \preceq \Sigma \quad (16)$$

(in the semi-definite positive sense). The next step consists in re-ordering the indices in order to group all the measurements from the same type and satellite in the same block. Because of our assumption of independence, this results in a block diagonal matrix:

$$C = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C_p \end{bmatrix}$$

where p is the number of measurement series (identified by satellite and type). The condition for positivity can then be expressed per measurement series as:

$$C_j \preceq \Sigma_j$$

Structure of covariance matrix for a stationary process

We will now drop the index corresponding to the measurement series, and add an index to indicate the size of the matrix (that is, how many time steps are being considered). For stationary processes, the covariance matrix $C^{(n)}$ has the following structure:

$$C^{(n)} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_n & \cdots & c_1 & c_0 \end{bmatrix} \quad (17)$$

that is, it is defined by the autocovariance series: $c_k = \text{cov}(e_{j+k}, e_j)$ (note again that we are only considering one measurement series here). For this report, we will restrict the search for the bounding matrix $S^{(n)}$ to stationary processes. Therefore, it will have a structure identical to $C^{(n)}$, so it is also defined by a series, which we will note σ_k .

Condition of semidefinite positivity of a series (t_n) using Fourier transform

Let us consider a series (t_n) . Under some convergence conditions (absolutely summable is sufficient), we can define its Fourier transform as follows:

$$f_{(t_n)}(\lambda) = \sum_{k=-\infty}^{+\infty} t_k e^{-ik\lambda}$$

Under some conditions on f (Riemann integrable [11]), we can recover the coefficients from the transform:

$$t_k = \frac{1}{2\pi} \int_0^{2\pi} f_{(t_n)}(\lambda) e^{-ik\lambda} d\lambda$$

This means that the series (t_n) is entirely defined by its Fourier transform. A key result is that the associated covariance matrix $T^{(n)}$ (defined like in (17)) is semi-definite positive for all n if and only if its Fourier transform is positive for all λ (except for a set of total length 0, which is a condition we should not be encountering) [11]. Because f is symmetric and periodic with period 2π , the condition can be written:

$$f_{(t_n)}(\rho) \geq 0 \text{ for } \rho \in [0, \rho] \text{ except for a set of total length 0}$$

Now, because the Fourier transform is a linear operation, the condition for $C^{(n)} \in S^{(n)}$ is given by:

$$f_{(c_n)}(\rho) \leq f_{(s_n)}(\rho) \text{ for } \rho \in [0, \rho] \text{ except for a set of total length 0}$$

We therefore need our overbound to be such its power spectral density (PSD) (the Fourier transform of the autocovariance) bounds the PSD of the actual error process.

Summary of the method

1. Obtain the autocovariance of the temporal series, or at least a set of constraints on it, based on analysis or empirically
2. Compute the Fourier transform of the autocovariance (or the family of possible ones), which is the power spectral density
3. Find a power spectral density corresponding to a process that can be easily modeled in a Kalman filter and that bounds the original one.
4. Use state augmentation to model the overbounding process

BOUNDING NOISE DEFINED BY A STATIONARY AUTOREGRESSIVE MODEL

In this section, we derive simple relationships that guarantee that a given autocovariance overbounds another one in terms of the process parameters, when the noise (either measurement or process) is defined by an autoregressive model of any order with uncertain parameters. This is an important case, because this the most common way of modeling error processes in in Kalman filters. In particular, we will look at the conditions under which the actual process noise can be bounded by white noise, and if so, how much we need to inflate the nominal covariance (the covariance of the noise at each epoch).

First order model

Let us consider a simple first order model:

$$e_k = ae_{k-1} + h_k \text{ with } \eta_k \sim N(0, \sigma^2(1-\alpha^2))$$

The PSD can be computed analytically. It is given by (note that we have switched to frequency):

$$S_{(x_n)}(e^{j2\pi f}) = S^2 \frac{1 - a^2}{1 - 2a \cos(2\pi f) + a^2}$$

Figure 1 shows the resulting curve for two values of α .

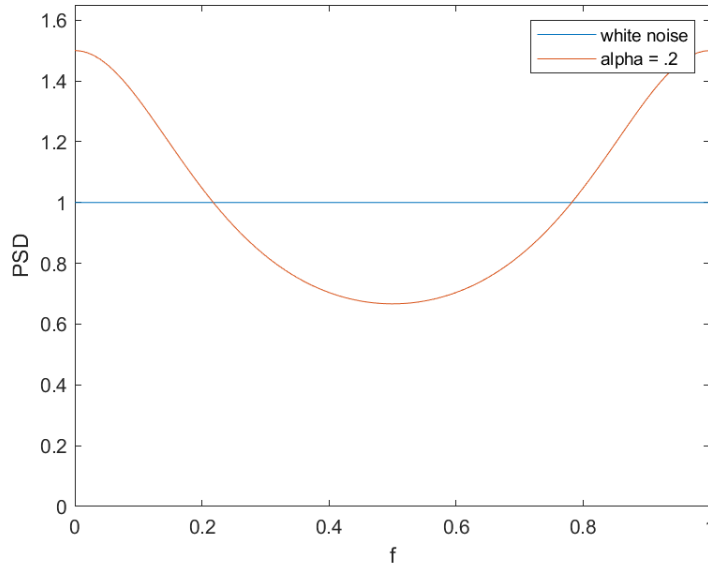


Figure 3. PSD for an AR(1) process

If, for example, we wish to bound an AR(1) process (or a class of them) by white noise, then we need to inflate the blue line (which represents the white noise) until it bounds all the points of the red curve. Since we have the PSD formula for both, we can derive the inflation factor. We need:

$$S_1^2 \frac{1 - a_1^2}{1 - 2a_1 \cos(2\pi f) + a_1^2} \leq S_2^2 \text{ for all values of } f$$

It is sufficient to bound at $f=0$. We get the simple condition:

$$S_1^2 \frac{1 + a_1}{1 - a_1} \leq S_2^2 \quad (18)$$

This very simple result provides a rule of thumb to inflate the white noise model when the error is actually temporally correlated and we have a bound on the correlation.

In some cases it might be better to assume some correlation (and therefore add states to the propagation and observation equations, as mentioned before), because it can result in smaller error bounds. In this case, the condition is:

$$S_1^2 \frac{1 - a_1^2}{1 - 2a_1 \cos(2\pi f) + a_1^2} \leq S_2^2 \frac{1 - a_2^2}{1 - 2a_2 \cos(2\pi f) + a_2^2} \text{ for all values of } f \quad (19)$$

These results are remarkable because they provide a method to order different error models.

Autoregressive model of any order p

Let us consider an autoregressive model of order p :

$$\varepsilon_k = \sum_{i=1}^p \alpha_i \varepsilon_{k-i} + \eta_k \quad (20)$$

The PSD of an autoregressive model of order p is given by:

$$S_{(\varepsilon_n)}(e^{j2\pi f}) = \frac{\sigma_\eta^2}{\left| 1 + \sum_{i=1}^p \alpha_i e^{-j2\pi f i} \right|} \quad (21)$$

Just as for the first order model, if we consider another autoregressive error process (defined by p' , α'_k , and σ'_η). A sufficient condition for this second error model to bound the first one is that:

$$\frac{\sigma_\eta^2}{\left| 1 + \sum_{i=1}^p \alpha_i e^{-j2\pi f i} \right|} \leq \frac{\sigma'^2_\eta}{\left| 1 + \sum_{i=1}^{p'} \alpha'_i e^{-j2\pi f i} \right|} \quad (22)$$

Bounding processes with unstructured temporal correlation

The same principle can be applied for an arbitrary stationary process with a known PSD. It is sufficient to find an autoregressive model (white noise, first order, etc) whose PSD bounds our target PSD. This seems easy in principle.

The problem here is that obtaining the PSD of a given process from data is not a trivial problem (this is a problem that is treated extensively in signal processing textbooks). There are several ways of approaching this problem. One is by computing the periodogram, which is essentially an empirical PSD. Another one consists in computing an empirical autocovariance and then computing the PSD of the process by taking its Fourier transform. In all these approaches, the problem tends to be that the estimates of the PSD become less accurate at low frequencies. This is due to fundamental sampling limitations. For this reason, the approach taken often consists in fitting a known parametric model (AR(k) where k is the degree). This in particular allows us to introduce known features in the model (like for example that the noise is expected to be decorrelated beyond a certain lag).

Finally, we stress that we have only treated the covariance propagation problem here. For non-gaussian effects, we would first need to develop multivariate Gaussian overbounds of the temporal error series. Techniques to bound multivariate random variables have been developed (for example in [22]). Once we have a bounding distribution that is a stationary gaussian process, we can apply the techniques described here.

EXAMPLE: FIRST ORDER ERROR MODEL REPRESENTING GPS CLOCK AND EPHEMERIS ERRORS

As an example, we show how this method could be applied to the determination of an error model for GPS clock and ephemeris errors. Please note that this is a very incomplete analysis and it is only shown here to illustrate the method.

Figure 4 shows the autocorrelation function of the clock and ephemeris errors of two GPS satellites (derived from the analysis of 9 years of data).

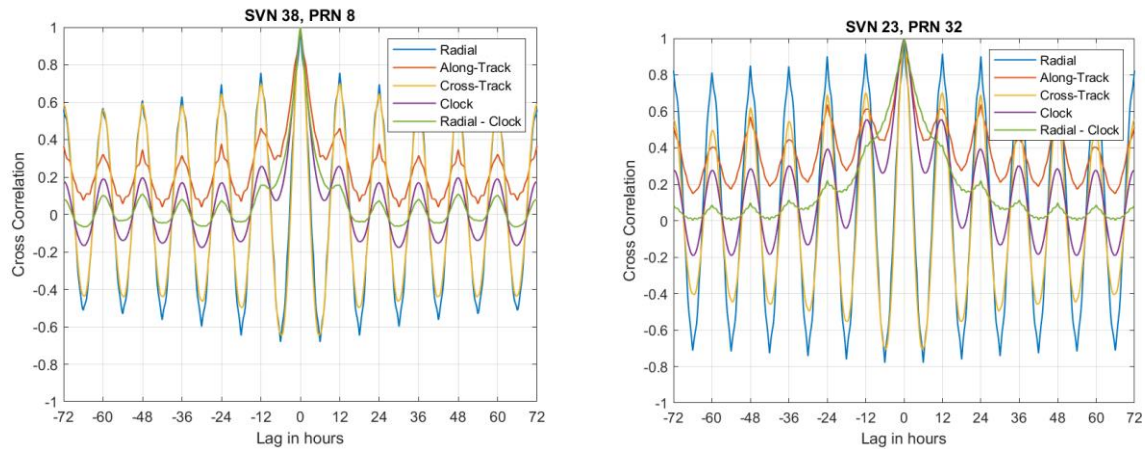


Figure 4. Autocorrelation function of the clock and ephemeris errors for two GPS satellites

Let us suppose that we would like to develop a first error model that we could use for all GPS satellites, for estimation filters using up to 12 hours. Because we are interested in the effect on a user, we will focus on the radial + clock error process. Also, we only need to be concerned with the autocorrelation between -12 and 12 hours. In Figure 5, we show a first order fit of the autocorrelation function between -12 and 12 hours for three GPS satellites.

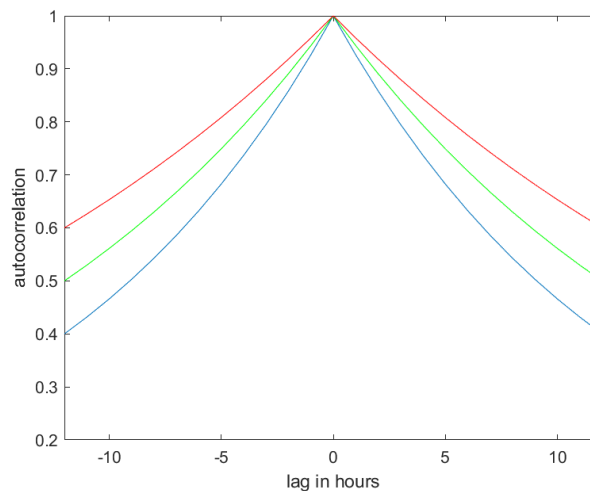


Figure 5. First order fit of the autocorrelation function between -12 and 12 hours of the radial + clock component for three GPS satellites

These processes are characterized by: $\alpha = 0.93, \alpha = 0.94, \alpha = 0.96$ (and a variance of one). In Figure 6 we show the corresponding PSD, as well as a process that bounds all three error models.

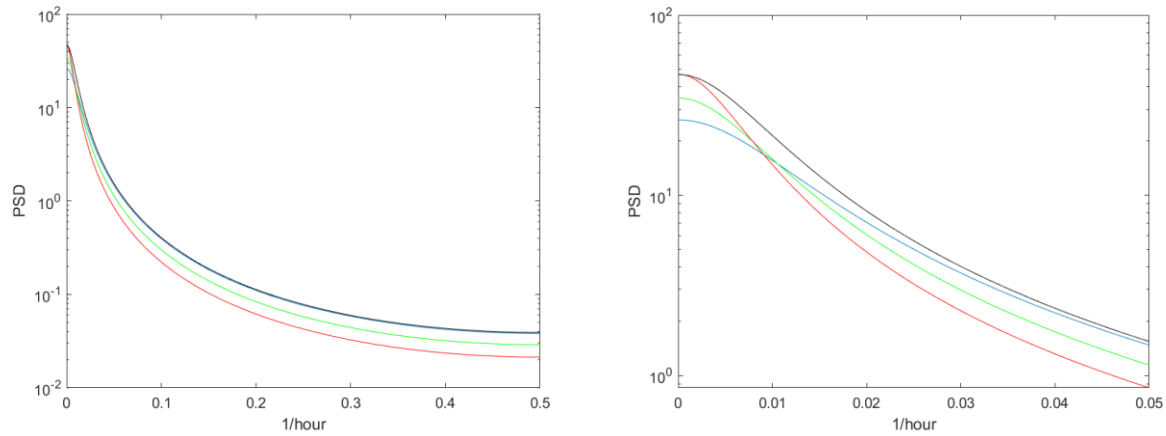


Figure 6. PSD corresponding to the three examined GPS satellites as well as from the bounding process (black curve)

The bounding process is defined by: $\alpha = 0.94$ and $\sigma^2 = 1.3$. Because the PSD of this process bounds the PSD of the three considered GPS satellites, it can be used conservatively for all three satellites.

SUMMARY

In this work, we have first described a method to determine simple temporal error models that account for temporal processes that are either uncertain or complex, developed a simple set of bounding criteria for first order processes, and, finally, demonstrated how to apply the method for the determination of a first order process bounding GPS clock and ephemeris errors.

REFERENCES

- [1] Kouba, J. and Heroux, P. (2001). Precise Point Positioning Using IGS Orbit and Clock Products, GPS Solutions, vol. 5, no. 2, pp. 12-28.
- [2] Madrid, P. F. Navarro, Fernández, L. Martínez, López, M. Alonso, Samper, M.D. Laínez, Merino, M.M. Romy, "PPP Integrity for Advanced Applications, Including Field Trials with Galileo, Geodetic and Low-Cost Receivers, and a Preliminary Safety Analysis," *Proceedings of the 29th International Technical Meeting of The Satellite Division of the Institute of Navigation (ION GNSS+ 2016)*, Portland, Oregon, September 2016, pp. 3332-3354.
- [3] K. Gunning, J. Blanch, T. Walter, L. de Groot, L. Norman, "Design and Evaluation of Integrity Algorithms for PPP in Kinematic Applications" *Proceedings of the 31th International Technical Meeting of The Satellite Division of the Institute of Navigation (ION GNSS+ 2018)*, Miami, Florida, September 2018.

[4] Joerger, Mathieu, Pervan, Boris, "Multi-Constellation ARAIM Exploiting Satellite Geometry Change," *Proceedings of the 28th International Technical Meeting of The Satellite Division of the Institute of Navigation (ION GNSS+ 2015)*, Tampa, Florida, September 2015, pp. 2688-2704.

[5] Joerger, Mathieu, Pervan, Boris, "Exploiting Satellite Motion in ARAIM: Measurement Error Model Refinement Using Experimental Data," *Proceedings of the 29th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS+ 2016)*, Portland, Oregon, September 2016, pp. 1696-1712.

<https://doi.org/10.33012/2016.14552>

[6] Phelts, R. Eric, Gunning, Kazuma, Blanch, Juan, Walter, Todd, "Evaluating the Application of PPP Techniques to ARAIM Using Flight Data," *Proceedings of the 2020 International Technical Meeting of The Institute of Navigation*, San Diego, California, January 2020, pp. 379-385.

<https://doi.org/10.33012/2020.17151>

[7] Gunning, Kazuma, Blanch, Juan, Walter, Todd, "SBAS Corrections for PPP Integrity with Solution Separation," *Proceedings of the 2019 International Technical Meeting of The Institute of Navigation*, Reston, Virginia, January 2019, pp. 707-719.

<https://doi.org/10.33012/2019.16739>

[8] Langel, Steven, Crespillo, Omar García, Joerger, Mathieu, "Bounding Sequential Estimation Errors Due to Gauss-Markov Noise with Uncertain Parameters," *Proceedings of the 32nd International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS+ 2019)*, Miami, Florida, September 2019, pp. 3079-3098.

<https://doi.org/10.33012/2019.17014>

[9] Tupysev, V. A., Stepanov, O. A., Loparev, A. V., and Litvinenko, Y. A. "Guaranteed Estimation in the Problems of Navigation Information Processing." *Proceedings of the 18th IEEE International Conference on Control Applications*, St. Petersburg, Russia, 2009, pp. 1672 –1677

[10] Steven Langel, Omar Garcia Crespillo, Mathieu Joerger. "A New Approach for Modeling Correlated Gaussian Errors Using PSD Overbounding" (abstract) Paper to be presented in PLANS 2020, April 2020.

[11] Robert M. Gray (2006), "Toeplitz and Circulant Matrices: A Review", *Foundations and Trends® in Communications and Information Theory*: Vol. 2: No. 3, pp 155-239.

<http://dx.doi.org/10.1561/01000000006>

[12] M. Reed and [Barry Simon](#), *Methods of Modern Mathematical Physics*, vol. II, Academic Press, 1975.

[13] "Bochner's theorem" https://en.wikipedia.org/wiki/Bochner%27s_theorem

[14] *Cressie, Noel A.C. (1993). Statistics for Spatial Data. Wiley-Interscience.*

[15] Walter, Todd, Blanch, Juan, "KEYNOTE - Characterization of GNSS Clock and Ephemeris Errors to Support ARAIM," *Proceedings of the ION 2015 Pacific PNT Meeting*, Honolulu, Hawaii, April 2015, pp. 920-931.

[16] Global Positioning System Standard Positioning Service Performance Standard. 2008.

[17] Blanch, Juan, Walter, Todd, Phelts, R. Eric, "Development and Evaluation of Airborne Multipath Error Bounds for L1-L5," *Proceedings of the 2019 International Technical Meeting of The Institute of Navigation*,

Reston, Virginia, January 2019, pp. 53-61.
<https://doi.org/10.33012/2019.16735>

[18] Amielh, Capucine, Chabory, Alexandre, Macabiau, Christophe, Azoulai, Laurent, "GNSS Multipath Error Model for Aircraft Surface Navigation Based on Canonical Scenarios for Class F Airports," *Proceedings of the 32nd International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS+ 2019)*, Miami, Florida, September 2019, pp. 1418-1437.
<https://doi.org/10.33012/2019.16907>

[19] L. Montloin, L. Azoulai, A. Chen, A. Martineau, C. Milner, A. Chabory, and C. Macabiau. GNSS multipath error model for airport surface operations. 25th International Technical Meeting of The Satellite Division of the Institute of Navigation, ION GNSS+ 2012, September 2012

[20] Leslie Montloin. Impact of GNSS singular events on the integrity of airport navigation systems. Signal and Image processing. INPT, 2014. English. tel-01377380

[21] "Over-bounding Integrated INS/GNSS Output Errors", Z. Xing, University of Minnesota, October 2010

[22] Rife, J. and Gebre-Egziabher, D. (2007), Symmetric Overbounding of Correlated Errors. *Navigation*, 54: 109-124. doi:[10.1002/j.2161-4296.2007.tb00398.x](https://doi.org/10.1002/j.2161-4296.2007.tb00398.x)