Lecture 4: Solving in Sequence Space

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Introduction

- **Lectures so far**: MIT shocks (nonlinear and linear)
- Fine object for very rare events, but for business-cycle-like shocks, **rational expectations** is more useful benchmark
  - How useful are MIT shocks for rational expectations solution?
- Our answer: **very useful!**
  - In fact, **MIT shocks are all you need**! ("certainty correspondence")
  - We'll start simple: MIT shocks are all you need **to first order** ("certainty equivalence"). Apply this to **simulations** + **estimation**.
  - Then we'll see how this generalizes to higher order

First-order certainty equivalence

Setup

- Consider general macro model:

$$F\left(X_{t-1}, X_t, \mathbb{E}_t\left[X_{t+1}\right], \epsilon_t\right) = 0$$

  - $X_t \in \mathbb{R}^n$ are endogenous variables [het-agent applications: n large, eg $n = 2{,}000$]
  - $\epsilon_t \in \mathbb{R}$ are exogenous innovations (realized shock process) [naturally extends to >1 shocks]
    - Distributed $\epsilon_t = \sigma_R \cdot \bar{\epsilon}_t$ with $\bar{\epsilon}_t$ iid, mean 0, variance 1, and symmetric density
- Expectations $\mathbb{E}_t[\,\cdot\,]$ are taken assuming future distribution of ϵ_t is $\sigma \cdot \bar{\epsilon}_t$
  - **Rational expectations**: $\sigma = \sigma_R$ (perceived distribution of future ϵ_t = truth)
  - **Perfect foresight**: $\sigma = 0$ (agents think all future ϵ_t will realize to 0 for sure)
- **Deterministic steady state** $X \in \mathbb{R}^n$ solves $F(X, X, X, 0) = 0$ (i.e. $\sigma = \sigma_R = 0$)

PE SIM example

- For concreteness, consider the PE SIM model

$$V_t(e,a) = \max_{c,a'} u(c) + \beta \mathbb{E}_{e'}[V_{t+1}(e',a') \,|\, e]$$

$$\text{s.t.} \qquad a' + c = (1+r)a + Z_t e \qquad a' \geq 0$$

- Aggregate income Z_t follows AR(1) process: $Z_t - 1 = \rho(Z_{t-1} - 1) + \epsilon_t$
- Here, $X_t \equiv (\boldsymbol{v}_t, \boldsymbol{\phi}_t, C_t, Z_t)$ where $\boldsymbol{v}_t$ stores marginal value $V_{a,t}$ over gridpoints (e,a), $\boldsymbol{\phi}_t$ stores the p.d.f, and C_t aggregates consumption policy over distribution
- Equations describing F: the law of motion of Z_t, plus

Dynamic programming: expectations enter via $\mathbb{E}_t[v_{t+1}]$

$$\boldsymbol{v}_t = \mathscr{V}(Z_t, \mathbb{E}_t \boldsymbol{v}_{t+1}) \qquad \boldsymbol{\phi}_t = \Lambda(Z_t, \mathbb{E}_t \boldsymbol{v}_{t+1}, \boldsymbol{\phi}_{t-1}) \qquad C_t = \mathscr{C}(Z_t, \mathbb{E}_t \boldsymbol{v}_{t+1}, \boldsymbol{\phi}_{t-1})$$

(Stochastic) sequence space solution

- Given σ, consider **nonlinear sequence-space solution** to F:

$$X_t = \mathcal{X}\left(\sigma; \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots\right) \quad \text{[Lan-Meyer-Gohde, Lombardo-Uhlig]}$$

[eg, obtained from stable state-space solution $X_t = g\left(X_{t-1}, \epsilon_t; \sigma\right)$]

- We are going to relate this to the **perfect-foresight solution**

$$X_t^{PF} = \mathcal{X}\left(0; \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots\right)$$

[eg, obtained from ∞-long sequence of repeated-surprise MIT shocks]

- Use **perturbation approach**: linearize $\mathcal{X}$ around $\sigma = \sigma_R = 0$ (i.e. $\boldsymbol{\epsilon} = \mathbf{0}$)
- Helpful result: $\mathcal{X}\left(\sigma; \epsilon_t, \epsilon_{t-1}, \ldots\right) = \mathcal{X}\left(-\sigma; \epsilon_t, \epsilon_{t-1}, \ldots\right)$, so, e.g., $\mathcal{X}_\sigma(\mathbf{0}) = 0$

MIT shock impulse response

- Special case of X^{PF} is the MIT shock impulse response

$$X_j^{MIT}(\nu) \equiv \mathscr{X}(0; 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{position } j}}{\nu}, 0 \ldots)$$

- Can compute $\{X_j^{MIT}(\nu)\}_j$ with nonlinear perfect foresight starting from s.s.
  - cf methods from Lecture 1!
- **Next**: why this is sufficient to get the first-order solution with aggregate risk!

Certainty equivalence

- Expand $X_t = \mathscr{X}\left(\sigma, \epsilon_t, \epsilon_{t-1}, \ldots\right)$ to first order in $(\sigma, \boldsymbol{\epsilon})$:

No anticipated risk,
no past realization

$$X_t = X + \cancel{\mathscr{X}_\sigma} \sigma + \frac{\partial \mathscr{X}}{\partial \epsilon_0}\left(\mathbf{0}\right)\epsilon_t + \frac{\partial \mathscr{X}}{\partial \epsilon_{-1}}\left(\mathbf{0}\right)\epsilon_{t-1} + \cdots + \mathcal{O}(\sigma^2)$$

Symmetry:
Steady state unchanged

- So, to first order, X_t follows **linear MA** process, with coefficients equal to

$$\frac{\partial \mathscr{X}}{\partial \epsilon_{-j}}\left(\mathbf{0}\right) = \frac{dX_j^{MIT}}{d\nu}(0) \equiv \mathscr{X}_j$$

Certainty equivalence. To first order, X_t follows a linear MA process whose coefficients are given by the first order MIT shock impulse response of X to ϵ [Boppart-Krusell-Mitman]

Implementation using SSJ

- To first-order, X_t (so all its elements!) follows linear $MA(\infty)$

$$X_t = X + \mathscr{X}_0\epsilon_t + \mathscr{X}_1\epsilon_{t-1} + \mathscr{X}_2\epsilon_{t-2} + \cdots + \mathcal{O}(\sigma^2) \qquad \mathscr{X}_j \equiv \frac{dX_j^{MIT}}{d\nu}(0)$$

- Convenient: we can use SSJ to very effectively get $\mathscr{X}_j$!

- For instance, suppose that we want the behavior of aggregate outcome X_t to monetary shock r_s in the canonical HANK model, with $MA(\infty)$ shock process

$$r_t = r + \mathscr{R}_0\epsilon_t + \mathscr{R}_1\epsilon_{t-1} + \mathscr{R}_2\epsilon_{t-2} + \cdots +$$

- Then we get the SSJ $G^{X,r}$ of X to r, then apply vector-matrix multiplication

$$(\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2, \cdots)' = G^{X,r} \cdot (\mathscr{R}_0, \mathscr{R}_1, \mathscr{R}_2, \cdots)'$$

Applications: simulation, second moments, estimation

Application to simulation

- Suppose we want to do stochastic simulations of X_t. Since

$$X_t \simeq X + \mathscr{X}_0 \epsilon_t + \mathscr{X}_1 \epsilon_{t-1} + \mathscr{X}_2 \epsilon_{t-2} + \cdots \qquad (\star)$$

- Once we have $\mathscr{X}_t$, we can just simulate by applying $(\star)$!

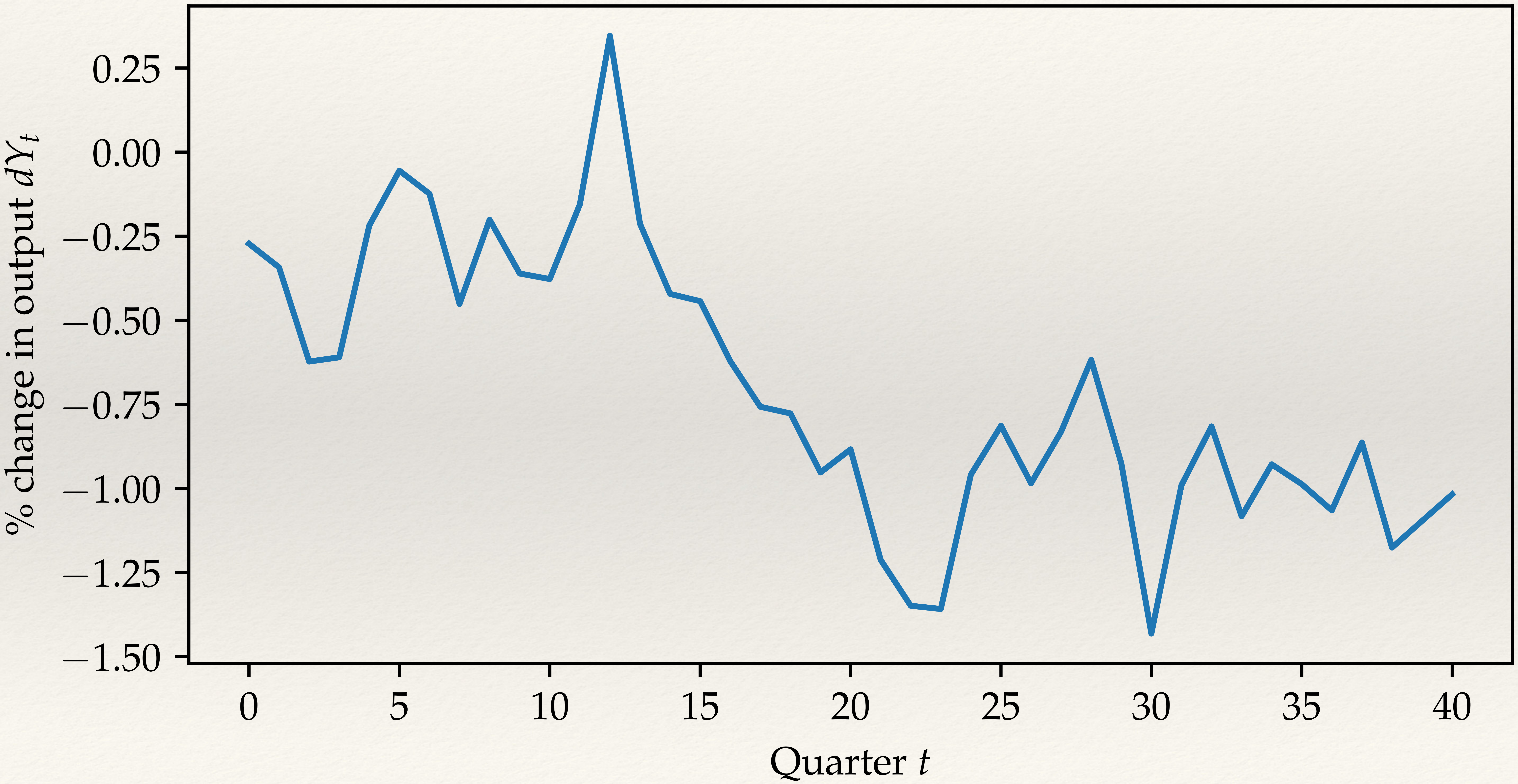
```python
rho = 0.9
dishock = rho**np.arange(T)
dY = G['Y', 'ishock'] @ dishock
```

✓ 0.0s

```python
sigma = 0.005
Tsim = 1_000_000                  # will simulate for 1 million quarters
np.random.seed(40)
eps = np.random.randn(Tsim+T)     # draw epsilons for T additional pre-per
dY_sim = np.empty(Tsim)
for t in range(Tsim):
    # at each t, take window of current up to T-1 previous epsilons
    # take dot product with MA coefficients implied by MIT shock
    dY_sim[t] = np.dot(dY, eps[t:t+T][::-1]) * sigma
```

✓ 0.7s Python

Simulating output response to monetary shock



Application to second moments

- Suppose we want to obtain second moments of X_t (with $\sigma_R = \sigma$). Have

$$X_t \simeq X + \mathscr{X}_0 \epsilon_t + \mathscr{X}_1 \epsilon_{t-1} + \mathscr{X}_2 \epsilon_{t-2} + \cdots \qquad (\star)$$

and we can just apply time series formulas to get these!

- For instance ergodic variance is area under squared impulse response

$$\mathrm{Var}(X_t) = \sigma^2 \sum_{j=0}^{\infty} \left(\mathscr{X}_j\right)^2$$

- More generally

$$\mathrm{Cov}(X_t, X_{t-k}) = \sigma^2 \sum_{j=0}^{\infty} \mathscr{X}_j \mathscr{X}_{j+k}$$

Implementation: second moments in HANK model

```python
var_dY = sigma**2 * (dY @ dY)
```

Python

Compare to the sample variance to make sure this is right. Note that it's close but not exact, reflecting Monte Carlo error:

```python
sample_var_dY = (dY_sim - dY_sim.mean()) @ (dY_sim - dY_sim.mean()) / Tsim
print(f"Sample variance: {sample_var_dY:.4e} | theoretical : {var_dY:.4e}")
```

Python

```
Sample variance: 3.1761e-05 | theoretical : 3.1870e-05
```

Application to impulse-response matching

- Straightforward to estimate by **matching impulse responses**
- Suppose model parameters are θ, giving impulse response $IRF_t(\theta)$, and identified empirical impulse response $\hat{irf}_t$
- Then we can just search for

$$\hat{\theta} = \arg\min \left(IRF(\theta) - \hat{irf} \right)' V^{-1} \left(IRF(\theta) - \hat{irf} \right)$$

where, e.g. V has sample variances of $\hat{irf}_t$ (Christiano, Eichenbaum, Evans)

- This is especially fast if we do not have to recompute all Jacobians for $IRF(\theta)$, such as we estimate shock process parameters (G unchanged!)

Application to method of moments

- Also straightforward to estimate by **matching moments**
- Suppose model parameters are θ, giving model moments $M(\theta)$
  - eg second moments, HP-filtered moments, regression coefficients…
- Then we can confront to empirical counterparts $\hat{m}$ by taking

$$\hat{\theta} = \arg\min \left(M(\theta) - \hat{m}\right)' V^{-1} \left(M(\theta) - \hat{m}\right)'$$

where, e.g. V is model's $\mathbb{E}\left[M(\theta) M(\theta)'\right]$

(Minimum distance estimation, aka "simulated" method of moments)

Application to likelihood-based estimation

- Let's assume that model innovations ϵ_t are normally distributed
- Given parameter θ, first-order process followed by X_t is

$$X_t \simeq X(\theta) + \mathcal{X}_0(\theta)\epsilon_t + \mathcal{X}_1(\theta)\epsilon_{t-1} + \mathcal{X}_2(\theta)\epsilon_{t-2} + \cdots$$

- Given data $\mathbf{X}$, we know the **(log) likelihood function**

$$\mathcal{L}(\mathbf{X};\theta) = -\frac{1}{2}\log\det\mathbf{V}(\theta) - \frac{1}{2}\mathbf{X}'\mathbf{V}(\theta)^{-1}\mathbf{X}$$

where $\mathbf{V}(\theta)$ is variance-covariance matrix of X_t at all lags

- In practice, use Cholesky decomposition to calculate $\det\mathbf{V}$ and $\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$

Application to likelihood-based estimation

- Given likelihood function $\mathscr{L}(\mathbf{X};\theta)$ we can in turn do:
  - Maximum likelihood estimation

$$\hat{\theta} = \arg\max \mathscr{L}\left(\boldsymbol{Y}, \theta\right)$$

  - Bayesian estimation given prior $p\left(\theta\right)$: find posterior distribution

$$p\left(\theta | \mathbf{Y}\right) = \frac{\mathscr{L}\left(\mathbf{Y} | \theta\right) p\left(\theta\right)}{p\left(\mathbf{Y}\right)} \propto \mathscr{L}\left(\mathbf{Y} | \theta\right) p\left(\theta\right)$$

- For instance, posterior mode just solves

$$\hat{\theta} = \arg\max p\left(\theta | \mathbf{Y}\right)$$

[More complex: simulate from posterior using MCMC]

Practical implementation: estimate HANK model!

- Online notebook estimates HANK model with
  - 3 shocks (government spending, TFP, monetary policy)
  - 3 observables (output, inflation, interest rates)
- Look for
  - shock process parameters (assumed AR(1)'s: 3 σ + 3 ρ)
  - coefficient on Taylor rule ϕ_π, fiscal rule ϕ_τ, slope of Phillips curve κ_w
- Done in less than a minute on a laptop!

Second-Order Certainty Correspondence

Consecutive MIT shocks

- Recall special case of X^{PF} is the MIT shock impulse response

$$X_j^{MIT}(\nu) \equiv \mathscr{X}(0; 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{position } j}}{\nu}, 0 \ldots)$$

- Similarly, for $l \geq 1$ we have the sequence of two MIT impulse responses

$$X_j^{MIT,l}(\nu, \epsilon) \equiv \mathscr{X}(0; 0, \ldots 0, \underset{\substack{\uparrow \\ \text{position } j}}{\nu}, 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{position } j+l}}{\epsilon}, 0 \ldots)$$

- For each l, can compute $\{X_j^{MIT,l}(\nu, \epsilon)\}_j$ starting from $X_l^{MIT}(\epsilon)$

- **Next**: why these are sufficient to get the second-order solution!

Second-order sequence space perturbation

- Now expand to second order in $(\sigma, \boldsymbol{\epsilon})$ around $(0,\mathbf{0})$ ("Volterra expansion")
- Use symmetry result again: $\dfrac{\partial^2 \mathscr{X}}{\partial\sigma\partial\epsilon_{-j}}(\mathbf{0}) = 0$
- Find:

Still no anticipated risk
or past realization

$$X_t = X + \sum_{j=0}^{\infty} \mathscr{X}_j \epsilon_{t-j} + \frac{1}{2}\left(\mathscr{X}_{\sigma\sigma}(\mathbf{0})\,\sigma^2 + \sum_{j,k\geq 0} \frac{\partial^2 \mathscr{X}}{\partial\epsilon_{-j}\partial\epsilon_{-k}}(\mathbf{0})\,\epsilon_{t-j}\epsilon_{t-k} \right) + \mathcal{O}(\sigma^3)$$

quadratic MA process

- Rearrange and interpret…

Interpreting the quadratic MA process

Anticipated risk **Size dependence** **State dependence**

$$X_t \simeq X + \sum_{j=0}^{\infty} \mathcal{X}_j \epsilon_{t-j} + \boxed{\frac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2} + \boxed{\frac{1}{2}\sum_{j=0}^{\infty} \frac{\partial^2 \mathcal{X}}{\partial \epsilon_{-j}^2}\epsilon_{t-j}^2} + \boxed{\sum_{j=0}^{\infty}\sum_{l=1}^{\infty} \frac{\partial^2 \mathcal{X}}{\partial \epsilon_{-j}\partial \epsilon_{-(j+l)}}\epsilon_{t-j}\epsilon_{t-(j+l)}}$$

- Size and state dependence are second-order MIT shock impulses

$$\frac{\partial^2 \mathcal{X}}{\partial \epsilon_{-j}^2} = \frac{d^2 X_j^{MIT}}{d\nu^2}(0) \equiv \mathcal{X}_{jj} \qquad \frac{\partial^2 \mathcal{X}}{\partial \epsilon_{-j}\partial \epsilon_{-(j+l)}} = \frac{\partial^2 X_j^{MIT,l}}{\partial \epsilon \partial \nu}(0,0) \equiv \mathcal{X}_{jj+l}$$

- **Next:** How to solve for $\mathcal{X}_{\sigma\sigma}$ using only perfect foresight as well!

Anticipated risk term

- How do we compute **anticipated risk term** $\frac{1}{2}\mathscr{X}_{\sigma\sigma}\sigma^2$?
- Idea: solve for the **risky steady state** with $\sigma_R = 0$ but $\sigma > 0$
- If all shocks have realized to $\epsilon_{t-j} = 0$ in the past then we have:

$$X_t = X_{t-1} \simeq \boxed{X + \frac{1}{2}\mathscr{X}_{\sigma\sigma}\sigma^2} \qquad \mathbb{E}_t X_{t+1} \simeq \boxed{X + \frac{1}{2}\mathscr{X}_{\sigma\sigma}\sigma^2} + \boxed{\frac{1}{2}\mathscr{X}_{00}\sigma^2}$$

Risky steady state (unknown)

Anticipation of impact size dependence (known)

- Find this X^* by using steady state code to solve $F\left(X^*, X^*, X^* + \frac{1}{2}\mathscr{X}_{00}\sigma^2, 0\right) = 0$

i.e. add impact size dependence $\mathscr{X}_{00}\sigma^2/2$ to $\boldsymbol{v}$ in equations where $\mathbb{E}_t[\boldsymbol{v}_{t+1}]$ appears

Bottom line: solving the quadratic MA process

Anticipated risk **Size dependence** **State dependence**

$$X_t \simeq X + \sum_{j=0}^{\infty} \mathcal{X}_j \epsilon_{t-j} + \boxed{\frac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2} + \boxed{\frac{1}{2}\sum_{j=0}^{\infty} \mathcal{X}_{jj}\epsilon_{t-j}^2} + \boxed{\sum_{j=0}^{\infty}\sum_{l=1}^{\infty} \mathcal{X}_{j,j+l}\epsilon_{t-j}\epsilon_{t-(j+l)}}$$

Compute by modifying steady state equations

Compute with second-order MIT shock

Compute this with two successive MIT shocks, $\times L$

- **Next:** Implications for model impulse responses and moments

Generalized impulse response

- Define generalized IRF of X as

$$IRF_j\left(\nu, \{\epsilon_{-k}\}_{k=1}^{\infty}\right) \equiv \mathbb{E}\left[X_j \mid \epsilon_0 = \nu, \{\epsilon_{-k}\}_{k=1}^{\infty}\right] - \mathbb{E}\left[X_j \mid \{\epsilon_{-k}\}_{k=1}^{\infty}\right] \quad j \geq 0$$

- We have:

First-order IRF **Size dependence** **State dependence**

$$IRF_j\left(\epsilon_0 = \nu, \{\epsilon_{-k}\}_{k=1}^{\infty}\right) = \boxed{x_j \nu} + \boxed{\frac{1}{2}\mathcal{X}_{jj}\left(\nu^2 - \sigma^2\right)} + \boxed{\sum_{l=1}^{\infty} \mathcal{X}_{j,j+l}\nu\epsilon_{-l}}$$

Linear, not state dependent

Subtracts *expected size dependence*

Interaction of date-0 shock with shock *l* periods ago

Model moments

- Ergodic (long-run) mean under rational expectations ($\sigma_R = \sigma$)

$$\mathbb{E}[X] = \boxed{X + \frac{1}{2}\mathcal{X}_{\sigma\sigma}\sigma^2} + \boxed{\frac{1}{2}\sum_{j=0}^{\infty}\mathcal{X}_{jj}\sigma^2}$$

Risky steady state **Expected size dependence**

- Ergodic variance from 2nd-order *MA*:

$$\mathrm{Var}(X) = \boxed{\sigma^2\sum_{j=0}^{\infty}\left(\mathcal{X}_j\right)^2} + \boxed{\frac{\sigma^4}{2}\sum_{j,k=0}^{\infty}\left(\mathcal{X}_{jk}\right)^2}$$

Implied by 1st-order MA (3rd order accurate) **Same as in pruned 2nd-order state-space**

- In paper: 4th-order accurate Var $\left(\mathcal{X}(\sigma; \sigma\bar{\epsilon}_t, \sigma\bar{\epsilon}_{t-1}, \ldots)\right)$ adds terms from 3rd-o. *MA*

Application to smooth SIM model

Smoothing the SIM model

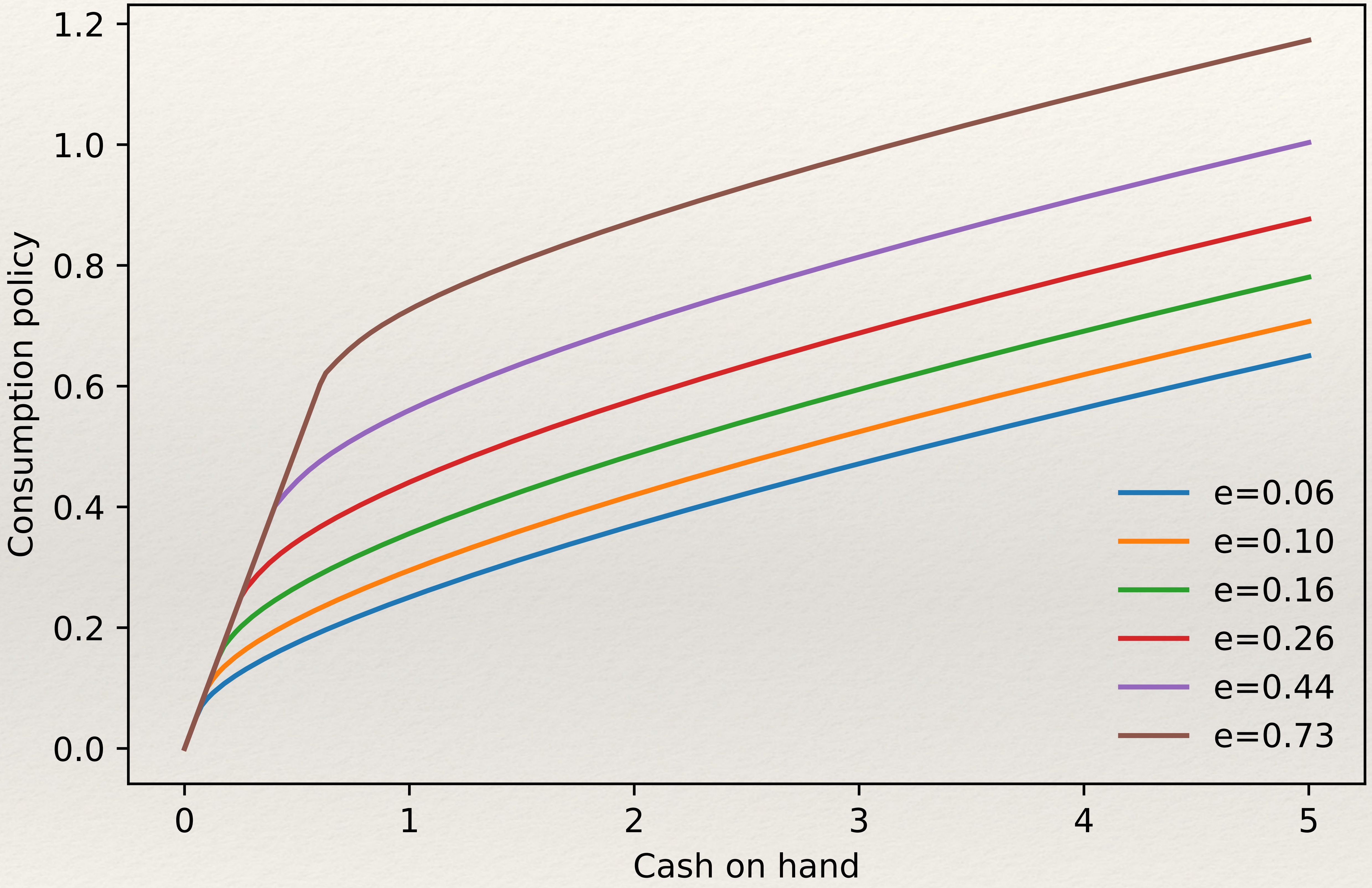
- Problem with SIM model so far: $C_t(\{Z_s\})$ it is not differentiable to 2nd order!
- This is due to non-smooth underlying model: MPC jumps + mass points
- Solution: make the model smoother! One simple way:

$$V_t(e,a) = \mathbb{E}_\xi \max_{c,a'} u(c) + \beta \mathbb{E}_{e'}[V_{t+1}(e',a') \mid e]$$

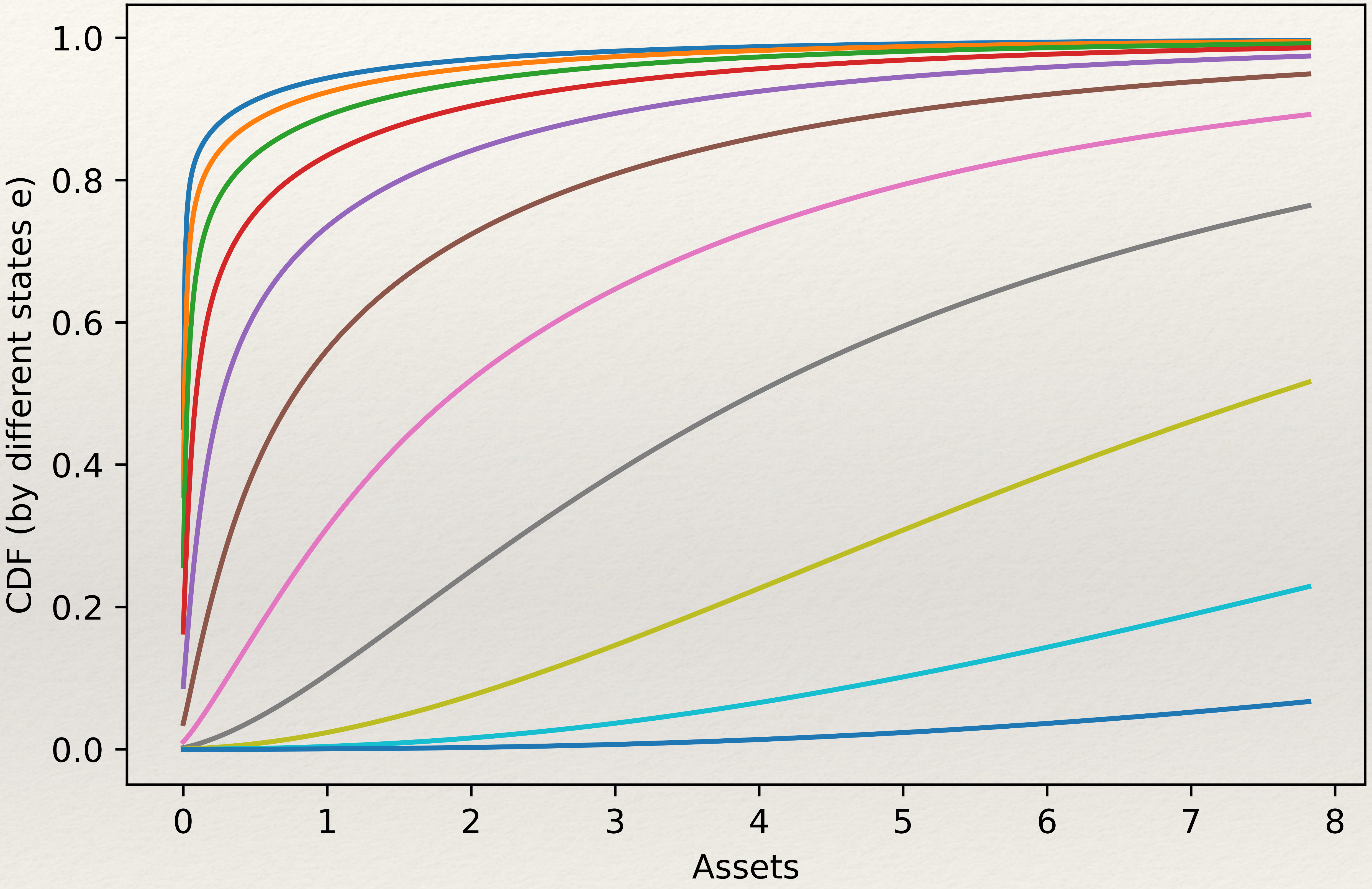
$$\text{s.t.} \qquad a' + c = (1+r)a + Z_t \xi e \qquad a' \geq 0$$

  - e follows discrete Markov chain, ξ is **iid, continuous, mean 1**
- Rest of setup same as before: $X_t \equiv (\boldsymbol{v}_t, \boldsymbol{\phi}_t, C_t, Z_t)$ where $\boldsymbol{v}_t$ stores marginal value $V_{a,t}$ over gridpoints (e,a), $\boldsymbol{\phi}_t$ now stores the c.d.f over (e,a')

What ξ shocks give us: smooth steady state



Single kink in consumption policy for each income state

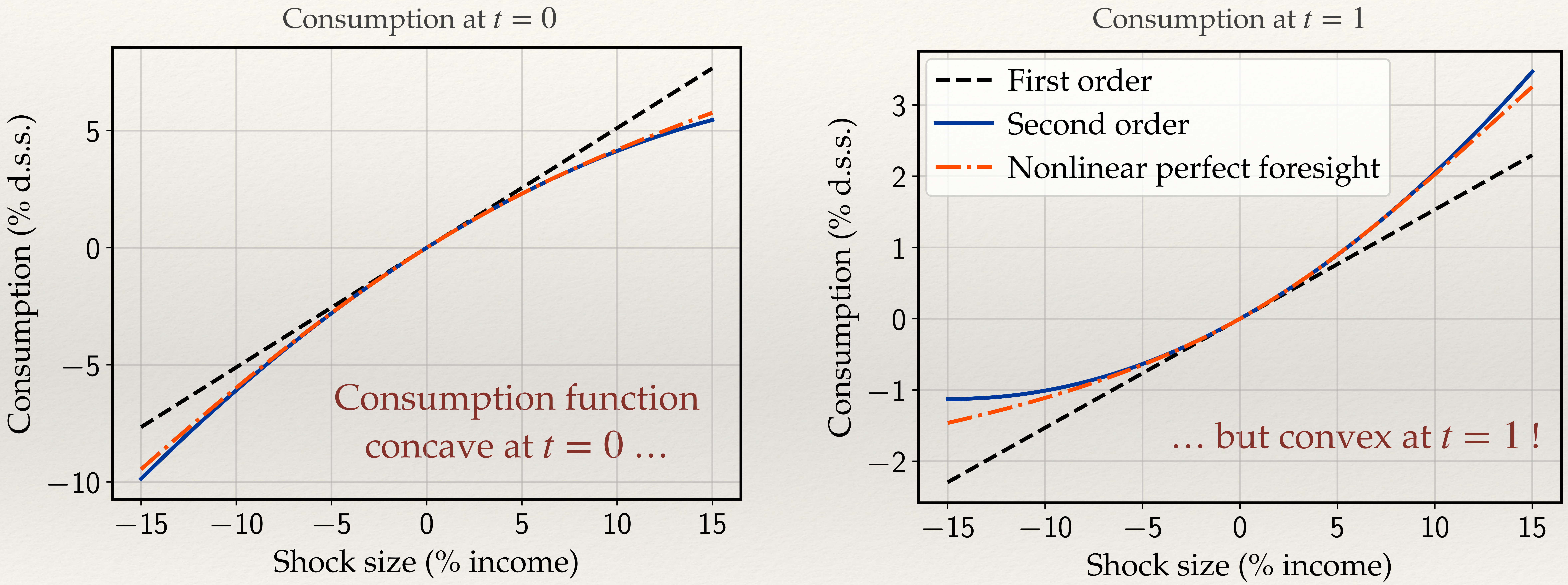


No jumps in CDF except at the constraint

- Now we can take 2nd+ order derivatives of model !

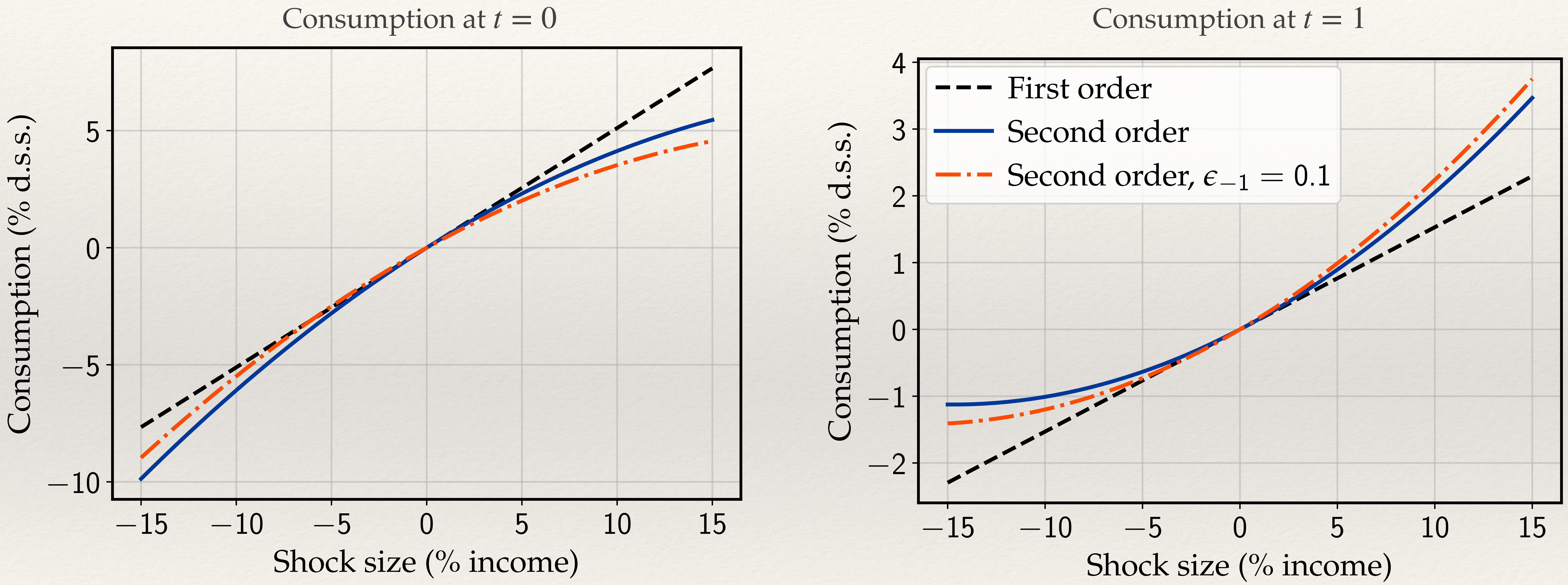
[cf Bhandari-Bourani-Evans-Golosov, Bianchi-Kaplan]

Application: size dependence in consumption



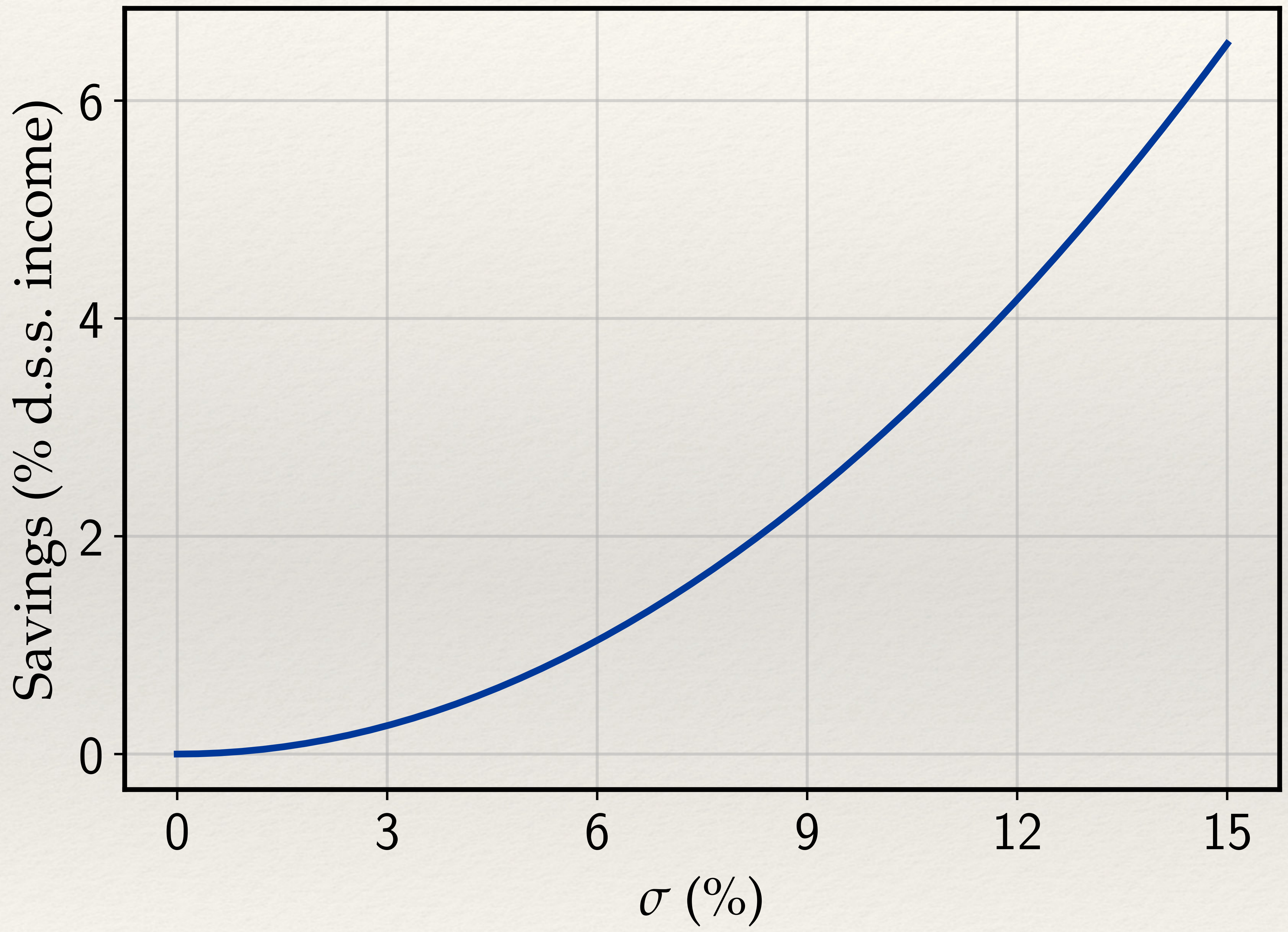
- Nonlinear iMPCs! Large shock → lower MPC today … but higher tomorrow!

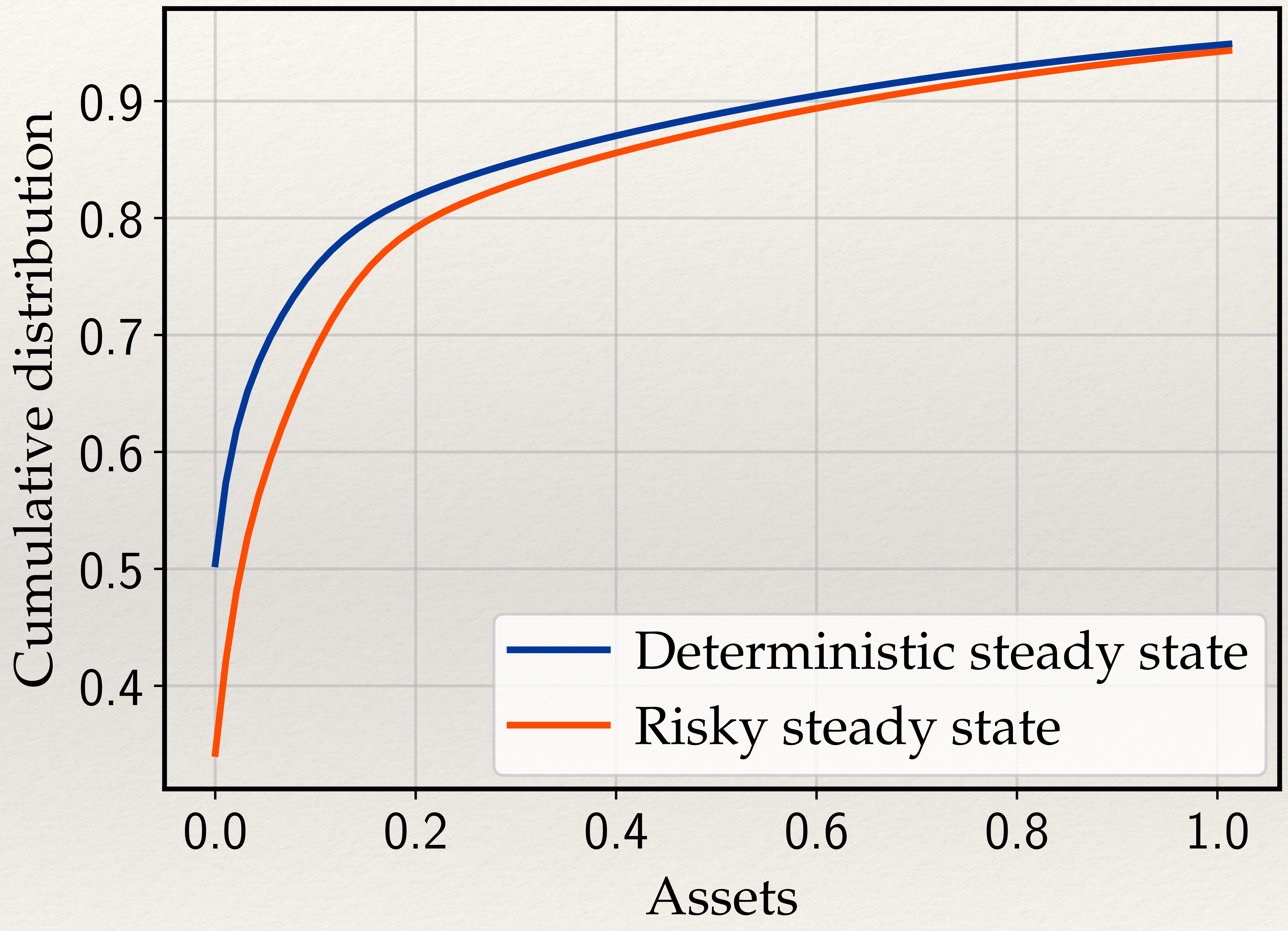
Application: history dependence in consumption



- High income shock in past implies more assets today, lower MPCs

Application: aggregate precautionary savings





- Perceived aggregate income risk raises risky steady-state savings, lowers MPCs

Third-Order Certainty Correspondence

Third-order sequence space perturbation

- Continue expanding to 3rd order, with symmetry implying $\mathcal{X}_{\sigma jk} = 0$
- Obtain **cubic** MA process:

$$X_t \approx X_t^{(2)} + \boxed{\frac{1}{6}\sum_{j=0}^{\infty} \frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-j} \partial \sigma^2} \epsilon_{t-j} \sigma^2} + \boxed{\frac{1}{6}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-j} \partial \epsilon_{-k} \partial \epsilon_{-l}} \epsilon_{t-j}\epsilon_{t-k}\epsilon_{t-l}}$$

- All **triple interactions** can be obtained from 1, 2, or 3 consecutive MIT shocks
- **New term** includes 2 effects:
  - How concerns about future risk affects first-order impulse response
  - How past shocks affect strength of precautionary motive

New term in $\epsilon_{t-j}\sigma^2$

- Split the new term into these two effects

$$\frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-j} \partial \sigma^2} = \boxed{\left(\frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-j} \partial \sigma^2}\right)^{rss}} + \boxed{\left(\frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-j} \partial \sigma^2}\right)^{tvps}}$$

Effect of risky steady state on impulse response

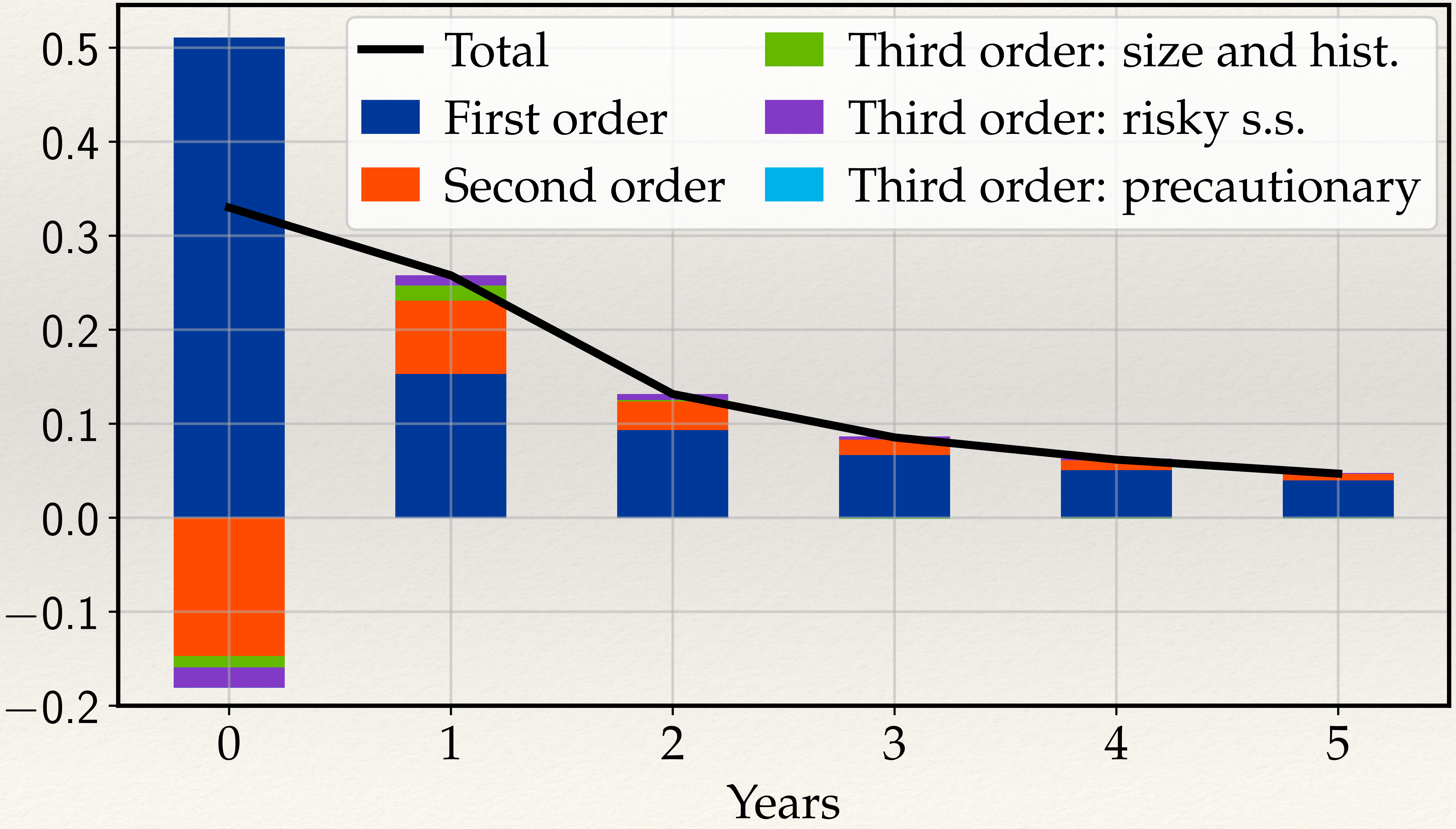
Effect of shock at date $-j$ on precautionary motive at date 0

- Get **1st** using impulse routines starting from the RSS distribution and expectations
- Get **2nd** by constructing a deterministic sequence X_t of perturbed expectations:

$$F\left(X_{t-1}, X_t, X_{t+1} + \frac{1}{6}\frac{\partial^3 \mathcal{X}}{\partial \epsilon_{-t} \partial \epsilon_0^2}\sigma^2, 0\right) = 0$$

Application: third order iMPCs

- Generalized impulse response of C_t to innovation in income:



Summary: certainty correspondence

- **Certainty correspondence:** can obtain 2nd and 3rd(+) order perturbation solutions to rational expectation solutions with perfect foresight methods!
  - … despite agents being forward looking and anticipating future aggregate risk!
- Opens up lots of possibilities for research
  - state-dependent or size-dependent policies (e.g. fiscal or monetary)
  - welfare cost of business cycles
  - steady-state and time variation in precautionary savings or risk premia…

Brief application to GE HANK model

Speeding up the GE solution using Jacobians

- Asset market clearing error: $H_t\left(\{Y_s\},\{B_s\}\right) \equiv A_t\left(\{Y_s - T_s(B_s)\}\right) - B_t$
- Given $\{B_s\}$, output solves

$$\mathbf{H}(\mathbf{Y},\mathbf{B}) = 0$$

- First-order solution: $d\mathbf{Y} = -\mathbf{H}_{\mathbf{Y}}^{-1}\mathbf{H}_{\mathbf{B}}d\mathbf{B}$ [ABRS 2021]
- Size-dependent impulse response, avoiding solving nonlinearly for $Y_t(\{B_s\})$!

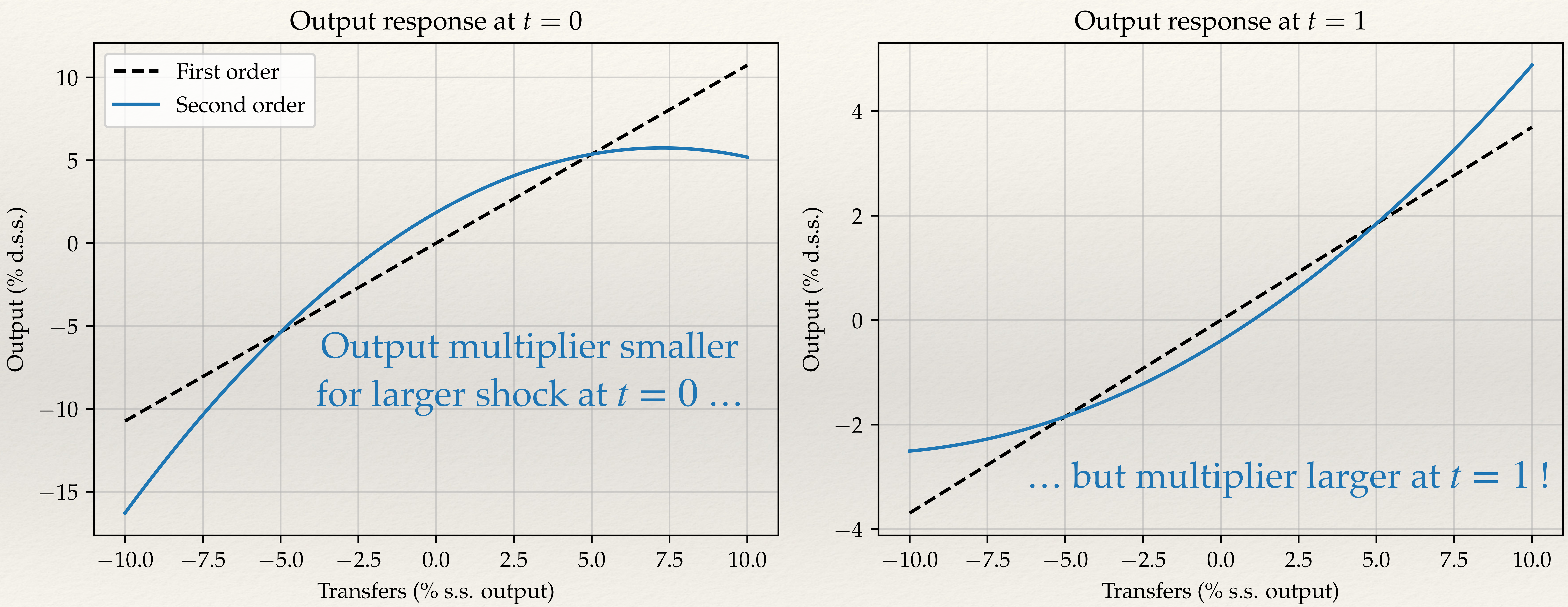
$$d^2\mathbf{Y} = -\mathbf{H}_{\mathbf{Y}}^{-1}\left(\lim_{\nu\to 0}\frac{\mathbf{H}\left(Y+\nu d\mathbf{Y}, B+\nu d\mathbf{B}\right)}{\nu^2}\right)$$

same as for $d\mathbf{Y}$

nonlinear error from $d\mathbf{Y}$

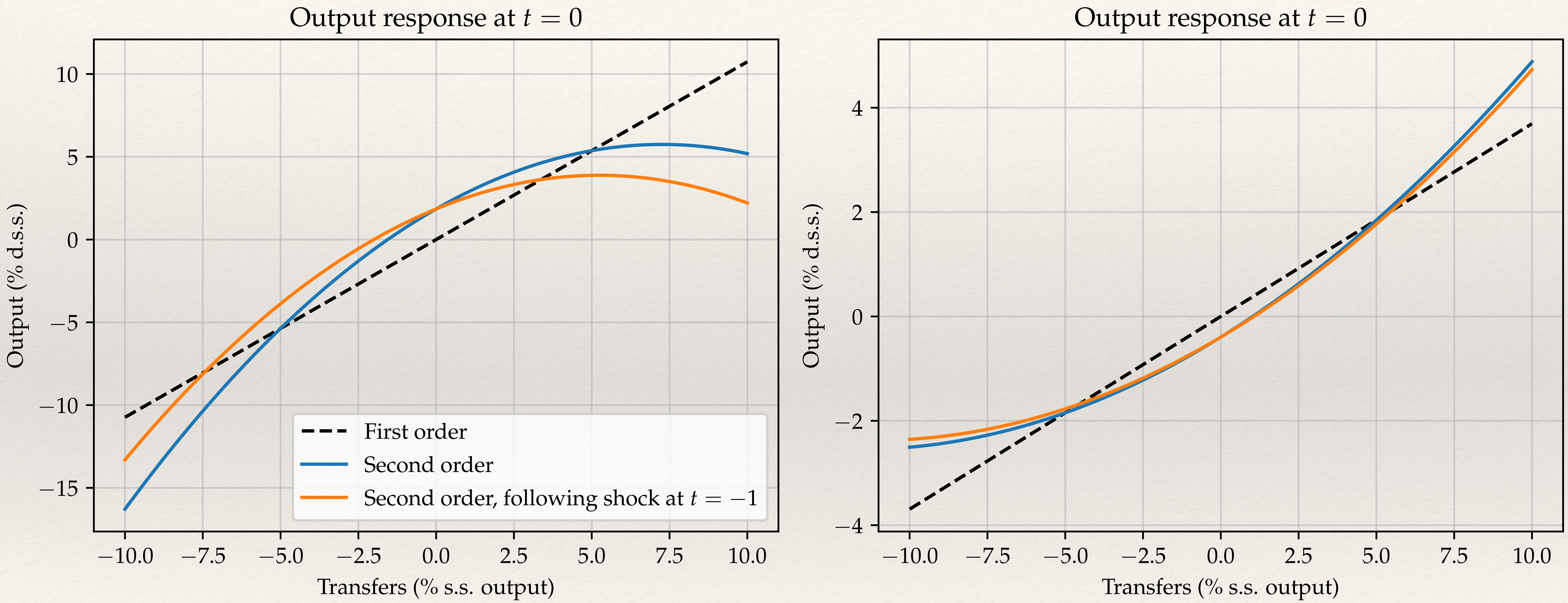
Use similar approach for state dependence
(just using different initial distribution)

Generalized impulse response: size dependence



- Concavity / convexity of responses translate to GE!

Generalized impulse response: state dependence



- Same with history: past transfers imply more excess savings today, lower MPCs

Takeaway

Summary: certainty correspondence

Order in agg. risk	Perfect foresight solution *Agents perceive no aggregate risk*		Rational expectations solution *Agents have correct expectations of aggregate risk*
0	Deterministic steady state	⟷	Deterministic steady state
1	1*st* order response	⟷ **Certainty equivalence**	Linear $MA(\infty)$
2	{2*nd* order responses}	⟷ **Certainty correspondence**	Quadratic $MA(\infty)$
3	{3*rd* order responses}	⟷	Cubic $MA(\infty)$
n	{nth order responses}	etc…	nth order Volterra expansion

Takeaway

Certainty Correspondence:

MIT shocks (on a smooth model) are all you need!