Directory

The Shape of Production Functions and the Direction of Technical Change

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Introduction

- Macro/growth literatures: strong assumptions on PF and direction of technical change. Justification?
- What is a production function? $y = f(k, t)$
  - Leontief example.
  - Switching from low k to high k may involve very different production techniques/ideas
  - A production function is not a single technology, but rather represents the substitution possibilities across different techniques
- The global shape of the production function is determined by the distribution of ideas.

Overview

- Kortum (1997) meets Houthakker (1955): Growth and Pareto Distributions
- Results:
  1. A production function with
     - low EofS for any given technique
     - Cobb-Douglas global production function.
  2. A theory of LATC
     - Possibility of KATC in model, but
     - Economy “chooses” LATC only in LR.
     - cf Acemoglu (2003)

Outline

1. Baseline Model
2. Model w/ Microfoundations
3. Discussion: Role of Pareto
4. Embed in a growth model: LATC
5. Simulation Results

Baseline Model: Preliminaries

- Idea = (a_i, b_i). Production with technique i:

$$Y = \tilde{F}(b_i K, a_i L) \quad \leftarrow \text{the local production function}$$

where $\tilde{F}$ is a neoclassical PF with EofS$<$1.

- Rewrite in per worker terms as

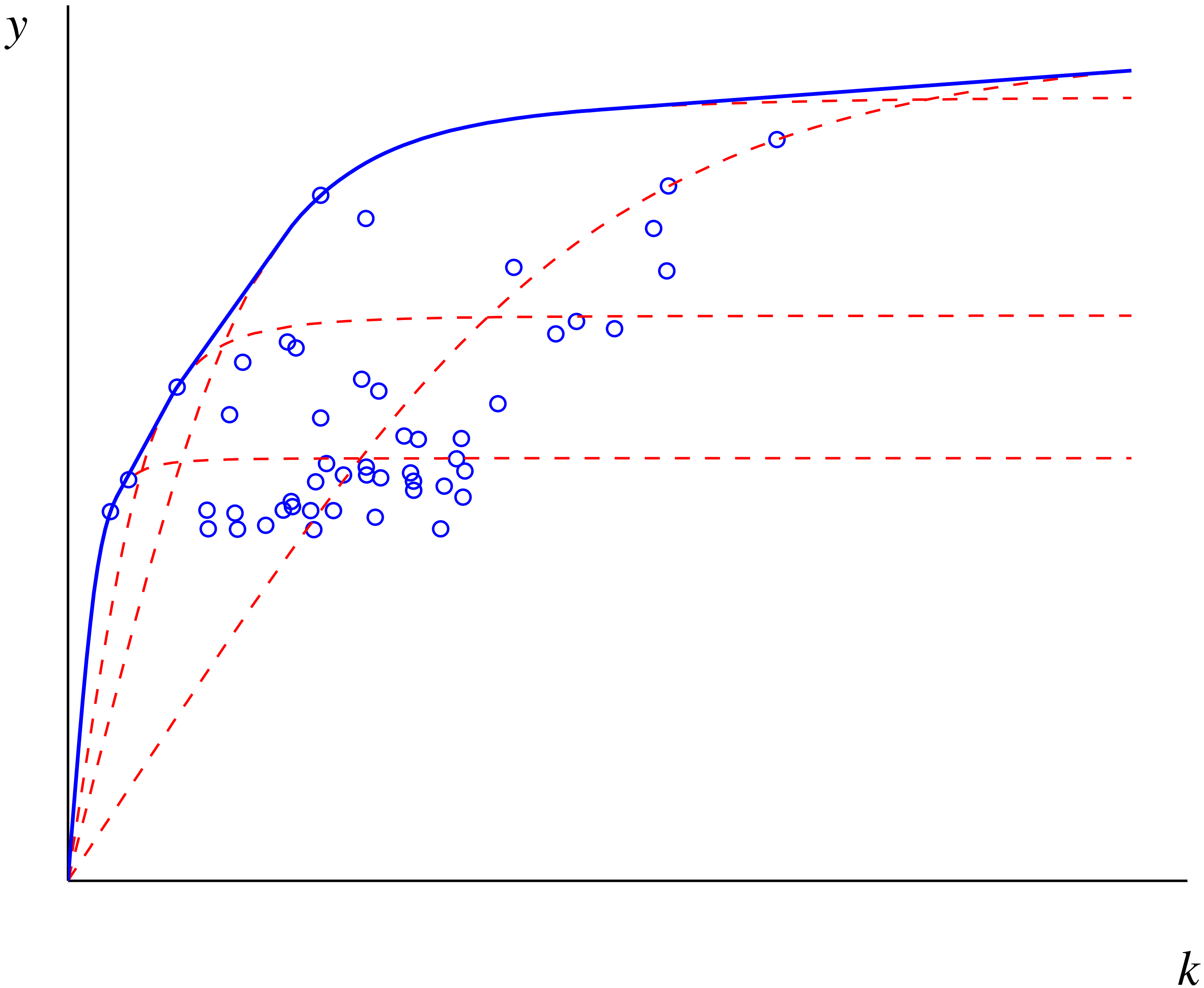
$$y = a_i \tilde{F}(\frac{b_i}{a_i} k, 1),$$

- Define $y_i = a_i$ and $k_i = a_i / b_i$. Then

$$y = y_i \tilde{F}\left(\frac{k}{k_i}, 1\right)$$

so that $k = k_i \Rightarrow y = y_i$.

The Global Production Function



Simple Model

- Firm has a stock of knowledge, N, that generates a menu of ideas

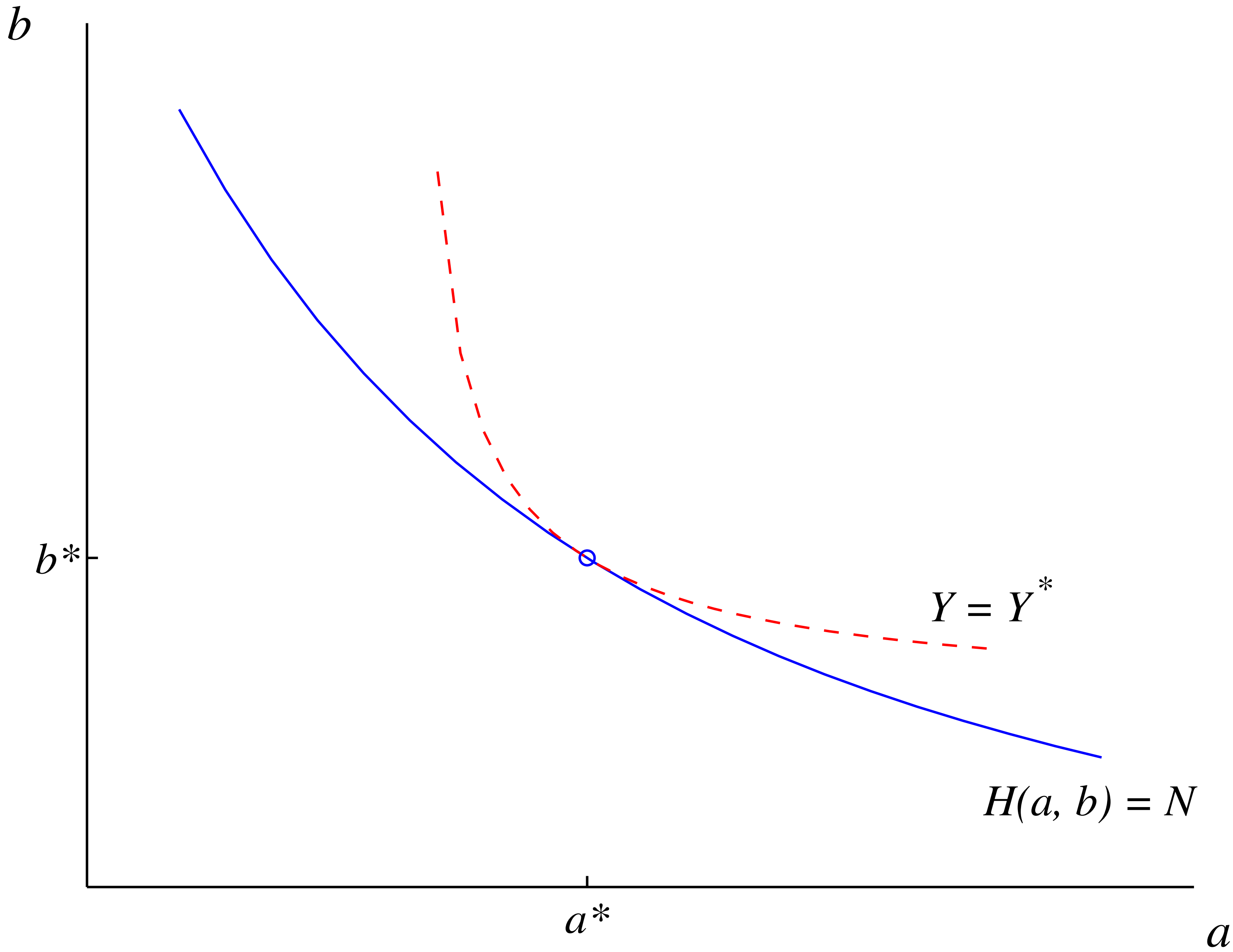
$$H(a,b) = N, \quad H_a > 0, \;\; H_b > 0. \tag{1}$$

- Associated with any idea (a,b) is a local production technique, as above.
- The global production function gives the highest output that can be produced using this menu:

$$Y = F(K,L;N) \equiv \max_{b,a} \; \tilde{F}(bK, aL)$$

subject to the technology menu constraint in (1).

Fig. 2: Direction of Technical Change



Solution

- First-order condition:

$$\frac{\theta_K}{\theta_L} = \frac{\eta_b}{\eta_a},$$

where $\theta_K(a, b; K, L) \equiv \tilde{F}_1 bK/Y$, $\theta_L = 1 - \theta_K$, $\eta_x \equiv \frac{\partial H}{\partial x}\frac{x}{H}$.

- Key special case: Constant elasticity menu

$$H(a, b) \equiv a^{\alpha} b^{\beta} = N.$$

$$\Rightarrow \theta_K = \beta/\alpha + \beta.$$

i.e. Capital share is constant for any K, L, and N.

- This leads to two results.

Result 1. Cobb-Douglas

- The capital share is constant for any K, L, N

  $\Rightarrow$ The global production function is Cobb-Douglas.
- Derive exact form:

$$y_i \equiv a_i$$

$$k_i \equiv \frac{a_i}{b_i}$$

  Technology menu then implies:

$$y_i = (N k_i^{\beta})^{\frac{1}{\alpha+\beta}}.$$

- The global production function equals this menu:

$$Y = \left(N K^{\beta} L^{\alpha}\right)^{\frac{1}{\alpha+\beta}}.$$

Result 2. LATC

- Embed this production setup in a standard neoclassical growth model
- Global Cobb-Douglas implies BGP exists if N grows exponentially.
- Steady-State Growth Theorem: In a steady state, either
  - Production is Cobb-Douglas, or
  - Technical change is labor augmenting.
- Production always occurs with some local PF, and the local is not Cobb-Douglas. Therefore LATC.

Proving LATC

- Rewrite the FOC as

$$\frac{bK\tilde{F}_1(bK, aL)}{aL\tilde{F}_2(bK, aL)} = \frac{\beta}{\alpha}.$$

- Define $x \equiv bK/aL$. $\tilde{F}$ CRS $\Rightarrow$ the marginal products are HD0:

$$\frac{x\tilde{F}_1(x, 1)}{\tilde{F}_2(x, 1)} = \frac{\beta}{\alpha}.$$

$\Rightarrow x$ must be constant.

Proof (continued)

- To show: x constant requires b constant in SS. Recall

$$Y_t = F(K_t, L_t; N_t) = \tilde{F}(b_t K_t, a_t L_t),$$

  where b_t and a_t are the optimal choices of the technology levels.

- Because $\tilde{F}$ exhibits constant returns, we have

$$\frac{Y_t}{a_t L_t} = \tilde{F}\left(\frac{b_t K_t}{a_t L_t}, 1\right).$$

- $x = bK/aL$ constant $\Rightarrow Y/aL$ constant $\Rightarrow bK/Y$ constant.
- K/Y is constant in SS $\Rightarrow b$ constant. QED.

Intuition

- Because local PF is not Cobb-Douglas, balanced growth requires bK and aL to grow at the same rate.
  - $Y = \tilde{F}(bK, aL)$ suggests new interpretation of "balanced"
  - bK and aL must balance to keep factor shares stable.
- Can only happen with b constant.
  - Recall, b constant means K/aL constant.
  - If b grew, so would bK/aL...

Clarifying the Result

- Well-known that with Cobb-Douglas production, the direction of technical change has no meaning.
- So how can we have both?
- Recall:

$$Y_t = \underbrace{F(K_t, L_t; N_t)}_{\text{global pf}} = \underbrace{\tilde{F}(b_t K_t, a_t L_t)}_{\text{local pf}}.$$

- Global production function $F(K, L; N)$ is Cobb-Douglas. Local production function $\tilde{F}(bK, aL)$ has LATC.

Discussion

- Related to World Technology Frontier problem in Caselli-Coleman (2004).
- Early literature on direction of TC chose growth rates: Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966).
- Acemoglu (2003) has related results in a Romer-type model:
  - LATC if production function for ideas is “just so”
  - Capital share in LR is invariant to policy

Model with Microfoundations

- Assume the local production function is Leontief:

$$Y = \tilde{F}(b_i K, a_i L) = \min\{b_i K, a_i L\}$$

- Ideas drawn from independent Pareto distributions:

$$\text{Prob}\left[a_i \le a\right] = 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha}, \quad a \ge \gamma_a > 0$$

$$\text{Prob}\left[b_i \le b\right] = 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta}, \quad b \ge \gamma_b > 0.$$

- Then, $G(b,a) \equiv \text{Prob}\left[b_i > b, a_i > a\right] = \left(\frac{b}{\gamma_b}\right)^{-\beta} \left(\frac{a}{\gamma_a}\right)^{-\alpha}$

Distribution of Output from Idea i

- Let $Y_i(K, L)$ denote output with idea i. Since $\tilde{F}$ is Leontief, the distribution of Y_i is

$$
\begin{aligned}
H(\tilde{y}) \equiv \text{Prob}\left[Y_i > \tilde{y}\right] &= \text{Prob}\left[b_i K > \tilde{y}, a_i L > \tilde{y}\right] \\
&= G\left(\frac{\tilde{y}}{K}, \frac{\tilde{y}}{L}\right) \\
&= \gamma K^{\beta} L^{\alpha} \tilde{y}^{-(\alpha+\beta)},
\end{aligned}
$$

  where $\gamma \equiv \gamma_a^{\alpha} \gamma_b^{\beta}$.

- That is, the distribution of Y_i is also Pareto.

The Global Production Function

- Assume only one technique can be used at a time.
- Let N denote the number of ideas, drawn independently.
- The global production function $F(K, L; N)$ is given as

$$F(K, L; N) \equiv \max_{i \in \{1,\ldots,N\}} \tilde{F}(b_i K, a_i L).$$

- Let $Y = F(K, L; N)$. Then

$$\begin{aligned} \text{Prob}\left[Y \leq \tilde{y}\right] &= \left(1 - H(\tilde{y})\right)^N . \\ &= \left(1 - \gamma K^\beta L^\alpha \tilde{y}^{-(\alpha+\beta)}\right)^N . \end{aligned}$$

(continued)

$$\text{Prob}\,[Y \le \tilde{y}] = \left(1 - \gamma K^{\beta} L^{\alpha} \tilde{y}^{-(\alpha+\beta)}\right)^{N}.$$

- As N gets large, this probability goes to zero.
  $\Rightarrow$ normalize to get a stable distribution

$$z_N \equiv \left(\gamma N K^{\beta} L^{\alpha}\right)^{\frac{1}{\alpha+\beta}}.$$

- Then,

$$\begin{aligned}\text{Prob}\,[Y \le z_N \tilde{y}] &= \left(1 - \gamma K^{\beta} L^{\alpha} (z_N \tilde{y})^{-(\alpha+\beta)}\right)^{N} \\ &= \left(1 - \frac{\tilde{y}^{-(\alpha+\beta)}}{N}\right)^{N}.\end{aligned}$$

The Cobb-Douglas Result

- Now, let N get large

$$\lim_{N \to \infty} \text{Prob}\left[Y \leq z_N \tilde{y}\right] = \exp(-\tilde{y}^{-(\alpha+\beta)})$$

- Or,

$$\frac{Y}{(\gamma N K^\beta L^\alpha)^{1/\alpha+\beta}} \overset{a}{\sim} \text{Fréchet}(\alpha+\beta).$$

- And therefore, for large N,

$$Y \approx \left(\gamma N K^\beta L^\alpha\right)^{\frac{1}{\alpha+\beta}} \epsilon$$

Remarks

$$Y \approx \left(\gamma N K^{\beta} L^{\alpha}\right)^{\frac{1}{\alpha+\beta}} \epsilon$$

1. Appendix: Poisson process for discovery of ideas yields the result for finite N.
2. Cobb-Douglas exponent depends on parameters of search distributions
   - Easier to find ideas $\rightarrow$ lower exponent.
   - Intuition: EofS$<$ 1.
3. ϵ is an iid shock drawn from a Fréchet distribution.
4. Higher N implies Higher Y.
5. Obviously Pareto assumption is crucial to result. More on this shortly.

Discussion: 1. Baseline Model

- Baseline model: constant elasticity in technology menu.
- Here, stochastic version. Consider iso-probability curve:

$$\text{Prob}\left[b_i > b, a_i > a\right] \equiv G(b,a) = C.$$

With Pareto,

$$b^\beta a^\alpha = \frac{\gamma}{C}.$$

- Stochastic version of the baseline technology menu.
  - Pareto delivers $\eta_b = \beta$ and $\eta_a = \alpha$
  - $1/C$ plays the role of N
  - Get the same form for the production function.

2. Comparison to Houthakker (1955)

- Pareto+Leontief = Cobb-Douglas is Houthakker
- Houthakker's result is an aggregation result
  - Continuum of firms with capacity constraints.
  - Firm PF: Leontief, with requirements $\sim$ Pareto.
  - Aggregate PF: Cobb-Douglas with DRS
- Result here:
  - Result applies for a firm/industry/country
  - Applies to global production function, i.e. across techniques.
  - No restriction to Leontief for SR PF (technique)
  - Nonrivalry of ideas $\Rightarrow$ CRS

3. Evidence for Pareto Distributions

- Key property: Prob $[X \geq \gamma x \mid X \geq x]$ for $\gamma > 1$ is independent of x.
- Empirical evidence for incomes, patent values, profitability, citations, firm size, stock returns.
  - Benchmark in literature is to test Pareto
  - Findings: Pareto (sometimes hard to distinguish from Lognormal)
- Kortum (1997):
  - Assume a production function and draw a_i only
  - Iff ideas are from a Pareto distribution, then we get exponential growth

(continued)

- Why is Pareto so important?
  - Steady-state growth requires probability the new best idea exceeds frontier by 5% is invariant to y.
  - Gabaix (1999) shows the reverse. Exponential growth delivers a Pareto distribution for city sizes (Zipf).
- This suggests that Pareto Distributions and exponential growth are two sides of the same coin.
  - What I add is that this same basic assumption delivers two additional results:
    1. Cobb-Douglas production
    2. Labor-augmenting technical change (next).

The Direction of Technical Change

- Embed this setup in a neoclassical growth model

$$Y_t = \left(\gamma N_t K_t^{\beta} L_t^{\alpha}\right)^{\frac{1}{\alpha+\beta}} \epsilon_t.$$

$$K_{t+1} = (1-\delta)K_t + sY_t$$

$$N_t = N_0 e^{gt}$$

- Therefore, steady-state growth in Y/L:

$$E[\log \frac{y_{t+1}}{y_t}] \approx g/\alpha.$$

Note: depends on α but not β.

(continued)

- Model exhibits a stable balanced growth path, because of global Cobb-Douglas production.
- However, production at date t occurs with some technique $i(t)$:

$$Y_t = \tilde{F}(b_{i(t)} K_t, a_{i(t)} L_t).$$

- Now use Steady-State Growth Theorem:
  - The production function for a technique is *not* Cobb-Douglas,
  - so Steady State implies that $b_{i(t)}$ is stationary!
- That is, technical change in this model is (asymptotically) labor-augmenting.
  - This is true even though $\max_i b_i \to \infty$.

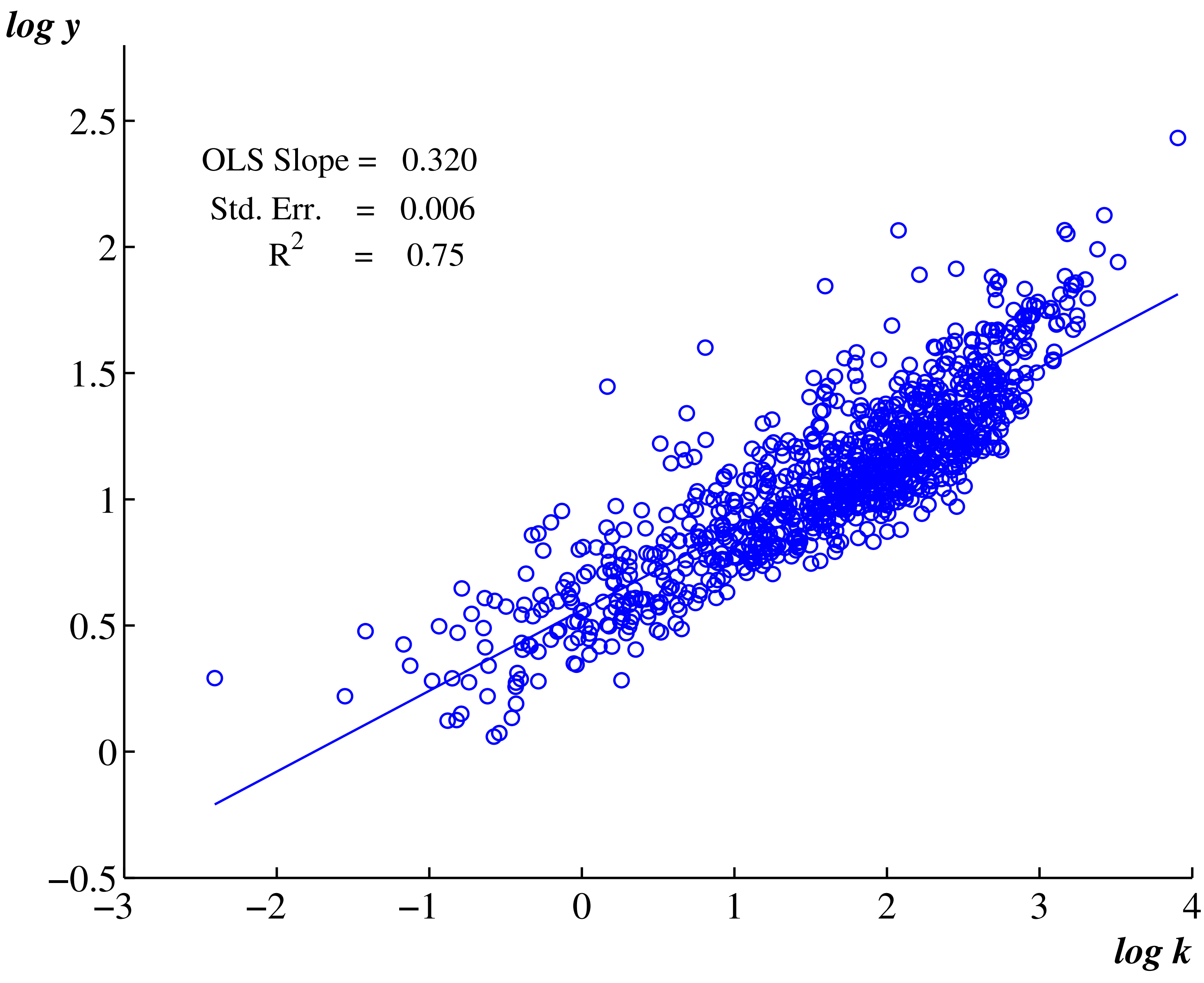
Simulating the Model

- Relax Leontief and allow multiple techniques
- CES production technique:

$$Y_t = \tilde{F}(b_i K_t, a_i L_t) = (\lambda(b_i K_t)^\rho + (1-\lambda)(a_i L_t)^\rho)^{1/\rho}$$

- First, show Cobb-Douglas.
  - $N = 500$, $\alpha = 5$, $\beta = 2.5$, $\rho = -1$.
    $\Rightarrow \frac{\beta}{\alpha+\beta} = 1/3$.
  - Compute convex hull and sample a (k, y) point randomly.
  - Repeat 1000 times and plot the sample.

Fig. 3: The Cobb-Douglas Result



Full Dynamic Simulation

- Parameter Values: $N_0 = 50$, $g = .10$, $\alpha = 5$, $\beta = 2.5$, $\gamma_a = 1$, $\gamma_b = 0.2$, $k_0 = 2.5$, $s = 0.2$, $\lambda = 1/3$, $\delta = .05$, and $\rho = -1$.
- Growth should average 2 percent
- Cobb-Douglas capital share 1/3

Fig. 4: Production

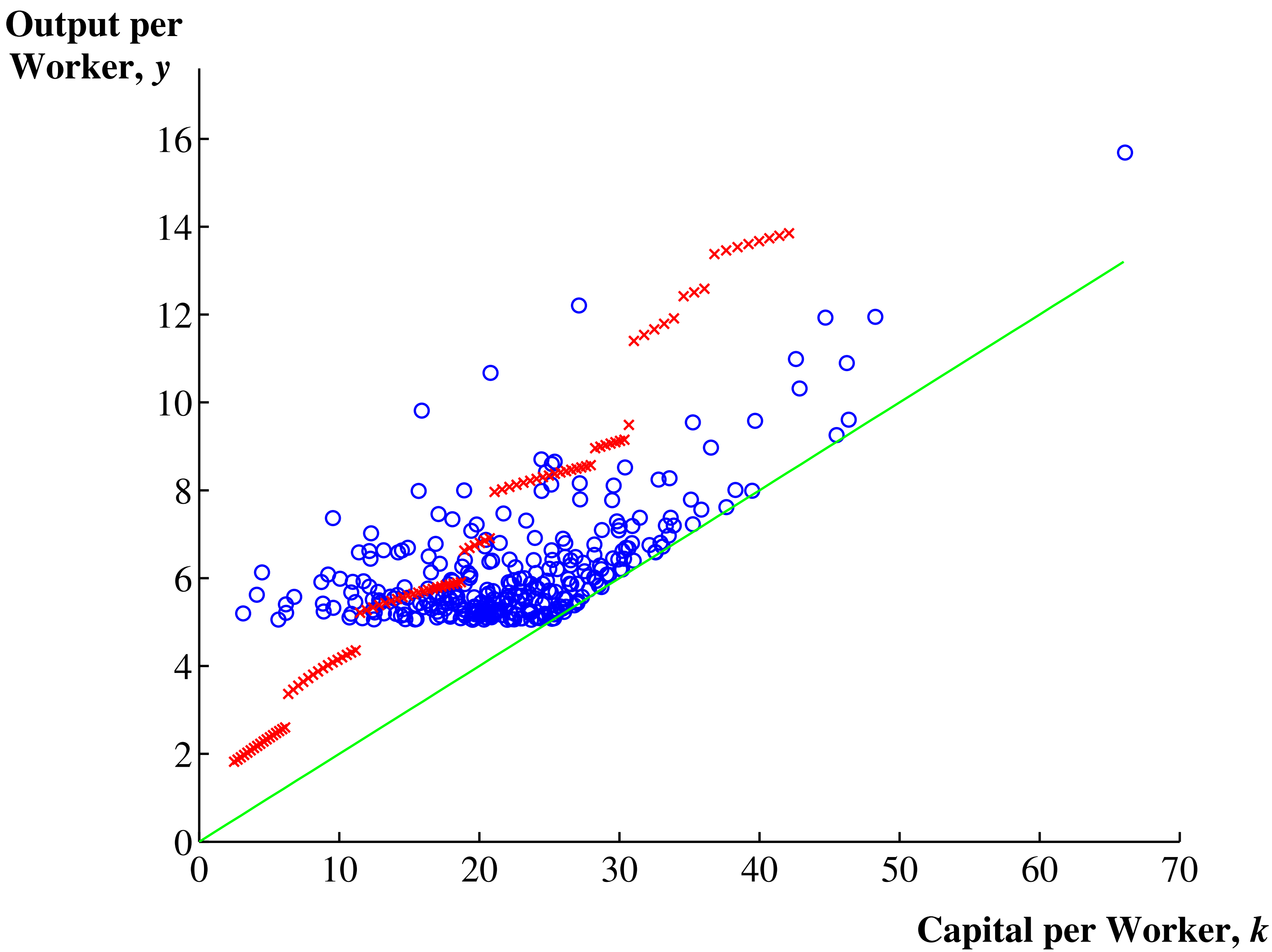


Fig. 5: Output per Worker

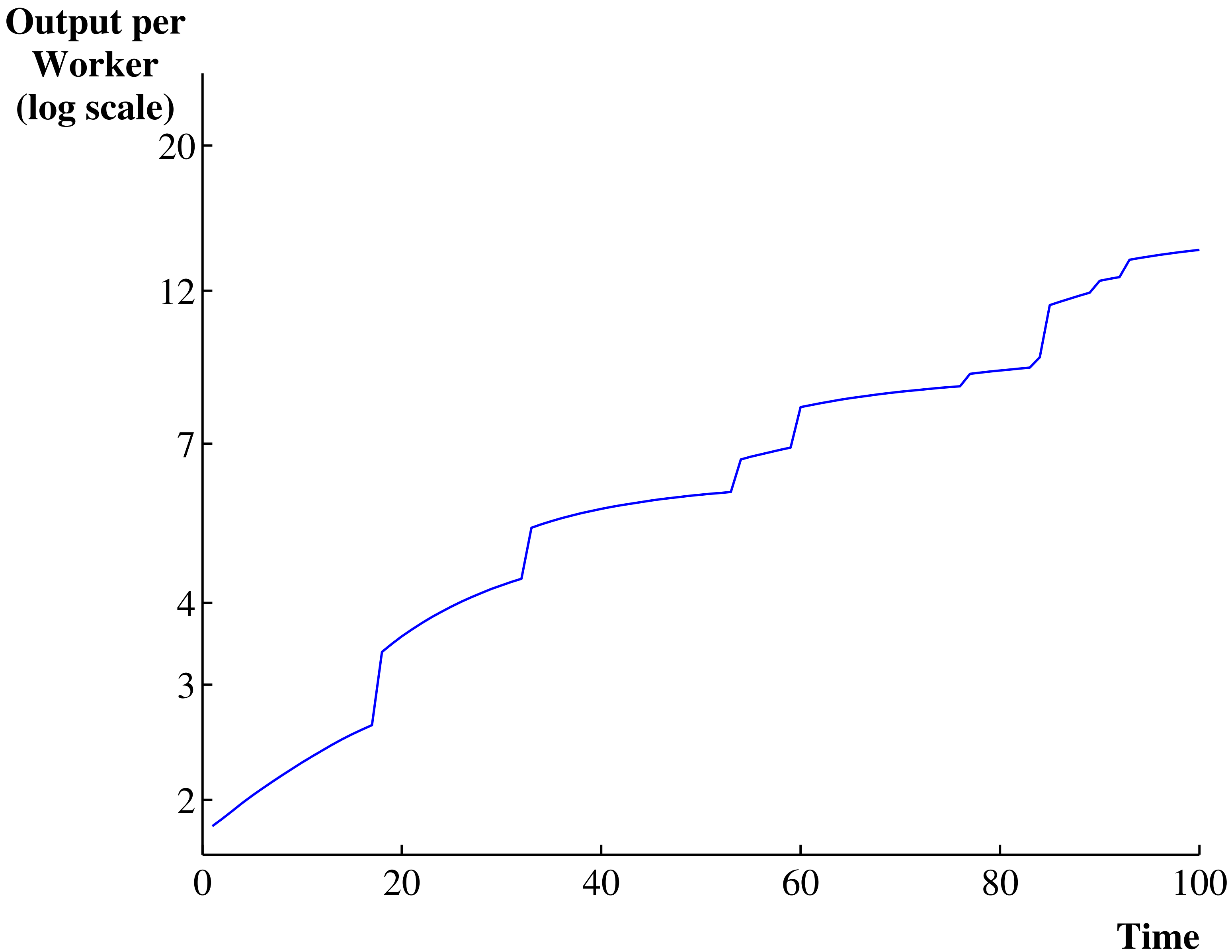


Fig. 6: The Capital Share over Time

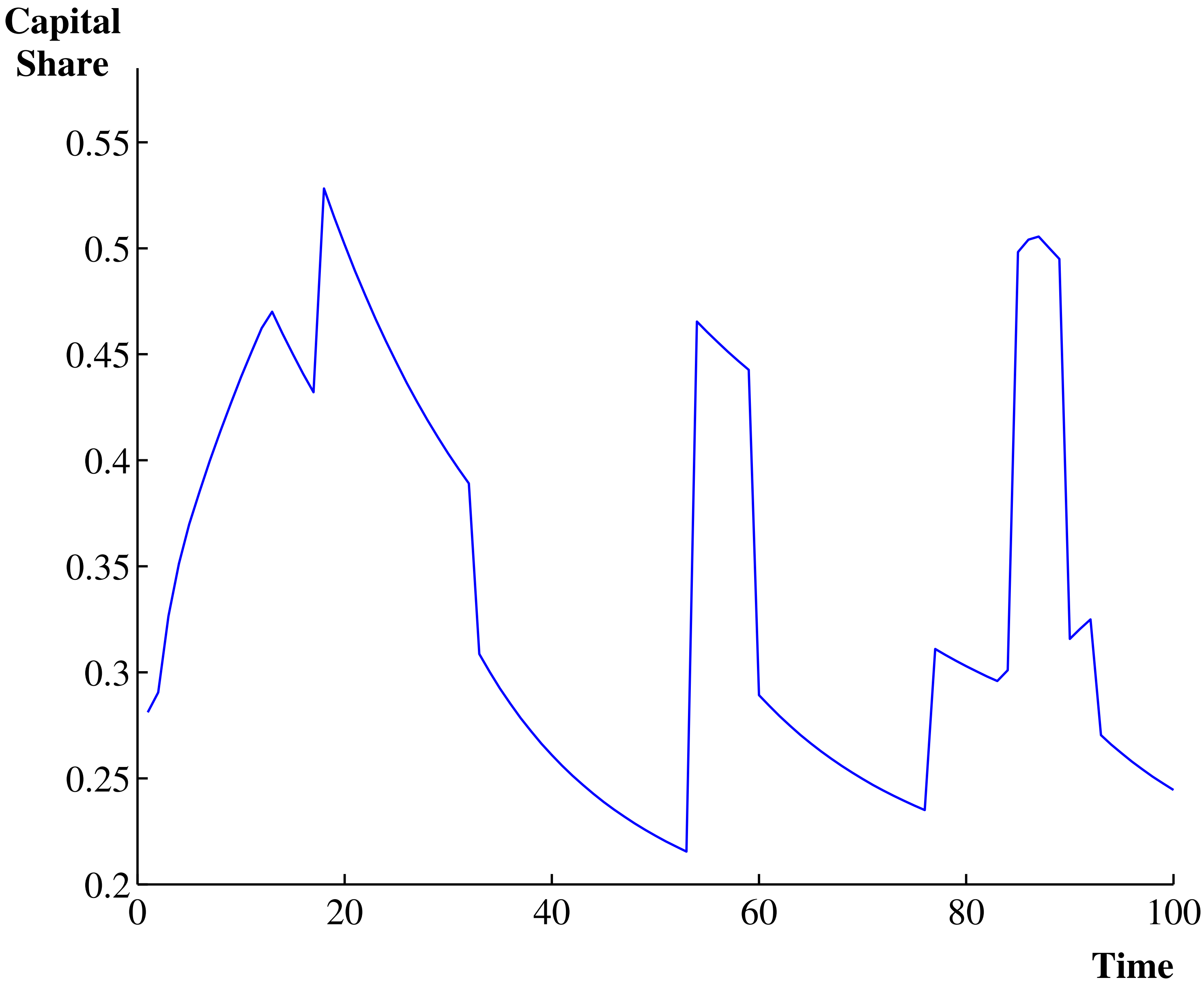
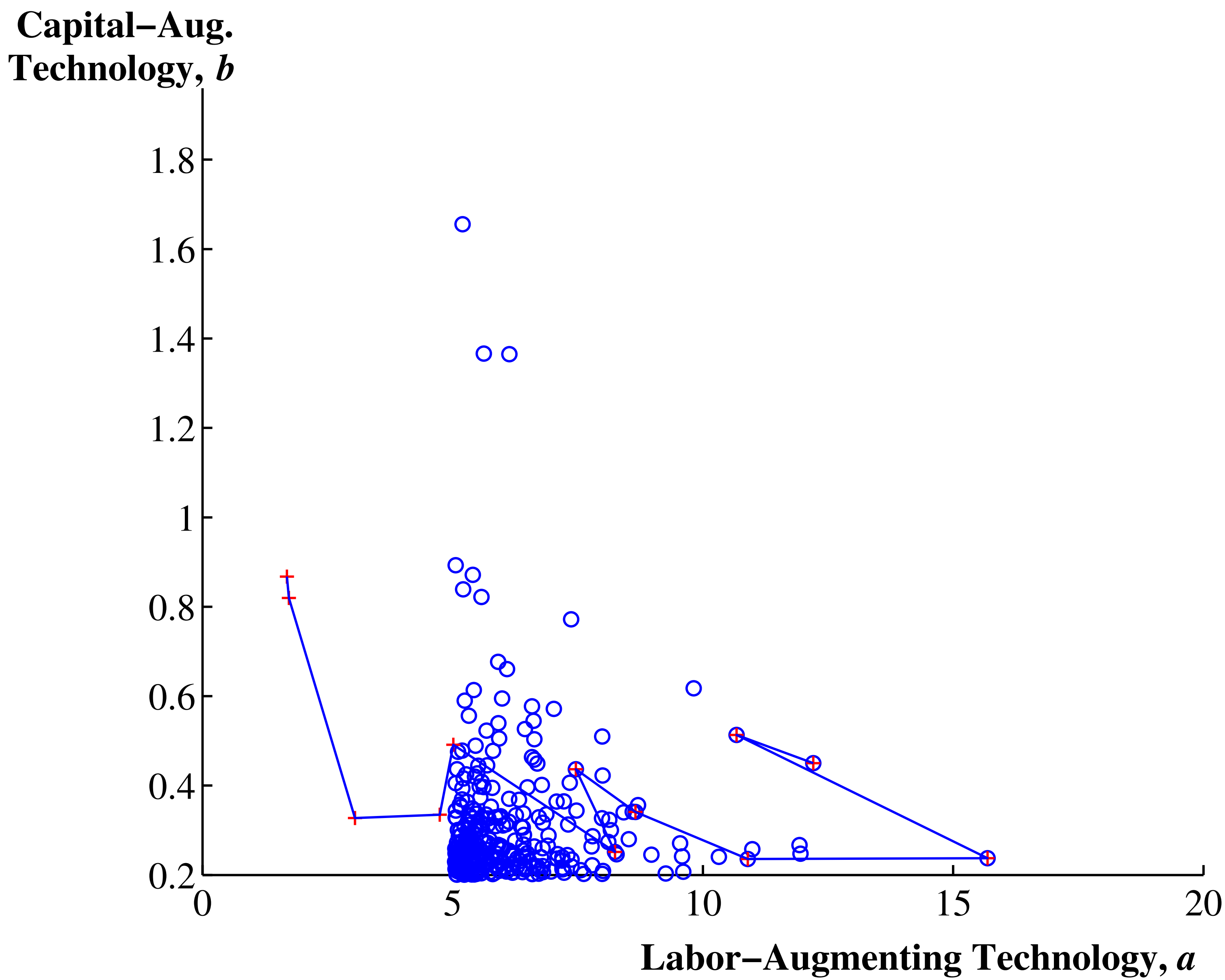


Fig. 7: Technology Choices



Conclusions

- Houthakker + Kortum =
  - Exponential growth
  - Cobb-Douglas (global) production function
  - Labor-augmenting technical change.

  The Pareto distribution buys us a lot!
- Extensions and future work:
  - Skilled versus unskilled labor?
  - What about computers and ISTC? GHK, Whelan, etc.?