Degrees of Freedom in Multiple-Antenna Channels: A Signal Space Approach

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***Abstract*—Multiple-antenna systems that are limited by the area and geometry of antenna arrays, are considered. Given these physical constraints, the limit on the available number of spatial degrees of freedom is derived. The commonly used statistical multiple-input multiple-output (MIMO) model is inadequate. Antenna theory is applied to take into account the area and geometry constraints, and to define the spatial signal space so as to interpret experimental channel measurements in an array-independent but manageable description of the physical environment. Based on these modeling strategies, for a spherical array of effective aperture $\mathcal{A}$ in a physical environment of angular spread $|\Omega|$ in solid angle, the number of spatial degrees of freedom is shown to be $\mathcal{A}|\Omega|$ for uni-polarized antennas and $2\mathcal{A}|\Omega|$ for tri-polarized antennas. Together with the $2WT$ degrees of freedom for a system of bandwidth W transmitting in an interval T, the total degrees of freedom of a multiple-antenna channel is therefore $4WT\mathcal{A}|\Omega|$.**

***Index Terms*—Antenna theory, degrees of freedom, multiple antennas, multiple-input multiple-output (MIMO) systems, physical channel modeling.**

I. INTRODUCTION

In multiple-antenna channels, the channel capacity grows linearly with the number of spatial degrees of freedom, which is therefore a key performance measure. A fundamental question arises: given an *area* limitation on the transmit and receive antenna arrays, what is the intrinsic number of degrees of freedom available in the channel? Statistical multiple-input multiple-output (MIMO) models are insufficient to answer this question. Early results [1], [2] focus on the model where the fading is independent and identically distributed (i.i.d.) across all antenna pairs, and show that the number of degrees of freedom is the minimum of the number of transmit and receive antennas. Packing more antennas in a given area will, however, make the fading correlated and therefore cannot increase the capacity indefinitely. This is analogous to the waveform channel where given the bandwidth constraint W and transmission interval T, increasing the number of time samples will also not increase the capacity indefinitely. The available degrees of freedom is fundamentally limited to $2WT$. In this paper, we will demonstrate that the physical constraint of antenna arrays and propagation environment put forth a deterministic limit to the spatial degrees of freedom underlying the statistical MIMO approach.

Let us review the reasoning involved to derive the $2WT$ formula [3, Ch. 8]. In waveform channels, the transmit and receive signals are represented either in the time domain as waveforms or in the frequency domain as spectra. The mapping between these two representations is the Fourier transform. The waveform is then approximately time-limited to $[-T/2, T/2]$ and its spectrum is frequency-limited to $[-W, W]$. The dimension of the subspace satisfying these two physical constraints gives the number of degrees of freedom $2WT$. To demonstrate an analogous result for multiple-antenna channels, the corresponding spatial signal domains and the mappings between them are required which is based on electromagnetic theory considerations.

In this paper, we incorporate antenna theory with experimental channel modeling to obtain a mathematical model that allows us to derive a more fundamental limit to the number of spatial degrees of freedom given a constraint on the areas of the transmit and receive antenna arrays. Physically, there are two signal domains of interest: the *array domain* used to describe the excitation current distributions and the *wavevector domain* (also known as *angular domain*) to describe the radiated field patterns. The mapping between them depends on the geometry of the antenna array as it imposes different boundary conditions on the set of Maxwell equations. In the case of linear arrays, the mapping between the array and wavevector domains is the familiar Fourier transform, same as the time/frequency counterparts. The mappings for circular and spherical arrays are however different. The current distribution (in array domain) is then limited by the size of the antenna array while the scattering of the physical environment limits the amount of radiated field (in wavevector or angular domain) reaching the receiver. In urban and indoor environments, the transmitter–receiver separation is typically comparable to the size of channel objects, so propagation paths are no longer discrete but are more appropriately analyzed as clusters, as illustrated in Fig. 1. The scattering in the physical environment is then characterized by the number of these clusters and the solid angles subtended. We will show that for a spherical array of effective aperture $\mathcal{A}$ in an environment with scattering clusters spanning over a total solid angle of $|\Omega|$, the number of degrees of freedom is $\mathcal{A}|\Omega|$. When polarization is taken into account, the signal space becomes a

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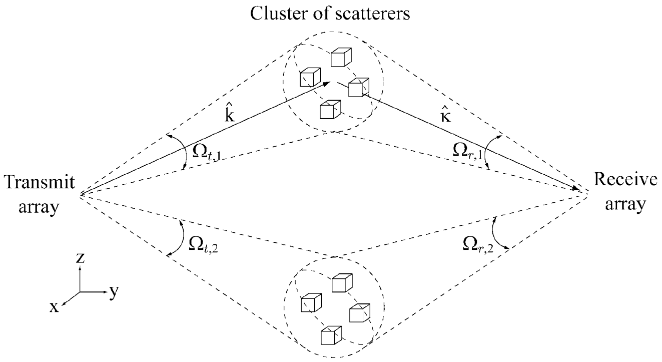

Fig. 1. Clustering of transmit and receive signals. The scattering intervals are $\Omega_t = \Omega_{t,1} \cup \Omega_{t,2} \cup \cdots$ and $\Omega_r = \Omega_{r,1} \cup \Omega_{r,2} \cup \cdots$.

set of vector fields. The number of degrees of freedom is shown to be $2\mathcal{A}|\Omega|$.

The degree-of-freedom formula gives insight on the number of antennas that should be put on the given transmit and receive areas so as to optimize the tradeoff between capacity and cost. As the number of RF/analog chains scales linearly with the number of antennas and the complexity in digital baseband increases at least quadratically [4], [5], it is tempting to use the optimal number of antennas given the physical environment and the shape of wireless device. We will ascertain that $\mathcal{A}|\Omega|$ number of tri-polarized antennas suffices.

In addition, the wavevector domain provides an appropriate coordinate to describe the physical position of other users. Extending the mathematical model to multiuser environments is straightforward. If users are cooperative and relay signals, the channel solid angles will likely increase and so does the spatial degrees of freedom. On the other hand, if users are noncooperative and create interference, it can be encapsulated as a decrease in the channel solid angles. It turns out that the optimal number of antennas is different in both channels.

Recent work such as [6]–[8] also brings in the array and wavevector domains as a means to relate the scattering of physical environment to the fading correlation among antennas on uni-polarized linear arrays. In [6], the physical environment is characterized by the number of propagation paths in the wavevector domain, and the spatial degrees of freedom is the minimum of the numbers of propagation paths, transmit antennas, and receive antennas. Though the number of propagation paths that can be resolved depends on the area of the transmit and receive arrays, [6] does not take this into account. Therefore, it does not answer our question of number of degrees of freedom per unit area. In [7], [8], the physical environment is also modeled as clusters but in a statistical manner. The decrease in fading correlation with increasing antenna spacing is investigated. It gives an intuitive explanation on how the capacity of linear arrays increases with increasing antenna spacing in different scattering environments. However, if the transmit and receive areas are constrained, then increasing the antenna spacing will reduce the number of antennas and hence will likely decrease the capacity. It is this physical constraint that is the focus of this paper. Furthermore, we use a deterministic signal space approach to look at the scattering in physical environments instead of statistical approaches.

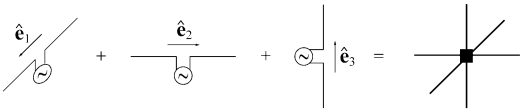

Fig. 2. A tripole antenna.

Finally, we have applied the mathematical model to study the degrees of freedom on linear arrays in [9]. Independent work such as [10]–[14] also considers the physical constraint of antenna arrays. In [10], an upper bound on the number of spatial degrees of freedom for fixed radius of circular arrays and ideal fully scattered environment is derived. In [11]–[13], numerical examples are given to illustrate the existence of a capacity limit for fixed-array apertures using hypothetical correlation models. In [14], a comprehensive channel model taking into account the area constraint of antenna arrays and characteristics of propagation environment is proposed for numerical analysis.

The paper is organized as follows. Section II presents the system model. Section III derives the spatial degrees of freedom. Section IV elaborates the physical insight of the main results. Section V extends the model from point-to-point to multiuser environments. Section VI elucidates the optimal number of antennas. Finally, we will conclude this paper in Section VII.

The following notation will be used in this paper. We will use boldface calligraphic letters for electromagnetic entities $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{J}}, \ldots)$, boldface capital letters for matrices $(\boldsymbol{C}, \boldsymbol{H}, \ldots)$, and boldfaced lower case letters for vectors $(\boldsymbol{p}, \boldsymbol{q}, \ldots)$. For a given vector $\boldsymbol{p}$, $\hat{\boldsymbol{p}}$ is a unit vector denoting its direction and p denotes its magnitude. i denotes square root of -1. $\nabla_{\boldsymbol{p}} \times$ denotes the curl operation with respect to $\boldsymbol{p}$. $\boldsymbol{I}$ is the identity matrix. $(\cdot)^*$, $(\cdot)^\dagger$, and $\mathsf{E}[\cdot]$ denote conjugate, conjugate-transpose, and expectation operations, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes its Lebesgue measure. $\Re^n$, $\mathcal{C}^n$, and $\mathcal{C}^{n \times m}$ denote the set of n-dimensional real numbers, n-dimensional complex numbers, and $n \times m$ complex matrices. $\lceil x \rceil$ gives the smallest integer equal to or greater than x. $(x)^+$ denotes $\max\{0, x\}$.

II. SYSTEM MODELS

We consider continuous arrays which are composed of an infinite number of antennas separated by infinitesimal distances. This eliminates the need to specify *a priori* the number of antennas and their relative positions on antenna arrays. Each antenna is composed of three orthogonal dipoles oriented along Euclidean directions $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ as pictured in Fig. 2. This antenna topology is often referred as a tripole where arbitrarily polarized electric fields can be generated. In a frequency-nonselective fading channel, the transmit and receive signals at a particular time are related by

$$\boldsymbol{y}(\boldsymbol{q}) = \int \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{p}) \boldsymbol{x}(\boldsymbol{p}) d\boldsymbol{p} + \boldsymbol{z}(\boldsymbol{q}). \tag{1}$$

The transmit signal $\boldsymbol{x}(\cdot)$ is a vector field on $\mathcal{R}^3$, a function that assigns each point $\boldsymbol{p} \in \mathcal{R}^3$ of the transmit array to a vector $\boldsymbol{x}(\boldsymbol{p}) \in \mathcal{C}^3$. Similarly, $\boldsymbol{y}(\cdot)$ is the receive vector field. The *channel response* $\boldsymbol{C}(\cdot, \cdot)$ is a 3×3 complex integral kernel where its domain is the set of transmit vector fields and its range is the set of receive vector fields. The matrix $\boldsymbol{C}(\boldsymbol{q}, \boldsymbol{p})$

gives the channel gain and polarization between the transmit position $\boldsymbol{p}$ and receive position $\boldsymbol{q}$. The vector field $\boldsymbol{z}(\cdot)$ is the additive noise.

The channel response can be decomposed into three responses

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) = \int\!\!\int \boldsymbol{A}_r(\boldsymbol{q},\hat{\boldsymbol{\kappa}})\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\boldsymbol{A}_t(\hat{\boldsymbol{k}},\boldsymbol{p})d\hat{\boldsymbol{k}}d\hat{\boldsymbol{\kappa}} \qquad (2)$$

where $\boldsymbol{A}_t(\cdot,\cdot)$, $\boldsymbol{A}_r(\cdot,\cdot)$, and $\boldsymbol{H}(\cdot,\cdot)$ are 3×3 complex integral kernels. The transmit array response $\boldsymbol{A}_t(\cdot,\cdot)$ maps the excitation current distribution to the radiated field pattern. Similarly, the receive array response $\boldsymbol{A}_r(\cdot,\cdot)$ maps the incident field pattern to the induced current distribution. The *scattering response* $\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})$ gives the channel gain and polarization between the transmit direction $\hat{\boldsymbol{k}}$ and receive direction $\hat{\boldsymbol{\kappa}}$ (see Fig. 1). We will next model these responses.

A. Array Responses

From Maxwell equations, the electric field $\boldsymbol{\mathcal{E}}(\cdot)$ due to the current density $\boldsymbol{\mathcal{J}}(\cdot)$ satisfies [15]

$$\left(-\nabla_{\boldsymbol{p}} \times \nabla_{\boldsymbol{p}} \times +k_0^2\right)\boldsymbol{\mathcal{E}}(\boldsymbol{p}) = \mathrm{i}k_0\eta\boldsymbol{\mathcal{J}}(\boldsymbol{p}) \qquad (3)$$

where $k_0 = 2\pi/\lambda$, λ is the wavelength, and η is the intrinsic impedance. The inverse map is

$$\boldsymbol{\mathcal{E}}(\boldsymbol{k}) = \int \boldsymbol{\mathcal{G}}(\boldsymbol{k},\boldsymbol{p})\boldsymbol{\mathcal{J}}(\boldsymbol{p})d\boldsymbol{p} \qquad (4)$$

for some integral kernel $\boldsymbol{\mathcal{G}}(\cdot,\cdot)$. The kernel $\boldsymbol{\mathcal{G}}(\cdot,\cdot)$ is often referred as the Green function in electromagnetic theory where it is commonly derived with a given coordinate system.[1] As we want to investigate the effect of array geometry in addition to array size, a coordinate-free version is derived in Appendix I and is given by

$$\boldsymbol{\mathcal{G}}(\boldsymbol{k},\boldsymbol{p}) = \frac{\mathrm{i}\eta e^{\mathrm{i}2\pi r/\lambda}}{2\lambda r}\left[\left(\boldsymbol{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}^\dagger\right) + \frac{\mathrm{i}}{2\pi r/\lambda}\left(\boldsymbol{I} - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}^\dagger\right) - \frac{1}{(2\pi r/\lambda)^2}\left(\boldsymbol{I} - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}^\dagger\right)\right] \qquad (5)$$

where $\boldsymbol{r} = \boldsymbol{k} - \boldsymbol{p}$. The Green function $\boldsymbol{\mathcal{G}}(\boldsymbol{k},\boldsymbol{p})$ gives the electric field at the observation point $\boldsymbol{k}$ due to a current source at point $\boldsymbol{p}$. It has three terms: far field, intermediate field, and near field. Only the far-field term corresponds to the radiated field as it falls off inversely as the distance apart r, and hence its power follows the inverse square law. The power of the remaining two terms falls off much faster than r^{-2} (r^{-4} and r^{-6}, respectively) so they do not contribute to electromagnetic radiation.

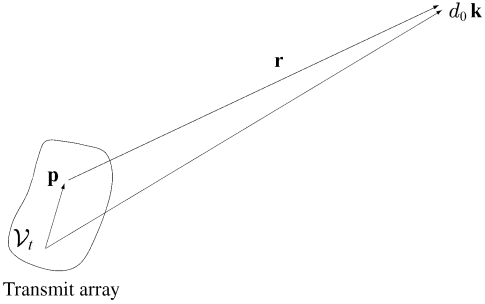

Fig. 3. A transmitting continuous array.

TABLE I
SUMMARY OF INDOOR SPATIAL CHANNEL MEASUREMENTS

	Frequency (GHz)	No. of Clusters	Cluster Angle (°)
USC UWB [17]	0 – 3	2 – 5	37
Intel UWB [18]	2 – 8	1 – 4	10 – 15
Spencer [19]	6.75 – 7.25	3 – 5	25.5
COST 259 [20]	24	3 – 5	18.5

As the transmit array response characterizes the radiated field to the current distribution, it is therefore given by the first term of the Green function

$$\boldsymbol{A}_t(\hat{\boldsymbol{k}},\boldsymbol{p}) = \frac{\mathrm{i}\eta e^{\mathrm{i}2\pi r}}{2\lambda^2 r}\left(\boldsymbol{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}^\dagger\right), \qquad \boldsymbol{p} \in \mathcal{V}_t \qquad (6)$$

and $\boldsymbol{r} = d_0\hat{\boldsymbol{k}} - \boldsymbol{p}$ (see Fig. 3) where d_0 is the reference distance and $\mathcal{V}_t$ denotes the transmit space. In the expression, the position vector $\boldsymbol{p}$, the reference distance d_0, and the separation r are normalized by the wavelength.[2] The reference distance d_0 is chosen such that $d_0 \gg p$ for all position vector $\boldsymbol{p} \in \mathcal{V}_t$, the far-field region. Then, we have the following approximations:

$$1/r \approx 1/d_0, \quad r \approx d_0 - \hat{\boldsymbol{k}}^\dagger\boldsymbol{p}, \quad \text{and} \quad \hat{\boldsymbol{r}} \approx \hat{\boldsymbol{k}}$$

and the array response can be approximated by

$$\boldsymbol{A}_t(\hat{\boldsymbol{k}},\boldsymbol{p}) \approx \frac{\mathrm{i}\eta e^{\mathrm{i}2\pi d_0}}{2\lambda^2 d_0}\left(\boldsymbol{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger\right)\exp\left(-\mathrm{i}2\pi\hat{\boldsymbol{k}}^\dagger\boldsymbol{p}\right), \quad \boldsymbol{p} \in \mathcal{V}_t. \qquad (7)$$

It gives the radiated field in the direction $\hat{\boldsymbol{k}}$ at the reference distance d_0 due to a current source at point $\boldsymbol{p}$. The $\boldsymbol{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger$ is a 3×3 rank 2 matrix and constrains the oscillation direction of the radiated field to be perpendicular to its propagation direction. The propagation term $\exp(-\mathrm{i}2\pi\hat{\boldsymbol{k}}^\dagger\boldsymbol{p})$ relates the propagation direction $\hat{\boldsymbol{k}}$ of the radiated field to the excitation position $\boldsymbol{p}$ on the array. By the reciprocity theorem for antennas [16], the receive array response is

$$\boldsymbol{A}_r(\boldsymbol{q},\hat{\boldsymbol{\kappa}}) \approx \frac{-\mathrm{i}\eta^* e^{-\mathrm{i}2\pi d_0}}{2\lambda^2 d_0}\left(\boldsymbol{I} - \hat{\boldsymbol{\kappa}}\hat{\boldsymbol{\kappa}}^\dagger\right)\exp\left(\mathrm{i}2\pi\hat{\boldsymbol{\kappa}}^\dagger\boldsymbol{q}\right), \quad \boldsymbol{q} \in \mathcal{V}_r \qquad (8)$$

for all-normalized position vector $\boldsymbol{q}$ in the receive space $\mathcal{V}_r$ and incident field from direction $\hat{\boldsymbol{\kappa}}$.

B. Clustered Scattering Responses

Recent indoor channel measurements show that physical paths are clustered around the transmit and receive directions as illustrated in Fig. 1. In the indoor environment, clustering can be the result of reflection from walls and ceilings, scattering from furniture, diffraction from doorway openings, and transmission through soft partitions. Table I summaries measurement results in the literature. They show that the number of clusters ranges from one to five, and the cluster azimuth/elevation angles vary from 10° to 30°. In general, the clustering phenomenon occurs when the transmitter–receiver separation is comparable

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[1] For example, Cartesian coordinates are used for rectangular arrays and cylindrical coordinates are used for circular arrays.

[2] The normalization is for conciseness at the later development of main results.

to the size of scattering sources which is typical in both indoor and urban environments.

Following the ray-tracing model in [21] and grouping the paths into clusters as in [20], yields the scattering response

$$\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})=\frac{1}{\sqrt{N_l}}\sum_{i=1}^{M}\sum_{j\in\mathcal{S}_i}\boldsymbol{\Gamma}_j\delta(\hat{\boldsymbol{\kappa}}-\hat{\boldsymbol{\kappa}}_j)\delta(\hat{\boldsymbol{k}}-\hat{\boldsymbol{k}}_j) \qquad (9)$$

where $\boldsymbol{\Gamma}_j$ denotes the attenuation and polarization on the jth path, $\mathcal{S}_i$ denotes the set of propagation paths in the ith cluster, and N_l is the total number of these paths. The response has the desired property of array-independent; however, there is an arbitrary number of paths in each cluster and hence lessens its analytical tractability. As the scattering response is sandwiched between the array responses, a common practice is to smooth out the scattering response by the array responses; however, the characteristics of the channel are then mixed up with that of antenna arrays.

Instead, we zoom out the granularity of the channel description and characterize the channel by the set of cluster boundaries. Refer to Fig. 1; Ω_t being the union of $\Omega_{t,i}$'s is the angular interval subtended by the scattering clusters being illuminated by the transmit array. Similarly, Ω_r is the scattering interval as observed from the receive array.[3] Then, the scattering response satisfies

$$\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\neq 0 \quad \text{only if} \quad (\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\in\Omega_r\times\Omega_t. \qquad (10)$$

To make sure the well-conditionedness of $\boldsymbol{H}(\cdot,\cdot)$, we assume that 1) $\int\|\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\|^2 d\hat{\boldsymbol{\kappa}}\neq 0$ for all $\hat{\boldsymbol{k}}\in\Omega_t$ and $\int\|\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\|^2 d\hat{\boldsymbol{k}}\neq 0$ for all $\hat{\boldsymbol{\kappa}}\in\Omega_r$; 2) the point spectrum (set of eigenvalues) of $\boldsymbol{H}(\cdot,\cdot)$ excluding 0 is infinite; and 3) $\|\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\|^2\neq 0\Rightarrow\operatorname{rank}\big(\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\big)=3$ for all $(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})\in\Omega_r\times\Omega_t$.

Define the channel solid angles as

$$|\Omega_t|=\int_{\Omega_t}\sin\theta d\theta d\phi \quad \text{and} \quad |\Omega_r|=\int_{\Omega_r}\sin\theta d\theta d\phi \qquad (11)$$

which are solid angles subtended by the scattering clusters as viewed from the transmitter and the receiver, respectively. At the transmitter (receiver), $|\Omega_t|$ ($|\Omega_r|$) is the area of projection of the scattering clusters onto the unit sphere enclosing the transmit (receive) array as pictured in Fig. 4. For example, in the ideal fully scattered environment the channel solid angle is 4π, the surface area of a unit sphere.

III. MAIN RESULTS

Now, the channel response in the far-field region can be approximated by

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{p})\approx\frac{\eta^2}{4\lambda^4 d_0^2}\int_{\Omega_r}\int_{\Omega_t}a^*(\hat{\boldsymbol{\kappa}},\boldsymbol{q})\left(\boldsymbol{I}-\hat{\boldsymbol{\kappa}}\hat{\boldsymbol{\kappa}}^\dagger\right)\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}}) \cdot\left(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger\right)a(\hat{\boldsymbol{k}},\boldsymbol{p})d\hat{\boldsymbol{k}}d\hat{\boldsymbol{\kappa}}, \quad (\boldsymbol{q},\boldsymbol{p})\in\mathcal{V}_r\times\mathcal{V}_t \qquad (12)$$

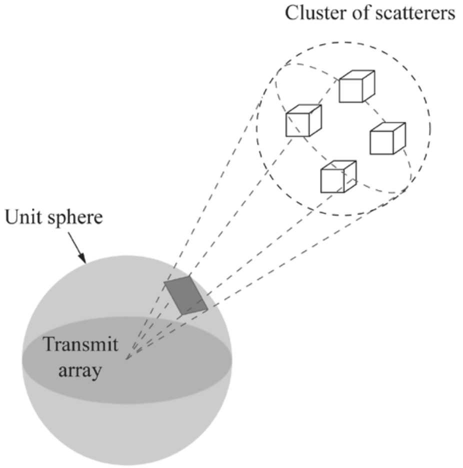

Fig. 4. The shaded area on the unit sphere is the channel solid angle subtended by the scattering cluster being illuminated by the transmit array.

where

$$a(\hat{\boldsymbol{k}},\boldsymbol{p})=\exp\left(-\mathrm{i}2\pi\hat{\boldsymbol{k}}^\dagger\boldsymbol{p}\right). \qquad (13)$$

As $\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})$ is well conditioned in $\Omega_r\times\Omega_t$, the spatial degrees of freedom is constrained by integral kernels

$$a^*(\hat{\boldsymbol{\kappa}},\boldsymbol{q})\left(\boldsymbol{I}-\hat{\boldsymbol{\kappa}}\hat{\boldsymbol{\kappa}}^\dagger\right), \quad (\hat{\boldsymbol{\kappa}},\boldsymbol{q})\in\Omega_r\times\mathcal{V}_r$$

$$\left(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger\right)a(\hat{\boldsymbol{k}},\boldsymbol{p}), \quad (\hat{\boldsymbol{k}},\boldsymbol{p})\in\Omega_t\times\mathcal{V}_t.$$

For uni-polarized antennas, it is equivalent to studying the kernel

$$a(\hat{\boldsymbol{k}},\boldsymbol{p}), \quad (\hat{\boldsymbol{k}},\boldsymbol{p})\in\Omega\times\mathcal{V} \qquad (14)$$

where $\mathcal{V}$ can be the transmit or receive spaces, and Ω is the corresponding scattering interval. We consider the following decomposition (equivalent to singular value decomposition on finite-dimensional matrices):

$$a(\hat{\boldsymbol{k}},\boldsymbol{p})=\sum_n\sigma_n\eta_n(\hat{\boldsymbol{k}})\xi_n^*(\boldsymbol{p}), \qquad (\hat{\boldsymbol{k}},\boldsymbol{p})\in\Omega\times\mathcal{V} \qquad (15)$$

where $\{\sigma_n\}$ is a sequence of nonnegative numbers (singular values), and $\{\eta_n(\cdot)\}$ and $\{\xi_n(\cdot)\}$ are orthonormal sets satisfying

$$\int_\Omega\eta_n(\hat{\boldsymbol{k}})\eta_m^*(\hat{\boldsymbol{k}})d\hat{\boldsymbol{k}}=\int_\mathcal{V}\xi_n(\boldsymbol{p})\xi_m^*(\boldsymbol{p})d\boldsymbol{p}=\delta_{nm}. \qquad (16)$$

The σ_n, $\eta_n(\cdot)$, and $\xi_n(\cdot)$ are related by

$$\int a(\hat{\boldsymbol{k}},\boldsymbol{p})\xi_n(\boldsymbol{p})d\boldsymbol{p}=\sigma_n\eta_n(\hat{\boldsymbol{k}}). \qquad (17)$$

The subspace spanned by $\int a(\hat{\boldsymbol{k}},\boldsymbol{p})\xi_n(\boldsymbol{p})d\boldsymbol{p}$ for all n corresponding to significant σ_n's, gives the set of radiated field patterns that are array limited to $\mathcal{V}$ and approximately wavevector limited to Ω. The number of significant singular values gives the dimension of this subspace. The minimum of the dimensions of the transmit and receive subspaces yields

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[3] Note that the clusters illuminated by the transmit array need not be the same as those observed from the receive array.

the spatial degrees of freedom for uni-polarized antennas. For tri-polarized antennas, we consider the 3×3 integral kernel

$$\left(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^{\dagger}\right) a(\hat{\boldsymbol{k}},\boldsymbol{p}), \qquad (\hat{\boldsymbol{k}},\boldsymbol{p}) \in \Omega \times \mathcal{V} \tag{18}$$

instead. To ease the understanding, we consider linear arrays first, followed by circular and spherical arrays.

A. Uni-Polarized Linear Arrays

In spherical coordinates, the propagation direction $\hat{\boldsymbol{k}}$ can be expressed as

$$\hat{\boldsymbol{k}} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

and $0 \leq \theta < \pi, 0 \leq \phi < 2\pi$. Assume Ω is separable over the elevation (θ) and azimuth (ϕ) directions, that is, $\Omega = \Theta \times \Phi$. Consider a linear array of length $2L$ oriented along the z-axis and centered at the origin. Then, the integral kernel $a(\cdot,\cdot)$ in (14) becomes

$$a(\cos\theta, p_z) = \exp(-\mathrm{i}2\pi p_z \cos\theta) \tag{19}$$

and corresponds to the Fourier transform, same as the mapping between the time and frequency domains in waveform channels.

In waveform channels, transmitted signals will be first bandlimited to $[-W,W]$ and then time-limited to $[-T/2,T/2]$. If $s(t)$ is the transmit signal, then the receive signal will be

$$r(t) = \begin{cases} \int_{-W}^{W} \exp(\mathrm{i}2\pi f t) S(f) df, & t \in [-T/2, T/2] \\ 0, & \text{otherwise} \end{cases}$$

where $S(f)$ is the Fourier transform of $s(t)$. This operation is equivalent to that performed by the kernel

$$\exp(-\mathrm{i}2\pi f t), \quad (t,f) \in [-T/2,T/2] \times [-W,W]. \tag{20}$$

It can be decomposed into a sum of shifted sinc dyads

$$\exp(-\mathrm{i}2\pi f t) = \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] e^{-\mathrm{i}\pi n f/W}, \quad (t,f) \in [-T/2,T/2] \times [-W,W]. \tag{21}$$

As most of the energy of the sinc function is concentrated within $\pm 1/2W$, so when $T \gg 1/(2W)$, we can make the following approximations:

$$\exp(-\mathrm{i}2\pi f t) \approx \sum_{n=-WT}^{WT} \mathrm{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] e^{-\mathrm{i}\pi n f/W}, \quad (t,f) \in [-T/2,T/2] \times [-W,W] \tag{22}$$

and

$$\int_{-T/2}^{T/2} 2W \mathrm{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] \cdot \mathrm{sinc}\left[2W\left(t - \frac{m}{2W}\right)\right] dt \approx \delta_{nm}, \qquad |n|, |m| \leq WT. \tag{23}$$

Compared to the decomposition defined in (15) and (16), the number of significant singular values is $2WT$ for $WT \gg 1$. The subspace spanned by

$$\left\{\sqrt{2W}\mathrm{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] : n = 0, \ldots, \pm WT\right\} \tag{24}$$

gives the set of frequency-limited and approximately time-limited waveforms. Cast onto the integral kernel for linear arrays, yields

$$a(\cos\theta, p_z) = \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left[2L\left(\cos\theta - \frac{n}{2L}\right)\right] e^{-\mathrm{i}\pi n p_z/L}, \quad (\cos\theta, p_z) \in \Omega_\theta \times [-L, L] \tag{25}$$

where $\Omega_\theta = \{\cos\theta : \theta \in \Theta\}$. The array-limited sinc functions have a resolution of $1/(2L)$ over Ω_θ. Therefore, the dimension of the array-limited and approximately wavevector-limited subspace is $2L|\Omega_\theta|$ for $L|\Omega_\theta| \gg 1$.

Now, we justify the approximation made in (22) and (23), and hence justify the $2L|\Omega_\theta|$ formula. Slepian *et al.* [22]–[24] showed that the integral kernel for waveform channels (20) can be decomposed into

$$\exp(-\mathrm{i}2\pi f t) = \sum_{n=0}^{\infty} \sigma_n \varphi_n(t) \Psi_n(f), \quad (t,f) \in [-T/2,T/2] \times [-W,W]$$

and the orthonormal sets $\{\varphi_n(\cdot)\}$ and $\{\Psi_n(\cdot)\}$ are the prolate spheroidal wave functions. The frequency-limited function $\Psi_n(\cdot)$ satisfies

$$\int_{-W}^{W} \Psi_n(f) \Psi_m^*(f) df = \delta_{nm},$$
$$\int_{-T/2}^{T/2} \psi_n(t) \psi_m^*(t) dt = \sigma_n^2 \delta_{nm}$$

where $\psi_n(\cdot)$ is the inverse Fourier transform of $\Psi_n(\cdot)$. Therefore, $\Psi_n(\cdot)$ contains σ_n^2 of its energy within the time interval $[-T/2, T/2]$. Thus, the behavior of σ_n with $n(1 > \sigma_0 > \sigma_1 > \cdots \geq 0)$ determines the dimension of the subspace of frequency-limited and approximately time-limited waveforms. Slepian *et al.* have shown that when $n \ll 2WT$, σ_n^2 closes to 1; and when $n \gg 2WT$, σ_n^2 closes to 0. The transition occurs in an interval of n centered at $2WT$ with width growing at rate $\ln(2WT)$. Therefore, the dimension of the subspace should be

$$2WT + c_1 \ln(2WT) + o(\ln(2WT)) \tag{26}$$

for $WT \gg 1$, and c_1 is a constant. Cast onto linear arrays, when Ω_θ contains a single interval, the dimension of the array-limited and approximately wavevector-limited subspace is

$$2L|\Omega_\theta| + c_1 \ln(2L|\Omega_\theta|) + o(\ln(2L|\Omega_\theta|)) \tag{27}$$

for $L|\Omega_\theta| \gg 1$. However, Ω_θ likely contains more than one interval (see Table I). Fortunately, Landau *et al.* [25] showed that when Ω_θ contains M subintervals, the dimension is simply

$$2L|\Omega_\theta| + c_2 M \ln(2L|\Omega_\theta|) + o(\ln(2L|\Omega_\theta|)) \tag{28}$$

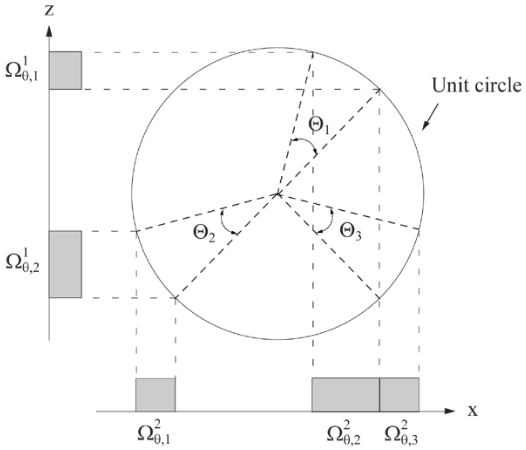

Fig. 5. Dependence of $|\Omega_\theta|$ on the orientation of a linear array. For a fixed scattering environment with $\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3$, $|\Omega_\theta|$ is $|\Omega^1_{\theta,1}| + |\Omega^1_{\theta,2}|$ when the array is oriented along the z-axis, and becomes $|\Omega^2_{\theta,1}| + |\Omega^2_{\theta,2}| + |\Omega^2_{\theta,3}|$ when the array is oriented along the x-axis.

for $L|\Omega_\theta| \gg 1$, and c_2 is a constant. Consequently, the dimension of the subspaces heuristically derived based on resolvability agrees in first order with this mathematically more rigorous approach.

B. Uni-Polarized Circular Arrays

Suppose Θ contains a single interval. If the linear array is oriented such that the scattering cluster is at the broadside,[4] then

$$|\Omega_\theta| = \int_\Theta d\cos\theta \approx \int_\Theta d\theta = |\Theta|.$$

On the other extreme, if the scattering cluster is at the endfire,[5] then

$$|\Omega_\theta| \approx \int_\Theta \theta d\theta = \frac{1}{2}|\Theta|^2.$$

As a result, the spatial degrees of freedom depend on the orientation of the linear array in addition to its size and the scattering of physical environment. Fig. 5 further illustrates this dependence for Θ containing multiple subintervals. To achieve a viable separation of array and channel characteristics, circular arrays are considered next.

Consider a circular array lying on the xy-plane of radius R normalized to a wavelength. Expressing the position vector $\boldsymbol{p}$ in spherical coordinates

$$\boldsymbol{p} = R\begin{bmatrix}\sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta'\end{bmatrix}$$

and $0 \le \theta' < \pi$, $0 \le \phi' < 2\pi$, the integral kernel in (14) becomes

$$a(\phi,\phi') = \exp[-\mathrm{i}2\pi R\cos(\phi-\phi')], \quad (\phi,\phi') \in \Phi \times [0,2\pi) \tag{29}$$

at $\theta = \pi/2$. As

$$e^{\mathrm{i}z\cos\phi} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(z) e^{\mathrm{i}n\phi}$$

where $J_n(\cdot)$ is the nth-order Bessel function of the first kind (referred as the Jacobi–Anger expansion [26]), it yields

$$a(\phi,\phi') = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(2\pi r) e^{\mathrm{i}n\phi} e^{-\mathrm{i}n\phi'}, \quad (\phi,\phi') \in \Phi \times [0,2\pi). \tag{30}$$

Obviously, the set $\{\frac{1}{\sqrt{2\pi}}e^{in\phi'}\}$ is orthonromal. If $\Phi = [0,2\pi)$ (fully scattered in the azimuth direction), the set $\{\frac{1}{\sqrt{2\pi}}e^{in\phi}, \phi \in \Phi\}$ is orthonormal as well, and the expansion in (30) corresponds to the decomposition defined in (15) and (16). Furthermore, when $R \gg 1$, $J_n(2\pi R) \approx 0$ for $|n| > 2\pi R$ [27]. Therefore, the dimension of the array-limited and approximately wavevector-limited subspace is $4\pi R$ for $R \gg 1$ and $\Phi = [0,2\pi)$.

When $\Phi \subset [0,2\pi)$, the set $\{e^{in\phi}, \phi \in \Phi\}$ is no longer orthogonal. Still, the subspace of array-limited radiated field patterns is spanned by

$$\left\{\frac{1}{\sqrt{2\pi}}e^{\mathrm{i}n\phi} : n = -2\pi R, \ldots, 2\pi R\right\} \tag{31}$$

for $R \gg 1$. Therefore, finding the number of significant singular values of $a(\phi,\phi')$, $\phi \in \Phi$ is equivalent to finding the dimension of the subspace of functions that are spanned by the set in (31) and contain most of their energy in Φ.

When Φ contains a single interval, Slepian [28] has shown that the subspace of functions spanned by $\{e^{\mathrm{i}n\phi}\}_{n=0}^{N-1}$ and contained most of their energy within Φ has a dimension of $N|\Phi|/(2\pi)$ for $N \gg 1$. This is also the dimension of the subspace of index-limited and approximately frequency-limited discrete-time sequences. The orthonormal sets is the set of discrete prolate spheriodal wave functions. As a result, the dimension of the array-limited and approximately wavevector-limited subspace would be $4\pi R \cdot |\Phi|/(2\pi) = 2R|\Phi|$ for $R \gg 1$.

When Φ contains more than one subinterval, we use the heuristic approach based on resolvability. In linear arrays, the radiated field pattern that has the narrowest beam width is $\mathrm{sinc}(2L\cos\theta)$ and hence, has a resolution of $1/(2L)$ over Ω_θ. For circular arrays, the radiated field pattern (spanned by the set in (31)) that has the narrowest beam width is given by

$$g(\phi) = \sum_{n=-2\pi R}^{2\pi R} e^{\mathrm{i}n\phi} = \frac{\sin(\pi 2R\phi)}{\sin\phi} \tag{32}$$

which is the periodic sinc (Dirichlet) function. It attains one main lobe at $\phi = 0$ and has zeros at multiples of $1/(2R)$. Therefore, the resolution of circular arrays over Φ is $1/(2R)$. Hence, the dimension of the array-limited and approximately wavevector-limited subspace is $2R|\Phi|$ for $R|\Phi| \gg 1$. Still, $|\Phi|$ depends on the orientation of the plane of the circular array so spherical arrays are considered next.

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[4] It refers to the direction normal to the linear array.

[5] It refers to the direction along the line of the linear array.

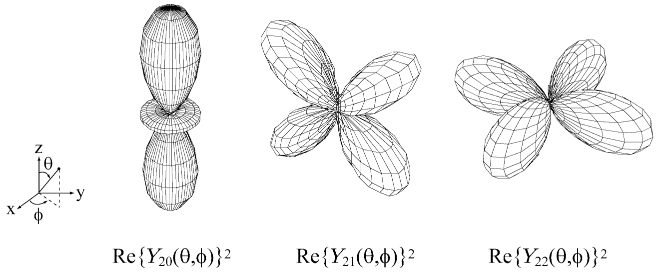

Fig. 6. Plots of spherical harmonic functions.

C. Uni-Polarized Spherical Arrays

Suppose the radius of the spherical array is R normalized to a wavelength. Then, the integral kernel in (14) becomes

$$a(\hat{\boldsymbol{k}},\hat{\boldsymbol{p}})=\exp\left\{-\mathrm{i}2\pi R[\sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta']\right\},\quad \hat{\boldsymbol{k}}\in\Omega. \quad (33)$$

As both position vector and propagation direction attain spherical geometries, the kernel can be decomposed into a sum of dyads of spherical harmonics

$$a(\hat{\boldsymbol{k}},\hat{\boldsymbol{p}})=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\mathrm{i})^l j_l(2\pi R)Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi'),\quad \hat{\boldsymbol{k}}\in\Omega \quad (34)$$

where $j_l(\cdot)$ is the spherical Bessel function of the first kind and order l. The set of spherical harmonics $\{Y_{lm}(\theta,\phi)\}$ is a complete orthonormal set on the surface of a unit sphere. The proof is included in Appendix II. When $\Omega=[0,\pi)\times[0,2\pi)$ (fully scattered in the entire propagation space), the decomposition agrees with that defined in (15) and (16). Furthermore, when $R\gg 1$, $j_l(2\pi R)\approx 0$ for $l>2\pi R$ [27]. Therefore, the dimension of the array-limited and approximately wavevector-limited subspace is $4\pi^2R^2$ for $R\gg 1$ and $\Omega=[0,\pi)\times[0,2\pi)$.

When $\Omega\subset[0,\pi)\times[0,2\pi)$, the set $\{Y_{lm}(\theta,\phi),\hat{\boldsymbol{k}}\in\Omega\}$ is no longer orthogonal. But the subspace of array-limited radiated field patterns is still spanned by

$$\left\{Y_{lm}(\theta,\phi): l=0,1,\ldots,2\pi R \text{ and } m=-l,\ldots,l\right\} \quad (35)$$

for $R\gg 1$. Fig. 6 plots $Y_{2m}(\theta,\phi)$ for $m=0,1,2$, in contrast to sinc functions of linear arrays and complex exponentials of circular arrays. Now, we want to find the dimension of the subspace of functions that are spanned by the set defined in (35) and contain most of their energy in Ω. To solve it, we reiterate the heuristic approach based on resolvability.

The spherical harmonics are separable

$$Y_{lm}(\theta,\phi)=c_{lm}P_l^m(\cos\theta)e^{\mathrm{i}m\phi}$$

where $c_{lm}=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$ and $P_l^m(\cdot)$ is the associated Legendre function. At $\theta=\pi/2$, the radiated field pattern that has the narrowest beam width in the azimuth direction, is a linear combination of $Y_{ll}(\pi/2,\phi)$ for $l=0,\ldots,2\pi R$ and is given by

$$g_\phi(\phi)=\sum_{l=-2\pi R}^{2\pi R}e^{\mathrm{i}l\phi}=\frac{\sin(\pi 2R\phi)}{\sin\phi} \quad (36)$$

same as circular arrays of radius R. It attains a main lobe at $\phi=0$ and has a beam width of $1/(2R)$. At the main lobe of $g_\phi(\phi)$, the radiated field pattern that has the narrowest beam width in the elevation direction, is a linear combination of $Y_{2\pi R,m}(\theta,0)$ for $m=-2\pi R,\ldots,2\pi R$. Among them, the beam width around $\theta=\pi/2$ is determined by $Y_{2\pi R,m}(\theta,0)$ for $m=\pm 2\pi R$ and the corresponding pattern is

$$g_\theta(\theta)=P_{2\pi R}^{2\pi R}(\cos\theta)=\frac{(-1)^{2\pi R}(2\pi R)!}{2^{2\pi R}(2\pi R)!}(\sin\theta)^{2\pi R}. \quad (37)$$

Since $\sin^n\theta\approx 1-n/2(\pi/2-\theta)^2$ around $\theta=\pi/2$, so the beam width of $g_\theta(\theta)$ is $4/(2\pi R)$. Hence, the resolution of spherical arrays over Ω is $\frac{1}{2R}\frac{4}{2\pi R}$ equal to $1/(\pi R^2)$. Defining the effective aperture of a spherical array as

$$\mathcal{A}=\pi R^2 \quad (38)$$

which is its area of projection onto a two–dimensional (2D) plane normalized to a square wavelength, the resolution is then expressed as $1/\mathcal{A}$. Consequently, the dimension of the array-limited and approximately wavevector-limited subspace is $\mathcal{A}|\Omega|$ for $\mathcal{A}|\Omega|\gg 1$.

D. Wavevector–Aperture–Polarization Product

Suppose $\mathcal{A}_t$ and $\mathcal{A}_r$ denote the effective aperture of the transmit and the receive spherical arrays, respectively. Then, the dimensions of the array-limited and approximately wavevector-limited subspace at the transmitter and the receiver are $\mathcal{A}_t|\Omega_t|$ and $\mathcal{A}_r|\Omega_r|$, respectively. The minimum of them gives the number of spatial degrees of freedom for uni-polarized antennas and is equal to

$$\min\{\mathcal{A}_t|\Omega_t|,\mathcal{A}_r|\Omega_r|\} \quad (39)$$

for $\mathcal{A}_t|\Omega_t|,\mathcal{A}_r|\Omega_r|\gg 1$. We refer to this quantity as the *wavevector–aperture product.*

To fully utilize the scattering in physical environments, tri-polarized antennas are needed. Now, we consider the 3 × 3 integral kernel in (18)

$$\left(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger\right)a(\hat{\boldsymbol{k}},\boldsymbol{p}),\qquad (\hat{\boldsymbol{k}},\boldsymbol{p})\in\Omega\times\mathcal{V}. \quad (40)$$

This kernel maps the vectored current distribution on the array to the polarized radiated field. The polarization matrix $(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger)$ is of rank 2 only. Immaterial to the rank of the scattering response $\boldsymbol{H}(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{k}})$, the rank of the system response can only be equal to or less than 2. Therefore, there is only a two-fold increase in the dimension of the transmit and receive subspaces, that is, $2\mathcal{A}|\Omega|$ for tri-polarized spherical arrays. Consequently, the number of spatial degrees of freedom is given by

$$\min\{2\mathcal{A}_t|\Omega_t|,2\mathcal{A}_r|\Omega_r|\} \quad (41)$$

for $\mathcal{A}_t|\Omega_t|,\mathcal{A}_r|\Omega_r|\gg 1$. We refer to this quantity as the wavevector–aperture–polarization product which gives the ultimate spatial degrees of freedom.

IV. PHYSICAL INTERPRETATIONS AND IMPLICATIONS

In this section, we will elucidate the physical insights of the main results, and attempt to connect antenna theory to information theory on concepts of antenna arrays.

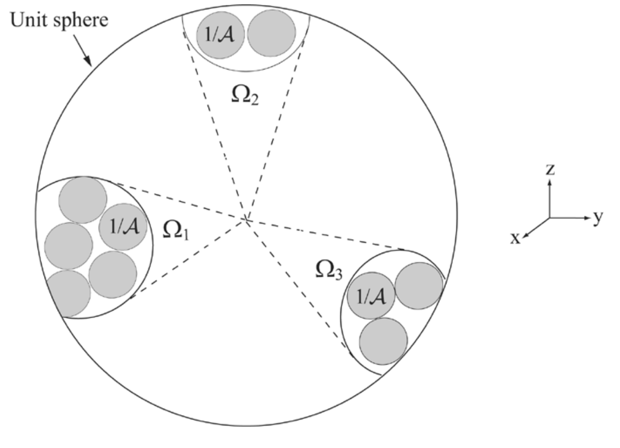

Fig. 7. Wavevector–aperture product. The shaded circle is of area $1/\mathcal{A}$. The number of these circles used to fill up the channel solid angles gives the spatial degrees of freedom (times a factor of 2 for polarization).

A. From Power Gain to Multiplexing Gain

From antenna theory, a larger antenna array is able to form a narrower beam.[6] At the transmitting side, it therefore increases the fraction of transmit power reaching the receiver while at the receiving side, it reduces the noise and hence increases the overall signal-to-noise ratio. As the transmit power is also relayed by scattering sources, the fraction of transmit power reaching the receiver also depends on the channel solid angles. The larger the angles are, the more the receiving power is. As a result, the power gain is limited by the product of the array size and the channel solid angle, $\mathcal{A}_t|\Omega_t|$ and $\mathcal{A}_r|\Omega_r|$.

When the size of arrays and/or channel solid angles are substantial, the transmit power can be split up to support parallel data streams. Fig. 7 gives a pictorial description on how the antenna array resolves the channel solid angles and creates parallel spatial channels. Now, the number of spatial channels is also limited by the product $\mathcal{A}_t|\Omega_t|$ and $\mathcal{A}_r|\Omega_r|$. Thus, the transition from a pure power gain perspective as in antenna theory to the spatial multiplexing gain considered in information theory is crucially determined by whether the wavevector–aperture product is substantial or not.

The literature of information theory, in contrast, infers the power gain from the array gain which is proportional to the number of antennas. It therefore claimed that increasing the number of antennas increases the power gain even for a fixed-array aperture such as in [10], [11], [13]. However, for a fixed-array aperture, increasing the number of antennas decreases the space occupied by each antenna and therefore decreases the individual antenna gain. Eventually, the power gain which is the product of the array gain and the antenna gain, remains unchanged and depends on the size of antenna array only.

B. Measure of Scattering

The wavevector domain provides an appropriate coordinate to describe the scattering of radio waves by physical objects in the environment. The total solid angles subtended by these objects, $|\Omega_t|$ and $|\Omega_r|$, reflect the amount of channel resources. In the statistical MIMO model, the channel correlation matrix is used to capture these channel resources. The scattering of physical environment is then measured by the rank of this matrix. An inadequate of this measure is its dependence on the number of antennas and their relative positions on antenna arrays. As an example, if we said that the delay spread of a channel is five taps without mentioning the bandwidth of the system, this would not be a very useful parameter about the channel. The channel solid angles, on the other hand, do not depend on characteristics of antenna arrays and hence provide a more intrinsic measure of the channel. Just like delay spread, we would say a typical indoor channel has a delay spread of 10–100 ns and based on this parameter, we design practical systems.

C. Geometry and Size of Arrays

The propagation space ($\hat{\boldsymbol{k}}$) is spanned by the elevation (θ) and the azimuth (ϕ) directions. The geometry of antenna arrays determines the efficiency in resolving these two directions and the resolution uniformity over the propagation space. For example, linear arrays can resolve the elevation direction and have a nonuniform resolution over it; circular arrays can resolve either the azimuth or the elevation directions, and have uniform resolution over the azimuth but nonuniform resolution over the elevation; and spherical arrays can resolve both the elevation and the azimuth, and have uniform resolution over them. The array geometry also determines the appropriate coordinate system to solve the set of Maxwell equations or, equivalently, the integral kernel $a(\cdot,\cdot)$ in (14). This results in different mapping between the array and the wavevector domains. The radiated field is then spanned by different basis functions: sinc functions for linear arrays, complex exponentials for circular arrays, and spherical harmonics for spherical arrays.

The array size then determines how many of these basis functions are significant or easy to excite. In other words, the array size determines the resolvability over the propagation space. For example, the number of significant modes are $4L$ for linear arrays of length $2L$, $2\pi R$ for circular arrays of radius R, and $4\pi\mathcal{A}$ for spherical arrays of effective aperture $\mathcal{A}$. Given the size of arrays, these numbers form the upper bound on the number of spatial degrees of freedom independent of the number of antennas. Packing more antennas beyond these limits will not increase the channel capacity significantly.

D. Polarization Benefits

The spatial degrees of freedom obtained from resolving the propagation space ($\hat{\boldsymbol{k}}$) is determined by the size of antenna arrays. Now, for each resolved propagation direction, the spatial degrees of freedom obtained from resolving the oscillation direction ($\hat{\boldsymbol{e}}_i$) of the propagating field is determined by the rank of the system response $\boldsymbol{C}(\cdot,\cdot)$ in (2). This is the spatial degrees of freedom from polarization. Since the system response is a composition of the array responses, $\boldsymbol{A}_r(\cdot,\cdot)$ and $\boldsymbol{A}_t(\cdot,\cdot)$, and the scattering response $\boldsymbol{H}(\cdot,\cdot)$, its rank is the minimum of the rank of these three responses. Therefore, just looking at the rank of the scattering response is not sufficient. For example, there is a

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[6] For example, a linear array of length $2L$ has a resolution of $1/(2L)$ over the $\cos\theta$-coordinate; a circular array of radius R has a resolution of $1/(2R)$ over the ϕ-coordinate; and a spherical array of radius R has a resolution of $1/(\pi R^2)$ over the (θ,ϕ)-coordinates.

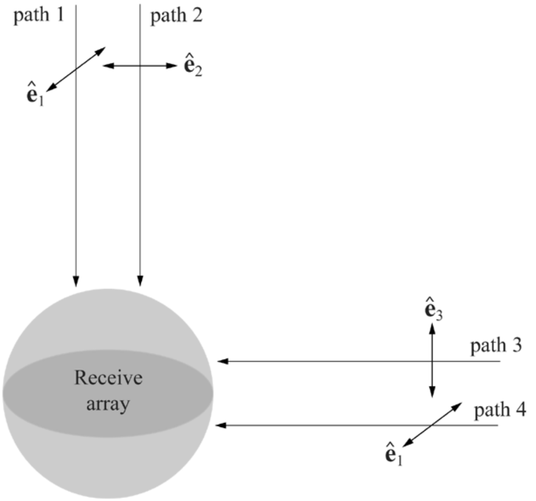

Fig. 8. A channel with four paths incident on a receive array. The propagation directions of path 1 and 2 are the same, but the electric field of path 1 oscillates along $\hat{\boldsymbol{e}}_1$ whereas that of path 2 oscillates along $\hat{\boldsymbol{e}}_2$. The propagation directions of path 3 and 4 are the same, and are perpendicular to that of path 1 and 2. The electric field of path 3 oscillates along $\hat{\boldsymbol{e}}_3$ while that of path 4 oscillates along $\hat{\boldsymbol{e}}_1$.

claim of six-fold increase in degrees of freedom from polarization [29]. The claim is based on the assumption that the electric field and the magnetic field are oscillating independently, and hence there is a total of six-fold increase. This independence may hold on the scattering response; however, it does not hold in the array responses where the electric field is given by the curl of the magnetic field. As a result, there is at most three-fold increase in degrees of freedom. In the intermediate- and near-field regions, the Green function in (5) is of rank 3, so in those two regions the three-fold increase is attainable. However, in the more important far-field region, the rank of the array responses is 2 and hence places a bottleneck on the available degrees of freedom.

The mathematical model presented makes substantial effort to distinguish the propagation direction and the oscillation direction of propagating fields. This helps us understand better the claim of four-fold increase in degrees of freedom from polarization [30]. The extra factor of 2 in the claim accounts for resolving the propagation direction, not the oscillation direction, and therefore should not be attributed as a polarization benefit. For example, consider the simplified scenario in Fig. 8 where there are four paths,[7] path 1 and 2 have the same propagation direction, and is orthogonal to that of path 3 and 4. The oscillation direction of the electric field is $\hat{\boldsymbol{e}}_1$ for path 1, $\hat{\boldsymbol{e}}_2$ for path 2, $\hat{\boldsymbol{e}}_3$ for path 3, and $\hat{\boldsymbol{e}}_1$ for path 4. Now if there is a single tri-pole antenna, its three orthogonal dipoles will be able to resolve signals from path 1 (or path 4), path 2, and path 3, and yield three spatial degrees of freedom. But path 1 and path 3 have different propagation directions. If we put more antennas on the array so as to distinguish paths 1 and 2 from paths 3 and 4, the total spatial degrees of freedom become four, a factor of 2 from resolving the propagation directions and another factor of 2 from resolving the oscillation directions. To avoid any ambiguity, only the degrees of freedom obtained from resolving the oscillation direction are considered as the polarization benefit.

E. Effects of Carrier Frequency

The array apertures in the wavevector–aperture–polarization product are normalized quantities. Let $\mathcal{A}_{t0}$ and $\mathcal{A}_{r0}$ denote the absolute effective aperture of the transmit and receive arrays, respectively. Then, the number of degrees of freedom can be expressed as

$$\frac{1}{\lambda^2}\min\{2\mathcal{A}_{t0}|\Omega_t|, 2\mathcal{A}_{r0}|\Omega_r|\}. \tag{42}$$

At first glance, the degrees of freedom increases with carrier frequency. Therefore, it is commonly believed that by operating at a higher frequency it is possible to increase the degrees of freedom by packing more antennas on the same wireless device. However, the channel solid angles decrease with increasing frequency. The reasons are twofold: 1) electromagnetic waves of higher frequency attenuate more after passing through or bouncing off channel objects which reduces the number of scattering clusters; and 2) at high frequency, the wavelength is small relative to the feature size of typical channel objects, so scattering appears to be more specular in nature and results in smaller solid angles. These factors together magnify the decrease in the channel solid angles. As a result, it is more conclusive to consider the entities $|\Omega_t|/\lambda^2$ and $|\Omega_r|/\lambda^2$ when judging the performance of multiple-antenna systems in different physical environments and frequency bands.

The exact relationship between the channel solid angle and carrier frequency varies between propagation scenarios, and is difficult to abstract into analytically tractable models. We believe that such a relationship would depend on the dimension and dielectric property of scattering sources, and the sensitivity of transceivers. As an example, Fig. 9(a) reproduces the measured $|\Omega_\theta|$ at different frequencies reported in [18] where linear arrays were used in the measurement. The $|\Omega_\theta|$ decreases with increasing frequency. As linear arrays can only resolve the elevation direction, we plot the entity $|\Omega_\theta|/\lambda$ in Fig. 9(b). The graphs show that $|\Omega_\theta|/\lambda$ initially increases with frequency and after passing the optimal frequencies, then decreases. Interestingly, the optimal frequencies for that particular office and residential environments are between 5 and 6 GHz where the IEEE 802.11a standard is located.

V. EXTENSION TO MULTIUSER ENVIRONMENTS

The scattering response in (10) is nonzero at transmit direction $\hat{\boldsymbol{k}}$ and receive direction $\hat{\boldsymbol{\kappa}}$, whenever there is a scatterer providing the connectivity between these directions. In a network, users can be viewed as another type of channel object affecting this connectivity and the wavevector domain provides an appropriate coordinate to describe the physical positions of network users.

Due to the broadcast nature of a wireless channel, signals can be captured and processed by any nearby user. If the user is *cooperative*, it will behave like a scatterer and relay the captured signals to the intended destination. But it can be more sophisticated: it can spatially demodulate the captured signals,

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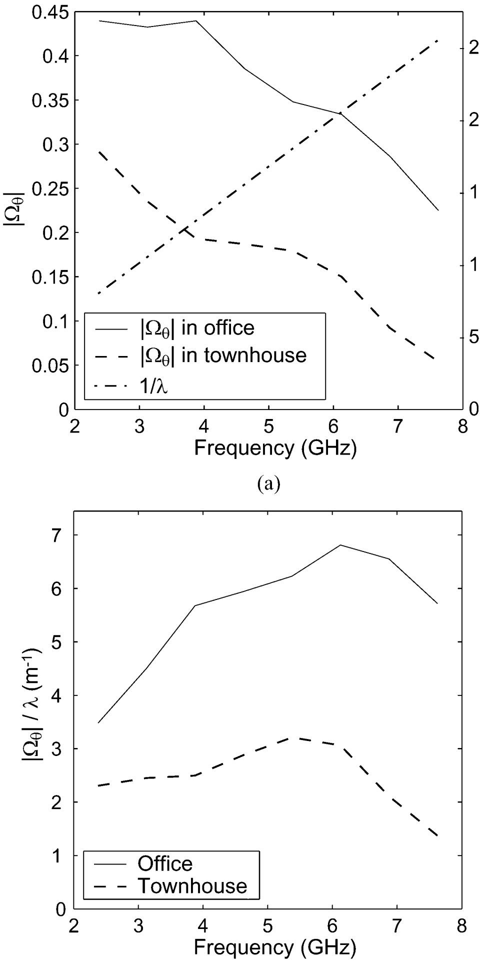
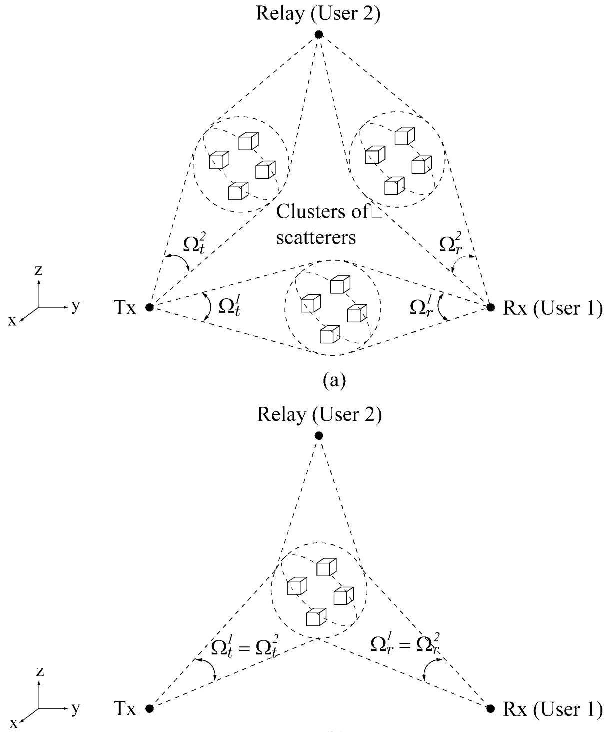
[7] Here, we use discrete paths for ease of exposition. In dense environments, they are forming clusters.



(a)

(b)

Fig. 9. (a) Reproduces the plot of $|\Omega_\theta|$ versus frequency in an office and residential environments from [18]. (b) Plots the corresponding $|\Omega_\theta|/\lambda$ versus frequency.



(a)

(b)

Fig. 10. A network with three nodes: (a) the channel solid angle subtended by user 2 is nonoverlapping with that subtended by the scattering clusters, and (b) the channel solid angle subtended by user 2 exactly overlaps with that of the scattering clusters.

then spatially modulate and forward them. Suppose Ω_t^1 and Ω_r^1 are the scattering intervals subtended by the scattering clusters, and Ω_t^i and $\Omega_r^i (i = 2, \dots, K)$ are the scattering intervals subtended by the ith nearby users from the transmitter and the receiver, respectively (see Fig. 10). Then, the scattering response is nonzero only on $\left(\bigcup_{i=1}^K \Omega_r^i\right) \times \left(\bigcup_{i=1}^K \Omega_t^i\right)$. Therefore, the channel solid angles in the wavevector–aperture–polarization product changes from $\left|\Omega_t^1\right|$ and $\left|\Omega_r^1\right|$ to $\left|\bigcup_{i=1}^K \Omega_t^i\right|$ and $\left|\bigcup_{i=1}^K \Omega_r^i\right|$.

Depending on the extent of overlapping among Ω_t^i's and among Ω_r^i's, they can be totally nonoverlapped as in Fig. 10(a) or completely overlapped as in Fig. 10(b). Except in the later case, the number of spatial degrees of freedom increases with the existence of network users. Consequently, we can think of "relay users" as an additional signal dimension. This is different from the time/frequency counterparts in waveform channels where having multiple users does not provide additional degrees of freedom. They are just sharing the existing time/frequency degrees of freedom.

On the contrary, if the nearby user is *noncooperative* and interfering, then it will reduce the signal-to-noise ratio on that particular subset of transmit and receive directions. Hence, its effect on the transmitted signal is additive interfering. When $\Omega_r^1 \cap \left(\bigcup_{i=2}^K \Omega_r^i\right)$ is nonempty, the receiver can avoid the interferers and abandon this subset of channel resource. This will incur a loss in the receive channel solid angle by $\left|\Omega_r^1 \cap (\bigcup_{i=2}^K \Omega_r^i)\right|$ and, hence, decrease the spatial degrees of freedom. In the extreme case, as shown in Fig. 10(b), all the degrees of freedom created by the scatterers will have been lost. Furthermore, if the scattering intervals Ω_t and Ω_r are coupled such as in the single-bounce channel, the transmitter should abandon the subset $\Omega_t^1 \cap \left(\bigcup_{i=2}^K \Omega_t^i\right)$ as well. This results in a loss of the transmit channel solid angle by $\left|\Omega_t^1 \cap \left(\bigcup_{i=2}^K \Omega_t^i\right)\right|$.

VI. OPTIMAL NUMBER OF ANTENNAS

To ease the understanding, we will use linear arrays as the mapping between the array and wavevector domains is the familiar Fourier transform. For other array geometries, the analysis continues to apply but with different mappings.

A. *Single-User Channels*

When Ω_θ contains a single interval, we can apply the Whittaker–Shannon sampling theorem. Now, $2L|\Omega_\theta|$ number of uniformly spaced antennas are adequate in the respective transmit and receive spaces. For example, in the fully scattered environment, we have $\Omega_\theta = (-1, 1]$, so $4L$, the number of antennas with $\lambda/2$ spacing suffices for optimal performance. Packing more antennas beyond $2L|\Omega_\theta|$ is unnecessary and corresponds to oversampling in the array domain. Putting fewer

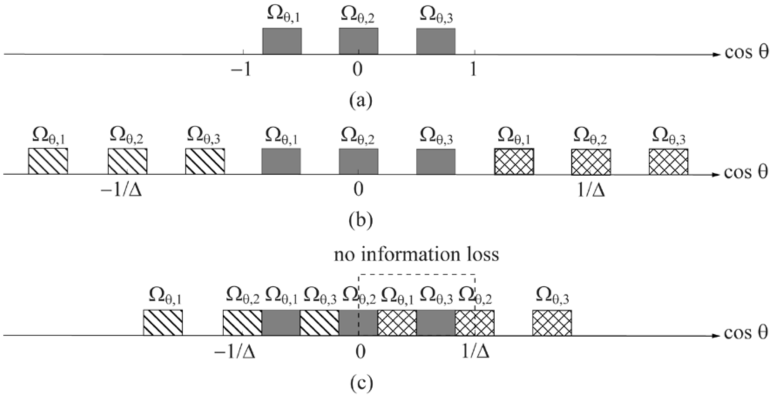


Fig. 11. The physical environment with $\Omega_\theta = \Omega_{\theta,1} \cup \Omega_{\theta,2} \cup \Omega_{\theta,3}$ as seen by (a) a continuous linear array; (b) a discrete linear array with uniform antenna spacing of $\Delta < 1/2$ (normalized to λ); and (c) a discrete linear array with uniform antenna spacing of $\Delta = \frac{1}{|\Omega_\theta|} \geq 1/2$.

antennas, on the other hand, is equivalent to decimating the signal on the array and hence it increases the periodicity of the radiated field in the wavevector domain. Grating lobes occur where the radiated field repeats itself within the interval Ω_θ. At the transmitter, the same information will be sent over more than one direction while at the receiver, signals from different directions will be aliased and perceived as from the same direction.

When Ω_θ contains multiple subintervals and is packable without gap,[8] $2L|\Omega_\theta|$, the number of uniformly spaced antennas is still adequate, as illustrated in Fig. 11. When Ω_θ is arbitrary, the problem is equivalent to finding the minimum rate to sample multiband signals[9] with spectral support of $\mathcal{W}$. Landau [31] has derived the lower bound which is $|\mathcal{W}|$, that is, the minumum number of samples taken over a period of T for perfect reconstruction is $T|\mathcal{W}|$. It has been shown that periodic nonuniform sampling can approach this lower bound [32]. This implies that the optimal number of antennas remains $2L|\Omega_\theta|$ but the position of the antennas depends on the structure of Ω_θ.

Averaging over the array orientation, $\mathsf{E}[|\Omega_\theta|]$ is approximately equal to $\frac{2}{\pi}|\Theta|$. If the parameters, $|\Theta_t|$ and $|\Theta_r|$, in the targeted environment are known *a priori*, we will demonstrate numerically that the respective $2L\mathsf{E}[|\Omega_\theta|]$ number of uniformly spaced antennas at the transmitter and the receiver are adequate to approach the optimal ergodic capacity. That is, $\frac{4}{\pi}L_t|\Theta_t|$ number of transmit and $\frac{4}{\pi}L_r|\Theta_r|$ number of receive antennas suffice where $2L_t$ and $2L_r$ are the length of the transmit and receive arrays, respectively. This is in analogy to knowing the delay spread in the targeted environment so as to design channel equalizers with the optimal number of taps.

B. Multiuser Environments

In a cooperative (relaying) network, simply replacing Ω in the single-user case by $|\bigcup_{i=1}^{K}\Omega_t^i|$ at the transmitter and by $|\bigcup_{i=1}^{K}\Omega_r^i|$ at the receiver, will give the optimal number of transmit and receive antennas. In a noncooperative (interfering) network, on the contrary, the performance of any interference management scheme depends on the gaps between the interfering and the desired scattering intervals. Larger gaps result in better performance. For example, if $\Omega_{\theta,2}$ in Fig. 11(a) corresponds to the interfering interval while $\Omega_{\theta,1}$ and $\Omega_{\theta,3}$ are the desired intervals, increasing the antenna spacing beyond $1/2$ (normalized to a wavelength) reduces the gaps between $\Omega_{\theta,2}$ and $\Omega_{\theta,1}$, and between $\Omega_{\theta,2}$ and $\Omega_{\theta,3}$ as illustrated in Fig. 11(c). This will degrade the performance. Likewise, decreasing the antenna spacing smaller than $1/2$ will not change the gaps (see Fig. 11(b)) and hence will not improve the performance. Therefore, the number of antennas in noncooperative networks is insensitive to the physical environment but depends on the size of arrays.

C. Numerical Examples

To verify the results, numerical examples are given. The transmit and receive signals on the arrays are related by

$$y(q_z) = \int c(q_z, p_z)x(p_z)dp_z + z(q_z) \tag{43}$$

where $z(p_z)$ is the additive white complex Gaussian noise of zero mean and unit variance. The transmit signal $x(\cdot)$ is normalized such that $\mathsf{E}[\|x(p_z)\|^2] = \mathsf{SNR}$. The system response $c(q_z, p_z)$ can be decomposed into

$$c(q_z, p_z) = \int\int e^{i2\pi q_z \cos\vartheta} h(\vartheta, \theta) e^{-i2\pi p_z \cos\theta} d\cos\vartheta d\cos\theta \tag{44}$$

where θ and ϑ are the transmit and receive angles, respectively. We assume that the scattering response $h(\vartheta, \theta)$ is uncorrelated at different (ϑ, θ), and is a complex Gaussian random process with zero mean and unit variance within the scattering intervals.

Suppose antennas are uniformly spaced with separation of Δ_t on the transmit array and Δ_r on the receive array. Then, the (n, m)th element of the channel matrix is given by

$$\boldsymbol{C}_{nm} = \sqrt{\Delta_r\Delta_t}c(n\Delta_r, m\Delta_t). \tag{45}$$

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[8] There exists α_0 such that translating Ω_θ by multiples of α_0 introduces no overlap, and tiling a number of them covers the entire axis.

[9] Signals have nonvanishing power spectral density over a finite union of arbitrary nonoverlapping intervals in the frequency domain.

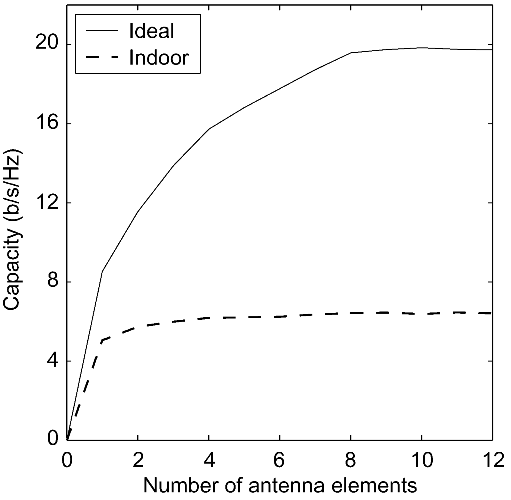

Fig. 12. Capacity versus number of antenna elements for four-wavelength long arrays at SNR of 10 dB.

The ergodic capacity with full channel state information at both transmitter and receiver is used for comparison, and is given by

$$\mathsf{E}_{\boldsymbol{C}}\Big[\sum_i \left(\log_2 \upsilon\sigma_i^2\right)^+\Big]$$

where υ satisfies

$$\sum_i \left(\upsilon - \sigma_i^{-2}\right)^+ = \mathsf{SNR}$$

and σ_i's are the singular values of $\boldsymbol{C}$.

Two propagation environments will be studied: fully scattered (ideal) and indoor. In the fully scattered environment, there are scatterers all around the transmitter and the receiver, which result in their respective $|\Omega_\theta|$ being 2. Parameters for the indoor environment are inferred from Table I. There are three scattering clusters each of angle 20° (0.35 rad) randomly placed which result in $\mathsf{E}[|\Omega_\theta|] \approx \frac{2}{\pi} \times 0.35 \times 3 = 0.67$. Fig. 12 plots the channel capacity versus the number of antennas on four-wavelength long arrays $(2L = 4)$ at SNR of 10 dB. The graphs show that $\lceil 2L\mathsf{E}[|\Omega_\theta|]\rceil$ number of transmit and receive antennas suffice to approach the optimal throughput in both environments. To further reinforce the observation, Fig. 13 plots the channel capacities for different array sizes in the indoor environment. The numbers of transmit and receive antennas in the discrete arrays are kept to $\lceil 2L\mathsf{E}[|\Omega_\theta|]\rceil$. Continuous arrays are used as upper bounds for comparison. The graphs show that the channel capacities acheived by discrete arrays approach closely to those of continuous arrays.

Taking into account the interfering signals, the additive noise term is no longer white but is colored with the (n, m)th element in the covariance matrix being

$$\delta_{n-m} + \sum_{i=2}^{K} \mathsf{INR}_i \Delta_r \int_{\Theta_r^i} e^{j2\pi \cos\theta \Delta_r (n-m)} d\cos\theta \qquad (46)$$

where INR_i denotes the average transmit interference-to-noise ratio of the ith user. Suppose $\mathsf{INR}_i = \mathsf{SNR}$, for $i = 2, \dots, K$. The scattering intervals Θ_t^i's and Θ_r^i's are assumed to be independently and uniformly distributed over $[0, \pi)$. Fig. 14 plots the waterfilling capacity versus the number of receive antennas for discrete four-wavelength long arrays at SNR of 10 dB and the number of transmit antennas is fixed to $\lceil 2L\mathsf{E}[|\Omega_\theta|]\rceil = 3$. The graphs show that invariant to the physical environment and the number of interferers, the optimal number of antennas is always $9 = 4L + 1$. Similar obervations are reported when the number of transmit antennas is increased.

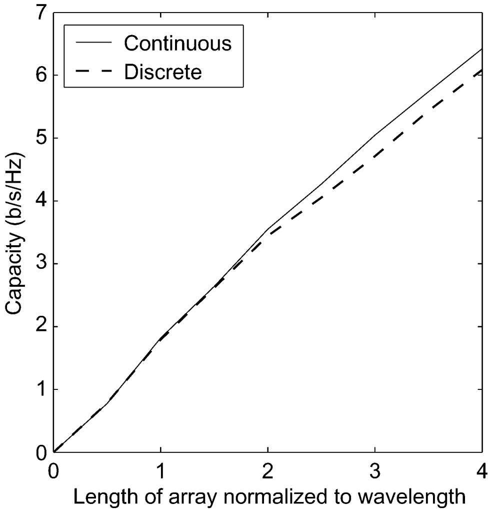

Fig. 13. Capacity versus length of array at SNR of 10 dB in the indoor environment. In the discrete array, the number of transmit and receive antenna elements are both equal to $\lceil 2L\mathsf{E}[|\Omega_\theta|]\rceil$.

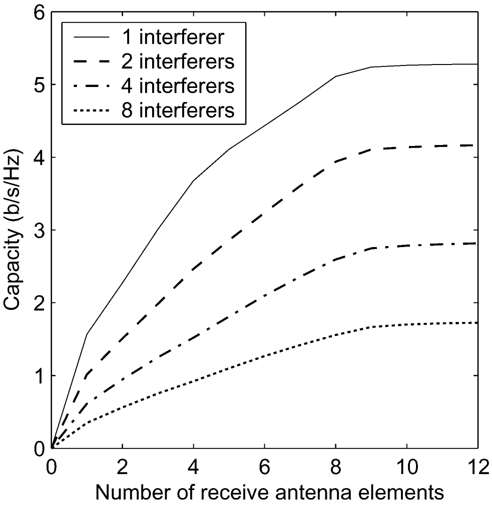

Fig. 14. Capacity versus number of antenna elements versus number of interferers for four-wavelength long arrays at SNR of 10 dB in the indoor environment.

VII. CONCLUSION

Previous studies on the capacity of multiple-antenna channels are based on a statistical MIMO approach. Given a constraint on the areas of the transmit and receive arrays, there is a deterministic limit to the number of spatial degrees of freedom underlying the statistical approach. In this paper, we incorporate antenna theory with observations from spatial channel measurements to obtain a mathematical model. Based on this model, we derive the limit to the spatial degrees of freedom given the space constraints. The results help assess the optimal number of antennas that should be packed on a given wireless device in a given application environment. Nowadays major portion of the chipset for single-antenna systems is occupied by the RF/analog front end and is expected to increase due to the poor scaling of analog circuitry. Also, most of the power consumption in single-antenna systems goes to the RF/analog front end. Therefore, optimizing the number of antennas plus RF/analog front ends is practically important for multiple-antenna systems. The proposed mathematical framework would be useful in this aspect.

APPENDIX I
PROOF OF (5)

The Green function $\mathcal{G}(\mathbf{p},\mathbf{p}')$ satisfies

$$\left(-\nabla_{\mathbf{p}}\times\nabla_{\mathbf{p}}\times+k_0^2\right)\mathcal{G}(\mathbf{p},\mathbf{p}') = \mathrm{i}k_0\eta\delta(\mathbf{p}-\mathbf{p}'). \tag{47}$$

Notice that under three-dimensional Fourier transform, the curl operator transforms to

$$\nabla_{\mathbf{p}}\times \xleftrightarrow{\mathcal{F}} \mathrm{i}\mathbf{k}\times .$$

Taking the three-dimensional Fourier transform with respect to $\mathbf{p}$ on both sides of (47) and applying the following identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}^\dagger\mathbf{c})\mathbf{b} - (\mathbf{a}^\dagger\mathbf{b})\mathbf{c}$$

we obtain

$$\left[\mathbf{k}\mathbf{k}^\dagger + (k_0^2-k^2)\mathbf{I}\right]\mathsf{G}(\mathbf{k},\mathbf{p}') = \mathrm{i}k_0\eta e^{-\mathrm{i}\mathbf{k}^\dagger\mathbf{p}'}.$$

Recalling the matrix inversion lemma, the Green function is then given by

$$\mathsf{G}(\mathbf{k},\mathbf{p}') = \frac{\mathrm{i}\eta}{k_0\left(k_0^2-k^2\right)}\left(k_0^2\mathbf{I}-\mathbf{k}\mathbf{k}^\dagger\right)e^{-\mathrm{i}\mathbf{k}^\dagger\mathbf{p}'}.$$

Defining $\mathbf{r}=\mathbf{p}-\mathbf{p}'$ and performing the inverse Fourier transform results in

$$\mathcal{G}(\mathbf{p},\mathbf{p}') = \frac{\mathrm{i}\eta}{(2\pi)^3k_0}\int\left(k_0^2\mathbf{I}-\mathbf{k}\mathbf{k}^\dagger\right)\frac{e^{\mathrm{i}\mathbf{k}^\dagger\mathbf{r}}}{k_0^2-k^2}d^3k$$

which is equivalent to

$$\mathcal{G}(\mathbf{p},\mathbf{p}') = \frac{\mathrm{i}k_0\eta}{(2\pi)^3}\left(\mathbf{I}+\frac{1}{k_0^2}\nabla_{\mathbf{r}}\nabla_{\mathbf{r}}\right)\int\frac{e^{\mathrm{i}\mathbf{k}^\dagger\mathbf{r}}}{k_0^2-k^2}d^3k.$$

The integral on the right-hand side is a complex integral and is evaluated as $2\pi^2e^{\mathrm{i}k_0r}/r$. Its gradient is

$$\nabla_{\mathbf{r}}\frac{e^{\mathrm{i}k_0r}}{r} = \left(\mathrm{i}k_0-\frac{1}{r}\right)\frac{e^{\mathrm{i}k_0r}}{r}\hat{\mathbf{r}}$$

and thus, the second gradient is

$$\begin{aligned}\nabla_{\mathbf{r}}\nabla_{\mathbf{r}}&\frac{e^{\mathrm{i}k_0r}}{r}\\ &= \frac{e^{\mathrm{i}k_0r}}{r}\left[-k_0^2\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger + \frac{\mathrm{i}k_0}{r}\left(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right) - \frac{1}{r^2}\left(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right)\right].\end{aligned}$$

Consequently, the Green function is

$$\begin{aligned}\mathcal{G}(\mathbf{p},\mathbf{p}') = \frac{\mathrm{i}k_0\eta e^{\mathrm{i}k_0r}}{4\pi r}\Bigg[&\left(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right)+\frac{\mathrm{i}}{k_0r}\left(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right)\\ &-\frac{1}{k_0^2r^2}(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger)\Bigg].\end{aligned}$$

As $k_0=2\pi/\lambda$, where λ is the wavelength, so the Green function can also be written as

$$\begin{aligned}\mathcal{G}(\mathbf{p},\mathbf{p}') = \frac{\mathrm{i}\eta e^{\mathrm{i}2\pi r/\lambda}}{2\lambda r}\Bigg[&\left(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right)+\frac{\mathrm{i}}{2\pi r/\lambda}\left(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger\right)\\ &-\frac{1}{(2\pi r/\lambda)^2}(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}}^\dagger)\Bigg].\end{aligned}$$

APPENDIX II
PROOF OF (34)

Define

$$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta'$$

in which the argument γ is the angle between the position vector $\mathbf{p}$ and the direction vector $\hat{\mathbf{k}}$. As $-1\le\cos\gamma\le 1$, so the array response can be expanded in terms of Legendre polynomials which are orthogonal over $[-1,1]$

$$e^{-\mathrm{i}2\pi R\cos\gamma} = \sum_{l=1}^{\infty}a_l\sqrt{\frac{2l+1}{2}}P_l(\cos\gamma).$$

The coefficients in this expansion are given by

$$a_l = \sqrt{\frac{2l+1}{2}}\int_{-1}^{1}e^{-\mathrm{i}2\pi Rx}P_l(x)dx.$$

The Rodrigues formula [27] provides

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

which results in

$$a_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{2l+1}{2}}\int_{-1}^{1}(x^2-1)^l\frac{d^l}{dx^l}e^{-\mathrm{i}2\pi Lx}dx.$$

The integral can be further simplified to

$$\begin{aligned}\int_{-1}^{1}&(x^2-1)^l\frac{d^l}{dx^l}e^{-\mathrm{i}2\pi Rx}dx\\ &= (-\mathrm{i}2\pi R)^l\int_{-1}^{1}(x^2-1)^le^{-\mathrm{i}2\pi Rx}dx\\ &= (-\mathrm{i}2\pi R)^l\int_{-1}^{1}(x^2-1)^l\cos(2\pi Rx)dx\\ &= (-\mathrm{i}2\pi R)^l\frac{l!}{(-\pi R)^l\sqrt{R}}J_{l+1/2}(2\pi R)\\ &= \mathrm{i}2^{l+1}l!j_l(2\pi R)\end{aligned}$$

which yields

$$a_l = (-\mathrm{i})^l\sqrt{2(2l+1)}j_l(2\pi R).$$

Therefore, we obtain

$$e^{-\mathrm{i}2\pi R\cos\gamma} = \sum_{l=0}^{\infty}(-\mathrm{i})^l(2l+1)j_l(2\pi R)P(\cos\gamma).$$

Applying the addition theorem [27] yields

$$e^{-\mathrm{i}2\pi R\cos\gamma} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\mathrm{i})^l j_l(2\pi R)Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi')$$

where $j_l(\cdot)$ is the spherical bessel function of the first kind and lth order.

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