Sequence-Space Jacobians: Computation and Applications

Adrien Auclert (Stanford University)

Boston University, April 2024

Recap

- Equilibrium in heterogeneous-agent models can often be represented as the solution to a system of equations in the sequence space
- Let $\mathbf{U} = \{U_0, U_1, \cdots\}$ be the time path of unknowns, $\mathbf{Z} = \{Z_0, Z_1, \ldots\}$ be the time path of exogenous variables, a generic sequence space system is:

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Roadmap

- Steps to get sequence-space Jacobian $\mathbf{H}_X$:
  1. Solve the steady state of the model without aggregate risk
  2. Solve for effect of a small shock on policy functions at all dates
  3. Solve for impact of distribution shift on aggregate outcomes at all dates
  4. Efficiently combine this information to get $\mathbf{H}_X$
- We first review how to get the steady state, then cover 2–4.

Steady state

The standard incomplete markets model (SIM)

Core of the canonical HANK model: the SIM model

$$V_t(s,a) = \max_{c,a'} u(c) + \beta \mathbb{E}[V_{t+1}(s',a')|s]$$

$$\text{s.t. } a' + c = (1 + r_t)a + y_t(s)$$

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Natural starting point, since this model generates four key outcomes:

1. endogenous buffer stock saving
2. a well defined stationary wealth distribution
3. high MPCs, consistent with those estimated in the data
4. a finitely elastic long-run asset demand curve

Four steps to solving the steady state

1. **Discretizing the state space:** Markov chain for s & grid for a.
2. **Solving for steady-state policy function:**
   - get policies $a'(s, a)$ and $c(s, a)$ on the grid
3. **Solving for steady-state distribution:**
   - get distribution $D(s, a)$ of agents on grid
4. **Aggregating:** get aggregate assets and consumption

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We'll use **Rouwenhorst**'s method. This is classic and pre-programmed.

[Also common: Tauchen method, but worse for usual $\rho \simeq 1$ case.]

Step 2: Steady state policies

We'll use two key optimality conditions:

Envelope condition:

$$V_{a,t}(s,a) = (1 + r_t)u'(c_t(s,a)) \tag{1}$$

First order condition:

$$u'(c_t(s,a)) \geq \beta \mathbb{E}[V_{a,t+1}(s', a'_t(s,a))|s] \tag{2}$$

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This is what we'll do:

$$V_{a,t+1}(s',a') \overset{\text{FOC}}{\to} c_t^{endo}(s,a') \overset{\text{interpolate}}{\to} a'_t(s,a) \overset{\text{budget}}{\to} c_t(s,a) \overset{\text{Envelope}}{\to} V_{a,t}(s,a)$$

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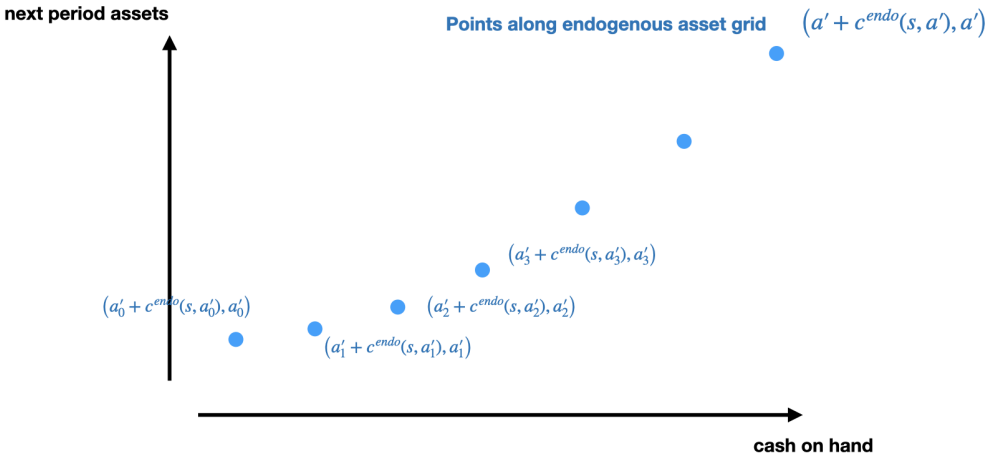
- Can do this with simple interpolation:
  - Plot $(c_t^{endo}(s, a') + a', a')$ points and interpolate at values of $\text{coh}_t(s, a)$...

Illustration of key EGM step



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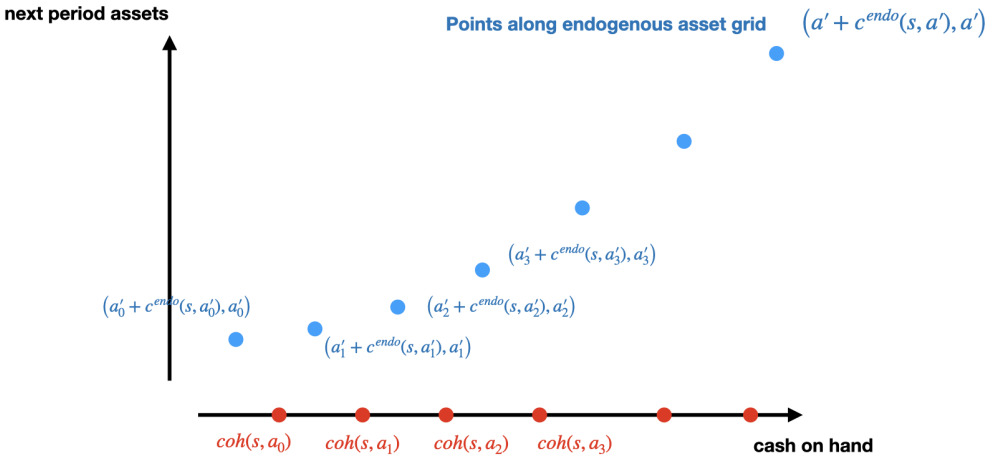

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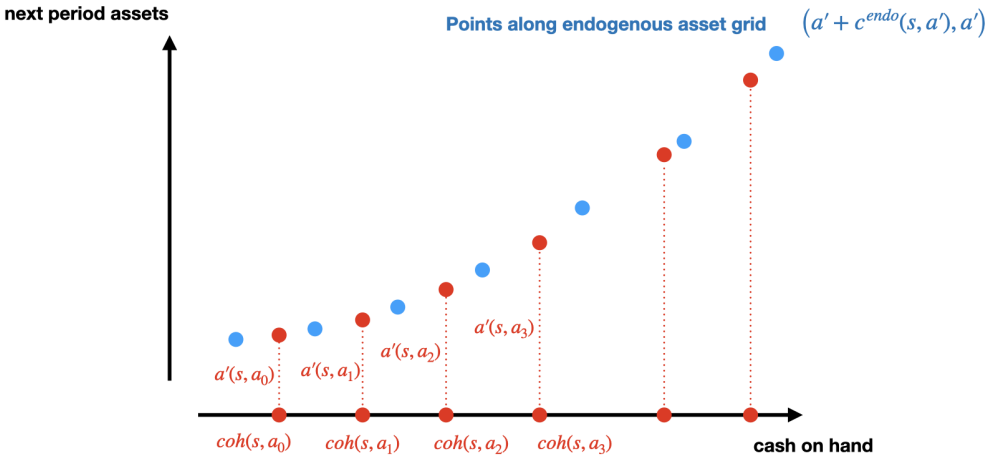

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- In the steady state: no $t \to$ just iterate until policies converge.

Step 2: Illustration

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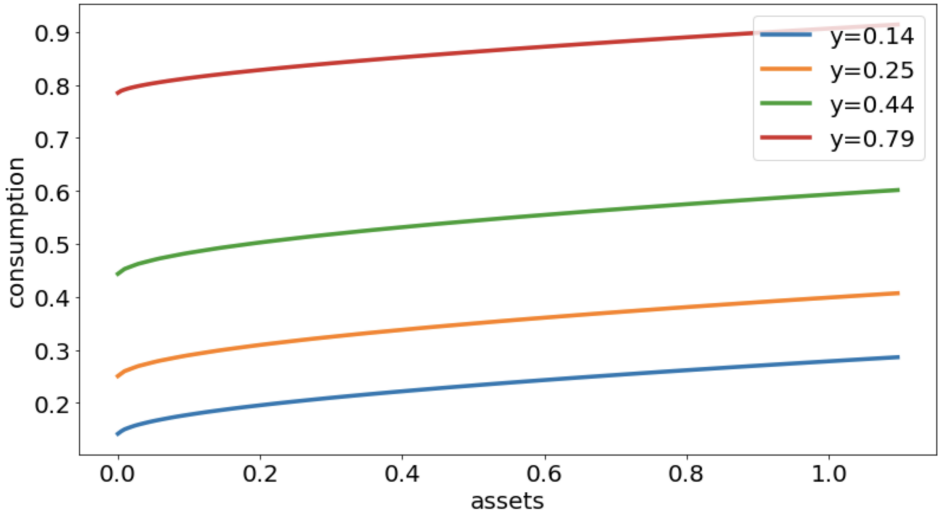

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Steady state: no t! Iterate to convergence...

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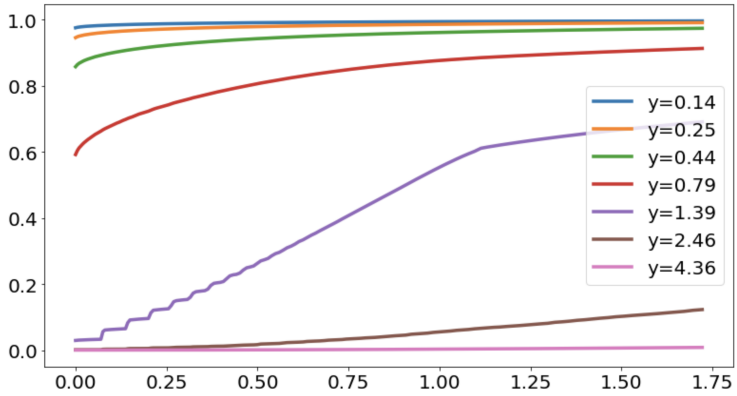

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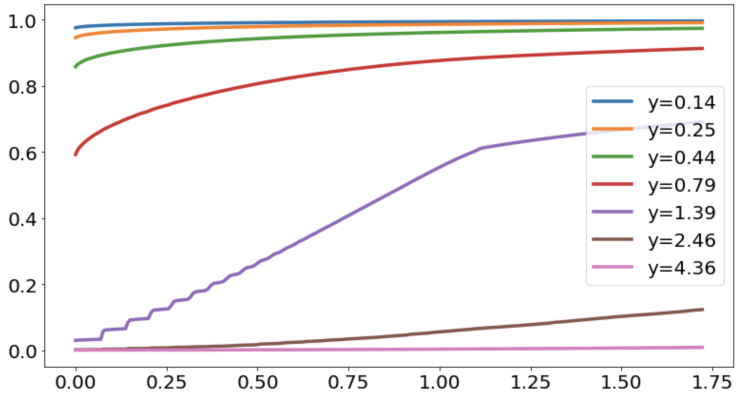


Wiggles: mass slowly moving away from borrowing constraint

Kink: target saving point for buffer stock saving

True distribution has these features, even with continuous distribution & continuous income process.

Step 4: Aggregation

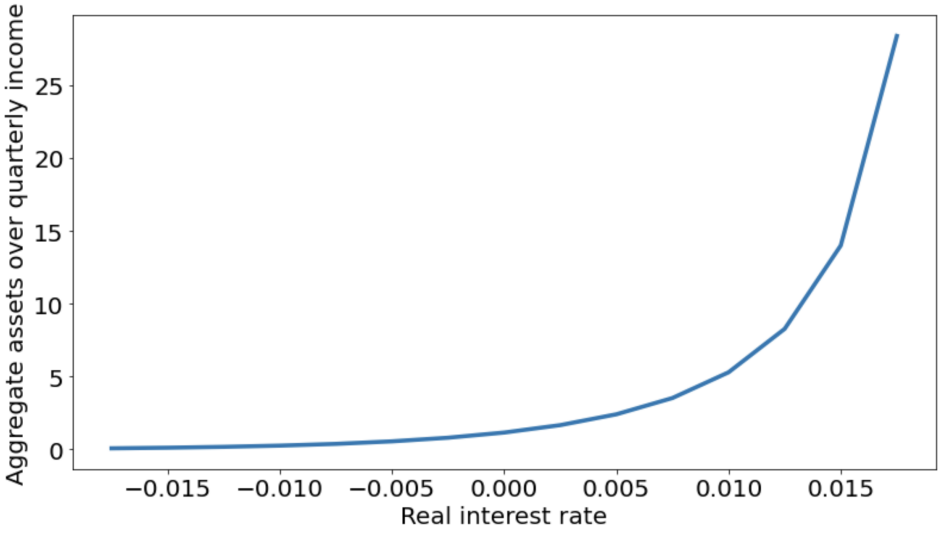
Aggregate consumption and assets

$$C_t = \sum_{s,a} c_t(s,a) \cdot D_t(s,a)$$

$$A_t = \sum_{s,a} a'_t(s,a) \cdot D_t(s,a)$$

Again: get these via simple dot product of vectors. Do not simulate.

Step 4: Steady state comparative statics



Dynamics

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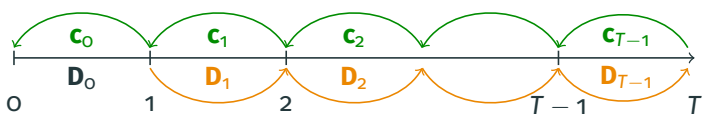

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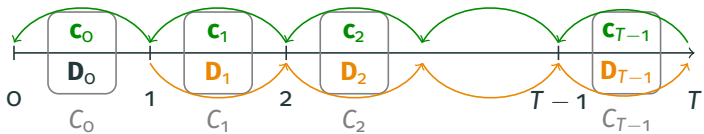

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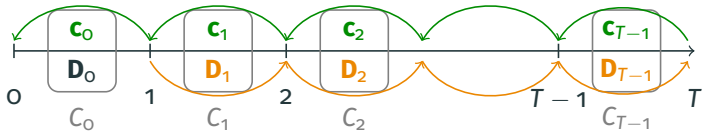
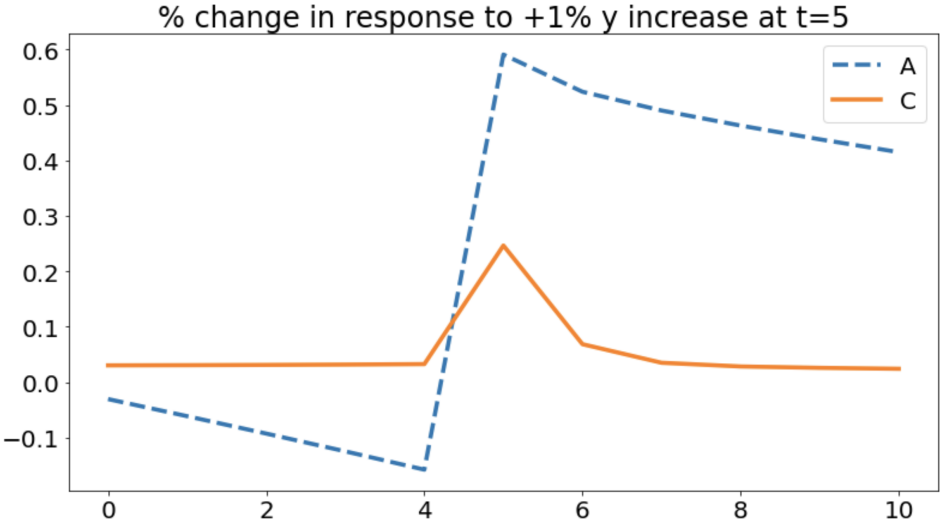


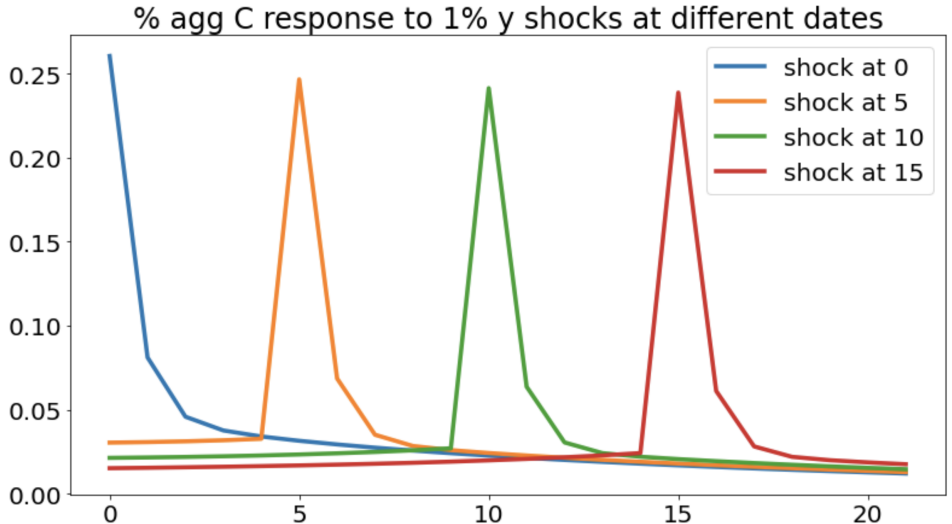
Let's look at an example ...

One time shock to income at date 5



Shocks at many dates s

[matrix]



Sequence-space Jacobians

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We can stack this into a matrix

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The $\mathcal{J}^{C,y}$ is our **sequence-space Jacobian**

Note this is a partial equilibrium object but, as we've seen, we will actually use it (combined with others) to solve for *general equilibrium* impulse responses

How do we compute sequence-space Jacobians?

To compute Jacobians, we have to truncate to some finite time T, say $T = 300$.

To compute T columns, we need to go backward $T \times T$ times and forward $T \times T$ times. This is doable (and a valid approach to getting the SSJ), but also slow...

How do we compute sequence-space Jacobians?

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Next: a method that can be used to compute the entire matrix in only T backward and T "forward iterations"! (Auclert et al., 2021)

Computing $\mathcal{J}$: Only T backward iterations

Denote by $c_t^s(s,a), a_t^s(s,a)$ the policies at date t in response to s date shock.

Why is a single backward iteration enough?

Key idea 1: Only the **time** $s-t$ **until the perturbation matters:**

$$c_t^s(s,a) = \begin{cases} c_{ss}(s,a) & s < t \\ c_{T-1-(s-t)}^{T-1}(s,a) & s \geq t \end{cases}$$

Thus, only need a single backward iteration with $s = T-1$ to get all the $c_t^s(s,a)$!

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From this we get:

- $C_0^s = \sum_{s,a} c_0^s(s,a) D_{ss}(s,a)$, so first row of Jacobian $\mathcal{J}_{0s} = \frac{\partial C_0}{\partial y_s}$
- $D_1^s(s,a)$, distribution at date 1 implied by new policy $c_0^s(s,a)$ at date 0

Computing $\mathcal{J}$: First column of Jacobian

Could keep going and compute $D_t^s(s, a)$, then $C_t^s = \mathbf{c}_t^{s\prime} \cdot \mathbf{D}_t^s$, for all s, t

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Consider first $s = 0$. For $t \geq 1$, only second term remains, with

$$d\mathbf{D}_t^0 = \left(\Lambda_{ss}'\right)^t d\mathbf{D}_1^0$$

where Λ_{ss} = s.s. transition matrix for states (s,a) (Why?).

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$$dC_t^0 = \mathbf{c}'_{ss} \cdot \left(\Lambda'_{ss}\right)^t d\mathbf{D}_1^0 \tag{3}$$

This gives us the first column of the Jacobian, $\mathcal{J}_{t,0}$. What about $s > 0$?

Computing $\mathcal{J}$: Other columns of Jacobian

Again, look at

$$dC_t^s = d\mathbf{c}_t^{s\prime} \cdot \mathbf{D}_{ss} + \mathbf{c}'_{ss} \cdot d\mathbf{D}_t^s$$

For $s > 0$, let's use our policy symmetry result. Since $d\mathbf{c}_t^s = d\mathbf{c}_{t-1}^{s-1}$, we have:

$$dC_t^s - dC_{t-1}^{s-1} = \mathbf{c}'_{ss} \cdot \left(d\mathbf{D}_t^s - d\mathbf{D}_{t-1}^{s-1}\right)$$

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Let's call this $\mathcal{F}_{t,s}dx$. Using policy symmetry again, it is simple to show:

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What's the meaning of $\mathcal{F}_{t,s}$? It's the effect of a "fake news" shock on C_t

- at $t = 0$: news shock that period $s > 0$ shock hits $\rightarrow$ distribution $d\mathbf{D}_1^s$
- at $t = 1$: news shock that there *won't* be a shock at $s \rightarrow$ use s.s. policies

Computing $\mathcal{J}$: Fake news matrix 1/2

Recall we want to know $\mathcal{J}_{t,s} \equiv \frac{\partial C_t}{\partial y_s}$ for $s, t \in \{0, \ldots, T-1\}$.

Can think of $\mathcal{J}$ as a **news matrix:**

- column s = response to news that shock is going to hit in period s

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We just saw how to get:

- the first row $\mathcal{J}_{0,s}$
- the first column $\mathcal{J}_{t,0}$
- differences in columns $\mathcal{F}_{t,s} \equiv \mathcal{J}_{t,s} - \mathcal{J}_{t-1,s-1}$ for all $s, t \geq 1$

Key idea 2: can recover $\mathcal{J}$ from first row, first column, and **"fake news" matrix** $\mathcal{F}$.

- News shock = sequence of fake news shocks

Computing $\mathcal{J}$: Fake news matrix 2/2

Expand $\mathcal{F}$ to include first row and column of $\mathcal{J}$:

$$\mathcal{F} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} - \mathcal{J}_{00} & \mathcal{J}_{12} - \mathcal{J}_{01} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} - \mathcal{J}_{10} & \mathcal{J}_{22} - \mathcal{J}_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathcal{J} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} & \mathcal{J}_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- can recover $\mathcal{J}$ from $\mathcal{F}$ by adding elements from top left diagonal
- ie, the formula implementing Key idea 2 is:

$$\mathcal{J}_{t,s} = \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{t-k,s-k} \tag{5}$$

Practical implementation: obtaining $\mathcal{J}_{to}$ and $\mathcal{F}_{t,s}$

Define **expectations function**

$$\mathcal{E}_t(s,a) \equiv \mathbb{E}\left[c(s_t,a_t)|s_0 = s, a_0 = a\right]$$

$=$ expected consumption in t periods for an agent in state (s,a) today.

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Expectations function is easy to compute: Stack as vector across (s,a), ...

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Key idea 3: $\mathcal{E}_t$'s are enough to get all rows $t \geq 1$ of $\mathcal{F}$

- First column s $=$ o: use (3), $\mathcal{J}_{t,o}dx = (\mathcal{E}_t)' \, d\mathbf{D}_1^o$
- Other columns: use (4), $\mathcal{F}_{t,s}dx = (\mathcal{E}_t)' \, d\mathbf{D}_1^s$

[Recall that we got the first row as well as $d\mathbf{D}_1^o$ from key idea 1]

Wrapping up

- **Fake news algorithm** for obtaining Jacobian $\mathcal{J}$ of aggregate (eg, C) to input of a heterogeneous-agent problem (eg, Y) relies on following steps:

1. Get all policies $\mathbf{c}_t^s$ and date-0 perturb. $d\mathbf{D}_1^0$ to dist. with one backward iteration
2. Calculate steady state expectation vectors $\mathcal{E}_t$
3. Build the Fake news matrix $\mathcal{F}_{t,s}$ using info from steps 1 and 2
4. Build Jacobian $\mathcal{J}$ from $\mathcal{F}$ using equation (5)

Applications

Taking stock

- How do you go about solving a rational expectations het. agent problem?
  1. Boil it down to a series of partial equilibrium problems
  2. Get the Jacobian of each partial equilibrium block
  3. Combine these Jacobians to form your H_U, H_Z, and get impulse responses
- This delivers the first-order impulse responses.
- From this can:
  1. Simulate the model for given policies
  2. Estimate the model on aggregate data
  3. Simulate the model for counterfactual policies

What else can you do?

- Other frontiers you can tackle with sequence-space Jacobians
  1. Solve for endogenous portfolios and risk premia
     - see “When do endogenous portfolios matter for HANK?”
  2. Solve for models where expectations are not rational
     - see “Micro Jumps, Macro Humps”
  3. Solve for optimal policy
     - see “Optimal Long-Run Fiscal Policy with Heterogeneous Agents”

  … and much more!
- The possibilities are nearly limitless, looking forward to your contributions!

References

Auclert, A., Bardóczy, B., Rognlie, M., and Straub, L. (2021). Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models. *Econometrica*, 89(5):2375–2408.