

CALCULUS TEST
2004 STANFORD MATHEMATICS TOURNAMENT
FEBRUARY 28, 2004

1. Evaluate $\lim_{x \rightarrow \infty} (\sqrt{4x^2 + 7x} - 2x)$.
2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.
3. An object moves along the x -axis with its position at any given time $t \geq 0$ given by $x(t) = 5t^4 - t^5$. During what time interval is the object slowing down?
4. For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.
5. The highway department of North Eulerina plans to construct a new road between towns Alpha and Beta. Town Alpha lies on a long abandoned road running east west. Town Beta lies 3 miles north and 5 miles east of Alpha. Instead of building a road directly between Alpha and Beta, the department proposes renovating part of the abandoned road (from Alpha to some point P) and then bulding a new road from P to Beta. If the cost of restoring each mile of old road is \$200,000 and the cost per mile of a new road is \$400,000, how much of the old road should be restored in order to minimize costs?
6. Consider the two graphs $y = 2x$ and $x^2 - xy + 2y^2 = 28$. What is the absolute value of the tangent of the angle between the two curves at the points where they meet?
7. A mouse is sitting in a toy car hooked to a spring launching device on a negligibly small turntable. The car has no way to turn, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 m/s. Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set of points the mouse can reach in his car within 1 second after the turntable is set in motion. What is the area of this set?
8. A spherical cow is being pulled out of a deep well. The bottom of the well is 100 feet down and the cow and all the water on him weighs 200 pounds. He is being hauled up at a constant rate with a chain which weighs 2 pounds per foot. The water on the cow drips off the cow at the rate of $\frac{1}{2}$ pounds per foot as he is being hauled up. How much work is required to rescue the cow in foot-pounds? Remember work is force times distance.
9. The base of a solid is the region between the parabolas $x = y^2$ and $2y^2 = 3 - x$. Find the volume of the solid if the cross-sections perpendicular to the x -axis are equilateral triangles.
10. Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

converges for $c > c_0$ and diverges for $0 < c < c_0$.