Lecture 1
Solving the Standard Incomplete Markets Model

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Overview

- In the het.-agent macro literature, models can rarely be solved analytically
- So we begin with a brief overview over how you can efficiently compute
  - steady states
  - transitional dynamics / impulse responses

  of the "standard incomplete markets" model
  (aka Bewley-Huggett-Aiyagari-Imrohoroglu-Zeldes-Deaton-Carroll model)
- This will be the backbone of everything we do in this course
- For the most part, this is automated in the sequence-space-jacobian toolkit, but it's important to know what's going on under the hood
- See Goethe summer course repo for code generating all results here

Roadmap

1. Steady state

2. Dynamics

Steady state

The standard incomplete markets model (SIM)

$$V_t(e, a) = \max_{c, a'} u(c) + \beta \mathbb{E}[V_{t+1}(e', a')|e]$$

$$\text{s.t. } a' + c = (1 + r_t^p)a + Z_t e$$

$$a' \geq \underline{a}$$

The model generates **four key outcomes**:

1. endogenous buffer stock saving
2. a well defined stationary wealth distribution
3. high MPCs, consistent with those estimated in the data
4. a finitely elastic long-run asset demand curve

Four steps to solving the steady state

1. **Discretizing the state space:** Markov chain for e & grid for a.
2. **Solving for steady-state policy function:**
   - get policies $a'(e, a)$ and $c(e, a)$ on the grid
3. **Solving for steady-state distribution:**
   - get distribution $D(e, a)$ of agents on grid
4. **Aggregating:** get aggregate assets and consumption

Asset grid: could assume uniform grid ... but most action / nonlinearity will be close to borrowing constraint $\underline{a}$!

Better choice: double exponential grid: If u_i is uniform on $[0, \overline{u}]$ then

$$a_i = \underline{a} + \exp\left(\exp u_i - 1\right) - 1$$

This makes the grid much denser at $\underline{a}$.

Markov chain for e: Typically want to approximate continuous Markov chain, e.g. AR(1) with persistence ρ. How do we discretize on n gridpoints?

We'll use **Rouwenhorst**'s method. Idea:

- $n_e - 1$ switches (0 or 1). Each flips with probability $p = (1-\rho)/2$.
- [Also common: Tauchen method, but worse for usual $\rho \simeq 1$ case.]

We'll use two key optimality conditions:

Envelope condition:

$$V_{a,t}(e,a) = (1+r_t^p)u'(c_t(e,a)) \tag{1}$$

First order condition:

$$u'(c_t(e,a)) \geq \beta \mathbb{E}[V_{a,t+1}(e',a_t'(e,a))|e] \tag{2}$$

with equality unless borrowing constraint binds.

Key insight of Carroll's (2006) Endogenous Gridpoints Method (EGM): More efficient & accurate to use (1) and (2) instead of value function iteration on V.

This is what we'll do:

$$V_{a,t+1}(e',a') \overset{\text{FOC}}{\rightarrow} c_t^{endo}(e,a') \overset{\text{interpolate}}{\longrightarrow} a_t'(e,a) \overset{\text{budget}}{\longrightarrow} c_t(e,a) \overset{\text{Envelope}}{\longrightarrow} V_{a,t}(e,a)$$

Step 2: Steady state policies: Details of EGM (backward iteration)

- First arrow is simple:

$$c_t^{endo}(e, a') = \left(u'\right)^{-1}\left(\beta\mathbb{E}[V_{a,t+1}(e', a')|e]\right)$$

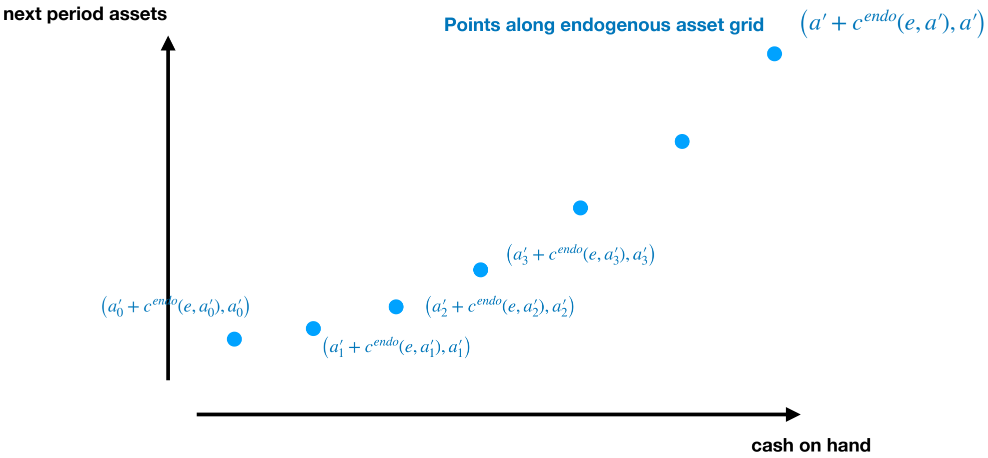
- But how to get to policy defined over current grid (e, a)? Note:

$$c_t^{endo}(e, a') + a' = \text{coh}_t(e, a) \equiv \left(1 + r_t^p\right) a + Z_t e$$

  We want to solve for a' as function of a.

- Can do this with simple interpolation:
  - Plot $(c_t^{endo}(e, a') + a', a')$ points and interpolate at values of $\text{coh}_t(e, a)$...

Illustration of key EGM step



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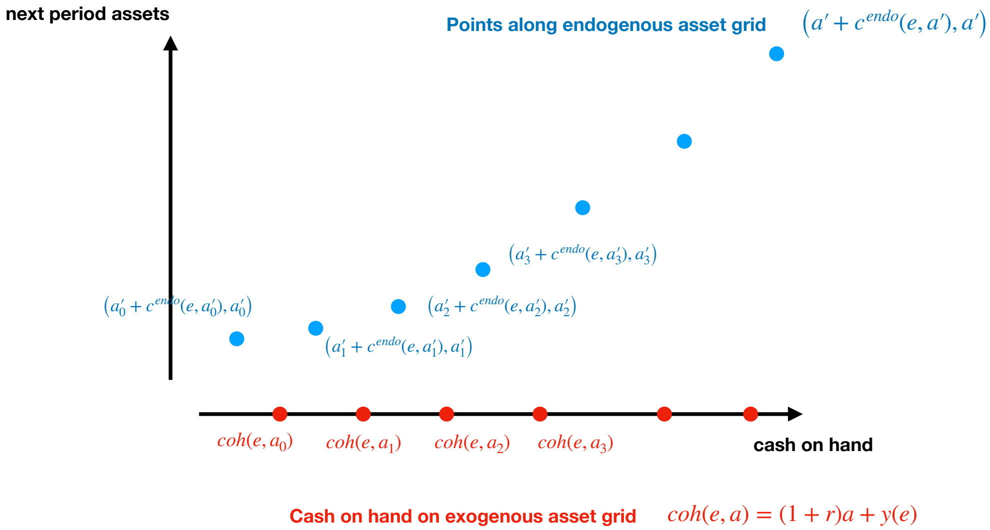
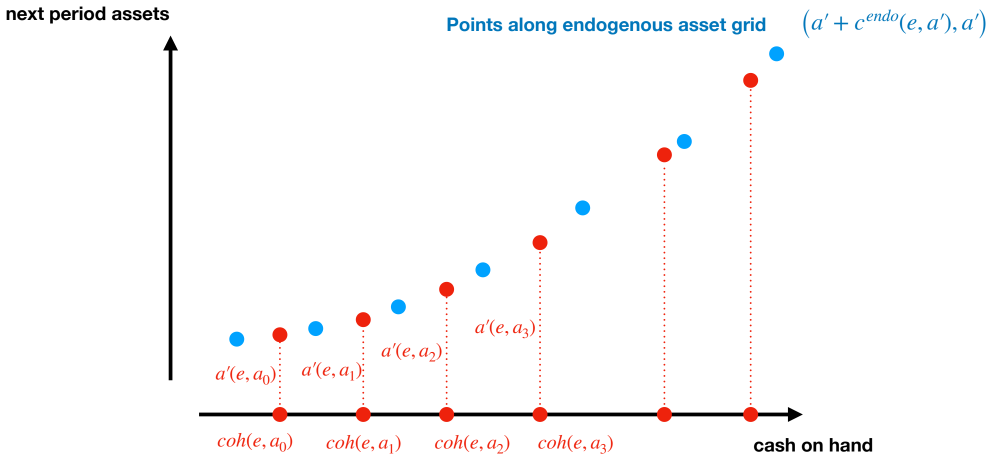


Illustration of key EGM step



Cash on hand on exogenous asset grid $coh(e, a) = (1 + r)a + y(e)$

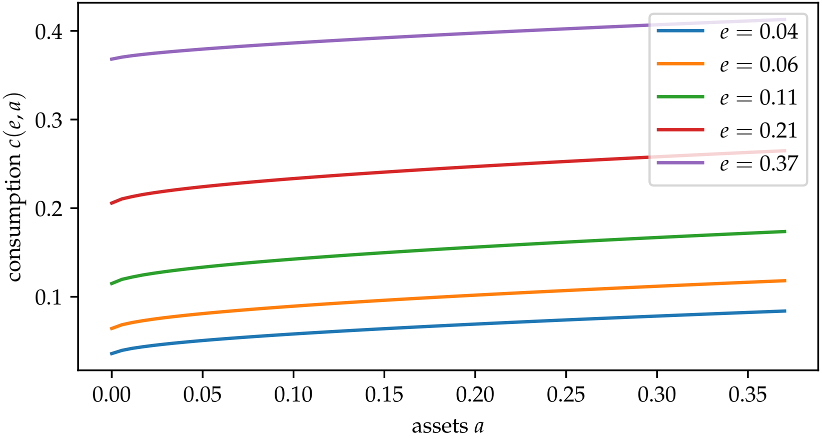
Step 2: Steady state policies: Details of EGM (backward iteration)

- Recall our steps:

$$V_{a,t+1}(e',a') \overset{\text{FOC}}{\to} c_t^{endo}(e,a') \overset{\text{interpolate}}{\to} a'_t(e,a) \overset{\text{budget}}{\to} c_t(e,a) \overset{\text{Envelope}}{\to} V_{a,t}(e,a)$$

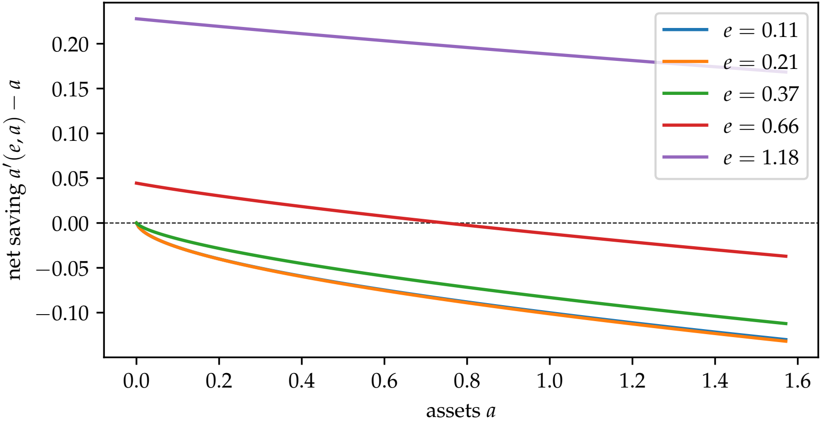
- We just found $a'_t(e,a)$, but that ignored the borrowing constraint.
    - Solution: just enforce it at this point, ie set $a'_t(e,a) \geq \underline{a}$
- Then compute $c_t(e,a)$ from $\text{coh}_t(e,a) - a'_t(e,a)$.
- Then use (1) for $V_{a,t}(e,a)$. Done!
- In the steady state: no $t \to$ just iterate until policies converge.

Step 2: Illustration



[Parameters: $Z = 0.7$, $\rho_e = (0.91)^{1/4}$, $\sigma_e = 0.92$, $n_e = 11$, $r = 0.02/4$, $\beta = 0.985$, $u = \log c$]

Step 2: Net saving policy



Step 3: Distribution (forward iteration)

Assume we have histogram (pdf) $D_t(e,a)$ over grid points for (e,a). (2-d array.)

Idea: use $a_t'(e,a)$ policy to send the mass on (e,a) to $(e',a_t'(e,a))$. Any issues?

- $a_t'(e,a)$ often falls in between grid points!

Solution: use lotteries! E.g. if for two grid points a_i, a_{i+1} we have

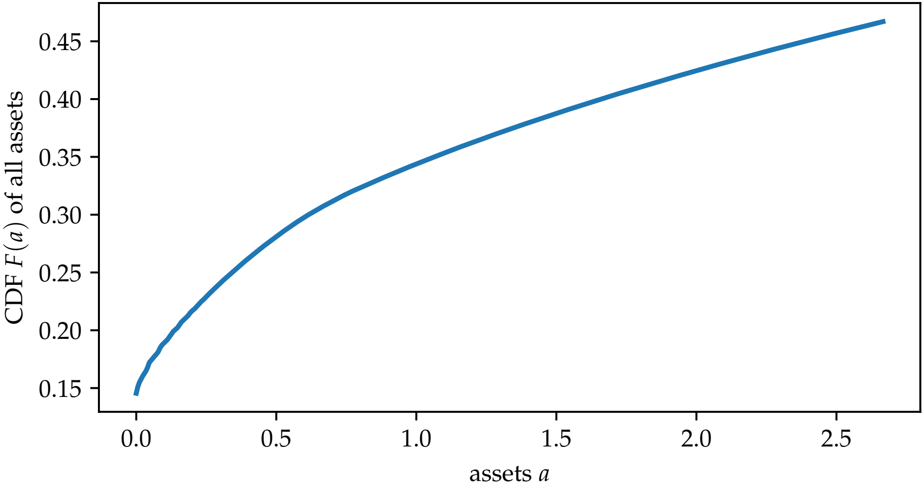
$$a_t'(e,a) = \pi a_i + (1-\pi)\, a_{i+1}$$

Then send mass $\pi D_t(e,a)$ to (e',a_i) and $(1-\pi)\, D_t(e,a)$ to (e',a_{i+1}).

- Often called “Young’s method”. Note: this is “non-stochastic simulation”
- Golden rule in this course: **avoid actual simulation if you can!**

Steady state: no t! Iterate to convergence...

Step 3: Illustrating the marginal asset distribution



Aggregate consumption and assets

$$C_t = \sum_{e,a} c_t(e,a) \cdot D_t(e,a)$$

$$A_t = \sum_{e,a} a_t'(e,a) \cdot D_t(e,a)$$

Again: get these via simple dot product of vectors. Do not simulate.

The marginal propensity to consume

Key object: the marginal propensity to consume (here in steady state)

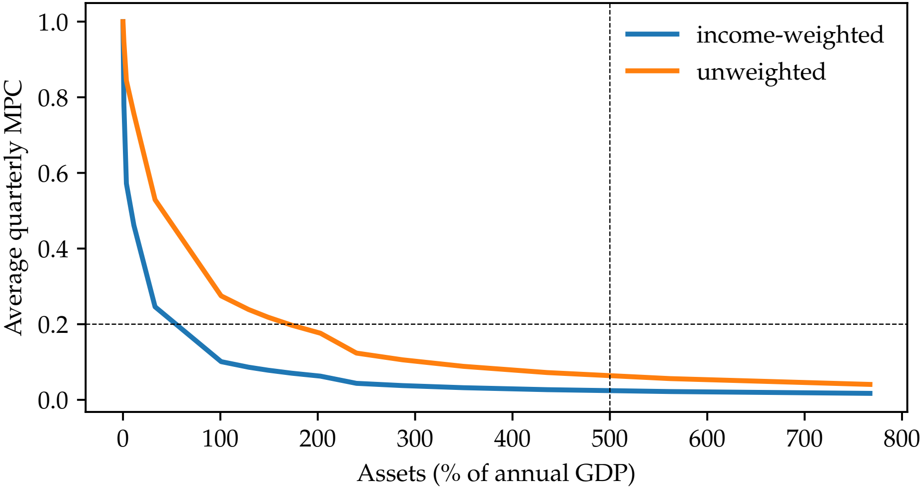
$$MPC\,(e,a) \equiv \frac{1}{1+r}\frac{\partial c(e,a)}{\partial a}$$

We'll that the average MPC matters a lot for the aggregate dynamics of the model

In the data, the average quarterly MPC is often estimated to be about 0.2 (Johnson et al. 2006, Jappelli and Pistaferri 2014, Fagereng et al. 2021, Boehm et al. 2025; see Auclert et al. 2023)

A key issue with the SIM model is that, without further changes, it cannot simultaneously match such a high MPC and high average assets (about 500×GDP in recent U.S. data)

Asset-MPC tradeoff visualized



Solving the asset-MPC tradeoff

The leading solutions to solve the asset-MPC tradeoff are:

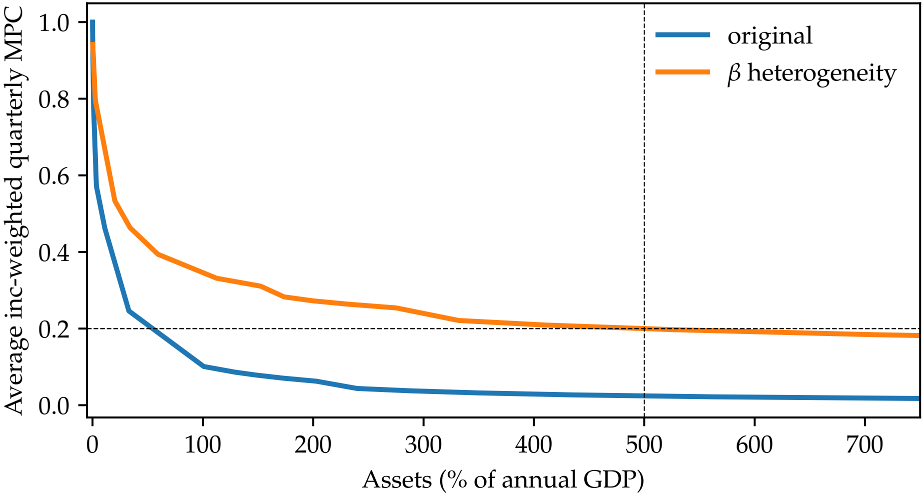
1. Introducing heterogeneity in discount rates β (eg, Krusell and Smith 1998)
2. Adding a illiquid account choice to the model (eg, Kaplan and Violante 2014, Kaplan et al. 2018, Bayer et al. 2019)

2 is nicer but more complicated. In this course we will follow 1 for simplicity.

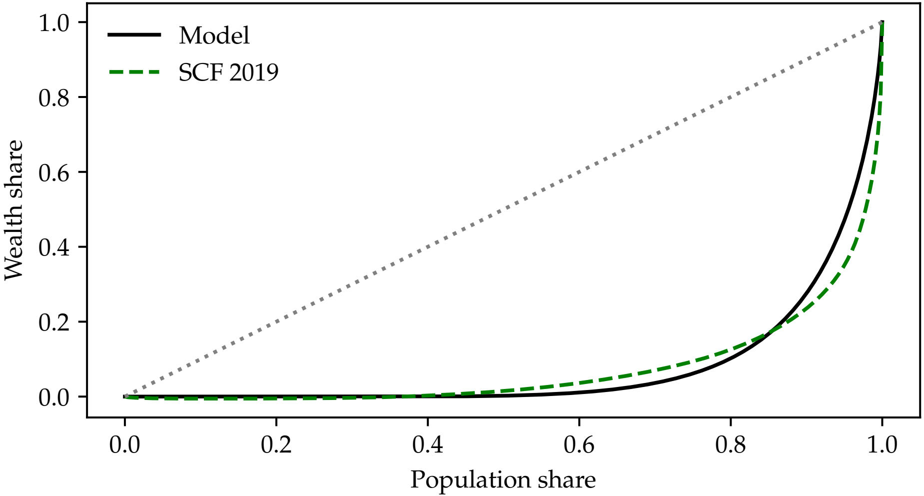
Implementation of β heterogeneity

- Add another exogenous discrete state β to $V(\beta, e, a)$
- Can make β either permanent or stochastic
  - latter limits polarization to "spenders" vs. "savers"
- Calibrate stochastic β process independent of e as in Auclert et al. (2025)
  - 4 equispaced βs with equal shares, each hh gets fresh β draw 1% of time
  - Loosely interpret as new draw of preferences every "generation" (25 years)
  - Calibrate to hit both $A/Y = 5$ and income-weighted MPC of 0.2
  - Calibrated quarterly βs: approximately 0.94, 0.96, 0.98, 1.0
- For next few lectures, will also consider "bonds only" calibration ($A/Y = 1$)
  - Calibrated βs: approximately 0.96, 0.97, 0.98 and 0.99x

Solving the MPC-asset tradeoff with β heterogeneity



Bonus: the model does a decent job at matching the wealth distribution!



Dynamics

Two kinds of dynamics ...

Partial equilibrium (PE): How do households respond to shock in interest rates or income

General equilibrium (GE): How do we find response of endogenous price (e.g. r_t) in response to shock (e.g. aggregate income Z_t)

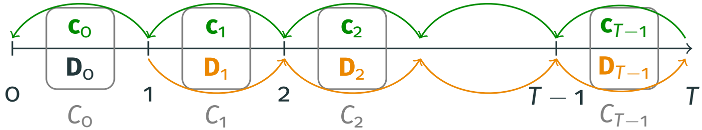
PE now, GE later!

Suppose households see time-varying $\{r_t^p\}$ or $\{Z_t\}$. What is C_t, A_t?

Our existing backward & forward iteration routines work:

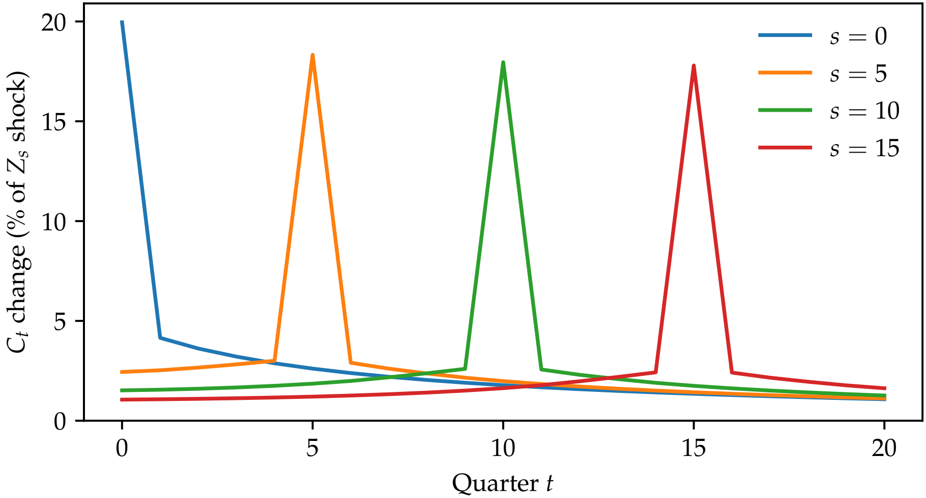
- From $V_{aT} = V_{a,ss}$, iterate backward$\rightarrow$ policies $a_t(e,a), c_t(e,a)$ for all $t \geq 0$.
- From $D_0(e,a) = D_{ss}(e,a)$, iterate forward using $a_t(e,a) \rightarrow D_t(e,a)$ for all $t \geq 1$.
- Compute $A_t = \sum_{e,a} a_t(e,a) D_t(e,a)$, $C_t = \sum_{e,a} c_t(e,a) D_t(e,a)$ as before!

Visualize this:

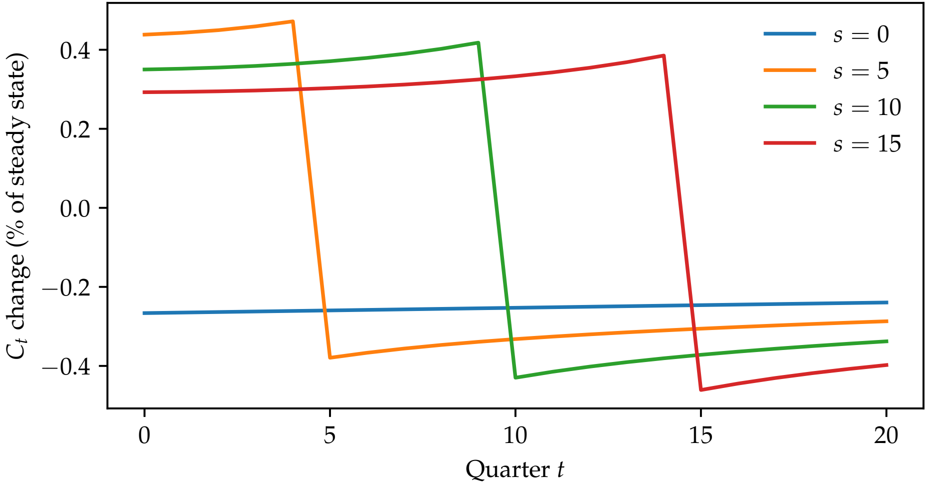


Let's look at an example ...

Shock to Z_s of 0.1% at various dates s



Shock to r_s^p of -1pp at various dates s



Denote the local impulse response to a one time income shock at date s by $\frac{\partial C_t}{\partial Z_s}$.

We can stack this into a matrix

$$\mathcal{J}^{C,Z} \equiv \left(\frac{\partial C_t}{\partial Z_s} \right)$$

Each column corresponds to the impulse response of C for a different s.

We will call $\mathcal{J}^{C,Z}$ **(sequence-space) Jacobians.** Those will turn out to be extremely useful objects for:

1. general equilibrium impulse responses
2. solving the model with recurring aggregate shocks

We'll discuss why in the next lectures

To compute Jacobians, we have to truncate to some finite time T, say $T = 300$.

To compute T columns, we need to go backward $T \times T$ times and forward $T \times T$ times. That can be slow...

Next: a method that can be used to compute the entire matrix in only T backward and T "forward iterations"! (Auclert et al., 2021)

Computing $\mathcal{J}$: Only T backward iterations

Denote by $c_t^s(s,a), a_t^s(s,a)$ the policies at date t in response to date s shock.

Why is a single backward iteration enough?

Key idea 1: Only the **time** $s-t$ **until the perturbation matters:**

$$c_t^s(e,a) = \begin{cases} c_{ss}(e,a) & s<t \\ c_{T-1-(s-t)}^{T-1}(e,a) & s \geq t \end{cases}$$

Thus, only need a single backward iteration with $s = T-1$ to get all the $c_t^s(e,a)$!

From this we get:

- $C_0^s = \sum_{e,a} c_0^s(e,a) D_{ss}(e,a)$, so first row of Jacobian $\mathcal{J}_{0s} = \frac{\partial C_0}{\partial y_s}$
- $D_1^s(e,a)$, distribution at date 1 implied by new policy $c_0^s(e,a)$ at date 0

Could keep going and compute $D_t^s(e,a)$, then $C_t^s = \mathbf{c}_t^{s\prime} \cdot \mathbf{D}_t^s$, for all s, t

But that's still time consuming. Can we do better? Yes!

Why? Observe that to first order,

$$dC_t^s = d\mathbf{c}_t^{s\prime} \cdot \mathbf{D}_{ss} + \mathbf{c}_{ss}' \cdot d\mathbf{D}_t^s$$

Consider first $s = 0$. For $t \geq 1$, only second term remains, with

$$d\mathbf{D}_t^0 = \left(\Lambda_{ss}'\right)^t d\mathbf{D}_1^0$$

where $\Lambda_{ss} =$ s.s. transition matrix for states (e, a) (Why?). So:

$$dC_t^0 = \mathbf{c}_{ss}' \cdot \left(\Lambda_{ss}'\right)^t d\mathbf{D}_1^0 \tag{3}$$

This gives us the first column of the Jacobian, $\mathcal{J}_{t,0}$. What about $s > 0$?

Computing $\mathcal{J}$: Other columns of Jacobian

Again, look at

$$dC_t^s = d\mathbf{c}_t^{s\prime} \cdot \mathbf{D}_{ss} + \mathbf{c}'_{ss} \cdot d\mathbf{D}_t^s$$

For $s > 0$, let's use our policy symmetry result. Since $d\mathbf{c}_t^s = d\mathbf{c}_{t-1}^{s-1}$, we have:

$$dC_t^s - dC_{t-1}^{s-1} = \mathbf{c}'_{ss} \cdot \left(d\mathbf{D}_t^s - d\mathbf{D}_{t-1}^{s-1}\right)$$

Let's call this $\mathcal{F}_{t,s}dx$. Using policy symmetry again, it is simple to show:

$$\mathcal{F}_{t,s}dx = \mathbf{c}'_{ss} \cdot \left(\Lambda'_{ss}\right)^t d\mathbf{D}_1^s \qquad \forall t, s \tag{4}$$

This looks exactly like (3)!

What's the meaning of $\mathcal{F}_{t,s}$? It's a "fake news shock"

- at $t = 0$: news shock that period $s > 0$ shock hits $\rightarrow$ distribution $d\mathbf{D}_1^s$
- at $t = 1$: news shock that there *won't* be a shock at s $\rightarrow$ use s.s. policies

Computing $\mathcal{J}$: Fake news matrix 1/2

Recall we want to know $\mathcal{J}_{t,s} \equiv \frac{\partial C_t}{\partial y_s}$ for $s, t \in \{0, \ldots, T-1\}$.

Can think of $\mathcal{J}$ as a **news matrix:**

- column s = response to news that shock is going to hit in period s

We just saw how to get:

- the first row $\mathcal{J}_{0,s}$
- the first column $\mathcal{J}_{t,0}$
- differences in columns $\mathcal{F}_{t,s} \equiv \mathcal{J}_{t,s} - \mathcal{J}_{t-1,s-1}$ for all $s, t \geq 1$

Key idea 2: can recover $\mathcal{J}$ from first row, first column, and **"fake news" matrix** $\mathcal{F}$.

- News shock = sequence of fake news shocks

Computing $\mathcal{J}$: Fake news matrix 2/2

Expand $\mathcal{F}$ to include first row and column of $\mathcal{J}$:

$$\mathcal{F} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} - \mathcal{J}_{00} & \mathcal{J}_{12} - \mathcal{J}_{01} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} - \mathcal{J}_{10} & \mathcal{J}_{22} - \mathcal{J}_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathcal{J} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} & \mathcal{J}_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- can recover $\mathcal{J}$ from $\mathcal{F}$ by adding elements from top left diagonal
- ie, the formula implementing Key idea 2 is:

$$\mathcal{J}_{t,s} = \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{t-k,s-k} \tag{5}$$

Define **expectations function**

$$\mathcal{E}_t(e,a) \equiv \mathbb{E}\left[c(e_t,a_t)|e_0 = e, a_0 = a\right]$$

$=$ expected consumption in t periods for an agent in state (e,a) today.

Expectations function is easy to compute: Stack as vector across (e,a), ...

$$\boldsymbol{\mathcal{E}}_0 = \mathbf{c}_{ss} \qquad \boldsymbol{\mathcal{E}}_t = \Lambda_{ss}\boldsymbol{\mathcal{E}}_{t-1}$$

Key idea 3: $\boldsymbol{\mathcal{E}}_t$'s are enough to get all rows $t \geq 1$ of $\mathcal{F}$

- First column $s = 0$: use (3), $\mathcal{J}_{t,0}dx = (\boldsymbol{\mathcal{E}}_t)' \, d\mathbf{D}_1^0$
- Other columns: use (4), $\mathcal{F}_{t,s}dx = (\boldsymbol{\mathcal{E}}_t)' \, d\mathbf{D}_1^s$

[Recall that we got the first row as well as $d\mathbf{D}_1^0$ from key idea 1]

Wrapping up

- **Fake news algorithm** for obtaining Jacobian $\mathcal{J}$ of aggregate (eg, C) to input of a heterogeneous-agent problem (eg, Z) relies on following steps:

1. Get all policies $\mathbf{c}_t^s$ and date-0 perturbation $d\mathbf{D}_1^0$ to distn with one backward iteration
2. Calculate steady state expectation vectors $\mathcal{E}_t$
3. Build the Fake news matrix $\mathcal{F}_{t,s}$ using info from steps 1 and 2
4. Build Jacobian $\mathcal{J}$ from $\mathcal{F}$ using equation (5)

Conclusion

Taking stock

- Introduced the “standard incomplete markets model”
- Computed steady state, discussed asset-MPC tradeoff
- Computed partial equilibrium impulse responses
- Collected impulse responses to one-time “unit” shocks in Jacobians
- Discussed “fake news algorithm” to compute those Jacobians very quickly

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