Information-Theoretically Optimal Compressed Sensing via Spatial Coupling and Approximate Message Passing

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Abstract

We study the compressed sensing reconstruction problem for a broad class of random, band-diagonal sensing matrices. This construction is inspired by the idea of spatial coupling in coding theory. As demonstrated heuristically and numerically by Krzakala et al. [KMS+11], message passing algorithms can effectively solve the reconstruction problem for spatially coupled measurements with undersampling rates close to the fraction of non-zero coordinates.

We use an approximate message passing (AMP) algorithm and analyze it through the state evolution method. We give a rigorous proof that this approach is successful as soon as the undersampling rate $\delta$ exceeds the (upper) Rényi information dimension of the signal, $\bar{d}(p_X)$. More precisely, for a sequence of signals of diverging dimension $n$ whose empirical distribution converges to $p_X$, reconstruction is with high probability successful from $\bar{d}(p_X) n + o(n)$ measurements taken according to a band diagonal matrix.

For sparse signals, i.e., sequences of dimension $n$ and $k(n)$ non-zero entries, this implies reconstruction from $k(n) + o(n)$ measurements. For ‘discrete’ signals, i.e., signals whose coordinates take a fixed finite set of values, this implies reconstruction from $o(n)$ measurements. The result is robust with respect to noise, does not apply uniquely to random signals, but requires the knowledge of the empirical distribution of the signal $p_X$.

Index Terms: Compressed sensing, approximate message passing, spatial coupling, information dimension, state evolution.

1 Introduction and main results

1.1 Background and contributions

Assume that $m$ linear measurements are taken of an unknown $n$-dimensional signal $x \in \mathbb{R}^n$, according to the model

$$y = Ax . \quad (1)$$
The reconstruction problem requires to reconstruct $x$ from the measured vector $y \in \mathbb{R}^m$, and the measurement matrix $A \in \mathbb{R}^{m \times n}$.

It is an elementary fact of linear algebra that the reconstruction problem will not have a unique solution unless $m \geq n$. This observation is however challenged within compressed sensing. A large corpus of research shows that, under the assumption that $x$ is sparse, a dramatically smaller number of measurements is sufficient [Don06a, CRT06a, Don06b]. Namely, if only $k$ entries of $x$ are non-vanishing, then roughly $m \gtrsim 2k \log(n/k)$ measurements are sufficient for $A$ random, and reconstruction can be solved efficiently by convex programming. Deterministic sensing matrices achieve similar performances, provided they satisfy a suitable restricted isometry condition [CT05]. On top of this, reconstruction is robust with respect to the addition of noise [CRT06b, DMM11], i.e., under the model

$$y = Ax + w,$$

with, say, $w \in \mathbb{R}^m$ a random vector with i.i.d. components $w_i \sim \text{N}(0, \sigma^2)$ (unless stated otherwise, $\sigma = 0$ is a valid choice). In this context, the notions of ‘robustness’ or ‘stability’ refers to the existence of universal constants $C$ such that the per-coordinate mean square error in reconstructing $x$ from noisy observation $y$ is upper bounded by $C \sigma^2$.

From an information-theoretic point of view it remains however unclear why we cannot achieve the same goal with far fewer than $2k \log(n/k)$ measurements. Indeed, we can interpret Eq. (1) as describing an analog data compression process, with $y$ a compressed version of $x$. From this point of view, we can encode all the information about $x$ in a single real number $y \in \mathbb{R}$ (i.e., use $m = 1$), because the cardinality of $\mathbb{R}$ is the same as the one of $\mathbb{R}^n$. Motivated by this puzzling remark, Wu and Verdú [WV10] introduced a Shannon-theoretic analogue of compressed sensing, whereby the vector $x$ has i.i.d. components $x_i \sim p_X$. Crucially, the distribution $p_X$ is available to, and may be used by the reconstruction algorithm. Under the mild assumptions that sensing is linear (as per Eq. (1)), and that the reconstruction mapping is Lipschitz continuous, they proved that compression is asymptotically lossless if and only if

$$m \geq n \overline{d}(p_X) + o(n).$$

Here $\overline{d}(p_X)$ is the (upper) Rényi information dimension of the distribution $p_X$. We refer to Section 1.2 for a precise definition of this quantity. Suffices to say that, if $p_X$ is $\varepsilon$-sparse (i.e., if it puts mass at most $\varepsilon$ on nonzeros) then $\overline{d}(p_X) \leq \varepsilon$. Also, if $p_X$ is the convex combination of a discrete part (sum of Dirac’s delta) and an absolutely continuous part (with a density), then $\overline{d}(p_X)$ is equal to the weight of the absolutely continuous part.

This result is quite striking. For instance, it implies that, for random $k$-sparse vectors, $m \geq k + o(n)$ measurements are sufficient. Also, if the entries of $x$ are random and take values in, say, $\{-10, -9, \ldots, -9, +10\}$, then a sublinear number of measurements $m = o(n)$, is sufficient! At the same time, the result of Wu and Verdú presents two important limitations. First, it does not provide robustness guarantees of the type described above. Second and most importantly, it does not provide any computationally practical algorithm for reconstructing $x$ from measurements $y$.

While this paper was about to be posted, we became aware of a paper by Wu and Verdú [WV11b] proving a robustness guarantee for $\delta > \overline{D}(p_X)$ for the case of probability distributions that do not contain singular continuous component. The reconstruction method is again not practical.

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In an independent line of work, Krzakala et al. [KMS +11] developed an approach that leverages the idea of spatial coupling. This idea was introduced for the compressed sensing literature by Kudekar and Pfister [KP10] (see [KRU11] and Section 1.5 for a discussion of earlier work on this topic). Spatially coupled matrices are, roughly speaking, random sensing matrices with a band-diagonal structure. The analogy is, this time, with channel coding.

In this context, spatial coupling, in conjunction with message-passing decoding, allows to achieve Shannon capacity on memoryless communication channels. It is therefore natural to ask whether an approach based on spatial coupling can enable to sense random vectors \( x \) at an undersampling rate \( m/n \) close to the Rényi information dimension of the coordinates of \( x \), \( \overline{d}(p_X) \). Indeed, the authors of [KMS +11] evaluate such a scheme numerically on a few classes of random vectors and demonstrate that it indeed achieves rates close to the fraction of non-zero entries. They also support this claim by insightful statistical physics arguments.

In this paper, we fill the gap between the above works, and present the following contributions:

**Construction.** We describe a construction for spatially coupled sensing matrices \( A \) that is somewhat broader than the one of [KMS +11] and give precise prescriptions for the asymptotic values of various parameters. We also use a somewhat different reconstruction algorithm from the one in [KMS +11], by building on the approximate message passing (AMP) approach of [DMM09, DMM10]. AMP algorithms have the advantage of smaller memory complexity with respect to standard message passing, and of smaller computational complexity whenever fast multiplication procedures are available for \( A \).

**Rigorous proof of convergence.** Our main contribution is a rigorous proof that the above approach indeed achieves the information-theoretic limits set out by Wu and Verdú [WV10]. Indeed, we prove that, for sequences of spatially coupled sensing matrices \( \{A(n)\}_{n \in \mathbb{N}} \), \( A(n) \in \mathbb{R}^{m(n) \times n} \) with asymptotic undersampling rate \( \delta = \lim_{n \to \infty} m(n)/n \), AMP reconstruction is with high probability successful in recovering the signal \( x \), provided \( \delta > \overline{d}(p_X) \).

**Robustness to noise.** We prove that the present approach is robust to noise in the following sense. For any signal distribution \( p_X \) and undersampling rate \( \delta \), there exists a constant \( C \) such that the output \( \hat{x}(y) \) of the reconstruction algorithm achieves a mean square error per coordinate \( n^{-1} \mathbb{E}\{\|\hat{x}(y) - x\|_2^2\} \leq C \sigma^2 \). This result holds under the noisy measurement model (2) for a broad class of noise models for \( w \), including i.i.d. noise coordinates \( w_i \) with \( \mathbb{E}\{w_i^2\} = \sigma^2 < \infty \).

**Non-random signals.** Our proof does not apply uniquely to random signals \( x \) with i.i.d. components, but indeed to more general sequences of signals \( \{x(n)\}_{n \in \mathbb{N}} \), \( x(n) \in \mathbb{R}^n \) indexed by their dimension \( n \). The conditions required are: (1) that the empirical distribution of the coordinates of \( x(n) \) converges (weakly) to \( p_X \); and (2) that \( \|x(n)\|_2^2 \) converges to the second moment of the asymptotic law \( p_X \).

There is a fundamental reason why this more general framework turns out to be equivalent to the random signal model. This can be traced back to the fact that, within our construction,

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2 Unlike [KMS +11], we follow here the terminology developed within coding theory.

3 This robustness bound holds for all \( \delta > \overline{D}(p_X) \), where \( \overline{D}(p_X) \) is the upper MMSE dimension of \( p_X \). (see Definition 1.4). It is worth noting that \( \overline{D}(p_X) = \overline{d}(p_X) \) for a broad class of distributions \( p_X \) including distributions without singular-continuous component.
the columns of the matrix $A$ are probabilistically exchangeable. Hence any vector $x(n)$ is equivalent to the one whose coordinates have been randomly permuted. The latter is in turn very close to the i.i.d. model. By the same token, the rows of $A$ are exchangeable and hence the noise vector $w$ does not need to be random either.

Interestingly, the present framework changes the notion of ‘structure’ that is relevant for reconstructing the signal $x$. Indeed, the focus is shifted from the *sparsity* of $x$ to the *information dimension* $\mathcal{d}(p_X)$. In other words, the signal structure that facilitates recovery from a small number of linear measurements is the low-dimensional structure in an information theoretic sense, quantified by the information dimension of the signal.

In the rest of this section we state formally our results, and discuss their implications and limitations, as well as relations with earlier work. Section 2.3 provides a precise description of the matrix construction and reconstruction algorithm. Section 4 reduces the proof of our main results to two key lemmas. One of these lemmas is a (quite straightforward) generalization of the state evolution construction and reconstruction algorithm. Section 4 reduces the proof of our main results to two
tions, as well as relations with earlier work. Section 2.3 provides a precise description of the matrix

### 1.2 Formal statement of the results

We consider the noisy model (2). An instance of the problem is therefore completely specified by the triple $(x, w, A)$. We will be interested in the asymptotic properties of sequence of instances indexed by the problem dimensions $S = \{(x(n), w(n), A(n))\}_{n \in \mathbb{N}}$. We recall a definition from [BM12]. (More precisely, [BM12] introduces the $B = 1$ case of this definition.)

**Definition 1.1.** The sequence of instances $S = \{x(n), w(n), A(n)\}_{n \in \mathbb{N}}$ indexed by $n$ is said to be a $B$-converging sequence if $x(n) \in \mathbb{R}^n$, $w(n) \in \mathbb{R}^m$, $A(n) \in \mathbb{R}^{m \times n}$ with $m = m(n)$ is such that $m/n \to \delta \in (0, \infty)$, and in addition the following conditions hold:

1. The empirical distribution of the entries of $x(n)$ converges weakly to a probability measure $p_X$ on $\mathbb{R}$ with bounded second moment. Further $n^{-1} \sum_{i=1}^n x_i(n)^2 \to \mathbb{E}\{X^2\}$, where the expectation is taken with respect to $p_X$.

2. The empirical distribution of the entries of $w(n)$ converges weakly to a probability measure $p_W$ on $\mathbb{R}$ with bounded second moment. Further $m^{-1} \sum_{i=1}^m w_i(n)^2 \to \mathbb{E}\{W^2\} = \sigma^2$, where the expectation is taken with respect to $p_W$.

3. If $\{e_i\}_{1 \leq i \leq n}$, $e_i \in \mathbb{R}^n$ denotes the canonical basis, then $\limsup_{n \to \infty} \frac{\max_{i \leq n} \|A(n)e_i\|_2}{\min_{i \leq n} \|A(n)e_i\|_2} \leq B$.

We further say that $\{(x(n), w(n))\}_{n \geq 0}$ is a converging sequence of instances, if they satisfy conditions (a) and (b). We say that $\{A(n)\}_{n \geq 0}$ is a $B$-converging sequence of sensing matrices if they satisfy

\footnote{If $(\mu_k)_{k \in \mathbb{N}}$ is a sequence of measures and $\mu$ is another measure, all defined on $\mathbb{R}$, the weak convergence of $\mu_k$ to $\mu$ along with the convergence of their second moments to the second moment of $\mu$ is equivalent to convergence in 2-Wasserstein distance $\mathcal{W}_{1}(\mu_k, \mu)$. Therefore, conditions (a)-(b) are equivalent to the following. The empirical distributions of the signal $x(n)$ and the empirical distributions of noise $w(n)$ converge in 2-Wasserstein distance.}
condition (c) above, and we call it a converging sequence if it is B-converging for some B. Similarly, we say S is a converging sequence if it is B-converging for some B.

Finally, if the sequence \( \{(x(n), w(n), A(n))\}_{n \geq 0} \) is random, the above conditions are required to hold almost surely.

Notice that standard normalizations of the sensing matrix correspond to \( \|A(n)e_i\|_2^2 = 1 \) (and hence \( B = 1 \)) or to \( \|A(n)e_i\|_2^2 = m(n)/n \). The former corresponds to normalized columns and the latter corresponds to normalized rows. Since throughout we assume \( m(n)/n \rightarrow \delta \in (0, \infty) \), these conventions only differ by a rescaling of the noise variance. In order to simplify the proofs, we allow ourselves somewhat more freedom by taking B a fixed constant.

Given a sensing matrix A, and a vector of measurements y, a reconstruction algorithm produces an estimate \( \hat{x}(A; y) \in \mathbb{R}^n \) of x. In this paper we assume that the empirical distribution \( p_X \), and the noise level \( \sigma^2 \) are known to the estimator, and hence the mapping \( \hat{x} : (A, y) \mapsto \hat{x}(A; y) \) implicitly depends on \( p_X \) and \( \sigma^2 \). Since however \( p_X, \sigma^2 \) are fixed throughout, we avoid the cumbersome notation \( \hat{x}(A, y, p_X, \sigma^2) \).

Given a converging sequence of instances \( S = \{(x(n), w(n), A(n))\}_{n \in \mathbb{N}} \), and an estimator \( \hat{x} \), we define the asymptotic per-coordinate reconstruction mean square error as

\[
\text{MSE}(S; \hat{x}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \| \hat{x}(A(n); y(n)) - x(n) \|^2. \tag{4}
\]

Notice that the quantity on the right hand side depends on the matrix \( A(n) \), which will be random, and on the signal and noise vectors \( x(n), w(n) \) which can themselves be random. Our results hold almost surely with respect to these random variables. In some applications it is more customary to take the expectation with respect to the noise and signal distribution, i.e., to consider the quantity

\[
\overline{\text{MSE}}(S; \hat{x}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \| \hat{x}(A(n); y(n)) - x(n) \|^2. \tag{5}
\]

It turns out that the almost sure bounds imply, in the present setting, bounds on the expected mean square error \( \overline{\text{MSE}} \), as well.

In this paper we study a specific low-complexity estimator, based on the AMP algorithm first proposed in [DMM09]. AMP is an iterative algorithm derived from the theory of belief propagation in graphical models [Mon12]. At each iteration \( t \), it keeps track of an estimate \( x^t \in \mathbb{R}^n \) of the unknown signal x. This is used to compute residuals \( (y - Ax^t) \in \mathbb{R}^m \). These correspond to the part of observations that is not explained by the current estimate \( x^t \). The residuals are then processed through a matched filter operator (roughly speaking, this amounts to multiplying the residuals by the transpose of A) and then applying a non-linear denoiser, to produce the new estimate \( x^{t+1} \).

Formally, we start with an initial guess \( x^1_i = \mathbb{E}\{X\} \) for all \( i \in [n] \) and proceed by

\[
\begin{align*}
x^{t+1} & = \eta_t(x^t + (Q^t \odot A)x^t), \tag{6} \\
r^{t} & = y - Ax^t + b^t \odot r^{t-1}. \tag{7}
\end{align*}
\]

The second equation corresponds to the computation of new residuals from the current estimate. The memory term (also known as ’Onsager term’ in statistical physics) plays a crucial role as emphasized in [DMM09 BM11 BLM12 JM12a]. The first equation describes matched filter, with multiplication by \((Q_t \odot A)^*\), followed by application of the denoiser \( \eta_t \). Throughout \( \odot \) indicates Hadamard (entrywise) product and \( X^* \) denotes the transpose of matrix X.
For each \( t \), the denoiser \( \eta_t : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a differentiable non-linear function that depends on the input distribution \( p_X \). Further, \( \eta_t \) is separable, namely, for a vector \( v \in \mathbb{R}^n \), we have \( \eta_t(v) = (\eta_{1,t}(v_1), \ldots, \eta_{n,t}(v_n)) \). The matrix \( Q^t \in \mathbb{R}^{m \times n} \) and the vector \( b^t \in \mathbb{R}^m \) can be efficiently computed from the current state \( x^t \) of the algorithm. Further \( Q^t \) does not depend on the problem instance and hence can be precomputed. Both \( Q^t \) and \( b^t \) are block-constants, i.e., they can be partitioned into blocks such that within each block all the entries have the same value. This property makes their evaluation, storage and manipulation particularly convenient.

We refer to the next section for explicit definitions of these quantities. A crucial element is the specific choice of \( \eta_{i,t} \). The general guiding principle is that the argument \( y^t = x^t + (Q^t \odot A)^* r^t \) in Eq. (8) should be interpreted as a noisy version of the unknown signal \( x \), i.e., \( y^t = x + \text{noise} \). The denoiser \( \eta_t \) must therefore be chosen as to minimize the mean square error at iteration \((t + 1)\). The papers \[ \text{DMM}09, \text{DJM}11 \] take a minimax point of view, and hence study denoisers that achieve the smallest mean square error over the worst case signal \( x \) in a certain class. For instance, coordinate-wise soft thresholding is nearly minimax optimal over the class of sparse signals \[ \text{DMM}11 \]. Here we instead assume that the prior \( p_X \) is known, and hence the choice of \( \eta_{i,t} \) is uniquely dictated by the objective of minimizing the mean square error at iteration \( t + 1 \). In other words \( \eta_{i,t} \) takes the form of a Bayes optimal estimator for the prior \( p_X \). In order to stress this point, we will occasionally refer to this as the Bayes optimal AMP algorithm. As shown in Appendix B, \( x^t \) is (almost surely) a local Lipschitz continuous function of the observations \( y \).

Finally notice that \[ \text{DMM}10, \text{Mon}12 \] also derived AMP starting from a Bayesian graphical models point of view, with the signal \( x \) modeled as random with i.i.d. entries. The algorithm in Eqs. (6), (7) differs from the one in \[ \text{DMM}10 \] in that the matched filter operation requires scaling \( A \) by the matrix \( Q^t \). This is related to the fact that we will use a matrix \( A \) with independent but not identically distributed entries and, as a consequence, the accuracy of each entry \( x^t_i \) depends on the index \( i \) as well as on \( t \).

We denote by \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \) the mean square error achieved by the Bayes optimal AMP algorithm, where we made explicit the dependence on \( \sigma^2 \). Since the AMP estimate depends on the iteration number \( t \), the definition of \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \) requires some care. The basic point is that we need to iterate the algorithm only for a constant number of iterations, as \( n \) gets large. Formally, we let

\[
\text{MSE}_{\text{AMP}}(S; \sigma^2) \equiv \lim_{t \to \infty} \limsup_{n \to \infty} \frac{1}{n} \| x^t(A(n); y(n)) - x(n) \|^2. \tag{8}
\]

As discussed above, limits will be shown to exist almost surely, when the instances \( (x(n), w(n), A(n)) \) are random, and almost sure upper bounds on \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \) will be proved. (Indeed \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \) turns out to be deterministic.) On the other hand, one might be interested in the expected error

\[
\overline{\text{MSE}}_{\text{AMP}}(S; \sigma^2) \equiv \lim_{t \to \infty} \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}\{ \| x^t(A(n); y(n)) - x(n) \|^2 \}. \tag{9}
\]

We will tie the success of our compressed sensing scheme to the fundamental information-theoretic limit established in \[ \text{WV}10 \]. The latter is expressed in terms of the Rényi information dimension of the probability measure \( p_X \).

\footnotemark

\footnotetext{We refer to \[ \text{DJM}11 \] for a study of non-separable denoisers in AMP algorithms.}
Definition 1.2. Let $p_X$ be a probability measure over $\mathbb{R}$, and $X \sim p_X$. The upper and lower information dimension of $p_X$ are defined as

\begin{align}
\overline{d}(p_X) &= \limsup_{\ell \to \infty} \frac{H([X]_\ell)}{\log \ell}, \\
\underline{d}(p_X) &= \liminf_{\ell \to \infty} \frac{H([X]_\ell)}{\log \ell}.
\end{align}

Here $H(\cdot)$ denotes Shannon entropy and, for $x \in \mathbb{R}$, $[x]_\ell \equiv \lfloor \ell x / \ell \rfloor$, and $\lfloor x \rfloor \equiv \max\{k \in \mathbb{Z} : k \leq x\}$.

If the lim sup and lim inf coincide, then we let $d(p_X) = \overline{d}(p_X) = \underline{d}(p_X)$.

Whenever the limit of $H([X]_\ell)/\log \ell$ exists and is finite\footnote{This condition can be replaced by $H([X]) < \infty$. A sufficient condition is that $\mathbb{E}[\log(1 + |X|)] < \infty$, which is certainly satisfied if $X$ has a finite variance [WV11a].}, the Rényi information dimension can also be characterized as follows. Write the binary expansion of $X$, $X = D_0.D_1D_2D_3...$ with $D_i \in \{0, 1\}$ for $i \geq 1$. Then $d(p_X)$ is the entropy rate of the stochastic process $\{D_1, D_2, D_3, ...\}$.

It is also convenient to recall the following result from [Rény59, WV10].

Proposition 1.3 ([Rény59, WV10]). Let $p_X$ be a probability measure over $\mathbb{R}$, and $X \sim p_X$. Assume $H([X])$ to be finite. If $p_X = (1 - \varepsilon)\nu_d + \varepsilon\tilde{\nu}$ with $\nu_d$ a discrete distribution (i.e., with countable support), then $\overline{d}(p_X) = \underline{d}(p_X) = d(p_X) = \varepsilon$. Further, if $\tilde{\nu}$ has a density with respect to Lebesgue measure, then $d(p_X) = \overline{d}(p_X) = \underline{d}(p_X) = \varepsilon$. In particular, if $\mathbb{P}\{X \neq 0\} \leq \varepsilon$ then $\overline{d}(p_X) \leq \varepsilon$.

In order to present our result concerning the robust reconstruction, we need the definition of MMSE dimension of the probability measure $p_X$.

Given the signal distribution $p_X$, we let $\text{mmse}(s)$ denote the minimum mean square error in estimating $X \sim p_X$ from a noisy observation in gaussian noise, at signal-to-noise ratio $s$. Formally

\[ \text{mmse}(s) \equiv \inf_{\eta: \mathbb{R} \to \mathbb{R}} \mathbb{E}\left\{ \left[ X - \eta(\sqrt{s}X + Z) \right]^2 \right\}, \]

where $Z \sim \mathcal{N}(0, 1)$. Since the minimum mean square error estimator is just the conditional expectation, this is given by

\[ \text{mmse}(s) = \mathbb{E}\left\{ \left[ X - \mathbb{E}[X|Y] \right]^2 \right\}, \quad Y = \sqrt{s}X + Z. \]

Notice that $\text{mmse}(s)$ is naturally well defined for $s = \infty$, with $\text{mmse}(\infty) = 0$. We will therefore interpret it as a function $\text{mmse} : \mathbb{R}_+ \to \mathbb{R}_+$ where $\mathbb{R}_+ \equiv [0, \infty]$ is the completed non-negative real line.

We recall the inequality

\[ 0 \leq \text{mmse}(s) \leq \frac{1}{s}, \]

obtained by the estimator $\eta(y) = y / \sqrt{s}$. A finer characterization of the scaling of $\text{mmse}(s)$ is provided by the following definition.
Theorem 1.7. Let \( \sigma \) there exists \( m \), then there exists a random converging sequence of sensing matrices \( \{ \} \), for which the following holds. For any converging sequence of instances \( \{ (x(n), w(n)) \} \), we have, almost surely

\[
\text{MSE}_{\text{AMP}}(S; \sigma^2) \leq \varepsilon.
\]  

Further, under the same assumptions, we have \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \leq \varepsilon \).

The second theorem characterizes the rate at which the mean square error goes to 0. In particular, we show that \( \text{MSE}_{\text{AMP}}(S; \sigma^2) = O(\sigma^2) \) provided \( \delta > \mathcal{D}(p_X) \).

Theorem 1.7. Let \( p_X \) be a probability measure on the real line and assume

\[
\delta > \mathcal{D}(p_X).
\]  

Then there exists a random converging sequence of sensing matrices \( \{ A(n) \} \), \( A(n) \in \mathbb{R}^{m \times n} \), \( m(n)/n \rightarrow \delta \) (with distribution depending only on \( \delta \)), for which the following holds. For any \( \varepsilon > 0 \), there exists \( \sigma_0 = \sigma_0(\varepsilon, \delta, p_X) \) such that for any converging sequence of instances \( \{ (x(n), w(n)) \} \) with parameters \( (p_X, \sigma^2, \delta) \) and \( \sigma \in [0, \sigma_0] \), we have, almost surely

\[
\text{MSE}_{\text{AMP}}(S; \sigma^2) \leq \varepsilon.
\]  

Further, under the same assumptions, we have \( \text{MSE}_{\text{AMP}}(S; \sigma^2) \leq C \sigma^2 \).

Finally, the sensitivity to small noise is bounded as

\[
\limsup_{\sigma \rightarrow 0} \frac{1}{\sigma^2} \text{MSE}_{\text{AMP}}(S; \sigma^2) \leq \frac{4\delta - 2\mathcal{D}(p_X)}{\delta - \mathcal{D}(p_X)}.
\]
The performance guarantees in Theorems 1.6 and 1.7 are achieved with special constructions of the sensing matrices $A(n)$. These are matrices with independent Gaussian entries with unequal variances (heteroscedastic entries), with a band diagonal structure. The motivation for this construction, and connection with coding theory is further discussed in Section 1.4, while formal definitions are given in Section 2.1 and 2.4.

Notice that, by Proposition 1.5, $\mathcal{D}(p_X) \geq \mathcal{d}(p_X)$, and $\mathcal{D}(p_X) = \mathcal{d}(p_X)$ for a broad class of probability measures $p_X$, including all measures that do not have a singular continuous component (i.e., decomposes into a pure point mass component and an absolutely continuous component).

The noiseless model (1) is covered as a special case of Theorem 1.6 by taking $\sigma^2 \downarrow 0$. For the reader’s convenience, we state the result explicitly as a corollary.

**Corollary 1.8.** Let $p_X$ be a probability measure on the real line. Then, for any $\delta > \mathcal{d}(p_X)$ there exists a random converging sequence of sensing matrices $\{A(n)\}_{n \geq 0}$, $A(n) \in \mathbb{R}^{m \times n}$, $m(n)/n \to \delta$ (with distribution depending only on $\delta$) such that, for any sequence of vectors $\{x(n)\}_{n \geq 0}$ whose empirical distribution converges to $p_X$, the Bayes optimal AMP asymptotically almost surely recovers $x(n)$ from $m(n)$ measurements $y = A(n)x(n) \in \mathbb{R}^{m(n)}$. (By ‘asymptotically almost surely’ we mean $\text{MSE}_{\text{AMP}}(S; 0) = 0$ almost surely, and $\overline{\text{MSE}}_{\text{AMP}}(S; 0) = 0$.)

Note that it would be interesting to prove a stronger guarantee in the noiseless case, namely $\lim_{n \to \infty} x^t(A(n); y(n)) = x(n)$ with probability converging to 1 as $n \to \infty$. The present paper does not lead to a proof of this statement.

### 1.3 Discussion

Theorem 1.6 and Corollary 1.8 are, in many ways, puzzling. It is instructive to spell out in detail a few specific examples, and discuss interesting features.

**Example 1 (Bernoulli-Gaussian signal).** Consider a Bernoulli-Gaussian distribution

$$p_X = (1 - \varepsilon) \delta_0 + \varepsilon \gamma_{\mu, \sigma}$$

where $\gamma_{\mu, \sigma}(dx) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} dx$ is the Gaussian measure with mean $\mu$ and variance $\sigma^2$. This model has been studied numerically in a number of papers, including [BSB10, KMS+11]. By Proposition 1.3, we have $\mathcal{d}(p_X) = \varepsilon$, and by Proposition 1.5, $\mathcal{D}(p_X) = \mathcal{D}(p_X) = \varepsilon$ as well.

Construct random signals $x(n) \in \mathbb{R}^n$ by sampling i.i.d. coordinates $x(n)_i \sim p_X$. Glivenko-Cantelli’s theorem implies that the empirical distribution of the coordinates of $x(n)$ converges almost surely to $p_X$, hence we can apply Corollary 1.8 to recover $x(n)$ from $m(n) = n\varepsilon + o(n)$ spatially coupled measurements $y(n) \in \mathbb{R}^{m(n)}$. Notice that the number of non-zero entries in $x(n)$ is, almost surely, $k(n) = n\varepsilon + o(n)$. Hence, we can restate the implication of Corollary 1.8 as follows. A sequence of vectors $x(n)$ with Bernoulli-Gaussian distribution and $k(n)$ nonzero entries can almost surely recovered by $m(n) = k(n) + o(n)$ spatially coupled measurements.

**Example 2 (Mixture signal with a point mass).** The above remarks generalize immediately to arbitrary mixture distributions of the form

$$p_X = (1 - \varepsilon) \delta_0 + \varepsilon q,$$

$$9$$
where \( q \) is a measure that is absolutely continuous with respect to Lebesgue measure, i.e., \( q(dx) = f(x)dx \) for some measurable function \( f \). Then, by Proposition 1.3, we have \( \overline{\mathcal{R}}(p_X) = \varepsilon \), and by Proposition 1.5, \( \overline{\mathcal{R}}(p_X) = \overline{\mathcal{D}}(p_X) = \varepsilon \) as well. Similar to Example 1, the number of non-zero entries in \( x(n) \) is, almost surely, \( k(n) = n\varepsilon + o(n) \), and we can recover \( x(n) \) from \( m(n) = n\varepsilon + o(n) \) spatially coupled measurements. This can be recast as follows.

**Corollary 1.9.** Let \( \{x(n)\}_{n \geq 0} \) be a sequence of vectors with i.i.d. components \( x(n)_i \sim p_X \) where \( p_X \) is a mixture distribution as per Eq. (24). Denote by \( k(n) \) the number of nonzero entries in \( x(n) \). Then, almost surely as \( n \to \infty \), Bayes optimal AMP recovers the signal \( x(n) \) from \( m(n) = k(n) + o(n) \) spatially coupled measurements.

Under the regularity hypotheses of [WY10], no scheme can do substantially better, i.e., reconstruct \( x(n) \) from \( m(n) \) measurements if \( \limsup_{n \to \infty} m(n)/k(n) < 1 \).

One way to think about this result is the following. If an oracle gave us the support of \( x(n) \), we would still need \( m(n) \geq k(n) - o(n) \) measurements to reconstruct the signal. Indeed, the entries in the support have distribution \( q \), and \( \overline{\mathcal{D}}(q) = 1 \). Corollary [L8] implies that the measurements overhead for estimating the support of \( x(n) \) is sublinear, \( o(n) \), even when the support is of order \( n \).

It is sometimes informally argued that compressed sensing requires at least \( \Theta(k \log(n/k)) \) for ‘information-theoretic reasons’, namely that specifying the support requires about \( k \log(n/k) \) bits. This argument is of course incomplete because it assumes that each measurement \( y_i \) is described by a bounded number of bits. Since it is folklore to say that sparse signal recovery requires \( m \geq Ck \log(n/k) \) measurement, it is instructive to survey the results of this type and explain why they do not apply to the present setting. This elucidates further the implications of our results.

Specifically, [Wai09, ASZ10] prove information-theoretic lower bounds on the required number of measurements, under specific constructions for the random sensing matrix \( A \). Further, these papers focus on the specific problem of exact support recovery. The paper [RWY09] proves minimax bounds for reconstructing vectors belonging to \( \ell_p \)-balls. Notice that these bounds are usually proved by exhibiting a least favorable prior, which is close to a signal with i.i.d. coordinates. However, as the noise variance tends to zero, these bounds depend on the sensing matrix in a way that is difficult to quantify. In particular, they provide no explicit lower bound on the number of measurements required for exact recovery in the noiseless limit. Similar bounds were obtained for arbitrary measurement matrices in [CD11]. Again, these lower bounds vanish as noise tends to zero as soon as \( m(n) \geq k(n) \).

A different line of work derives lower bounds from Gelfand’ width arguments [Don06a, KT07]. These lower bounds are only proved to be a necessary condition for a stronger reconstruction guarantee. Namely, these works require the vector of measurements \( y = Ax \) to enable recovery for all \( k \)-sparse vectors \( x \in \mathbb{R}^n \). This corresponds to the ‘strong’ phase transition of [DT05, Don06b], and is also referred to as the ‘for all’ guarantee in the computer science literature [BGI+08].

The lower bound that comes closest to the present setting is the ‘randomized’ lower bound [BIPW10]. In this work the authors consider a fixed signal \( x \) and a random sensing matrix as in our setting. In other words they do not assume a standard minimax setting. However they require an \( \ell_1 - \ell_1 \) error guarantee which is a stronger stability condition than what is achieved in Theorem [L7] allowing for a more powerful noise process. Indeed the same paper also proves that recovery is possible from \( m(n) = O(k(n)) \) measurements under stronger conditions.

**Example 3 (Discrete signal).** Let \( K \) be a fixed integer, \( a_1, \ldots, a_K \in \mathbb{R} \), and \( (p_1, p_2, \ldots, p_K) \) be a collection of non-negative numbers that add up to one. Consider the probability distribution
that puts mass $p_i$ on each $a_i$

$$p_X = \sum_{i=1}^{K} p_i \delta_{a_i}, \quad (25)$$

and let $x(n)$ be a signal with i.i.d. coordinates $x(n)_i \sim p_X$. By Proposition 1.3, we have $d(p_X) = 0$. As above, the empirical distribution of the coordinates of the vectors $x(n)$ converges to $p_X$. By applying Corollary 1.8 we obtain the following

**Corollary 1.10.** Let $\{x(n)\}_{n \geq 0}$ be a sequence of vectors with i.i.d. components $x(n)_i \sim p_X$ where $p_X$ is a discrete distribution as per Eq. (25). Then, almost surely as $n \rightarrow \infty$, Bayes optimal AMP recovers the signal $x(n)$ from $m(n) = o(n)$ spatially coupled measurements.

It is important to further discuss the last statement because the reader might be misled into too optimistic a conclusion. Consider any signal $x \in \mathbb{R}^n$. For practical purposes, this will be represented with finite precision, say as a vector of $\ell$-bit numbers. Hence, in practice, the distribution $p_X$ is always discrete, with $K = 2^\ell$ a fixed number dictated by the precision requirements. A sublinear number of measurements $m(n) = o(n)$ will then be sufficient to achieve this precision.

On the other hand, Theorem 1.6 and Corollary 1.8 are asymptotic statements, and the convergence rate is not claimed to be uniform in $p_X$. In particular, the values of $n$ at which it becomes accurate will likely increase with $K$.

**Example 4 (A discrete-continuous mixture).** Consider the probability distribution

$$p_X = \varepsilon_+ \delta_{\varepsilon_+} + \varepsilon_- \delta_{\varepsilon_-} + \varepsilon q, \quad (26)$$

where $\varepsilon_+ + \varepsilon_- + \varepsilon = 1$ and the probability measure $q$ has a density with respect to Lebesgue measure. Again, let $x(n)$ be a vector with i.i.d. components $x(n)_i \sim p_X$. We can apply Corollary 1.8 to conclude that $m(n) = n\varepsilon + o(n)$ spatially coupled measurements are sufficient. This should be contrasted with the case of sensing matrices with i.i.d. entries studied in [DT10] under convex reconstruction methods (namely solving the feasibility problem $y = Ax$ under the constraint $\|x\|_\infty \leq 1$). In this case $m(n) = n(1 + \varepsilon)/2 + o(n)$ measurements are necessary.

In the next section we describe the basic intuition behind the surprising phenomenon in Theorems 1.6 and 1.7, and why spatially coupled sensing matrices are so useful. We conclude by stressing once more the limitations of these results:

- The Bayes optimal AMP algorithm requires knowledge of the signal distribution $p_X$. Notice however that only a good approximation of $p_X$ (call it $p_X$, and denote by $\tilde{X}$ the corresponding random variable) is sufficient. Assume indeed that $p_X$ and $p_{\tilde{X}}$ can be coupled in such a way that $E\{(X - \tilde{X})^2\} \leq \tilde{\sigma}^2$. Then

$$x = \tilde{x} + u \quad (27)$$

where $\|u\|_2^2 \lesssim n\tilde{\sigma}^2$. This is roughly equivalent to adding to the noise vector $z$ further ‘noise’ $\tilde{z}$ with variance $\delta^2/\delta$. By this argument the guarantee in Theorem 1.7 degrades gracefully as $p_{\tilde{X}}$ gets different from $p_X$. Another argument that leads to the same conclusion consists in studying the evolution of the algorithm (6), (7) when $\eta_t$ is matched to the incorrect prior, see Appendix A.
Finally, it was demonstrated numerically in [VS11, KMS+11] that, in some cases, a good ‘proxy’ for $p_X$ can be learned through an Expectation-Maximization-style iteration. A rigorous study of this approach goes beyond the scope of present paper.

• In particular, the present approach does not provide uniform guarantees over the class of, say, sparse signals characterized by $p_X(\{0\}) \geq 1 - \varepsilon$. In particular, both the phase transition location, cf. Eq. (18), and the robustness constant, cf. Eq. (21), depend on the distribution $p_X$. This should be contrasted with the minimax approach of [DMM09, DMM11, DJM11] which provides uniform guarantees over sparse signals. See Table 1 for a comparison between the two schemes.

<table>
<thead>
<tr>
<th>Minimax setup [DMM11]</th>
<th>Bayesian setup (present approach)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensing matrix</td>
<td>i.i.d. Gaussian entries (homoscedastic)</td>
</tr>
<tr>
<td>Reconstruction algorithm</td>
<td>AMP with soft thresholding denoiser</td>
</tr>
<tr>
<td>Compression rate</td>
<td>$\delta &gt; M(\varepsilon)$</td>
</tr>
<tr>
<td>Sensitivity to noise</td>
<td>$\frac{M(\varepsilon)}{1-M(\varepsilon)/\delta}$</td>
</tr>
</tbody>
</table>

Table 1: Comparison between the minimax setup in [DMM11] and the Bayesian setup considered in this paper. (cf. [DMM11] Eq. (2.4)) for definition of $M(\varepsilon)$).

• As mentioned above, the guarantees in Theorems 1.6 and 1.7 are only asymptotic. It would be important to develop analogous non-asymptotic results.

• The stability bound (21) is non-uniform, in that the proportionality constant $C$ depends on the signal distribution. It would be important to establish analogous bounds that are uniform over suitable classes of distributions. (We do not expect Eq. (21) to hold uniformly over all distributions.)

1.4 How does spatial coupling work?

Spatial coupling was developed in coding theory to construct capacity achieving LDPC codes [FZ99, SLJZ04, KMIRU10, HMU10, KRU12]. The standard construction starts from the parity check matrix of an LDPC code that is sparse but unstructured apart from the degree sequence. A spatially coupled ensemble is then obtained by enforcing a band-diagonal structure, while keeping the degree sequence unchanged. Usually this is done by graph liftings, but the underlying principle is more general [HMU10].

Following the above intuition, spatially coupled sensing matrices $A$ are, roughly speaking, random band-diagonal matrices. The construction given below (as the one of [KMS+11]) uses matrices with independent zero-mean Gaussian entries, with non-identical variances (heteroscedastic entries). However, the simulations of [JM12b] suggest that a much broader set of matrices display similar performances. As discussed in Section 2.1 the construction is analogous to graph liftings. We start
by a matrix of variances $W = (W_{r,c})$ and obtain the sensing matrix $A$ by replacing each entry $W_{r,c}$
by a block with i.i.d. Gaussian entries with variance proportional to $W_{r,c}$.

It is convenient to think of the graph structure that they induce on the reconstruction problem. 

Associate one node (a variable node in the language of factor graphs) to each coordinate $i$ in the
unknown signal $x$. Order these nodes on the real line $\mathbb{R}$, putting the $i$-th node at location $i \in \mathbb{R}$.
Analogously, associate a node (a factor node) to each coordinate $a$ in the measurement vector $y$, and
place the node $a$ at position $a/\delta$ on the same line. Connect this node to all the variable nodes $i$ such
that $A_{ai} \neq 0$. If $A$ is band diagonal, only nodes that are placed close enough will be connected by
an edge. See Figure 1 for an illustration.

In a spatially coupled matrix, additional measurements are associated to the first few coordinates
of $x$, say coordinates $x_1, \ldots, x_{n_0}$ with $n_0$ much smaller than $n$. This has a negligible impact on the
overall undersampling ratio as $n/n_0 \to \infty$. Although the overall undersampling remains $\delta < 1$, the
coordinates $x_1, \ldots, x_{n_0}$ are oversampled. This ensures that these first coordinates are recovered
correctly (up to a mean square error of order $\sigma^2$). As the algorithm is iterated, the contribution
of these first few coordinates is correctly subtracted from all the measurements, and hence we can
effectively eliminate those nodes from the graph. In the resulting graph, the first few variables are
effectively oversampled and hence the algorithm will reconstruct their values, up to a mean square
error of order $\sigma^2$. As the process is iterated, variables are progressively reconstructed, proceeding
from left to right along the node layout.

While the above explains the basic dynamics of AMP reconstruction algorithms under spatial
coupling, a careful consideration reveals that this picture leaves open several challenging questions.
In particular, why does the overall undersampling factor $\delta$ have to exceed $\overline{d}(p_X)$ for reconstruction
to be successful? Our proof is based on a potential function argument. We will prove that there
exists a potential function for the AMP algorithm, such that, when $\delta > \overline{d}(p_X)$, this function has
its global minimum close to exact reconstruction. Further, we will prove that, unless this minimum
is essentially achieved, AMP can always decrease the function. This technique is different from the
one followed in [KRU11] for the LDPC codes over the binary erasure channel, and we think it is of
independent interest.
1.5 Further related work

The most closely related earlier work was already discussed above.

More broadly, message passing algorithms for compressed sensing where the object of a number of studies studies, starting with [BSB10]. As mentioned, we will focus on approximate message passing (AMP) as introduced in [DMM09, DMM10]. As shown in [DJM11] these algorithms can be used in conjunction with a rich class of denoisers \( \eta(\cdot) \). A subset of these denoisers arise as posterior mean associated to a prior \( p_X \). Several interesting examples were studied by Schniter and collaborators [Sch10, Sch11, SPS10], and by Rangan and collaborators [Ban11, KGR11].

Spatial coupling has been the object of growing interest within coding theory over the last few years. The first instance of spatially coupled code ensembles were the convolutional LDPC codes ofFelström and Zigangirov [FZ99]. While the excellent performances of such codes had been known for quite some time [SLJZ04], the fundamental reason was not elucidated until recently [KRU11] (see also [LF10]). In particular [KRU11] proved, for communication over the binary erasure channel (BEC), that the thresholds of spatially coupled ensembles under message passing decoding coincide with the thresholds of the base LDPC code under MAP decoding. In particular, this implies that spatially coupled ensembles achieve capacity over the BEC. The analogous statement for general memoryless symmetric channels was first elucidated in [KMRU10] and finally proved in [KRU12]. The paper [HMU10] discusses similar ideas in a number of graphical models.

The first application of spatial coupling ideas to compressed sensing is due to Kudekar and Pfister [KP10]. They consider a class of sparse spatially coupled sensing matrices, very similar to parity check matrices for spatially coupled LDPC codes. On the other hand, their proposed message passing algorithms do not make use of the signal distribution \( p_X \), and do not fully exploit the potential of spatially coupled matrices. The message passing algorithm used here belongs to the general class introduced in [DMM09]. The specific use of the minimum-mean square error denoiser was suggested in [DMM10]. The same choice is made in [KMS+11], which also considers Gaussian matrices with heteroscedastic entries although the variance structure is somewhat less general.

Finally, let us mention that robust sparse recovery of \( k \)-sparse vectors from \( m = O(k \log \log(n/k)) \) measurement is possible, using suitable ‘adaptive’ sensing schemes [IPW11].

2 Matrix and algorithm construction

In this section, we define an ensemble of random matrices, and the corresponding choices of \( Q^t, b^t, \eta_t \) that achieve the reconstruction guarantees in Theorems 1.6 and 1.7. We proceed by first introducing a general ensemble of random matrices. Correspondingly, we define a deterministic recursion named state evolution, that plays a crucial role in the algorithm analysis. In Section 2.3 we define the algorithm parameters and construct specific choices of \( Q^t, b^t, \eta_t \). The last section also contains a restatement of Theorems 1.6 and 1.7 in which this construction is made explicit.

2.1 General matrix ensemble

The sensing matrix \( A \) will be constructed randomly, from an ensemble denoted by \( \mathcal{M}(W, M, N) \). The ensemble depends on two integers \( M, N \in \mathbb{N} \), and on a matrix with non-negative entries \( W \in \mathbb{R}^{R \times C} \), whose rows and columns are indexed by the finite sets \( R, C \) (respectively ‘rows’ and ‘columns’). The band-diagonal structure that is characteristic of spatial coupling is imposed by a suitable choice of
the matrix $W$. In this section we define the ensemble for a general choice of $W$. In Section 2.4 we discuss a class of choices for $W$ that corresponds to spatial coupling, and that yields Theorems 1.6 and 1.7.

In a nutshell, the sensing matrix $A$ is obtained from $W$ through a suitable ‘lifting’ procedure. Each entry $W_{r,c}$ is replaced by an $M \times N$ block with i.i.d. entries $A_{ij} \sim \mathcal{N}(0, W_{r,c}/M)$. Rows and columns of $A$ are then re-ordered uniformly at random to ensure exchangeability. For the reader familiar with the application of spatial coupling to coding theory, it might be useful to notice the differences and analogies with graph liftings. In that case, the ‘lifted’ matrix is obtained by replacing each edge in the base graph with a random permutation matrix.

Passing to the formal definition, we will assume that the matrix $W$ is roughly row-stochastic, i.e.,

\[
\frac{1}{2} \leq \sum_{c \in C} W_{r,c} \leq 2, \quad \text{for all } r \in R.
\]  

(This is a convenient simplification for ensuring correct normalization of $A$.) We will let $|R| \equiv L_r$ and $|C| \equiv L_c$ denote the matrix dimensions. The ensemble parameters are related to the sensing matrix dimensions by $n = NL_c$ and $m = ML_r$.

In order to describe a random matrix $A \sim \mathcal{M}(W, M, N)$ from this ensemble, partition the columns and row indices in, respectively, $L_c$ and $L_r$ groups of equal size. Explicitly

\[
[n] = \bigcup_{s \in C} C(s), \quad |C(s)| = N,
\]

\[
[m] = \bigcup_{r \in R} R(r), \quad |R(r)| = M.
\]

Here and below we use $[k]$ to denote the set of first $k$ integers $[k] \equiv \{1, 2, \ldots, k\}$. Further, if $i \in R(r)$ or $j \in C(s)$ we will write, respectively, $r = g(i)$ or $s = g(j)$. In other words $g(\cdot)$ is the operator determining the group index of a given row or column.

With this notation we have the following concise definition of the ensemble.

**Definition 2.1.** A random sensing matrix $A$ is distributed according to the ensemble $\mathcal{M}(W, M, N)$ (and we write $A \sim \mathcal{M}(W, M, N)$) if the partition of rows and columns ($[m] = \bigcup_{r \in R} R(r)$ and $[n] = \bigcup_{s \in C} C(s)$) are uniformly random, and given this partitioning, the entries \{\(A_{ij}, i \in [m], j \in [n]\)\} are independent Gaussian random variables with

\[
A_{ij} \sim \mathcal{N}\left(0, \frac{1}{M W_{g(i),g(j)}}\right).
\]  

(29)

We refer to Fig. 2 for an illustration. Note that the randomness of the partitioning of row and column indices is only used in the proof of Lemma 4.1 (cf. [JM12a]), and hence this and other illustrations assume that the partitions are contiguous.

Within the applications of spatial coupling to LDPC codes, see [KMRU10, HMU10, KRU12], the spatially-coupled codes are constructed by ‘coupling’ or ‘chaining’ a sequence of sparse graphs. The indexes $r \in R, c \in C$ in the above construction correspond to the index of the graph along the chain in those constructions.

For proving Theorem 1.6 and Theorem 1.7 we will consider suitable sequences of ensembles $\mathcal{M}(W, M, N)$ with undersampling ratio converging to $\delta$. While a complete description is given below,

\footnote{As in many papers on compressed sensing, the matrix here has independent zero-mean Gaussian entries; however, unlike standard practice, here the entries are of widely different variances.}
Figure 2: Construction of the spatially coupled measurement matrix $A$ as described in Section 2.1. The matrix is divided into blocks with size $M$ by $N$. (Number of blocks in each row and each column are respectively $L_c$ and $L_r$, hence $m = M L_r$, $n = N L_c$). The matrix elements $A_{ij}$ are chosen as $N(0, \frac{1}{M} W_{g(i), g(j)})$. In this figure, $W_{i,j}$ depends only on $|i - j|$ and thus blocks on each diagonal have the same variance.

let us stress that we take the limit $M, N \to \infty$ (with $M = N \delta$) before the limit $L_r, L_c \to \infty$. Hence, the resulting matrix $A$ is essentially dense: the fraction of non-zero entries per row vanishes only after the number of groups goes to $\infty$.

2.2 State evolution

State evolution allows an exact asymptotic analysis of AMP algorithms in the limit of a large number of dimensions. As indicated by the name, it bears close resemblance to the density evolution method in iterative coding theory [RU08]. Somewhat surprisingly, this analysis approach is asymptotically exact despite the underlying factor graph being far from locally tree-like.

State evolution was first developed in [DMM09] on the basis of heuristic arguments, and substantial numerical evidence. Subsequently, it was proved to hold for Gaussian sensing matrices with i.i.d. entries, and a broad class of iterative algorithms in [BM11]. These proofs were further generalized in [Ran11], to cover ‘generalized’ AMP algorithms.

In the present case, state evolution takes the following form.  

\[8\]

\[8\]In previous work, the state variable concerned a single scalar, representing the mean-squared error in the current reconstruction, averaged across all coordinates. In this paper, the dimensionality of the state variable is much larger, because it contains $\psi$, an individualized MSE for each coordinate of the reconstruction and also $\phi$, a noise variance for
Definition 2.2. Given $W \in \mathbb{R}_{+}^{L_C \times L_C}$ roughly row-stochastic, and $\delta > 0$, the corresponding state evolution maps $T'_W : \mathbb{R}_{+}^{L_C} \to \mathbb{R}_+^C$, $T''_W : \mathbb{R}_+^C \to \mathbb{R}_+^L$, are defined as follows. For $\phi = (\phi_a)_{a \in \mathbb{R}} \in \mathbb{R}_+^L$, $\psi = (\psi_i)_{i \in \mathbb{C}} \in \mathbb{R}_+^C$, we let:

\[ T'_W(\phi)_i = \text{mmse}\left( \sum_{b \in \mathbb{R}} W_{b,i} \phi_b^{-1} \right), \tag{30} \]

\[ T''_W(\psi)_a = \sigma^2 + \frac{1}{\delta} \sum_{i \in \mathbb{C}} W_{a,i} \psi_i. \tag{31} \]

We finally define $T_W = T'_W \circ T''_W$.

In the following, we shall omit the subscripts from $T_W$ whenever clear from the context.

Definition 2.3. Given $W \in \mathbb{R}_{+}^{L_C \times L_C}$ roughly row-stochastic, the corresponding state evolution sequence is the sequence of vectors \( \{\phi(t), \psi(t)\}_{t \geq 0} \), $\phi(t) = (\phi_a(t))_{a \in \mathbb{R}} \in \mathbb{R}_+^L$, $\psi(t) = (\psi_i(t))_{i \in \mathbb{C}} \in \mathbb{R}_+^C$, defined recursively by $\phi(t) = T'_W(\psi(t))$, $\psi(t + 1) = T_W(\phi(t))$, with initial condition

\[ \psi_1(0) = \infty \text{ for all } i \in \mathbb{C}. \tag{32} \]

Hence, for all $t \geq 0$,

\[ \phi_a(t) = \sigma^2 + \frac{1}{\delta} \sum_{i \in \mathbb{C}} W_{a,i} \psi_i(t), \]

\[ \psi_i(t + 1) = \text{mmse}\left( \sum_{b \in \mathbb{R}} W_{b,i} \phi_b(t)^{-1} \right). \tag{33} \]

The quantities $\psi_i(t)$, $\phi_a(t)$ correspond to the asymptotic MSE achieved by the AMP algorithm. More precisely, $\psi_i(t)$ corresponds to the asymptotic mean square error $\mathbb{E}\{(x_j^t - x_j)^2\}$ for $j \in C(i)$, as $N \to \infty$. Analogously, $\phi_a(t)$ is the noise variance in residuals $r_j^t$ corresponding to rows $j \in R(a)$. This correspondence is stated formally in Lemma 4.1 below. The state evolution (33) describes the evolution of these quantities. In particular, the linear operation in Eq. (7) corresponds to a sum of noise variances as per Eq. (31) and the application of denoisers $\eta_t$ corresponds to a noise reduction as per Eq. (30).

As we will see, the definition of denoiser function $\eta_t$ involves the state vector $\phi(t)$. (Notice that the state vectors $\{\phi(t), \psi(t)\}_{t \geq 0}$ can be precomputed). Hence, $\eta_t$ is ‘tuned’ according to the predicted reconstruction error at iteration $t$.

### 2.3 General algorithm definition

In order to fully define the AMP algorithm\[6, 7\], we need to provide constructions for the matrix $Q^t$, the nonlinearities $\eta_t$, and the vector $b^t$. In doing this, we exploit the fact that the state evolution sequence $\{\phi(t)\}_{t \geq 0}$ can be precomputed.

We define the matrix $Q^t$ by

\[ Q^t_{ij} = \frac{\phi_{g(j)}(t)^{-1}}{\sum_{k=1}^{L_C} W_{k,g(j)} \phi_k(t)^{-1}}, \tag{34} \]

the residuals $r^t$ for each measurement coordinate.
Notice that \( Q^t \) is block-constant: for any \( r, s \in [L] \), the block \( Q^t_{(r), (s)} \) has all its entries equal.

As mentioned in Section 1, the function \( \eta_t : \mathbb{R}^n \to \mathbb{R}^n \) is chosen to be separable, i.e., for \( v \in \mathbb{R}^N \):

\[
\eta_t(v) = (\eta_{t,1}(v_1), \eta_{t,2}(v_2), \ldots, \eta_{t,N}(v_N)).
\]

(35)

We take \( \eta_{t,i} \) to be a conditional expectation estimator for \( X \sim p_X \) in gaussian noise:

\[
\eta_{t,i}(v_i) = \mathbb{E}\{X \mid X + s_{g(i)}(t)^{-1/2}Z = v_i\}, \quad s_r(t) \equiv \sum_{u \in \mathbb{R}} W_{u,r} \phi_u(t)^{-1}.
\]

(36)

Notice that the function \( \eta_{t,i}(\cdot) \) depends on \( i \) only through the group index \( g(i) \), and in fact only parametrically through \( s_{g(i)}(t) \). It is also interesting to notice that the denoiser \( \eta_{t,i}(\cdot) \) does not have any tuning parameter to be optimized over. This was instead the case for the soft-thresholding AMP algorithm studied in [DMM09] for which the threshold level had to be adjusted in a non-trivial manner to the sparsity level. This difference is due to the fact that the prior \( p_X \) is assumed to be known and hence the optimal denoiser is uniquely determined to be the posterior expectation as per Eq. (36).

Finally, in order to define the vector \( b_t^i \), let us introduce the quantity

\[
\langle \eta_t \rangle_u = \frac{1}{N} \sum_{i \in C(u)} \eta_{t,i}'(x_t^i + ((Q^t \circ A)^* r^t)_i).
\]

(37)

Recalling that \( Q^t \) is block-constant, we define matrix \( \tilde{Q}^t \in \mathbb{R}^{L_r \times L_c} \) as \( \tilde{Q}_{r,u}^t = Q_{i,j}^t \), with \( i \in R(r) \) and \( j \in C(u) \). In words, \( \tilde{Q}^t \) contains one representative of each block. The vector \( b_t^i \) is then defined by

\[
b_t^i \equiv \frac{1}{\delta} \sum_{u \in C(i)} W_{g(i),u} \tilde{Q}_{g(i),u}^{t-1} \langle \eta_{t-1} \rangle_u.
\]

(38)

Again \( b_t^i \) is block-constant: the vector \( b_{t,C(u)}^i \) has all its entries equal.

This completes our definition of the AMP algorithm. Let us conclude with a few computational remarks:

1. The quantities \( \tilde{Q}^t, \phi(t) \) can be precomputed efficiently iteration by iteration, because they are, respectively, \( L_r \times L_c \) and \( L_r \)-dimensional, and, as discussed further below, \( L_r, L_c \) are much smaller than \( m, n \). The most complex part of this computation is implementing the iteration (33), which has complexity \( O((L_r + L_c)^3) \), plus the complexity of evaluating the \text{mmse} function, which is a one-dimensional integral.

2. The vector \( b^i \) is also block-constant, so can be efficiently computed using Eq. (38).

3. Instead of computing \( \phi(t) \) analytically by iteration (33), \( \phi(t) \) can also be estimated from data \( x^t, r^t \). In particular, by generalizing the methods introduced in [DMM09] [Mon12], we get the estimator

\[
\hat{\phi}_u(t) = \frac{1}{M} \| r^t_{R(a)} \|^2_2.
\]

(39)
where \( r^t_{R(a)} = (r^t_j)_{j \in R(a)} \) is the restriction of \( r^t \) to the indices in \( R(a) \). An alternative more robust estimator (more resilient to outliers), would be

\[
\phi_a(t) = \frac{1}{\Phi^{-1}(3/4)} |r^t_{R(a)}(M/2)|,
\]

where \( \Phi(z) \) is the Gaussian distribution function, and, for \( v \in \mathbb{R}^K \), \( |v|_\ell \) is the \( \ell \)-th largest entry in the vector \( (|v_1|, |v_2|, \ldots, |v_K|) \). (See, e.g., [HR09] for background in robust estimation.) The idea underlying both of the above estimators is that the components of \( r^t_{R(a)} \) are asymptotically i.i.d. with mean zero and variance \( \phi_a(t) \).

### 2.4 Choices of parameters, and spatial coupling

In order to prove our main Theorem 1.6, we use a sensing matrix from the ensemble \( \mathcal{M}(W, M, N) \) for a suitable choice of the matrix \( W \in \mathbb{R}^{R \times C} \). Our construction depends on parameters \( \rho \in \mathbb{R}_+, L, L_0 \in \mathbb{N} \), and on the ‘shape function’ \( W \). As explained below, \( \rho \) will be taken to be small, and hence we will treat \( 1/\rho \) as an integer to avoid rounding (which introduces in any case a negligible error).

Here and below \( \cong \) denotes identity between two sets up to a relabeling.

**Definition 2.4.** A shape function is a function \( \mathcal{W} : \mathbb{R} \rightarrow \mathbb{R}_+ \) continuously differentiable, with support in \([-1, 1]\) and such that \( \int_{\mathbb{R}} \mathcal{W}(u) \, du = 1 \), and \( \mathcal{W}(-u) = \mathcal{W}(u) \).

We let \( C \cong \{-2\rho^{-1}, \ldots, 0, 1, \ldots, L - 1\} \), so that \( L_c = L + 2\rho^{-1} \). Also let \( C_0 = \{0, 1, \ldots, L - 1\} \).

The rows are partitioned as follows:

\[
R = R_0 \cup \left\{ \cup_{i=-2\rho^{-1}}^{-1} R_i \right\},
\]

where \( R_0 \cong \{-\rho^{-1}, \ldots, 0, 1, \ldots, L - 1 + \rho^{-1}\} \), and \( R_i = \{iL_0, \ldots, (i+1)L_0 - 1\} \), for \( i = -2\rho^{-1}, \ldots, -1 \). Hence, \( |R_i| = L_0 \), and \( L_r = L_c + 2\rho^{-1}L_0 \).

Finally, we take \( N \) so that \( n = NL_c \), and let \( M = N\delta \) so that \( m = ML_r = N(L_c + 2\rho^{-1}L_0)\delta \). Notice that \( m/n = \delta(L_c + 2\rho^{-1}L_0)/L_c \). Since we will take \( L_c \) much larger than \( L_0/\rho \), we in fact have \( m/n \) arbitrarily close to \( \delta \).

Given these inputs, we construct the corresponding matrix \( W = W(L, L_0, \mathcal{W}, \rho) \) as follows.

1. For \( i \in \{-2\rho^{-1}, \ldots, -1\} \), and each \( a \in R_i \), we let \( W_{a,i} = 1 \). Further, \( W_{a,j} = 0 \) for all \( j \in C \setminus \{i\} \).
2. For all \( a \in R_0 \cong \{-\rho^{-1}, \ldots, 0, \ldots, L - 1 + \rho^{-1}\} \), we let

\[
W_{a,i} = \rho \mathcal{W}(\rho(a - i)) \quad i \in \{-2\rho^{-1}, \ldots, L - 1\}.
\]

The role of the rows in \( \left\{ \cup_{i=-2\rho^{-1}}^{-1} R_i \right\} \) and the corresponding rows in \( A \) are to oversample the first few (namely the first \( 2\rho^{-1}N \)) coordinates of the signal as explained in Section 1.4. Furthermore, the restriction of \( W \) to the rows in \( R_0 \) is band diagonal as \( W \) is supported on \([-1, 1]\). See Fig. 3 for an illustration of the matrix \( W \).

In the following we occasionally use the shorthand \( W_{a^{-1}} = \rho \mathcal{W}(\rho(a - i)) \). Note that \( W \) is roughly row-stochastic. Also, the restriction of \( W \) to the rows in \( R_0 \) is roughly column-stochastic.
This follows from the fact that the function \( W(\cdot) \) has continuous (and thus bounded) derivative on the compact interval \([-1, 1]\), and \( \int_{\mathbb{R}} W(u) du = 1 \). Therefore, using the standard convergence of Riemann sums to Riemann integrals and the fact that \( \rho \) is small, we get the result.

We are now in position to restate Theorem 1.6 in a more explicit form.

**Theorem 2.5.** Let \( p_X \) be a probability measure on the real line with \( \delta > d(p_X) \), and let \( W : \mathbb{R} \to \mathbb{R}_+ \) be a shape function. For any \( \varepsilon > 0 \), there exist \( L_0, L, \rho, t_0, s_0 = \sigma_0(\varepsilon, \delta, p_X)^2 \) such that \( L_0/(L \rho) \leq \varepsilon \), and further the following holds true for \( W = W(L, L_0, W, \rho) \).

For \( N \geq 0 \), and \( A(n) \sim \mathcal{M}(W, M, N) \) with \( M = N \delta \), and for all \( \sigma^2 \leq s_0^2 \), \( t \geq t_0 \), we almost surely have

\[
\limsup_{N \to \infty} \frac{1}{n} \| x'(A(n); y(n)) - x(n) \|^2 \leq \varepsilon. \tag{42}
\]

Further, under the same assumptions, we have

\[
\limsup_{N \to \infty} \frac{1}{n} \mathbb{E}\{\| x'(A(n); y(n)) - x(n) \|^2\} \leq \varepsilon. \tag{43}
\]

In order to obtain a stronger form of robustness, as per Theorem 1.7, we slightly modify the sensing scheme. We construct the sensing matrix \( \tilde{A} \) from \( A \) by appending \( 2\rho^{-1}L_0 \) rows in the
where $I$ is the identity matrix of dimensions $2\rho^{-1}L_0$. Note that this corresponds to increasing the number of measurements; however, the asymptotic undersampling rate remains $\delta$, provided that $L_0/(L\rho) \to 0$, as $n \to \infty$.

The reconstruction scheme is modified as follows. Let $x_1$ be the vector obtained by restricting $x$ to entries in $\cup_i C(i)$, where $i \in \{-2\rho^{-1}, \ldots, L - 2\rho^{-1} - 1\}$. Also, let $x_2$ be the vector obtained by restricting $x$ to entries in $\cup_i C(i)$, where $i \in \{L - 2\rho^{-1}, \ldots, L - 1\}$. Therefore, $x = (x_1, x_2)^T$. Analogously, let $y = (y_1, y_2)^T$ where $y_1$ is given by the restriction of $y$ to $\cup_{i \in \mathbb{R}} R(i)$ and $y_2$ corresponds to the additional $2\rho^{-1}L_0$ rows. Define $w_1$ and $w_2$ from the noise vector $w$, analogously. Hence,

$$
\begin{pmatrix}
 y_1 \\
 y_2
\end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\
 x_2 \end{pmatrix} + \begin{pmatrix} w_1 \\
 w_2 \end{pmatrix}.
$$

(45)

Note that the sampling rate for vector $x_2$ is one, i.e., $y_2$ and $x_2$ are of the same length and are related to each other through the identity matrix $I$. Hence, we have a fairly good approximation of these entries. We use the AMP algorithm as described in the previous section to obtain an estimation of $x_1$. Formally, let $x^t$ be the estimation at iteration $t$ obtained by applying the AMP algorithm to the problem $y_1 = Ax_1 + w_1$. The modified estimation is then $\tilde{x}^t = (x_1^t, y_2^t)^T$.

As we will see later, this modification in the sensing matrix and algorithm, while not necessary, simplifies some technical steps in the proof.

**Theorem 2.6.** Let $p_X$ be a probability measure on the real line with $\delta > D(p_X)$, and let $W : \mathbb{R} \to \mathbb{R}_+$ be a shape function. There exist $L_0, L, \rho, t_0$ and a finite stability constant $C = C(p_X, \delta)$, such that $L_0/(L\rho) < \epsilon$, for any given $\epsilon > 0$, and the following holds true for the modified reconstruction scheme.

For $t \geq t_0$, we almost surely have,

$$
\limsup_{n \to \infty} \frac{1}{n} \left\| \tilde{x}^t(\tilde{A}(n); y(n)) - x(n) \right\|^2 \leq C \sigma^2.
$$

(46)

Further, under the same assumptions, we have

$$
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E} \left\{ \left\| \tilde{x}^t(\tilde{A}(n); y(n)) - x(n) \right\|^2 \right\} \leq C \sigma^2.
$$

(47)

Finally, in the asymptotic case where $\ell = L\rho \to \infty$, $\rho \to 0$, $L_0 \to \infty$, we have

$$
\limsup_{n \to \infty} \frac{1}{\sigma^2} \left\{ \lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \left\| \tilde{x}^t(\tilde{A}(n); y(n)) - x(n) \right\|^2 \right\} \leq \frac{4\delta - 2D(p_X)}{\delta - D(p_X)}.
$$

It is obvious that Theorems 2.5 and 2.6 respectively imply Theorems 1.6 and 1.7. We shall therefore focus on the proofs of Theorems 2.5 and 2.6 in the rest of the paper.

It is worth noting that as per Theorem 2.6, the sensitivity constant at small noise depends on the signal distribution $p_X$ only through its upper MMSE dimension $D(p_X)$. In particular, for signal distribution of the form 24, the robustness guarantee is independent of the $q$ component.
Notice that the results of Theorems 2.3 and 2.6 only deal with a linear subsequence $n = NL_c$ with $N \to \infty$. However, this is sufficient to prove the claim of Theorems 1.6 and 1.7. More specifically, suppose that $n$ is not a multiple of $L_c$. Let $n'$ be the smallest number greater than $n$ which is divisible by $L_c$, i.e., $n' = \lceil n/L_c \rceil L_c$, and let $\hat{x} = (x, 0)^T \in \mathbb{R}^{n'}$ be obtained by padding $x$ with zeros. Let $\hat{x}^t$ denote the Bayes optimal AMP estimate of $\hat{x}$ and $x^t$ be the restriction of $\hat{x}^t$ to the first $n$ entries. We have $(1/n)'\|x^t - x\|^2 \leq (n'/n')(1/n')\|\hat{x}^t - \hat{x}\|^2$. The result of Theorem 1.6 follows by applying Theorem 2.5 (for the sequence $n = NL_c$, $N \to \infty$), and noting that $n'/n \leq (1 + L_c/n) \to 1$, as $N \to \infty$. Similar comment applies to Theorems 2.6 and 1.7.

3 Advantages of spatial coupling

Within the construction proposed in this paper, spatially coupled sensing matrices have independent heteroscedastic entries (entries with different variances). In addition to this, we also oversample a few number of coordinates of the signal, namely the first $2\rho^{-1}N$ coordinates. In this section we informally discuss the various components of this scheme.

It can be instructive to compare this construction with the case of homoscedastic Gaussian matrices (i.i.d. entries). For the reader familiar with coding theory, this comparison is analogous to the comparison between regular LDPC codes and spatially coupled regular LDPC codes. Regular LDPC codes have been known since Gallager [Gal63, MMRU09] to achieve the channel capacity, as the degree gets large, under maximum likelihood decoding. However their performances under practical (belief propagation) decoding is rather poor. When the code ensemble is modified via spatial coupling, the belief propagation performances improve to become asymptotically equivalent to the maximum likelihood performances. Hence spatially coupled LDPC codes achieve capacity under practical decoding schemes.

Similarly, standard (non-spatially coupled) sensing matrices achieve the information theoretic limit under computationally unpractical recovery schemes [WV10], but do not perform ideally under practical reconstruction algorithms. Consider for instance Bayes optimal AMP. Within the standard ensemble, the state evolution recursion reads

$$\phi(t) = \sigma^2 + \frac{1}{\delta} \psi(t),$$

$$\psi(t + 1) = \text{mmse}(\phi(t)^{-1}).$$

Note that $\psi(t + 1)$ is the minimum mean square error at signal-to-noise ratio $\phi(t)^{-1}$, i.e., treating the residual part as noise. Let $\tilde{\delta}(p_X) \equiv \sup_{s \geq 0} s \cdot \text{mmse}(s) > \bar{d}(p_X)$. It is immediate to see that the last recursion develops two (or possibly more) stable fixed points for $\delta < \tilde{\delta}(p_X)$ and all $\sigma^2$ small enough. The smallest fixed point, call it $\phi_{\text{good}}$, corresponds to correct reconstruction and is such that $\phi_{\text{good}} = O(\sigma^2)$ as $\sigma \to 0$. The largest fixed point, call it $\phi_{\text{bad}}$, corresponds to incorrect reconstruction and is such that $\phi_{\text{bad}} = \Theta(1)$ as $\sigma \to 0$. A study of the above recursion shows that $\lim_{t \to \infty} \phi(t) = \phi_{\text{bad}}$. State evolution converges to the ‘incorrect’ fixed point, hence predicting a large MSE for AMP.

On the contrary, for $\bar{d}(p_X) < \delta < \tilde{\delta}(p_X)$ the recursion (48) converges (for appropriate choices of $W$ as in the previous section) to the ‘ideal’ fixed point $\lim_{t \to \infty} \phi_n(t) = \phi_{\text{good}}$ for all $a$ (except possibly those near the boundaries). This is illustrated in Fig. 4. We also refer to [HIMU10] for a survey of examples of the same phenomenon and to [KMS+11, JM12b] for further discussion in compressed sensing.
The above discussion also clarifies why the posterior expectation denoiser is useful. Spatially coupled sensing matrices do not yield better performances than the ones dictated by the best fixed point in the ‘standard’ recursion (48). In particular, replacing the Bayes optimal denoiser by another denoiser $\eta_t$ amounts, roughly, to replacing $\text{mmse}$ in Eq. (48) by the MSE of another denoiser, hence leading to worse performances.

In particular, if the posterior expectation denoiser is replaced by soft thresholding, the resulting state evolution recursion always has a unique stable fixed point for homoscedastic matrices [DMM09]. This suggests that spatial coupling does not lead to any improvement for soft thresholding AMP and hence (via the correspondence of [BM12]) for LASSO or $\ell_1$ reconstruction. This expectation is indeed confirmed numerically in [JM12b].

4 Key lemmas and proof of the main theorems

Our proof is based in a crucial way on state evolution. This effectively reduces the analysis of the algorithm (6), (7) to the analysis of the deterministic recursion (33).

Lemma 4.1. Let $W \in \mathbb{R}_+^{R \times C}$ be a roughly row-stochastic matrix (see Eq. (28)) and $\phi(t)$, $Q^t$, $b^t$ be defined as in Section 2.3. Let $M = M(N)$ be such that $M/N \to \delta$, as $N \to \infty$. Define $m = ML_c$, $n = NL_c$, and for each $N \geq 1$, let $A(n) \sim \mathcal{M}(W, M, N)$. Let $\{(x(n), w(n))\}_{n \geq 0}$ be a converging sequence of instances with parameters $(p_X, \sigma^2)$. Then, for all $t \geq 1$, almost surely we have

$$\limsup_{N \to \infty} \frac{1}{N} \|x_{C(i)}^t(A(n); y(n)) - x_{C(i)}^t\|_2^2 = \text{mmse}\left(\sum_{a \in \mathbb{R}} W_{a,i} \phi_a(t - 1)^{-1}\right),$$

for all $i \in C$.

This lemma is a straightforward generalization of [BM11]. Since a formal proof does not require new ideas, but a significant amount of new notations, it is presented in a separate publication [JM12a] which covers an even more general setting. In the interest of self-containment, and to develop useful intuition on state evolution, we present an heuristic derivation of the state evolution equations (33) in Section 6.

The next Lemma provides the needed analysis of the recursion (33).

Lemma 4.2. Let $\delta > 0$, and $p_X$ be a probability measure on the real line. Let $W : \mathbb{R} \to \mathbb{R}_+$ be a shape function.

(a) If $\delta > \overline{d}(p_X)$, then for any $\varepsilon > 0$, there exist $\sigma_0 = \sigma_0(\varepsilon, \delta, p_X)$, $\rho, L_0 > 0$, such that for any $\sigma^2 \in [0, \sigma_0^2]$, $L_0 > 3/\delta$, and $L > L_0$, the following holds for $W = W(L, L_0, W, \rho)$:

$$\lim_{t \to \infty} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L+\rho^{-1}-1} \phi_a(t) \leq \varepsilon.$$

(b) If further $\delta > \overline{d}(p_X)$, then there exist $\rho, L_0 > 0$, and a finite stability constant $C = C(p_X, \delta)$, such that for $L_0 > 3/\delta$, and $L > L_0$, the following holds for $W = W(L, L_0, W, \rho)$:

$$\lim_{t \to \infty} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} \phi_a(t) \leq C\sigma^2.$$
Finally, in the asymptotic case where \( \ell = L \rho \to \infty, \rho \to 0, L_0 \to \infty, \) we have

\[
\limsup_{\sigma \to 0} \lim_{t \to \infty} \frac{1}{\sigma^3 L} \sum_{a = -\rho}^{L - \rho - 1} \phi_a(t) \leq \frac{3\delta - \overline{D}(p_X)}{\delta - \overline{D}(p_X)}. \tag{52}
\]

The proof of this lemma is deferred to Section 7 and is indeed the technical core of the paper. Now, we have in place all we need to prove our main results.

Proof (Theorem 2.5). Recall that \( C \cong \{-2\rho^{-1} \cdots, L - 1\} \). Therefore,

\[
\limsup_{N \to \infty} \frac{1}{n} \left\| x^t(A(n); y(n)) - x(n) \right\|^2 \leq \frac{1}{Lc} \sum_{i \in C} \limsup_{N \to \infty} \frac{1}{N} \left\| x^t_{C(i)}(A(n); y(n)) - x_{C(i)}(n) \right\|^2
\]

\[
\begin{align*}
(a) & \leq \frac{1}{Lc} \sum_{i = -2\rho^{-1}}^{L-1} \text{mmse} \left( \sum_{a \in R} W_{a,i} \phi_a(t - 1)^{-1} \right) \\
(b) & \leq \frac{1}{Lc} \sum_{i = -2\rho^{-1}}^{L-1} \text{mmse} \left( \sum_{a \in R_0} W_{a,i} \phi_a(t - 1)^{-1} \right) \\
(c) & \leq \frac{1}{Lc} \sum_{i = -2\rho^{-1}}^{L-1} \text{mmse} \left( \frac{1}{2} \phi_{i+\rho^{-1}}(t - 1)^{-1} \right) \\
(d) & \leq \frac{1}{Lc} \sum_{a = -\rho}^{L+\rho^{-1}-1} 2\phi_a(t - 1). 
\end{align*}
\]

Here, (a) follows from Lemma 4.1; (b) follows from the fact that mmse is non-increasing; (c) holds because of the following facts: (i) \( \phi_a(t) \) is nondecreasing in \( a \) for every \( t \) (see Lemma 7.10 below). (ii) Restriction of \( W \) to the rows in \( R_0 \) is roughly column-stochastic. (iii) mmse is non-increasing; (d) follows from the inequality mmse\((s) \leq 1/s \). The result is immediate due to Lemma 4.2, Part (a).

Now, we prove the claim regarding the expected error. Let \( f_n = \frac{1}{n} \left\| x^t(A(n); y(n)) - x(n) \right\|^2 \). Since \( \limsup_{n} f_n \leq \varepsilon \), there exists \( n_0 \) such that \( f_n \leq 2\varepsilon \) for \( n \geq n_0 \). Applying reverse Fatou’s lemma to the bounded sequence \{\( f_n \)\}n≥n0, we have \( \limsup_{N \to \infty} \mathbb{E} f_n \leq \mathbb{E} \left[ \limsup_{N \to \infty} f_n \right] \leq \varepsilon. \)
As explained in Section 1.4, in the spatially coupled sensing matrix, additional measurements are empirical errors.

Our first set of experiments aims at illustrating the evolution of the profile \( \phi(t) \) versus iteration \( t \).

5.1 Evolution of the AMP algorithm

In the experiments, we use \( \varepsilon \)-coordinates (2).

We consider a Bernoulli-Gaussian distribution \( \mathcal{B}(a, b) \) with \( a \approx \frac{1}{2} \).

5 Numerical experiments

We construct a random signal \( x(n) \in \mathbb{R}^n \) by sampling i.i.d. coordinates \( x(n)_i \sim p_X \).

We have \( \tilde{d}(p_X) = \varepsilon \) by Proposition 1.3 and

\[
\eta_t(v_i) = \frac{\varepsilon \gamma^{1+s^{-1}(v_i)}(\varepsilon)}{\varepsilon \gamma^{1+s^{-1}(v_i)}(\varepsilon) + (1-\varepsilon)\gamma^{s^{-1}(v_i)}(\varepsilon)} \cdot \frac{1}{1 + s^{1-s^{-1}(v_i)}}.
\]

In the experiments, we use \( \varepsilon = 0.1 \), \( \sigma = 0.01 \), \( \rho = 0.1 \), \( M = 6 \), \( N = 50 \), \( L = 500 \), \( L_0 = 5 \).

5.1 Evolution of the AMP algorithm

Our first set of experiments aims at illustrating the evolution of the profile \( \phi(t) \) defined by state evolution versus iteration \( t \), and comparing the predicted errors by the state evolution with the empirical errors.

Figure 4 shows the evolution of profile \( \phi(t) \in \mathbb{R}^{L_t} \), given by the state evolution recursion (33). As explained in Section 1.4, in the spatially coupled sensing matrix, additional measurements are...
associated to the first few coordinates of $x$, namely, $2\rho^{-1}N = 1000$ first coordinates. This ensures that the values of these coordinates are recovered up to a mean square error of order $\sigma^2$. This is reflected in the figure as the profile $\phi$ becomes of order $\sigma^2$ on the first few entries after a few iterations (see $t = 5$ in the figure). As the iteration proceeds, the contribution of these components is correctly subtracted from all the measurements, and essentially they are removed from the problem. Now, in the resulting problem the first few variables are effectively oversampled and the algorithm reconstructs their values up to a mean square error of $\sigma^2$. Correspondingly, the profile $\phi$ falls to a value of order $\sigma^2$ in the next few coordinates. As the process is iterated, all the variables are progressively reconstructed and the profile $\phi$ follows a traveling wave with constant velocity. After a sufficient number of iterations ($t = 800$ in the figure), $\phi$ is uniformly of order $\sigma^2$.

Next, we numerically verify that the deterministic state evolution recursion predicts the performance of the AMP at each iteration. Define the empirical and the predicted mean square errors respectively by

$$\text{MSE}_{\text{AMP}}(t) = \frac{1}{n} \| x^t(y) - x \|_2^2, \quad (56)$$

$$\text{MSE}_{\text{SE}}(t) = \frac{1}{L_c} \sum_{i \in \mathcal{C}} \text{mmse} \left( \sum_{a \in \mathbb{R}} W_{a,i} \phi_a^{-1}(t-1) \right). \quad (57)$$

The values of $\text{MSE}_{\text{AMP}}(t)$ and $\text{MSE}_{\text{SE}}(t)$ are depicted versus $t$ in Fig. 5 (Values of $\text{MSE}_{\text{AMP}}(t)$ and the error bars correspond to $M = 30$ Monte Carlo instances). This verifies that the state evolution provides an iteration-by-iteration prediction of the AMP performance. We observe that $\text{MSE}_{\text{AMP}}(t)$ (and $\text{MSE}_{\text{SE}}(t)$) decreases linearly versus $t$.

5.2 Phase diagram

Consider a noiseless setting and let $\mathcal{A}$ be a sensing matrix–reconstruction algorithm scheme. The curve $\varepsilon \mapsto \delta_\mathcal{A}(\varepsilon)$ describes the sparsity-undersampling tradeoff of $\mathcal{A}$ if the following happens in the
large-system limit $n, m \to \infty$, with $m/n = \delta$. The scheme $\mathcal{A}$ does (with high probability) correctly recover the original signal provided $\delta > \delta_A(\varepsilon)$, while for $\delta < \delta_A(\varepsilon)$ the algorithm fails with high probability.

The goal of this section is to numerically compute the sparsity-undersampling tradeoff curve for the proposed scheme (spatially coupled sensing matrices and Bayes optimal AMP). We consider a set of sparsity parameters $\varepsilon \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$, and for each value of $\varepsilon$, evaluate the empirical phase transition through a logit fit (we omit details, but follow the methodology described in [DMM09]). As shown in Fig 6, the numerical results are consistent with the claim that this scheme achieves the information theoretic lower bound $\delta > \overline{d}(p_X) = \varepsilon$. (We indeed expect the gap to decrease further by taking larger values of $L$).

In [JM12b], we numerically show that the spatial coupling phenomenon is significantly more robust and general than suggested by constructions in the present paper. Namely, we consider the problem of sampling a signal with sparse support in frequency domain and propose a sampling scheme that acquires a random subset of Gabor coefficients of the signal. This scheme offers one venue (out of many) for implementing the idea of spatial coupling. Note that the corresponding sensing matrix, in this context, does not have gaussian entries. As shown numerically for the mixture model, the combination of this scheme and the Bayes optimal AMP achieves the fundamental lower bound $\delta > \overline{d}(p_X)$.

6 State evolution: an heuristic derivation

This section presents an heuristic derivation of the state evolution equations (33). Our objective is to provide some basic intuition: a proof in a more general setting will appear in a separate publication [JM12a]. An heuristic derivation similar to the present one, for the special cases of sensing matrices with i.i.d. entries was presented in [BM11].

Consider the recursion (6)-(7), and introduce the following modifications: (i) At each iteration,
replace the random matrix $A$ with a new independent copy $A^t$; (ii) Replace the observation vector $y$ with $y^t = A^t x + w$; (iii) Eliminate the last term in the update equation for $r^t$. Then, we have the following update rules:

$$
x^{t+1} = \eta_t (x^t + (Q^t \odot A^t)^* r^t),
$$

$$
r^t = y^t - A^t x^t,
$$

where $A^0, A^1, A^2, \cdots$ are i.i.d. random matrices distributed according to the ensemble $\mathcal{M}(W, M, N)$, i.e.,

$$
A^t_{ij} \sim \mathcal{N}\left(0, \frac{1}{M} W_{g(i), g(j)}\right).
$$

Rewriting the recursion by eliminating $r^t$, we obtain:

$$
x^{t+1} = \eta_t ((Q^t \odot A^t)^* y^t + (I - (Q^t \odot A^t)^* A^t)x^t)
= \eta_t (x + (Q^t \odot A^t)^* w + B^t (x^t - x)),
$$

where $B^t = I - (Q^t \odot A^t)^* A^t \in \mathbb{R}^{n \times n}$. Note that the recursion (61) does not correspond to the AMP update rules defined per Eqs. (6) and (7). In particular, it does not correspond to any practical algorithm since the sensing matrix $A$ is a fixed input to a reconstruction algorithm and is not resampled at each iteration. However, it is much easier to analyze, since $A^t$ is independent of $x^t$ and therefore the distribution of $(Q^t \odot A^t)^* r^t$ can be easily characterized. Also, it is useful for presenting the intuition behind the AMP algorithm and to emphasize the role of the term $b^t \odot r^{t-1}$ in the update rule for $r^t$. As it emerges from the proof of [BM11], this term does asymptotically cancel dependencies across iterations.

By virtue of the central limit theorem, each entry of $B^t$ is approximately normal. More specifically, $B^t_{ij}$ is approximately normal with mean zero and variance $(1/M) \sum_{r \in R} W_{r, g(i)} W_{r, g(j)} (Q^t_{r, g(i)})^2$, for
\(i, j \in [n]\). Define \(\hat{\eta}_t(s) = \lim_{N \to \infty} \|x_{C(s)}^t - x_{C(s)}\|^2/N\), for \(s \in \mathbb{C}\). It is easy to show that distinct entries in \(B^t\) are approximately independent. Also, \(B^t\) is independent of \(\{B^s\}_{1 \leq s \leq t-1}\), and in particular, of \(x^t - x\). Hence, \(B'(x^t - x)\) converges to a vector, say \(v\), with i.i.d. normal entries, and for \(i \in [n]\),

\[
\mathbb{E}\{v_i\} = 0, \quad \mathbb{E}\{v_i^2\} = \frac{N}{M} \sum_{u \in \mathbb{C}} \sum_{r \in \mathcal{R}} W_{r,g(i)} W_{r,u} (Q_{r,g(i)}^t)^2 \hat{\tau}_t(u). \tag{62}
\]

Conditional on \(w\), \((Q^t \odot A^t)^* w\) is a vector with i.i.d. zero-mean normal entries. Also, the variance of its \(i^{th}\) entry, for \(i \in [n]\), is

\[
\frac{1}{M} \sum_{r \in \mathcal{R}} W_{r,g(i)} (Q^t_{r,g(i)})^2 \|w_R(r)\|^2, \tag{63}
\]

which converges to \(\sum_{r \in \mathcal{R}} W_{r,g(i)} (Q^t_{r,g(i)})^2 \sigma^2\), by the law of large numbers. With slightly more work, it can be shown that these entries are approximately independent of the ones of \(B'(x^t - x)\).

Summarizing, the \(i^{th}\) entry of the vector in the argument of \(\eta_t\) in Eq. (61) converges to \(X + \tau_t(g(i))^{1/2} Z\) with \(Z \sim \mathcal{N}(0,1)\) independent of \(X\), and

\[
\tau_t(s) = \sum_{r \in \mathcal{R}} W_{r,s} (Q^t_{r,s})^2 \left\{\sigma^2 + \frac{1}{\delta} \sum_{u \in \mathbb{C}} W_{r,u} \hat{\tau}_t(u)\right\}, \tag{64}
\]

for \(s \in \mathbb{C}\). In addition, using Eq. (61) and invoking Eqs. (35), (36), each entry of \(x_{C(s)}^{t+1} - x_{C(s)}\) converges to \(\eta_{t,s}(X + \tau_t(s)^{1/2} Z) - X\), for \(s \in \mathbb{C}\). Therefore,

\[
\hat{\tau}_{t+1}(s) = \lim_{N \to \infty} \frac{1}{N} \|x_{C(s)}^{t+1} - x_{C(s)}\|^2 = \mathbb{E}\{[q_{t,s}(X + \tau_t(s)^{1/2} Z) - X]^2\} = \text{mmse}(\tau_t(s)^{-1}). \tag{65}
\]

Using Eqs. (64) and (65), we obtain:

\[
\tau_{t+1}(s) = \sum_{r \in \mathcal{R}} W_{r,s} (Q^t_{r,s})^2 \left\{\sigma^2 + \frac{1}{\delta} \sum_{u \in \mathbb{C}} W_{r,u} \text{mmse}(\tau_t(u)^{-1})\right\}. \tag{66}
\]

Applying the change of variable \(\tau_t(u)^{-1} = \sum_{b \in \mathcal{R}} W_{b,u} \phi_b(t)^{-1}\), and substituting for \(Q^t_{r,s}\) from Eq. (34), we obtain the state evolution recursion, Eq. (33).

In conclusion, we showed that the state evolution recursion would hold if the matrix \(A\) was resampled independently from the ensemble \(\mathcal{M}(W, M, N)\), at each iteration. However, in our proposed AMP algorithm, the matrix \(A\) is constant across iterations, and the above argument is not valid since \(x^t\) and \(A\) are dependent. The dependency between \(A\) and \(x^t\) cannot be neglected. Indeed, state evolution does not apply to the following naive iteration in which we dropped the memory term \(b^t \odot r^{t-1}\):

\[
x^{t+1} = \eta_t(x^t + (Q^t \odot A)^* r^t), \quad r^t = y^t - Ax^t. \tag{67}
\]

Indeed, the term \(b^t \odot r^{t-1}\) leads to an asymptotic cancellation of the dependencies between \(A\) and \(x^t\) as proved in [BM11, JM12a].
7 Analysis of state evolution: Proof of Lemma 4.2

This section is devoted to the analysis of the state evolution recursion for spatially coupled matrices $A$, hence proving Lemma 4.2.

In order to prove Lemma 4.2, we will construct a free energy functional $E_W(\phi)$ such that the fixed points of the state evolution are the stationary points of $E_W$. We then assume by contradiction that the claim of the lemma does not hold, i.e., $\phi(t)$ converges to a fixed point $\phi(\infty)$ with $\phi_a(\infty) \gg \sigma^2$ for a significant fraction of the indices $a$. We then obtain a contradiction by describing an infinitesimal deformation of this fixed point (roughly speaking, a shift to the right) that decreases its free energy.

7.1 Outline

A more precise outline of the proof is given below:

(i) We establish some useful properties of the state evolution sequence $\{\phi(t), \psi(t)\}_{t \geq 0}$. This includes a monotonicity property as well as a lower and an upper bound for the state vectors.

(ii) We define a modified state evolution sequence, denoted by $\{\phi^{\text{mod}}(t), \psi^{\text{mod}}(t)\}_{t \geq 0}$. This sequence dominates the original state vectors (see Lemma 7.8) and hence it suffices to focus on the modified state evolution to get the desired result. As we will see the modified state evolution is more amenable to analysis.

(iii) We next introduce continuum state evolution which serves as the continuous analog of the modified state evolution. (The continuum states are functions rather than vectors). The bounds on the continuum state evolution sequence lead to bounds on the modified state vectors.

(iv) Analysis of the continuum state evolution incorporates the definition of a free energy functional defined on the space of non-negative measurable functions with bounded support. The energy is constructed in a way to ensure that the fixed points of the continuum state evolution are the stationary points of the free energy. Then, we show that if the undersampling rate is greater than the information dimension, the solution of the continuum state evolution can be made as small as $O(\sigma^2)$. If this were not the case, the (large) fixed point could be perturbed slightly in such a way that the free energy decreases to the first order. However, since the fixed point is a stationary point of the free energy, this leads to a contradiction.

7.2 Properties of the state evolution sequence

Throughout this section $p_X$ is a given probability distribution over the real line, and $X \sim p_X$. Also, we will take $\sigma > 0$. The result for the noiseless model (Corollary 1.8) follows by letting $\sigma \downarrow 0$. Recall the inequality

$$\text{mmse}(s) \leq \min(\text{Var}(X), \frac{1}{s}). \tag{69}$$

Definition 7.1. For two vectors $\phi, \tilde{\phi} \in \mathbb{R}^K$, we write $\phi \geq \tilde{\phi}$ if all $\phi_r \geq \tilde{\phi}_r$ for $r \in \{1, \ldots, K\}$.

Proposition 7.2. For any $W \in \mathbb{R}^{K \times C}_+$, the maps $T'_W : \mathbb{R}_+^K \rightarrow \mathbb{R}_+^C$ and $T''_W : \mathbb{R}_+^C \rightarrow \mathbb{R}_+^K$, as defined in Definition 2.2, are monotone; i.e., if $\phi \geq \tilde{\phi}$ then $T'_W(\phi) \geq T'_W(\tilde{\phi})$, and if $\psi \geq \tilde{\psi}$ then $T''_W(\psi) \geq T''_W(\tilde{\psi})$. Consequently, $T_W$ is also monotone.
Proof. It follows immediately from the fact that $s \mapsto \text{mmse}(s)$ is a monotone decreasing function and the positivity of the matrix $W$. \hfill \Box

**Proposition 7.3.** The state evolution sequence $\{\phi(t), \psi(t)\}_{t \geq 0}$ with initial condition $\psi_i(0) = \infty$, for $i \in C$, is monotone decreasing, in the sense that $\phi(0) \geq \phi(1) \geq \phi(2) \geq \ldots$ and $\psi(0) \geq \psi(1) \geq \psi(2) \geq \ldots$.

Proof. Since $\psi_i(0) = \infty$ for all $i$, we have $\psi(0) \geq \psi(1)$. The thesis follows from the monotonicity of the state evolution map. \hfill \Box

**Proposition 7.4.** The state evolution sequence $\{\phi(t), \psi(t)\}_{t \geq 0}$ is monotone increasing in $\sigma^2$. Namely, let $0 \leq \sigma_1 \leq \sigma_2$ and $\{\phi^{(1)}(t), \psi^{(1)}(t)\}_{t \geq 0}$, $\{\phi^{(2)}(t), \psi^{(2)}(t)\}_{t \geq 0}$ be the state evolution sequences corresponding to setting, respectively, $\sigma^2 = \sigma_1^2$ and $\sigma^2 = \sigma_2^2$ in Eq. (33), with identical initial conditions. Then $\phi^{(1)}(t) \geq \phi^{(2)}(t)$, $\psi^{(1)}(t) \leq \psi^{(2)}(t)$ for all $t$.

Proof. Follows immediately from Proposition 7.2 and the monotonicity of the one-step mapping in Eq. (33). \hfill \Box

**Lemma 7.5.** Assume $\delta L_0 > 3$. Then there exists $t_0$ (depending only on $p_X$), such that, for all $t \geq t_0$ and all $i \in \{-2p^{-1}, \ldots, -1\}$, $a \in R_i$, we have

\begin{align*}
\psi_i(t) &\leq \text{mmse}\left(\frac{L_0}{2\sigma^2}\right) \leq 2\sigma^2 \frac{L_0}{L_0}, \\
\phi_a(t) &\leq \sigma^2 + \frac{1}{\delta}\text{mmse}\left(\frac{L_0}{2\sigma^2}\right) \leq \left(1 + \frac{2}{\delta L_0}\right)\sigma^2.
\end{align*}

(70) (71)

Proof. Take $i \in \{-2p^{-1}, \ldots, -1\}$. For $a \in R_i$, we have $\phi_a(t) = \sigma^2 + (1/\delta)\psi_i(t)$. Further from $\text{mmse}(s) \leq 1/s$, we deduce that

\begin{align*}
\psi_i(t + 1) &= \text{mmse}\left(\sum_{b \in R} W_{b,i} \phi_b(t)^{-1}\right) \\
&\leq \left(\sum_{a \in R_i} W_{a,i} \phi_a(t)^{-1}\right)^{-1} = \left(\sum_{b \in R} W_{b,i} \phi_b(t)^{-1}\right)^{-1} = \phi_a(t)\frac{L_0}{L_0}.
\end{align*}

(72)

Here we used the facts that $W_{a,i} = 1$, for $a \in R_i$ and $|R_i| = L_0$. Substituting in the earlier relation, we get $\psi_i(t + 1) \leq (1/L_0)(\sigma^2 + (1/\delta)\psi_i(t))$. Recalling that $\delta L_0 > 3$, we have $\psi_i(t) \leq 2\sigma^2/L_0$, for all $t$ sufficiently large. Now, using this in the equation for $\phi_a(t)$, $a \in R_i$, we obtain

\begin{align*}
\phi_a(t) &= \sigma^2 + \frac{1}{\delta}\psi_i(t) \leq \left(1 + \frac{2}{\delta L_0}\right)\sigma^2.
\end{align*}

(73)

We prove the other claims by repeatedly substituting in the previous bounds. In particular,

\begin{align*}
\psi_i(t) &= \text{mmse}\left(\sum_{b \in R} W_{b,i} \phi_b(t - 1)^{-1}\right) \leq \text{mmse}\left(\sum_{a \in R_i} W_{a,i} \phi_a(t)^{-1}\right) \\
&= \text{mmse}\left(L_0 \phi_a(t)^{-1}\right) \leq \text{mmse}\left(\frac{L_0}{(1 + \frac{2}{\delta L_0})\sigma^2}\right) \leq \text{mmse}\left(\frac{L_0}{2\sigma^2}\right),
\end{align*}

(74)
where we used Eq. (73) in the penultimate inequality. Finally,

\[ \phi_a(t) \leq \sigma^2 + \frac{1}{\delta} \psi_i(t) \leq \sigma^2 + \frac{1}{\delta} \text{mmse}(\frac{L_0}{2\sigma^2}), \tag{75} \]

where the inequality follows from Eq. (74).

Next we prove a lower bound on the state evolution sequence. Here and below \( C_0 \equiv C \setminus \{-2\rho^{-1}, \ldots, -1\} \cong \{0, \ldots, L - 1\} \). Also, recall that \( R_0 \equiv \{-\rho^{-1}, \ldots, 0, \ldots, L - 1 + \rho^{-1}\} \). (See Fig. 3).

**Lemma 7.6.** For any \( t \geq 0 \), and any \( i \in C_0 \), \( \psi_i(t) \geq \text{mmse}(2\sigma^{-2}) \). Further, for any \( a \in R_0 \) and any \( t \geq 0 \) we have \( \phi_a(t) \geq \sigma^2 + (2\delta)^{-1}\text{mmse}(2\sigma^2) \).

**Proof.** Since \( \phi_a(t) \geq \sigma^2 \) by definition, we have, for \( i \geq 0 \), \( \psi_i(t) \geq \text{mmse}(\sigma^{-2} \sum_b W_{bi}) \geq \text{mmse}(2\sigma^{-2}) \), where we used the fact that the restriction of \( W \) to columns in \( C_0 \) is roughly column-stochastic. Plugging this into the expression for \( \phi_a \), we get

\[ \phi_a(t) \geq \sigma^2 + \frac{1}{\delta} \sum_{i \in C} W_{ai} \text{mmse}(2\sigma^{-2}) \geq \sigma^2 + \frac{1}{2\delta} \text{mmse}(2\sigma^{-2}). \tag{76} \]

Notice that for \( L_{0,*} \geq 4 \) and for all \( L_0 > L_{0,*} \), the upper bound for \( \psi_i(t) \), \( i \in \{-2\rho^{-1}, \ldots, -1\} \), given in Lemma 7.5 is below the lower bound for \( \psi_i(t) \), with \( i \in C_0 \), given in Lemma 7.6, i.e., for all \( \sigma \),

\[ \text{mmse}\left(\frac{L_0}{2\sigma^2}\right) \leq \text{mmse}\left(\frac{2}{\sigma^2}\right). \tag{77} \]

### 7.3 Modified state evolution

First of all, by Proposition 7.4 we can assume, without loss of generality \( \sigma > 0 \).

Motivated by the monotonicity properties of the state evolution sequence mentioned in Lemmas 7.5 and 7.6, we introduce a new state evolution recursion that dominates the original one and yet is more amenable to analysis. Namely, we define the modified state evolution maps \( F_W : R_0^+ \to R_0^C \), \( F_W'' : R_0^C \to R_0^R \). For \( \phi = (\phi_a)_{a \in R_0} \in R_0^R \), \( \psi = (\psi_i)_{i \in C_0} \in R_0^C \), and for all \( i \in C_0 \), \( a \in R_0 \), let:

\[ F_W(\phi)_i = \text{mmse}\left(\sum_{b \in R_0} W_{b-i} \phi_b^{-1}\right), \tag{78} \]

\[ F_W''(\psi)_a = \sigma^2 + \frac{1}{\delta} \sum_{i \in Z} W_{a-i} \psi_i. \tag{79} \]

where, in the last equation we set by convention, \( \psi_i(t) = \text{mmse}(L_0/(2\sigma^2)) \) for \( i \leq -1 \), and \( \psi_i = \infty \) for \( i \geq L \), and recall the shorthand \( W_{a-i} \equiv \rho W(\rho(a - i)) \) introduced in Section 2.4. We also let \( F_W = F_W'' \circ F_W' \).

**Definition 7.7.** The modified state evolution sequence is the sequence \( \{\phi(t), \psi(t)\}_{t \geq 0} \) with \( \phi(t) = F_W''(\psi(t)) \) and \( \psi(t + 1) = F_W'(\phi(t)) \) for all \( t \geq 0 \), and \( \psi_i(0) = \infty \) for all \( i \in C_0 \). We also adopt the convention that, for \( i \geq L \), \( \psi_i(t) = +\infty \) and for \( i \leq -1 \), \( \psi_i(t) = \text{mmse}(L_0/(2\sigma^2)) \), for all \( t \).
Lemma 7.8 then implies the following.

**Lemma 7.8.** Let \( \{\phi(t), \psi(t)\}_{t \geq 0} \) denote the state evolution sequence as per Definition 2.3 and \( \{\phi^{mod}(t), \psi^{mod}(t)\}_{t \geq 0} \) denote the modified state evolution sequence as per Definition 7.7. Then, there exists \( t_0 \) (depending only on \( p_X \)), such that, for all \( t \geq t_0, \phi(t) \leq \phi^{mod}(t-t_0) \) and \( \psi(t) \leq \psi^{mod}(t-t_0) \).

**Proof.** Choose \( t_0 = t(L_0, \delta) \) as given by Lemma 7.5. We prove the claims by induction on \( t \). For the induction basis (\( t = t_0 \)), we have from Lemma 7.5, \( \psi_i(t_0) \leq \text{mmse}(L_0/(2\sigma^2)) = \psi_i^{mod}(0), \) for \( i \leq -1 \). Also, we have \( \psi_i^{mod}(0) = \infty \geq \psi_i(t_0), \) for \( i \geq 0 \). Further, \( \phi_a^{mod}(0) = F_W^{''}(\psi^{mod}(0))a \geq T_W^{''}(\psi^{mod}(0))a \geq T_W^{''}(\psi(t_0))a = \phi_a(t_0), \) (80) for \( a \in R_0 \). Here, the last inequality follows from monotonicity of \( T_W^{''} \) (Proposition 7.2). Now, assume that the claim holds for \( t \); we prove it for \( t+1 \). For \( i \in C_0 \), we have

\[
\psi_i^{mod}(t+1-t_0) = F_W^{''}(\phi^{mod}(t-t_0))i = T_W^{''}(\phi^{mod}(t-t_0))i \\
\geq T_W^{''}(\phi(t))i = \psi_i(t+1),
\]

where the inequality follows from monotonicity of \( T_W^{''} \) (Proposition 7.2) and the induction hypothesis. In addition, for \( a \in R_0 \),

\[
\phi_a^{mod}(t+1-t_0) = F_W^{''}(\psi^{mod}(t+1-t_0))a \geq T_W^{''}(\psi^{mod}(t+1-t_0))a \\
\geq T_W^{''}(\psi(t+1))a = \phi_a(t+1). \tag{82}
\]

Here, the last inequality follows from monotonicity of \( T_W^{''} \) and Eq. (81). \( \square \)

By Lemma 7.8, we can now focus on the modified state evolution sequence in order to prove Lemma 4.2. Notice that the mapping \( F_W \) has a particularly simple description in terms of a shift-invariant state evolution mapping. Explicitly, define \( T_{W,\infty} : R_z^Z \to R_z^Z, T_{W,\infty}'' : R_z^Z \to R_z^Z \), by letting, for \( \phi, \psi \in R_z^Z \) and all \( i, a \in Z \):

\[
T_{W,\infty}^{'}(\phi)_i = \text{mmse}(\sum_{b \in Z} W_{b-i} \phi_{b}^{-1}),
\]

\[
T_{W,\infty}^{''}(\psi)_a = \sigma^2 + \frac{1}{\delta} \sum_{i \in Z} W_{a-i} \psi_i. \tag{84}
\]

Further, define the embedding \( H : R_{C_0} \to R_z^Z \) by letting

\[
(H\psi)_i = \begin{cases} 
\text{mmse}(L_0/(2\sigma^2)) & \text{if } i < 0, \\
\psi_i & \text{if } 0 \leq i \leq L - 1, \\
+\infty & \text{if } i \geq L,
\end{cases} \tag{85}
\]

And the restriction mapping \( H_{a,b}^{'} : R_z^Z \to R_z^{b-a+1} \) by \( H_{a,b}^{'} \psi = (\psi_a, \ldots, \psi_b) \).

**Lemma 7.9.** With the above definitions, \( F_W = H_{0,L-1}^{'} \circ T_{W,\infty} \circ H \).
Proof. Clearly, for any \( \psi = (\psi_i)_{i \in C_0} \), we have \( T_W'' \circ H(\psi)_a = F_W'' \circ H(\psi)_a \) for \( a \in R_0 \), since the definition of the embedding \( H \) is consistent with the convention adopted in defining the modified state evolution. Moreover, for \( i \in C_0 \approx \{0, \ldots, L - 1\} \), we have
\[
T_{W,\infty}(\phi)_i = \text{mmse}\left( \sum_{b \in \mathbb{Z}} W_{b-i} \phi_b^{-1} \right) = \text{mmse}\left( \sum_{-\rho^{-1} \leq b \leq L-1+\rho^{-1}} W_{b-i} \phi_b^{-1} \right)
= \text{mmse}\left( \sum_{b \in R_0} W_{b-i} \phi_b^{-1} \right) = F_W''(\phi)_i.
\]

Hence, \( T_{W,\infty}' \circ T_{W,\infty}'' \circ H(\psi)_i = F_W' \circ F_W'' \circ H(\psi)_i, \) for \( i \in C_0 \). Therefore, \( H_{0,L-1} \circ T_{W,\infty} \circ H(\psi) = F_W \circ H(\psi) \), for any \( \psi \in R_{C_0}^\infty \), which completes the proof.

We will say that a vector \( \psi \in R^K \) is nondecreasing if, for every \( 1 \leq i < j \leq K, \psi_i \leq \psi_j \).

Lemma 7.10. If \( \psi \in R_{C_0}^\infty \) is nondecreasing, with \( \psi_i \geq \text{mmse}(L_0/(2\sigma^2)) \) for all \( i \), then \( F_W(\psi) \) is nondecreasing as well. In particular, if \( \{\phi(t), \psi(t)\}_{t \geq 0} \) is the modified state evolution sequence, then \( \phi(t) \) and \( \psi(t) \) are nondecreasing for all \( t \).

Proof. By Lemma 7.9, we know that \( F_W = H_{0,L-1}' \circ T_{W,\infty} \circ H \). We first notice that, by the assumption \( \psi_i \geq \text{mmse}(L_0/(2\sigma^2)) \), we have that \( H(\psi) \) is nondecreasing.

Next, if \( \psi \in R^\infty \) is nondecreasing, \( T_{W,\infty}(\psi) \) is nondecreasing as well. In fact, the mappings \( T_{W,\infty}' \) and \( T_{W,\infty}'' \) both preserve the nondecreasing property, since both are shift invariant, and \( \text{mmse}(\cdot) \) is a decreasing function. Finally, the restriction of a nondecreasing vector is obviously nondecreasing.

This proves that \( F_W \) preserves the nondecreasing property. To conclude that \( \psi(t) \) is nondecreasing for all \( t \), notice that the condition \( \psi_i(t) \geq \text{mmse}(L_0/(2\sigma^2)) \) is satisfied at all \( t \) by Lemma 7.6 and condition (77). The claim for \( \psi(t) \) follows by induction.

Now, since \( F_W' \) preserves the nondecreasing property, we have \( \phi(t) = F_W'(\psi(t)) \) is nondecreasing for all \( t \), as well.

7.4 Continuum state evolution

We start by defining the continuum state evolution mappings. For \( \Omega \subseteq \mathbb{R} \), let \( \mathcal{M}(\Omega) \) be the space of non-negative measurable functions on \( \Omega \) (up to measure-zero redefinitions). Define \( F_W : \mathcal{M}([-1, \ell + 1]) \rightarrow \mathcal{M}([0, \ell]) \) and \( F_W' : \mathcal{M}([0, \ell]) \rightarrow \mathcal{M}([-1, \ell + 1]) \) as follows. For \( \phi \in \mathcal{M}([-1, \ell + 1]), \psi \in \mathcal{M}([0, \ell]), \) and for all \( x \in [0, \ell], y \in [-1, \ell + 1] \), we let
\[
F_W(\phi)(x) = \text{mmse}\left( \int_{-1}^{\ell} W(x-z)\phi(z)^{-1}dz \right),
\]
(87)
\[
F_W'\psi(y) = \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(y-x)\psi(x)dx,
\]
(88)
where we adopt the convention that \( \psi(x) = \text{mmse}(L_0/(2\sigma^2)) \) for \( x < 0 \), and \( \psi(x) = \infty \) for \( x > \ell \).

Definition 7.11. The continuum state evolution sequence is the sequence \( \{\phi(\cdot; t), \psi(\cdot; t)\}_{t \geq 0} \), with \( \phi(t) = F_W'(\psi(t)) \) and \( \psi(t+1) = F_W(\phi(t)) \) for all \( t \geq 0 \), and \( \psi(x;0) = \infty \) for all \( x \in [0, \ell] \).
Recalling Eq. [69], we have $\psi(x; t) = F^t_W(\phi(t - 1))(x) \leq \text{Var}(X)$, for $t \geq 1$. Also, $\phi(x; t) = F^t_W(\psi(t))(x) \leq \sigma^2 + (1/\delta)\text{Var}(X)$, for $t \geq 1$. Define,

$$\Phi_M = 1 + \frac{1}{\delta}\text{Var}(X).$$

(89)

Assuming $\sigma < 1$, we have $\phi(x; t) < \Phi_M$, for all $t \geq 1$.

The point of introducing continuum state evolution is that by construction of the matrix $W$ and the continuity of $W$, when $\rho$ is small, one can approximate summation by integration and study the evolution of the continuum states which are represented by functions rather than vectors. This observation is formally stated in lemma below.

**Lemma 7.12.** Let $\{\phi(\cdot; t), \psi(\cdot; t)\}_{t \geq 0}$ be the continuum state evolution sequence and $\{\phi(t), \psi(t)\}_{t \geq 0}$ be the modified discrete state evolution sequence, with parameters $\rho$ and $L = \ell/\rho$. Then for any $t \geq 0$

$$\lim_{\rho \to 0} \frac{1}{L} \sum_{i=0}^{L-1} |\psi_i(t) - \psi(\rho \cdot t)| = 0,$$

(90)

$$\lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi_a(t) - \phi(\rho \cdot t)| = 0.$$  

(91)

Lemma 7.12 is proved in Appendix C.

**Corollary 7.13.** The continuum state evolution sequence $\{\phi(\cdot; t), \psi(\cdot; t)\}_{t \geq 0}$, with initial condition $\psi(x) = \text{mmse}(L_0/(2\sigma^2))$ for $x < 0$, and $\psi(x) = \infty$ for $x > \ell$, is monotone decreasing, in the sense that $\phi(x; 0) \geq \phi(x; 1) \geq \phi(x; 2) \geq \cdots$ and $\psi(x; 0) \geq \psi(x; 1) \geq \psi(x; 2) \geq \cdots$, for all $x \in [0, \ell]$.

**Proof.** Follows immediately from Lemmas 7.3 and 7.12.

**Corollary 7.14.** Let $\{\phi(\cdot; t), \psi(\cdot; t)\}_{t \geq 2}$ be the continuum state evolution sequence. Then for any $t$, $x \mapsto \psi(x; t)$ and $x \mapsto \phi(x; t)$ are nondecreasing Lipschitz continuous functions.

**Proof.** Nondecreasing property of functions $x \mapsto \psi(x; t)$, and $x \mapsto \phi(x; t)$ follows immediately from Lemmas 7.10 and 7.12. Further, since $\psi(x; t)$ is bounded for $t \geq 1$, and $W(\cdot)$ is Lipschitz continuous, recalling Eq. (88), the function $x \mapsto \phi(x; t)$ is Lipschitz continuous as well, for $t \geq 1$. Similarly, since $\sigma^2 < \phi(x; t) < \Phi_M$, invoking Eq. (87), the function $x \mapsto \psi(x; t)$ is Lipschitz continuous for $t \geq 2$.

7.4.1 Free energy

A key role in our analysis is played by the free energy functional. In order to define the free energy, we first provide some preliminaries. Define the mutual information between $X$ and a noisy observation of $X$ at signal-to-noise ratio $s$ by

$$I(s) \equiv I(X; \sqrt{s}X + Z),$$

(92)

with $Z \sim N(0, 1)$ independent of $X \sim p_X$. Recall the relation

$$\frac{d}{ds} I(s) = \frac{1}{2} \text{mmse}(s).$$

(93)

Furthermore, the following identities relate the scaling law of mutual information under weak noise to Rényi information dimension [WV11a].
Proposition 7.15. Assume $H(\lfloor X \rfloor) < \infty$. Then

\[
\liminf_{s \to \infty} \frac{1}{2} \frac{l(s)}{\log s} = \underline{d}(p_X),
\]

\[
\limsup_{s \to \infty} \frac{1}{2} \frac{l(s)}{\log s} = \overline{d}(p_X).
\]

(94)

Now we are ready to define the free energy functional.

Definition 7.16. Let $W(\cdot)$ be a shape function, and $\sigma, \delta > 0$ be given. The corresponding free energy is the functional $E_W : \mathcal{M}([-1, \ell + 1]) \to \mathbb{R}$ defined as follows for $\phi \in \mathcal{M}([-1, \ell + 1])$:

\[
E_W(\phi) = \frac{\delta}{2} \int_{-1}^{\ell-1} \left\{ \frac{\varsigma^2(x)}{\phi(x)} + \log \phi(x) \right\} dx + \int_{0}^{\ell} \left( \int W(x - z) \phi(z)^{-1} dz \right) dx,
\]

(95)

where

\[
\varsigma^2(x) = \sigma^2 + \frac{1}{\delta} \left( \int_{y \leq 0} W(y - x) dy \right) \text{mmse} \left( \frac{L_0}{2\sigma^2} \right). \tag{96}
\]

The name ‘free energy’ is motivated by the connection with statistical physics, whereby $E_W(\phi)$ is the asymptotic log-partition function for the Gibbs-Boltzmann measure corresponding to the posterior distribution of $x$ given $y$. (This connection is however immaterial for our proof and we will not explore it further, see for instance [KMS+11].)

Notice that this is where the R"enyi information comes into the picture. The mutual information appears in the expression of the free energy and the mutual information is related to the R"enyi information via Proposition 7.15.

Viewing $E_W$ as a function defined on the Banach space $L_2([-1, \ell])$, we will denote by $\nabla E_W(\phi)$ its Fréchet derivative at $\phi$. This will be identified, via standard duality, with a function in $L_2([-1, \ell])$. It is not hard to show that the Fréchet derivative exists on $\{ \phi : \phi(x) \geq \sigma^2 \}$ and is such that

\[
\nabla E_W(\phi)(y) = \frac{\delta}{2\phi^2(y)} \left\{ \phi(y) - \varsigma^2(y) - \frac{1}{\delta} \int_0^\ell W(x - y) \text{mmse} \left( \int W(x - z) \phi(z)^{-1} dz \right) dx \right\},
\]

(97)

for $-1 \leq y \leq \ell - 1$. Note that the condition $\phi(x) \geq \sigma^2$ is immediately satisfied by the state evolution sequence since, by Eq. (88), $F''_W(\psi)(y) \geq \sigma^2$ for all $y$ (because $W(y - x)$, $\psi(x; t) \geq 0$); see also Definition 7.11.

The specific choice of the free energy in Eq. (95) ensures that the fixed points of the continuum state evolution are the stationary points of the free energy.

Corollary 7.17. If $\{ \phi, \psi \}$ is the fixed point of the continuum state evolution, then $\nabla E_W(\phi)(y) = 0$, for $-1 \leq y \leq \ell - 1$. 


Proof. We have $\phi = F_W'(\psi)$ and $\psi = F_W(\phi)$, whereby for $-1 \leq y \leq \ell - 1$,

$$
\phi(y) = \sigma^2 + \frac{1}{\delta} \int \mathcal{W}(y - x)\psi(x)dx
= \sigma^2 + \frac{1}{\delta} \left( \int_{x \leq 0} \mathcal{W}(y - x)dx \right) \text{mmse}\left( \frac{L_0}{2\sigma^2} \right) + \frac{1}{\delta} \int_0^\ell \mathcal{W}(y - x)\text{mmse}\left( \int_{-1}^{\ell+1} \mathcal{W}(x - z)\phi(z)^{-1}dz \right)dx
= \varsigma^2(y) + \frac{1}{\delta} \int_0^\ell \mathcal{W}(y - x)\text{mmse}\left( \int_{-1}^{\ell+1} \mathcal{W}(x - z)\phi(z)^{-1}dz \right)dx.
$$

(98)

The result follows immediately from Eq. (97). \quad \Box

Definition 7.18. Define the potential function $V : \mathbb{R}_+ \to \mathbb{R}_+$ as follows.

$$
V(\phi) = \frac{\delta}{2} \left( \frac{\sigma^2}{\phi} + \log \phi \right) + I(\phi^{-1}).
$$

(99)

As we will see later, the analysis of the continuum state evolution involves a decomposition of the free energy functional into three terms and a careful treatment of each term separately. The definition of the potential function $V$ is motivated by that decomposition.

Using Eq. (94), we have for $\phi \to 0$,

$$
V(\phi) = \frac{\delta}{2} \left( \frac{\sigma^2}{\phi} + \log \phi \right) + \frac{1}{2} \delta \overline{d}(p_X) \log(\phi^{-1})(1 + o(1))
= \frac{\delta \sigma^2}{2\phi} + \frac{1}{2} \left\{ \delta - \overline{d}(p_X)(1 + o(1)) \right\} \log(\phi).
$$

(100)

Define

$$
\phi^* = \sigma^2 + \frac{1}{\delta} \text{mmse}\left( \frac{L_0}{2\sigma^2} \right).
$$

(101)

Notice that $\sigma^2 < \phi^* \leq (1 + 2/(\delta L_0))\sigma^2 < 2\sigma^2$, given that $\delta L_0 > 3$. The following proposition upper bounds $V(\phi^*)$ and its proof is deferred to Appendix D.

Proposition 7.19. There exists $\sigma_2 > 0$, such that, for $\sigma \in (0, \sigma_2]$, we have

$$
V(\phi^*) \leq \frac{\delta}{2} + \frac{\delta - \overline{d}(p_X)}{4} \log(2\sigma^2).
$$

(102)

Now, we write the energy functional in terms of the potential function.

$$
\mathcal{E}_W(\phi) = \int_{-1}^{\ell-1} V(\phi(x)) dx + \frac{\delta}{2} \int_{-1}^{\ell-1} \varsigma^2(x) - \frac{\sigma^2}{\phi(x)} \, dx + \tilde{\mathcal{E}}_W(\phi),
$$

(103)

with,

$$
\tilde{\mathcal{E}}_W(\phi) = \int_0^\ell \left\{ I(W * \phi(y)^{-1}) - I(\phi(y-1)^{-1}) \right\} dy.
$$

(104)

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7.4.2 Analysis of the continuum state evolution

Now we are ready to study the fixed points of the continuum state evolution.

Lemma 7.20. Let \( \delta > 0 \), and \( p_X \) be a probability measure on the real line with \( \delta > \bar{d}(p_X) \). For any \( \kappa > 0 \), there exist \( \ell_0, \sigma_0^2 = \sigma_0(\kappa, \delta, p_X)^2 \), such that, for any \( \ell > \ell_0 \) and \( \sigma \in (0, \sigma_0] \), and any fixed point of continuum state evolution, \( \{\phi, \psi\} \), with \( \psi \) and \( \phi \) nondecreasing Lipschitz functions and \( \psi(x) \geq \text{mmse}(L_0/(2\sigma^2)) \), the following holds.

\[
\int_{-1}^{\ell-1} |\phi(x) - \phi^*| \, dx \leq \kappa \ell. \tag{105}
\]

Proof. The claim is trivial for \( \kappa \geq \Phi_M \), since \( \phi(x) \leq \Phi_M \). Fix \( \kappa < \Phi_M \), and choose \( \sigma_1 \), such that \( \phi^* < \kappa/2 \), for \( \sigma \in (0, \sigma_1] \). Since \( \phi \) is a fixed point of continuum state evolution, we have \( \nabla E_W(\phi) = 0 \), on the interval \([-1, \ell - 1] \) by Corollary 7.17. Now, assume that \( \int_{-1}^{\ell-1} |\phi(x) - \phi^*| > \kappa \ell \). We introduce an infinitesimal perturbation of \( \phi \) that decreases the energy in the first order; this contradicts the fact \( \nabla E_W(\phi) = 0 \) on the interval \([-1, \ell - 1] \).

Claim 7.21. For each fixed point of continuum state evolution that satisfies the hypothesis of Lemma 7.20, the following holds. For any \( K > 0 \), there exists \( \ell_0 \), such that, for \( \ell > \ell_0 \) there exist \( x_1 < x_2 \in [0, \ell - 1] \), with \( x_2 - x_1 = K \) and \( \kappa/2 + \phi^* < \phi(x) \), for \( x \in [x_1, x_2] \).

Claim 7.21 is proved in Appendix F.

Fix \( K > 2 \) and let \( x_0 = (x_1 + x_2)/2 \). Thus, \( x_0 \geq 1 \). For \( a \in (0, 1] \), define

\[
\phi_a(x) = \begin{cases} 
\phi(x), & \text{for } x \leq x_0, \\
\phi\left(\frac{x - x_0}{x_2 - x_0 - a}\right) - \frac{a x_2 - x_0}{x_2 - x_0 - a}, & \text{for } x \in [x_0 + a, x_2), \\
\phi(x - a), & \text{for } x \in [-1 + a, x_0 + a), \\
\phi^*, & \text{for } x \in [-1, -1 + a). 
\end{cases} \tag{106}
\]

See Fig. 7 for an illustration. (Note that from Eq. (88), \( \phi(-1) = \phi^* \)). In the following, we bound the difference of the free energies of functions \( \phi \) and \( \phi_a \).

Proposition 7.22. For each fixed point of continuum state evolution, satisfying the hypothesis of Lemma 7.20, there exists a constant \( C(K) \), such that

\[
\int_{-1}^{\ell-1} \left\{ \frac{\kappa^2(x) - \sigma^2}{\phi_a(x)} - \frac{\kappa^2(x) - \sigma^2}{\phi(x)} \right\} \, dx \leq C(K) a. 
\]

We refer to Appendix F for the proof of Proposition 7.22.

Proposition 7.23. For each fixed point of continuum state evolution, satisfying the hypothesis of Lemma 7.20, there exists a constant \( C(\kappa, K) \), such that

\[
\hat{E}_W(\phi_a) - \hat{E}_W(\phi) \leq C(\kappa, K) a. 
\]

Proof of Proposition 7.23 is deferred to Appendix G.

Using Eq. (103) and Proposition 7.23, we have

\[
E_W(\phi_a) - E_W(\phi) \leq \int_{-1}^{\ell-1} \{V(\phi_a(x)) - V(\phi(x))\} \, dx + C(\kappa, K) a, \tag{107}
\]

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Figure 7: An illustration of function $\phi(x)$ and its perturbation $\phi_a(x)$.

where the constants $(\delta/2)C(K)$ and $C(\kappa, K)$ are absorbed in $C(\kappa, K)$.

We proceed by proving the following proposition. Its proof is deferred to Appendix H.

**Proposition 7.24.** For any $C = C(\kappa, K)$, there exists $\sigma_0$, such that for $\sigma \in (0, \sigma_0]$ the following holds.

$$\int_{-1}^{\ell-1} \{V(\phi_a(x)) - V(\phi(x))\} \, dx < -2C(\kappa, K)a.$$  \hfill (108)

Fix $C(\kappa, K) > 0$. As a result of Eq. (107) and Proposition 7.24,

$$E_W(\phi_a) - E_W(\phi) < \int_{-1}^{\ell-1} \{V(\phi_a(x)) - V(\phi(x))\} \, dx + C(\kappa, K)a$$  \hfill (109)

$$\leq -C(\kappa, K)a.

Since $\phi$ is a Lipschitz function by assumption, it is easy to see that $\|\phi_a - \phi\|_2 \leq C_a$, for some constant $C$. By Taylor expansion of the free energy functional around function $\phi$, we have

$$\langle \nabla E_W(\phi), \phi_a - \phi \rangle = E_W(\phi_a) - E_W(\phi) + o(\|\phi_a - \phi\|_2)$$

$$\leq -C(\kappa, K)a + o(a).$$  \hfill (110)

However, since $\{\phi, \psi\}$ is a fixed point of the continuum state evolution, we have $\nabla E_W(\phi) = 0$ on the interval $[-1, \ell - 1]$ (cf. Corollary 7.17). Also, $\phi_a - \phi$ is zero out of $[-1, \ell - 1]$. Therefore, $\langle \nabla E_W(\phi), \phi_a - \phi \rangle = 0$, which leads to a contradiction in Eq (110). This implies that our first assumption $\int_{-1}^{\ell-1} |\phi(x) - \phi^*| \, dx > \kappa \ell$ is false. The result follows.

### 7.4.3 Analysis of the continuum state evolution: robust reconstruction

Next lemma pertains to the robust reconstruction of the signal. Prior to stating the lemma, we need to establish some definitions. Due to technical reasons in the proof, we consider an alternative decomposition of $E_W(\phi)$ to Eq. (103).
Define the potential function $V_{\text{rob}}: \mathbb{R}_+ \to \mathbb{R}_+$ as follows.

$$V_{\text{rob}}(\phi) = \frac{\delta}{2} \left( \frac{\sigma^2}{\phi} + \log \phi \right),$$  \hspace{1cm} (111)

and decompose the Energy functional as:

$$E_W(\phi) = \int_{-1}^{\ell-1} V_{\text{rob}}(\phi(x)) \, dx + \frac{\delta}{2} \int_{-1}^{\ell-1} \frac{\varsigma^2(x) - \sigma^2}{\phi(x)} \, dx + \hat{E}_{W,\text{rob}}(\phi),$$  \hspace{1cm} (112)

with,

$$\hat{E}_{W,\text{rob}}(\phi) = \int_{0}^{\epsilon} (W * (\phi(y)) - \phi) \, dy.$$  \hspace{1cm} (113)

**Lemma 7.25.** Let $\delta > 0$, and $p_X$ be a probability measure on the real line with $\delta > \overline{D}(p_X)$. For any $0 < \alpha < 1$, there exist $\ell_0 = \ell_0(\alpha)$, $\sigma_0 = \sigma_0(p_X, \delta, \alpha)^2$, such that for any $\ell > \ell_0$ and $\sigma \in (0, \sigma_0]$, and for any fixed point of continuum state evolution, $\{\phi, \psi\}$, with $\psi$ and $\phi$ nondecreasing Lipschitz functions and $\psi(x) \geq \text{mmse}(L_0/(2\sigma^2))$, the following holds.

$$\int_{-1}^{\ell-1} |\phi(x) - \phi^*| \, dx \leq C\sigma^2 \ell,$$  \hspace{1cm} (114)

with $C = \frac{2\delta}{(1-\alpha)(\delta - \overline{D}(p_X))}$.  

**Proof.** Suppose $\int_{-1}^{\ell-1} |\phi(x) - \phi^*| \, dx > C\sigma^2 \ell$, for the given $C$. Similar to the proof of Lemma 7.20, we obtain an infinitesimal perturbation of $\phi$ that decreases the free energy in the first order, contradicting the fact $\nabla E_W(\phi) = 0$ on the interval $[-1, \ell - 1]$.

By definition of upper MMSE dimension (Eq. (15)), for any $\varepsilon > 0$, there exists $\phi_1$, such that, for $\phi \in [0, \phi_1]$,

$$\text{mmse}(\phi^{-1}) \leq (\overline{D}(p_X) + \varepsilon)\phi.$$  \hspace{1cm} (115)

Henceforth, fix $\varepsilon$ and $\phi_1$.

**Claim 7.26.** For each fixed point of continuum state evolution that satisfies the hypothesis of Lemma 7.25, the following holds. For any $K > 0$, $0 < \alpha < 1$, there exist $\ell_0 = \ell_0(\alpha)$ and $\sigma_0 = \sigma_0(\varepsilon, \alpha, p_X, \delta)$, such that for $\ell > \ell_0$ and $\sigma \in (0, \sigma_0]$, there exist $x_1 < x_2 \in [0, \ell - 1)$, with $x_2 - x_1 = K$ and $C\sigma^2(1 - \alpha) \leq \phi(x) \leq \phi_1$, for $x \in [x_1, x_2]$.

Claim 7.26 is proved in Appendix [1]. For positive values of $a$, define

$$\phi_a(x) = \begin{cases} 
\phi(x), & \text{for } x \leq x_1, x_2 \leq x, \\
(1-a)\phi(x) & \text{for } x \in (x_1, x_2). 
\end{cases}$$  \hspace{1cm} (116)

Our aim is to show that $E_W(\phi_a) - E_W(\phi) \leq -ca$, for some constant $c > 0$.

Invoking Eq. (103), we have

$$E_W(\phi_a) - E_W(\phi) = \int_{-1}^{\ell-1} \{V_{\text{rob}}(\phi_a(x)) - V_{\text{rob}}(\phi(x))\} \, dx$$

$$+ \frac{\delta}{2} \int_{-1}^{\ell-1} (\varsigma^2(x) - \sigma^2) \left( \frac{1}{\phi_a(x)} - \frac{1}{\phi(x)} \right) \, dx + \hat{E}_{W,\text{rob}}(\phi_a) - \hat{E}_{W,\text{rob}}(\phi).$$  \hspace{1cm} (117)

The following proposition bounds each term on the right hand side separately.
Proposition 7.27. For the function \( \phi(x) \) and its perturbation \( \phi_a(x) \), we have

\[
\int_{-1}^{\ell-1} \{ \hat{V}_{\text{rob}}(\phi_a(x)) - \hat{V}_{\text{rob}}(\phi(x)) \} \, dx \leq \frac{\delta}{2} K \log(1 - a) + K \frac{\delta a}{2C(1 - \alpha)(1 - a)}, \tag{118}
\]

\[
\int_{-1}^{\ell-1} (\sigma^2(x) - \sigma^2) \left( \frac{1}{\phi_a(x)} - \frac{1}{\phi(x)} \right) \, dx \leq K \frac{a}{C(1 - \alpha)(1 - a)}, \tag{119}
\]

\[
\hat{E}_{W, \text{rob}}(\phi_a) - \hat{E}_{W, \text{rob}}(\phi) \leq -\frac{\mathcal{D}(p_X) + \varepsilon}{2} (K + 2) \log(1 - a). \tag{120}
\]

We refer to Appendix I for the proof of Proposition 7.27.

Combining the bounds given by Proposition 7.27, we obtain

\[
E_W(\phi_a) - E_W(\phi) \leq \frac{K}{2} \log(1 - a) \left\{ \delta - (\mathcal{D}(p_X) + \varepsilon) (1 + \frac{2}{K}) \right\} + K \frac{\delta a}{C(1 - \alpha)(1 - a)}. \tag{121}
\]

Since \( \delta > \mathcal{D}(p_X) \) by our assumption, and \( C = \frac{2\delta}{(1 - \alpha)(\delta - \mathcal{D}(p_X))} \), there exist \( \varepsilon, a \) small enough and \( K \) large enough, such that

\[
c = \delta - (\mathcal{D}(p_X) + \varepsilon)(1 + \frac{2}{K}) - \frac{2\delta}{C(1 - \alpha)(1 - a)} > 0.
\]

Using Eq. (121), we get

\[
E_W(\phi_a) - E_W(\phi) \leq -\frac{cK}{2} a. \tag{122}
\]

By an argument analogous to the one in the proof of Lemma 7.20, this is in contradiction with \( \nabla E_W(\phi) = 0 \). The result follows.

7.5 Proof of Lemma 4.2

By Lemma 7.8, \( \phi_a(t) \leq \phi_a^{\text{mod}}(t - t_0) \), for \( a \in R_0 \cong \{ \rho^{-1}, \cdots, L - 1 + \rho^{-1} \} \) and \( t \geq t_1(L_0, \delta) \). Therefore, we only need to prove the claim for the modified state evolution. The idea of the proof is as follows. In the previous section, we analyzed the continuum state evolution and showed that at the fixed point, the function \( \phi(x) \) is close to the constant \( \phi^* \). Also, in Lemma 7.12, we proved that the modified state evolution is essentially approximated by the continuum state evolution as \( \rho \to 0 \). Combining these results implies the thesis.

Proof (Part(a)). By monotonicity of continuum state evolution (cf. Corollary 7.13), \( \lim_{t \to \infty} \phi(x; t) = \phi(x) \) exists. Further, by continuity of state evolution recursions, \( \phi(x) \) is a fixed point. Finally, \( \phi(x) \) is a nondecreasing Lipschitz function (cf. Corollary 7.14). Using Lemma 7.20 in conjunction with the Dominated Convergence theorem, we have, for any \( \varepsilon > 0 \)

\[
\lim_{t \to \infty} \frac{1}{\ell} \int_{-1}^{\ell-1} |\phi(x; t) - \phi^*| \, dx \leq \frac{\varepsilon}{4}, \tag{123}
\]

for \( \sigma \in (0, \sigma_0^a] \) and \( \ell > \ell_0 \). Therefore, there exists \( t_2 > 0 \) such that \( \frac{1}{\ell} \int_{-1}^{\ell-1} |\phi(x; t_2) - \phi^*| \, dx \leq \varepsilon/2 \). Moreover, for any \( t \geq 0 \),

\[
\frac{1}{\ell} \int_{-1}^{\ell-1} |\phi(x; t) - \phi^*| \, dx = \lim_{\rho \to 0} \frac{1}{\ell} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi(\rho a; t) - \phi^*| = \lim_{\rho \to 0} \frac{1}{\ell} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi(\rho a; t) - \phi^*|. \tag{124}
\]
By triangle inequality, for any \( t \geq 0, \)
\[
\lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi_a(t) - \phi^*| \leq \lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi_a(t) - \phi(\rho a; t)| + \lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} |\phi(\rho a; t) - \phi^*| \\
= \frac{1}{L} \int_{-1}^{1} |\phi(x; t) - \phi^*| dx,
\]
where the last step follows from Lemma \[7.12\] and Eq. \[124\]. Since the sequence \( \{\phi(t)\} \) is monotone decreasing in \( t \), we have
\[
\lim_{\rho \to 0} \lim_{t \to \infty} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} \phi_a(t) \leq \lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} \phi(t_2) \\
\leq \lim_{\rho \to 0} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} (|\phi_a(t_2) - \phi^*| + \phi^*) \\
\leq \frac{1}{L} \int_{-1}^{1} |\phi(x; t_2) - \phi^*| dx + \phi^* \\
\leq \frac{\varepsilon}{2} + \phi^*.
\]
Finally,
\[
\lim_{t \to \infty} \frac{L+\rho^{-1}-1}{L} \sum_{a=-\rho^{-1}}^{L+\rho^{-1}-1} \phi_a(t) \leq \frac{2\rho^{-1}}{L} \Phi_M + \frac{\varepsilon}{2} + \phi^* \\
\leq \frac{2\rho^{-1}}{L_*} \Phi_M + \frac{\varepsilon}{2} + 2\sigma_0.
\]
Clearly, by choosing \( L_* \) large enough and \( \sigma_0 \) sufficiently small, we can ensure that the right hand side of Eq. \[127\] is less than \( \varepsilon \).

\textit{Proof (Part (b)).} Consider the following two cases.

- \( \sigma \leq \sigma_0 \): In this case, proceeding along the same lines as the proof of Part (a), and using Lemma \[7.25\] in lieu of Lemma \[7.20\] we have
\[
\lim_{t \to \infty} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} \phi_a(t) \leq C\sigma^2 + \phi^* \leq \left( \frac{2\delta}{(1-\alpha)(\delta - D(p_X))} + 1 + \frac{2}{\delta L_0} \right) \sigma^2. \tag{128}
\]

- \( \sigma > \sigma_0 \): Since \( \phi_a(t) \leq \sigma^2 + (1/\delta) \text{Var}(X) \) for any \( t > 0 \), we have
\[
\lim_{t \to \infty} \frac{1}{L} \sum_{a=-\rho^{-1}}^{L-\rho^{-1}-1} \phi_a(t) \leq \sigma^2 + \frac{1}{\delta} \text{Var}(X). \tag{129}
\]

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Choosing
\[ C = \max \left\{ \frac{2\delta}{(1 - \alpha)(\delta - \bar{D}(p_X))} + 1 + \frac{2}{\delta L_0}, 1 + \frac{\text{Var}(X)}{\delta \sigma_0^2} \right\}, \]
proves the claim in both cases.

Finally, in the asymptotic case where \( \ell = L \rho \to \infty, \rho \to 0, L_0 \to \infty, \) we have \( \alpha \to 0 \) and using Eq. (128), we get

\[
\lim_{\sigma \to 0} \lim_{t \to \infty} \frac{1}{\sigma^2 L} \sum_{a = -\rho^{-1}}^{L - \rho^{-1} - 1} \phi_a(t) \leq \frac{3\delta - \bar{D}(p_X)}{\delta - \bar{D}(p_X)}.
\]

\[ \Box \]
A Dependence of the algorithm on the prior \( p_X \)

In this appendix we briefly discuss the impact of a wrong estimation of the prior \( p_X \) on the AMP algorithm. Namely, suppose that instead of the true prior \( p_X \), we have an approximation of \( p_X \) denoted by \( p_{\tilde{X}} \). The only change in the algorithm is in the posterior expectation denoiser. That is to say, the denoiser \( \eta \) in Eq. (6) will be replaced by a new denoiser \( \tilde{\eta} \). We will quantify the discrepancy between \( p_X \) and \( p_{\tilde{X}} \) through their Kolmogorov-Smirnov distance \( D_{KS}(p_X, p_{\tilde{X}}) \). Denoting by \( F_X(z) = p_X((−\infty, z]) \) and \( \tilde{F}_{\tilde{X}}(z) = p_{\tilde{X}}((−\infty, z]) \) the corresponding distribution functions, we have

\[
D_{KS}(p_X, p_{\tilde{X}}) = \sup_{z \in \mathbb{R}} |F_X(z) - \tilde{F}_{\tilde{X}}(z)|.
\]

The next lemma establishes a bound on the pointwise distance between \( \eta \) and \( \tilde{\eta} \) in terms of \( D_{KS}(p_X, p_{\tilde{X}}) \).

Note that state evolution \([33]\) applies also to the algorithm with the mismatched denoiser, provided the \( \text{mmse}(\cdot) \) function is replaced by the mean square error for the non-optimal denoiser \( \tilde{\eta} \). Hence the bound on \( |\eta(y) - \tilde{\eta}(y)| \) given below can be translated into a bound on the performance of AMP with the mismatched prior. A full study of this issue goes beyond the scope of this paper and will be the object of a forthcoming publication.

For the sake of simplicity we shall assume that \( p_X, p_{\tilde{X}} \) have bounded supports. The general case requires a more careful consideration.

**Lemma A.1.** Let \( \eta : \mathbb{R} \to \mathbb{R} \) be the Bayes optimal estimator for estimating \( X \sim p_X \) in Gaussian noise \( \eta(y) = \mathbb{E}(X|X + Z = y) \), with \( Z \sim \mathcal{N}(0, 1) \). Define denoiser \( \tilde{\eta} \) similarly, with respect to \( p_{\tilde{X}} \).
Assume that \( p_X \) is supported in \([−M, M]\). Then for any \( p_{\tilde{X}} \) supported in \([−M, M]\), we have

\[
|\eta(y) - \tilde{\eta}(y)| \leq \frac{M(15 + 10M|y|)}{\mathbb{E}\{e^{-X^2/2}\}} D_{KS}(p_X, p_{\tilde{X}}) e^{2M|y|}.
\]

**Proof.** Throughout the proof we let \( \Delta \equiv D_{KS}(p_X, p_{\tilde{X}}) \), and \( \Delta_1 \equiv \mathbb{E}\{e^{-X^2/2}\} \).

Let \( \gamma(z) = \exp(-z^2/2)/\sqrt{2\pi} \) be the Gaussian density. We then have \( \eta(y) = \mathbb{E}\{X \gamma(X - y)\}/\mathbb{E}\{\gamma(X - y)\} \). Let \( p_W \) be the probability measure with Radon-Nikodym derivative with respect to \( p_X \) given by

\[
\frac{dp_W(x)}{dp_X(x)} = \frac{e^{-x^2/2}}{\mathbb{E}\{e^{-X^2/2}\}}.
\]

We define \( p_{\tilde{W}} \) analogously from the measure \( p_{\tilde{X}} \) and let \( W, \tilde{W} \) be two random variables with law \( p_W \) and \( p_{\tilde{W}} \), respectively. We then have

\[
\eta(y) = \frac{\mathbb{E}\{W e^{yW}\}}{\mathbb{E}\{e^{yW}\}}.
\]

Letting \( F_W, F_{\tilde{W}} \) denote the corresponding distribution functions, we have

\[
F_W(x) = \int_{−\infty}^{x} dp_W(w) = \frac{\int_{−\infty}^{x} e^{-z^2/2} dp_X(z)}{\mathbb{E}\{e^{-X^2/2}\}} = \frac{e^{-x^2/2} F_X(x) + \int_{−\infty}^{x} ze^{-z^2/2} F_X(z) \, dz}{\int_{−\infty}^{\infty} ze^{-z^2/2} F_X(z) \, dz}.
\]

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Letting $N_W(x)$ be the numerator in this expression, we have
\[
|N_W(x) - N_{\tilde{W}}(x)| \leq |F_X(x) - F_{\tilde{X}}(x)| + \int_{-\infty}^{x} |z| e^{-z^2/2} |F_X(z) - F_{\tilde{X}}(z)| \, dz \leq 3\Delta.
\]
Proceeding analogously for the denominator, we have
\[
|\mathbb{E}\{e^{-X^2/2}\} - \mathbb{E}\{e^{-\tilde{X}^2/2}\}| \leq \int_{-\infty}^{\infty} |z| e^{-z^2/2} |F_X(z) - F_{\tilde{X}}(z)| \, dz \leq 2\Delta.
\]
Combining these bounds, we obtain
\[
|\frac{N_W(x)}{\mathbb{E}\{e^{-X^2/2}\}} - \frac{N_{\tilde{W}}(x)}{\mathbb{E}\{e^{-\tilde{X}^2/2}\}}| \leq \frac{3\Delta + 2\Delta}{\Delta} = \frac{5\Delta}{\Delta}.
\]
(131)

Since, the above inequality holds for any $x \in \mathbb{R}$, we get
\[
D_{KS}(p_W, p_{\tilde{W}}) \leq \frac{5\Delta}{\Delta}.
\]
(132)

Consider now Eq. (130). We have
\[
|\mathbb{E}\{e^{yW}\} - \mathbb{E}\{e^{y\tilde{W}}\}| = |y| \int e^{yx} |F_W(x) - F_{\tilde{W}}(x)| \, dx \\
\leq |y| D_{KS}(p_W, p_{\tilde{W}}) \int_{-M}^{M} e^{yx} \, dx \leq e^{M|y|} D_{KS}(p_W, p_{\tilde{W}}).
\]

We proceed analogously for the numerator, namely,
\[
|\mathbb{E}\{We^{yW}\} - \mathbb{E}\{\tilde{W}e^{y\tilde{W}}\}| = \int (1 + |yx|) e^{yx} |F_W(x) - F_{\tilde{W}}(x)| \, dx \\
\leq D_{KS}(p_W, p_{\tilde{W}}) \int_{-M}^{M} (1 + |yx|) e^{yx} \, dx \leq 2M(1 + M|y|) e^{M|y|} D_{KS}(p_W, p_{\tilde{W}}).
\]

Combining these bounds and proceeding along similar lines to Eq. (131), we obtain
\[
|\tilde{\eta}(y) - \eta(y)| \leq \frac{2M(1 + M|y|)}{\mathbb{E}\{e^{yW}\}} + \frac{\tilde{\eta}(y)}{\mathbb{E}\{e^{yW}\}} e^{M|y|} D_{KS}(p_W, p_{\tilde{W}}).
\]
(133)

Note that $\tilde{\eta}(y) \in [-M, M]$ since $p_{\tilde{X}}$ is supported on $[-M, M]$, and thus $|\tilde{\eta}(y)| \leq M$. Also, $p_W$ is supported on $[-M, M]$ since it is absolutely continuous with respect to $p_X$ and $p_X$ is supported on $[-M, M]$. Therefore, $\mathbb{E}\{e^{yW}\} \geq e^{-M|y|}$. Using these bounds in Eq. (133), we obtain
\[
|\tilde{\eta}(y) - \eta(y)| \leq M(3 + 2M|y|) e^{2M|y|} D_{KS}(p_W, p_{\tilde{W}}).
\]
(134)

The result follows by plugging in the bound given by Eq. (132).
B Lipschitz continuity of AMP

Let $x^t$ be the Bayes optimal AMP estimation at iteration $t$ as given by Eqs. (6), (7). We show that for each fixed iteration number $t$, the mapping $y \rightarrow x^t(y)$ is locally Lipschitz continuous.

**Lemma B.1.** For any $R, B > 0$, $t \in \mathbb{N}$, there exists $L = L(R, B; t) < \infty$ such that for any $y, \tilde{y} \in \mathbb{R}^m$ with $\|y\|, \|\tilde{y}\| \leq R$, and any matrix $A$ with $\|A\|_2 \leq B$ we have

$$\|x^t(y) - x^t(\tilde{y})\| \leq L \|y - \tilde{y}\|. \quad (135)$$

Note that in the statement we assume $\|A\|_2$ to be finite. This happens as long as the entries of $A$ are bounded and hence almost surely within our setting.

Also, we assume $\|y\|, \|\tilde{y}\| \leq R$ for some fixed $R$. In other words, we prove that the algorithm is locally Lipschitz. We can obtain an algorithm that is globally Lipschitz by defining $x^t(y)$ via the AMP iteration for $\|y\| \leq R$, and by an arbitrary bounded Lipschitz extension for $\|y\| \geq R$. Notice that $\|y\| \leq B \|x\| + \|w\|$, and, by the law of large numbers, $\|x\|^2 \leq (\mathbb{E}\{X^2\} + \epsilon)n$, $\|w\|^2 \leq (\sigma^2 + \epsilon)m$ with probability converging to 1. Hence, the globally Lipschitz modification of AMP achieves the same performance as the original AMP, almost surely. (Note that $R$ can depend on $n$).

**Proof (Lemma B.1).** Suppose that we have two measurement vectors $y$ and $\tilde{y}$. Note that the state evolution is completely characterized in terms of prior $p_X$ and noise variance $\sigma^2$, and can be precomputed (independent of measurement vector).

Let $(x^t, r^t)$ correspond to the AMP with measurement vector $y$ and $(\tilde{x}^t, \tilde{r}^t)$ correspond to the AMP with measurement vector $\tilde{y}$. (To clarify, note that $x^t \equiv x^t(y)$ and $\tilde{x}^t \equiv x^t(\tilde{y})$). Further define

$$\xi_t = \max(\|x^t - \tilde{x}^t\|, \|r^t - \tilde{r}^t\|, \|y - \tilde{y}\|).$$

We show that

$$\xi_t \leq C_t(1 + \|y\|) \xi_{t-1}, \quad (136)$$

for a constant $C_t$. This establishes the claim since

$$\|x^t - \tilde{x}^t\| \leq \xi_t \leq C_tC_{t-1} \ldots C_2 (1 + \|y\|)^{t-1} \xi_1 = C_tC_{t-1} \ldots C_2 (1 + \|y\|)^{t-1} \|y - \tilde{y}\|,$$

where the last step holds since $x_{t}^{t-1} = \tilde{x}_{t}^{t-1} = \mathbb{E}\{X\}$ and $r_{t}^{t-1} = y - \tilde{y}$.

In order to prove Eq. (136), we need to prove the following two claims.

**Claim B.2.** For any fixed iteration number $t$, there exists a constant $C_t$, such that

$$\|r^t\| \leq C_t \max(\|x^t\|, \|y\|).$$

**Proof (Claim B.2).** Define $\lambda_t = \max(\|x^{t+1}\|, \|r^t\|, \|y\|)$. Then,

$$\|r^t\| \leq \|y\| + \|A\|_2\|x^t\| + \|b^t\|_\infty \|r^{t-1}\|.$$

Note that $A$ has bounded operator by assumption. Also, the posterior mean $\eta$ is a smooth function with bounded derivative. Therefore, recalling the definition of $b^t$,

$$b^t \equiv \frac{1}{\delta} \sum_{u \in \mathcal{C}} W_{g(i),a} \tilde{r}^{t-1}_{g(i),a} (\eta^{t}_{-1}/u),$$
we have $\|b^t\|_\infty \leq C_{1,t}$ for some constant $C_{1,t}$. Hence, $\|r^t\| \leq C_{2,t} \lambda_{t-1}$. Moreover,
\[
\|x^{t+1}\| = \|\eta_t(x^t + (Q^t \circ A)^* r^t)\| \leq C(\|x^t\| + \|Q^t \circ A\|_2 \|r^t\|) \leq C_{3,t} \max(\|x^t\|, \|r^t\|),
\]
for some constant $C_{3,t}$. In the first inequality, we used the fact that $\eta$ is Lipschitz continuous. Therefore, $\lambda_t \leq C' C_{t-1}$, where $C' = \max(1, C_{2,t}, C_{3,t}, C_{2,t} C_{3,t})$, and
\[
\|r^t\| \leq \lambda_t \leq C' \cdot \cdots C' \cdot \lambda_0 \leq C' \cdot \cdots C' \cdot \max(\|x^1\|, \|y\|),
\]
with $x_i^1 = \mathbb{E}\{X\}$, for $i \in [n]$.

**Claim B.3.** For any fixed iteration number $t$, there exists a constant $C_t$, such that
\[
\|b^t \circ r^{t-1} - \tilde{b}^t \circ \tilde{r}^{t-1}\| \leq C_t(1 + \|y\|) \max(\|x^{t-1} - \tilde{x}^{t-1}\|, \|r^{t-1} - \tilde{r}^{t-1}\|).
\]

**Proof (Claim B.3).** Using triangle inequality, we have
\[
\|b^t \circ r^{t-1} - \tilde{b}^t \circ \tilde{r}^{t-1}\| \leq \|(b^t - \tilde{b}^t) \circ r^{t-1}\| + \|(b^t - \tilde{b}^t) \circ r^{t-1}\|.
\]

Since $\eta'$ is Lipschitz continuous, we have
\[
\|b^t - \tilde{b}^t\| \leq C_{1,t}(\|x^{t-1} - \tilde{x}^{t-1}\| + \|r^{t-1} - \tilde{r}^{t-1}\|),
\]
for some constant $C_{1,t}$. Also, as discussed in the proof of Claim B.2, the Onsager terms $b^t$ are uniformly bounded. Applying these bounds to the right hand side of Eq. (137), we obtain
\[
\|b^t \circ r^{t-1} - \tilde{b}^t \circ \tilde{r}^{t-1}\| \leq C_{1,t}(\|x^{t-1} - \tilde{x}^{t-1}\| + \|r^{t-1} - \tilde{r}^{t-1}\|) \|r^{t-1}\| + C_{2,t} \|r^{t-1} - \tilde{r}^{t-1}\|
\]
\[
\leq C_t (1 + \|y\|) \max(\|x^{t-1} - \tilde{x}^{t-1}\|, \|r^{t-1} - \tilde{r}^{t-1}\|),
\]
for some constants $C_{1,t}, C_{2,t}, C_t$. The last inequality here follows from the bound given in Claim B.2.

Now, we are ready to prove Eq. (136). We write
\[
\|x^t - \tilde{x}^t\| = \|\eta_{t-1}(x^{t-1} + (Q^{t-1} \circ A)^* r^{t-1}) - \eta_{t-1}(\tilde{x}^{t-1} + (Q^{t-1} \circ A)^* \tilde{r}^{t-1})\|
\]
\[
\leq C(\|x^{t-1} - \tilde{x}^{t-1}\| + \|Q^{t-1} \circ A\|_2 \|r^{t-1} - \tilde{r}^{t-1}\|)
\]
\[
\leq C_{1,t} \max(\|x^{t-1} - \tilde{x}^{t-1}\|, \|r^{t-1} - \tilde{r}^{t-1}\|, \|y - \tilde{y}\|) = C_{1,t} \xi_{t-1},
\]
for some constant $C_{1,t}$. Furthermore,
\[
\|r^t - \tilde{r}^t\| \leq \|y - \tilde{y}\| + \|A\|_2 \|x^t - \tilde{x}^t\| + \|b^t \circ r^{t-1} - \tilde{b}^t \circ \tilde{r}^{t-1}\|
\]
\[
\leq \|y - \tilde{y}\| + \|A\|_2 C_{1,t} C_{t} C_{t} (1 + \|y\|) \max(\|x^{t-1} - \tilde{x}^{t-1}\|, \|r^{t-1} - \tilde{r}^{t-1}\|)
\]
\[
\leq C_{2,t} (1 + \|y\|) \xi_{t-1},
\]
for some constant $C_{2,t}$ and using Eq. (138) and Claim B.3 in deriving the second inequality. Combining Eqs. (138) and (139), we obtain
\[
\xi_t \leq \max(1, C_{1,t}, C_{2,t}) (1 + \|y\|) \xi_{t-1}.
\]
C Proof of Lemma 7.12

We prove the first claim, Eq. [90]. The second one follows by a similar argument. The proof uses induction on \( t \). It is a simple exercise to show that the induction basis \( (t = 1) \) holds (the calculation follows the same lines as the induction step). Assuming the claim for \( t \), we write, for \( i \in \{0, 1, \ldots, L - 1\} \)

\[
|\psi_i(t + 1) - \psi(\rho i; t + 1)| = \left| \text{mmse} \left( \sum_{b \in \mathcal{R}_0} W_{b-i} [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi_j(t)]^{-1} \right) \right.
\]

\[
- \text{mmse} \left( \int_{-1}^{t+1} \mathcal{W}(z - \rho i) [\sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} \mathcal{W}(z - y) \psi(y; t) dy]^{-1} dz \right) \left| \leq \left| \text{mmse} \left( \sum_{b \in \mathcal{R}_0} W_{b-i} [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi_j(t)]^{-1} \right) \right.
\]

\[
- \text{mmse} \left( \sum_{b \in \mathcal{R}_0} W_{b-i} [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi(\rho j; t)]^{-1} \right) \left| + \left| \text{mmse} \left( \sum_{b \in \mathcal{R}_0} \rho \mathcal{W}(\rho(b - i)) [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} \rho \mathcal{W}(\rho(b - j)) \psi(\rho j; t)]^{-1} \right) \right.
\]

\[
- \text{mmse} \left( \int_{-1}^{t+1} \mathcal{W}(z - \rho i) [\sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} \mathcal{W}(z - y) \psi(y; t) dy]^{-1} dz \right) \right| \right.
\]

Now, we bound the two terms on the right hand side separately. Note that the arguments of \( \text{mmse}(\cdot) \) in the above terms are at most \( 2/\sigma^2 \). Since \( \text{mmse} \) has a continuous derivative, there exists a constant \( C \) such that \( \frac{d}{ds} \text{mmse}(s) \leq C \), for \( s \in [0, 2/\sigma^2] \). Then, considering the first term in the upper bound [140], we have

\[
\left| \text{mmse} \left( \sum_{b \in \mathcal{R}_0} W_{b-i} [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi_j(t)]^{-1} \right) - \text{mmse} \left( \sum_{b \in \mathcal{R}_0} W_{b-i} [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi(\rho j; t)]^{-1} \right) \right| \leq C \left| \sum_{b \in \mathcal{R}_0} W_{b-i} \left( [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi_j(t)]^{-1} - [\sigma^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}} W_{b-j} \psi(\rho j; t)]^{-1} \right) \right|
\]

\[
\leq \frac{C}{\sigma^4} \sum_{b \in \mathcal{R}_0} W_{b-i} \left| \sum_{j=-\infty}^{L-1} W_{b-j} (\psi(\rho j; t) - \psi_j(t)) \right|
\]

\[
\leq \frac{C}{\delta \sigma^4} \sum_{b \in \mathcal{R}_0} W_{b-i} \left| \sum_{j=-\infty}^{L-1} W_{b-j} |\psi(\rho j; t) - \psi_j(t)| \right|
\]

\[
= \frac{C}{\delta \sigma^4} \sum_{j=0}^{L-1} \left( \sum_{b \in \mathcal{R}_0} W_{b-j} W_{b-j} \right) |\psi(\rho j; t) - \psi_j(t)|
\]

\[
\leq \frac{C}{\delta \sigma^4} \sum_{i \in \mathbb{Z}} W_i^2 \left| \sum_{j=0}^{L-1} |\psi(\rho j; t) - \psi_j(t)| \right|
\]

\[
\leq \frac{C'\rho}{\delta \sigma^4} \sum_{j=0}^{L-1} |\psi(\rho j; t) - \psi_j(t)|.
\]
Here we used $\sum_{i \in Z} W_i^2 = \sum_{i \in Z} \rho^2 W(\rho i)^2 \leq C \sum_{|i| \leq \rho^{-1}} \rho^2 \leq C \rho$ (where the first inequality follows from the fact that $W$ is bounded).

To bound the second term in Eq. (140), note that

$$\left| \text{mmse} \left( \sum_{b \in R_0} \rho W(\rho(b - i)) \left[ \sigma^2 + \frac{1}{\delta} \sum_{j \in Z} \rho W(\rho(b - j)) \psi(\rho j; t) \right]^{-1} \right) \right|$$

$$- \text{mmse} \left( \int_{-1}^{\ell+1} W(z - \rho i) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(z - y) \psi(y; t) dy \right]^{-1} dz \right)$$

$$\leq C \left| \sum_{b \in R_0} \rho W(\rho(b - i)) \left[ \sigma^2 + \frac{1}{\delta} \sum_{j \in Z} \rho W(\rho(b - j)) \psi(\rho j; t) \right]^{-1} \right|$$

$$- \int_{-1}^{\ell+1} W(z - \rho i) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(\rho b - y) \psi(y; t) dy \right]^{-1} dz$$

$$\leq C \left| \sum_{b \in R_0} \rho W(\rho(b - i)) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(\rho b - y) \psi(y; t) dy \right]^{-1} dz \right|$$

$$+ C \left| \sum_{b \in R_0} \rho W(\rho(b - i)) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(\rho b - y) \psi(y; t) dy \right]^{-1} dz \right|$$

$$- \int_{-1}^{\ell+1} W(z - \rho i) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(\rho b - y) \psi(y; t) dy \right]^{-1} dz$$

$$\leq \frac{C}{\delta \sigma^4} \sum_{b \in R_0} \rho W(\rho(b - i)) \left| \sum_{j \in Z} \rho F_1(\rho b; \rho j) - \int_{\mathbb{R}} F_1(\rho b; y) dy \right|$$

$$+ C \left| \sum_{b \in R_0} \rho F_2(\rho b) - \int_{-1}^{\ell+1} F_2(z) dz \right|$$

where $F_1(x; y) = W(x - y) \psi(y; t)$ and $F_2(z) = W(z - \rho i) \left[ \sigma^2 + \frac{1}{\delta} \int_{\mathbb{R}} W(z - y) \psi(y; t) dy \right]^{-1}$. Since the functions $W(\cdot)$ and $\psi(\cdot)$ have continuous (and thus bounded) derivative on compact interval $[0, \ell]$, the same is true for $F_1$ and $F_2$. Using the standard convergence of Riemann sums to Riemann integrals, right hand side of Eq. (142) can be bounded by $C_3 \rho / \delta \sigma^4$, for some constant $C_3$. Let $\epsilon_i(t) = |\psi_i(t) - \psi(\rho i; t)|$. Combining Eqs. (141) and (142), we get

$$\epsilon_i(t + 1) \leq \frac{\rho}{\delta \sigma^4} \left( C' L \sum_{j=0}^{L-1} \epsilon_j(t) + C_3 \right). \quad (143)$$

Therefore,

$$\frac{1}{L} \sum_{i=0}^{L-1} \epsilon_i(t + 1) \leq \frac{\ell}{\delta \sigma^4} \left( \frac{C' L}{L} \sum_{j=0}^{L-1} \epsilon_j(t) \right) + \frac{C_3 \rho}{\delta \sigma^4}. \quad (144)$$

The claims follows from the induction hypothesis.
D Proof of Proposition 7.19

By Eq. (94), for any $\varepsilon > 0$, there exists $\phi_0$, such that for $0 \leq \phi \leq \phi_0$,

$$I(\phi^{-1}) \leq \frac{d(p_X) + \varepsilon}{2} \log(\phi^{-1}). \tag{145}$$

Therefore,

$$V(\phi) \leq \frac{\delta \sigma^2}{2 \phi} + \frac{\delta - d(p_X) - \varepsilon}{2} \log \phi. \tag{146}$$

Now let $\varepsilon = (\delta - d(p_X))/2$ and $\sigma_2 = \sqrt{\phi_0/2}$. Hence, for $\sigma \in (0, \sigma_2]$, we get $\phi^* < 2\sigma^2 \leq \phi_0$. Plugging in $\phi^*$ for $\phi$ in the above equation, we get

$$V(\phi^*) \leq \frac{\delta \sigma^2}{2 \phi^*} + \frac{\delta - d(p_X)}{4} \log \phi^*$$

$$< \frac{\delta}{2} + \frac{\delta - d(p_X)}{4} \log(2\sigma^2). \tag{147}$$

E Proof of Claim 7.21

Recall that $\kappa < \Phi_M$ and $\phi(x)$ is nondecreasing. Let

$$0 < \theta = \frac{\Phi_M - \kappa}{\Phi_M - \frac{\kappa}{2}} < 1.$$  

We show that $\phi(\theta \ell - 1) \geq \kappa/2 + \phi^*$. If this is not true, using the nondecreasing property of $\phi(x)$, we obtain

$$\int_{-1}^{\ell-1} |\phi(x) - \phi^*| \, dx = \int_{-1}^{\theta \ell - 1} |\phi(x) - \phi^*| \, dx + \int_{\theta \ell - 1}^{\ell - 1} |\phi(x) - \phi^*| \, dx$$

$$< \frac{\kappa}{2} \theta \ell + \Phi_M (1 - \theta) \ell$$

$$= \kappa \ell,$$  

contradicting our assumption. Therefore, $\phi(x) \geq \kappa/2 + \phi^*$, for $\theta \ell - 1 \leq x \leq \ell - 1$. For given $K$, choose $\ell_0 = K/(1 - \theta)$. Hence, for $\ell > \ell_0$, interval $[\theta \ell - 1, \ell - 1]$ has length at least $K$. The result follows.

F Proof of Proposition 7.22

We first establish some properties of function $\varsigma^2(x)$.

Remark F.1. The function $\varsigma^2(x)$ as defined in Eq. (96), is non increasing in $x$. Also, $\varsigma^2(x) = \sigma^2 + (1/\delta) \text{mmse}(L_0/(2\sigma^2))$, for $x \leq -1$ and $\varsigma^2(x) = \sigma^2$, for $x \geq 1$. For $\delta L_0 > 3$, we have $\sigma^2 \leq \varsigma^2(x) < 2\sigma^2$. 

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Remark F.2. The function $\zeta^2(x)/\sigma^2$ is Lipschitz continuous. More specifically, there exists a constant $C$, such that, $|\zeta^2(\alpha_1) - \zeta^2(\alpha_2)| < C\sigma^2|\alpha_2 - \alpha_1|$, for any two values $\alpha_1, \alpha_2$. Further, if $L_0 \delta > 3$ we can take $C < 1$.

The proof of Remarks F.1 and F.2 are immediate from Eq. (96).

To prove the proposition, we split the integral over the intervals $[-1, -1 + a], [-1 + a, x_0 + a], [x_0 + a, x_2], [x_2, \ell - 1]$, and bound each one separately. Firstly, note that

$$\int_{x_2}^{\ell - 1} \left\{ \frac{\zeta^2(x) - \sigma^2}{\phi_a(x)} - \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \right\} \, dx = 0,$$

(149)

since $\phi_a(x)$ and $\phi(x)$ are identical for $x \geq x_2$.

Secondly, let $\alpha = (x_2 - x_0)/(x_2 - x_0 - a)$, and $\beta = (ax_2)/(x_2 - x_0 - a)$. Then,

$$\int_{x_0 + a}^{x_2} \left\{ \frac{\zeta^2(x) - \sigma^2}{\phi_a(x)} - \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \right\} \, dx$$

$$= \int_{x_0}^{x_2} \frac{\zeta^2(x + \beta) - \sigma^2}{\phi(x)} \, dx - \int_{x_0 + a}^{x_2} \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \, dx$$

$$= \int_{x_0}^{x_2} \left\{ \frac{1}{\alpha} \frac{\zeta^2(x + \beta) - \sigma^2}{\phi(x)} - \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \right\} \, dx + \int_{x_0}^{x_0 + a} \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \, dx$$

$$\leq \frac{1}{\alpha^2} \int_{x_0}^{x_2} \left\{ \frac{1}{\alpha} \frac{\zeta^2(x + \beta) - \sigma^2}{\phi(x)} \right\} \, dx + \left( 1 - \frac{1}{\alpha} \right) \int_{x_0}^{x_2} \frac{\sigma^2}{\phi(x)} \, dx + \int_{x_0}^{x_0 + a} \frac{\sigma^2}{\phi(x)} \, dx$$

(150)

$$\leq \frac{1}{\alpha^2} \int_{x_0}^{x_2} \left( 1 - \frac{1}{\alpha} \right) \zeta^2(x + \beta) \, dx + \frac{1}{\alpha^2} \int_{x_0}^{x_2} \left| \zeta^2(x + \beta) - \zeta^2(x) \right| \, dx + \frac{K}{2} \left( 1 - \frac{1}{\alpha} \right) + a$$

(151)

$$\leq \left( 1 - \frac{1}{\alpha} \right) K + C K^2 \left( 1 - \frac{1}{\alpha} \right) + C K a + \frac{K}{2} \left( 1 - \frac{1}{\alpha} \right) + a$$

where (a) follows from the fact $\sigma^2 \leq \phi(x)$ and Remark F.1 (b) follows from Remark F.2.

Thirdly, recall that $\phi_a(x) = \phi(x - a)$, for $x \in [-1 + a, x_0 + a)$. Therefore,

$$\int_{x_0 + a}^{x_0 + a} \left\{ \frac{\zeta^2(x) - \sigma^2}{\phi_a(x)} - \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \right\} \, dx$$

$$= \int_{-1}^{x_0} \frac{\zeta^2(x + a) - \sigma^2}{\phi(x)} \, dx - \int_{-1 + a}^{x_0 + a} \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \, dx$$

$$= \int_{-1}^{x_0} \frac{\zeta^2(x + a) - \zeta^2(x)}{\phi(x)} \, dx - \int_{x_0}^{x_0 + a} \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \, dx + \int_{-1 + a}^{x_0 + a} \frac{\zeta^2(x) - \sigma^2}{\phi(x)} \, dx$$

$$\leq 0 + 0 + \int_{-1}^{-1 + a} \frac{\sigma^2}{\phi(x)} \, dx$$

$$\leq a,$$

where the first inequality follows from Remark F.1 and the second follows from $\phi(x) \geq \sigma^2$. 

51
Finally, using the facts $\sigma^2 \leq \varsigma^2(x) \leq 2\sigma^2$, and $\sigma^2 \leq \phi(x)$, we have
\[
\int_{-1}^{1+a} \left\{ \frac{\varsigma^2(x) - \sigma^2}{\phi_a(x)} - \frac{\varsigma^2(x) - \sigma^2}{\phi(x)} \right\} dx \leq a. \tag{152}
\]
Combining Eqs. (149), (150), (151), and (152) implies the desired result.

G Proof of Proposition 7.23

Proof. Let $\tilde{E}_W(\phi_a) = \tilde{E}_{W,1}(\phi_a) + \tilde{E}_{W,2}(\phi_a) + \tilde{E}_{W,3}(\phi_a)$, where
\[
\tilde{E}_{W,1}(\phi_a) = \int_{x_0+a}^{x_0+a} \{I(W \ast \phi_a(y)^{-1}) - I(\phi_a(y-1)^{-1})\} dy,
\]
\[
\tilde{E}_{W,2}(\phi_a) = \int_{0}^{x_0+a} \{I(W \ast \phi_a(y)^{-1}) - I(\phi_a(y-1)^{-1})\} dy, \tag{153}
\]
\[
\tilde{E}_{W,3}(\phi_a) = \int_{0}^{x_0+a} \{I(W \ast \phi_a(y)^{-1}) - I(\phi_a(y-1)^{-1})\} dy.
\]

Also let $\tilde{E}_W(\phi) = \tilde{E}_{W,1}(\phi) + \tilde{E}_{W,2,3}(\phi)$, where
\[
\tilde{E}_{W,1}(\phi) = \int_{x_0+a}^{x_0+a} \{I(W \ast \phi(y)^{-1}) - I(\phi(y-1)^{-1})\} dy,
\]
\[
\tilde{E}_{W,2,3}(\phi) = \int_{0}^{x_0+a} \{I(W \ast \phi(y)^{-1}) - I(\phi(y-1)^{-1})\} dy. \tag{154}
\]

The following remark is used several times in the proof.

Remark G.1. For any two values $0 \leq \alpha_1 < \alpha_2$,
\[
I(\alpha_2) - I(\alpha_1) = \int_{\alpha_1}^{\alpha_2} \frac{1}{2} \text{mmse}(z) dz \leq \int_{\alpha_1}^{\alpha_2} \frac{1}{2} \log \left( \frac{\alpha_2}{\alpha_1} \right) \leq \frac{1}{2} \left( \frac{\alpha_2}{\alpha_1} - 1 \right). \tag{155}
\]

- Bounding $\tilde{E}_{W,1}(\phi_a) - \tilde{E}_{W,1}(\phi)$.

Notice that the functions $\phi(x) = \phi_a(x)$, for $x_2 \leq x$. Also $\kappa/2 < \phi_a(x) \leq \phi(x) \leq \Phi_M$, for $x_1 < x < x_2$.

Let $\alpha = (x_2 - x_1)/(x_2 - x_1 - a)$, and $\beta = (ax_2)/(x_2 - x_1 - a)$. Then, $\phi_a(x) = \phi(\alpha x - \beta)$ for
We have
\[ \frac{\Phi(y)}{W(y-z)} \cdot \frac{1}{W(y-z)} \cdot (W(y-z) - W(y-z)) = \Phi(z)^{-1} \]
Hence,
\[ \int_{x_0 + a}^{x_2 + 1} \left\{ \int_{x_0 + a}^{x_2} W(y-z) \phi(z)^{-1} \, dz - \int_{x_0 + a - 1}^{x_2} W(y-z) \phi(z)^{-1} \, dz \right\} \, dy \]
\[ = \frac{\Phi_M}{2} \int_{x_0 + a}^{x_2 + 1} \left\{ \int_{x_0}^{x_2} \left( \frac{1}{\alpha} W(y-z) - W(y-z) \right) \phi(z)^{-1} \, dz + \int_{x_0}^{x_0 + a - 1} W(y-z) \phi(z)^{-1} \, dz + \int_{x_0 - 1}^{x_0 + a - 1} W(y-z) \phi(z)^{-1} \, dz \right\} \, dy \]
\[ \leq \frac{\Phi_M}{2} \int_{x_0 + a}^{x_2 + 1} \left\{ \int_{x_0}^{x_2} \left( \frac{1}{\alpha} W(y-z) - W(y-z) \right) \phi(z)^{-1} \, dz + \int_{x_0}^{x_0 + a - 1} W(y-z) \phi(z)^{-1} \, dz + \int_{x_0 - 1}^{x_0 + a - 1} W(y-z) \phi(z)^{-1} \, dz \right\} \, dy \]
\[ \leq C_1(1 - \frac{1}{\alpha}) + C_2 \frac{\beta}{\alpha} + C_3 a \leq C_4 a. \] (156)

Here \( C_1, C_2, C_3, C_4 \) are some constants that depend only on \( K \) and \( \kappa \). The last step follows from the facts that \( W(\cdot) \) is a bounded Lipschitz function and \( \phi(z)^{-1} \leq 2/\kappa \) for \( z \in [x_1, x_2] \). Also, note that in the first inequality, \( \int (\phi(y-1)^{-1}) - \int (\phi_\alpha(y-1)^{-1}) \leq 0 \), since \( \phi(y-1)^{-1} \leq \phi_\alpha(y-1)^{-1} \), and \( I(\cdot) \) is nondecreasing.

**Bounding \( \tilde{E}_{W,2}(\phi_a) - \tilde{E}_{W,2,3}(\phi) \).**

We have
\[ \tilde{E}_{W,2}(\phi_a) = \int_{x_0 + a - 1}^{x_0 + a} \{I(W * \phi_a(y)^{-1}) - I(\phi_a(y-1)^{-1})\} \, dy \]
\[ + \int_{x_0 + a - 1}^{x_0 + a} \{I(W * \phi_a(y)^{-1}) - I(\phi_a(y-1)^{-1})\} \, dy. \] (157)
We treat each term separately. For the first term,

\[
\int_0^{x_0 + a} \left\{ I(W \ast \phi_a(y)^{-1}) - I(\phi_a(y) - 1)^{-1} \right\} dy
\]

\[
= \int_0^{x_0 + a} \left\{ I \left( \int_{x_0}^{x_0 + a} W(y - z) \phi_a(z)^{-1} \, dz + \int_{x_0 + a - 2}^{x_0 + a} W(y - z) \phi_a(z)^{-1} \, dz \right) - I(\phi_a(y) - 1)^{-1} \right\} dy
\]

\[
= \int_0^{x_0} \left\{ I \left( \int_{x_0}^{x_0 + a} W(y - z) \phi_a(z)^{-1} \, dz + \int_{x_0 + a - 2}^{x_0} W(y - z) \phi_a(z)^{-1} \, dz \right) - I(\phi(y) - 1)^{-1} \right\} dy
\]

\[
= \int_0^{x_0} \left\{ I \left( (x_0 + \beta) \phi_a(z)^{-1} \, dz + \int_{x_0 + a - 2}^{x_0} W(y - z) \phi_a(z)^{-1} \, dz \right) - I(\phi(y) - 1)^{-1} \right\} dy
\]

\[
\leq C_5 a + \int_0^{x_0} \left\{ I \left( (x_0 + \beta) \phi_a(z)^{-1} \, dz + \int_{x_0 + a - 2}^{x_0} W(y - z) \phi_a(z)^{-1} \, dz \right) - I(\phi(y) - 1)^{-1} \right\} dy
\]

\[
= C_5 a + \int_0^{x_0} \left\{ I(W \ast \phi(y)^{-1}) - I(\phi(y) - 1)^{-1} \right\} dy,
\]

where the last inequality is an application of remark [G.1]. More specifically,

\[
I \left( \int_{x_0}^{x_0 + a} W(y + a - \frac{z + \beta}{\alpha}) \phi_a(z)^{-1} \, dz + \int_{x_0 + a - 2}^{x_0} W(y - z) \phi_a(z)^{-1} \, dz \right)
\]

\[
- I \left( \int_{x_0 + a - 2}^{x_0 + a} W(y - z) \phi_a(z)^{-1} \, dz \right)
\]

\[
\leq \frac{\Phi_M}{2} \left( \int_{x_0}^{x_0 + a} W(y + a - \frac{z + \beta}{\alpha}) \phi_a(z)^{-1} \, dz \right) - \int_{x_0}^{x_0 + a - 2} W(y - z) \phi_a(z)^{-1} \, dz
\]

\[
\leq \frac{\Phi_M}{2} \left( \int_{x_0}^{x_0 + a} W(y + a - \frac{z + \beta}{\alpha}) \phi_a(z)^{-1} \, dz \right) + \frac{\Phi_M}{2} \int_{x_0}^{x_0 + a - 2} W(y - z) \phi_a(z)^{-1} \, dz
\]

\[
\leq C_1 (1 - \frac{1}{\alpha}) + C_2 \frac{\beta}{\alpha} + C_3 a \leq C_5 a,
\]

where \( C_1, C_2, C_3, C_5 \) are constants that depend only on \( \kappa \). Here, the penultimate inequality follows from \( \alpha > 1 \), and the last one follows from the fact that \( W(\cdot) \) is a bounded Lipschitz function and that \( \phi(z)^{-1} \leq 2/2, \) for \( z \in [x_1, x_2] \).

To bound the second term on the right hand side of Eq. (158), notice that \( \phi_a(z) = \phi(z - a) \), for \( z \in [-1 + a, x_0 + a] \), whereby

\[
\int_0^{x_0 + a - 1} \left\{ I(W \ast \phi_a(y)^{-1}) - I(\phi_a(y) - 1)^{-1} \right\} dy = \int_0^{x_0 - 1} \left\{ I(W \ast \phi(y)^{-1}) - I(\phi(y) - 1)^{-1} \right\} dy.
\]

(159)
Now, using Eqs. (154), (157) and (159), we obtain

\[
\tilde{E}_{W,2}(\phi_a) - \tilde{E}_{W,2,3}(\phi) \leq C_5 a - \int_{x_0}^{x_0+a} \{l(W \ast \phi(y)^{-1}) - l(\phi(y-1)^{-1})\} dy \\
\leq C_5 a + \int_{x_0}^{x_0+a} \log \left( \frac{\phi(y-1)^{-1}}{W \ast \phi(y)^{-1}} \right) \\
\leq C_5 a + a \log \left( \frac{\Phi_M}{\kappa} \right) = C_6 a,
\]

(160)

where \( C_6 \) is a constant that depends only on \( \kappa \).

- **Bounding \( \tilde{E}_{W,3}(\phi_a) \).**

Notice that \( \phi_a(y) \geq \sigma^2 \). Therefore, \( l(W \ast \phi_a(y)^{-1}) \leq l(\sigma^{-2}) \), since \( l(\cdot) \) is nondecreasing. Recall that \( \phi_a(y) = \phi^* < 2\sigma^2 \), for \( y \in [-1, -1 + a] \). Consequently,

\[
\tilde{E}_{W,3}(\phi_a) \leq \int_{0}^{a} \{l(\sigma^{-2}) - l(\phi^{*-1})\} dy \leq \frac{a}{2} \log \left( \frac{\phi^*}{\sigma^2} \right) < \frac{a}{2} \log 2,
\]

(161)

where the first inequality follows from Remark G.1.

Finally, we are in position to prove the proposition. Using Eqs. (156), (160) and (161), we get

\[
\tilde{E}_{W}(\phi_a) - \tilde{E}_{W}(\phi) \leq C_4 a + C_6 a + \frac{a}{2} \log 2 = C(\kappa, K) a.
\]

(162)

**H Proof of Proposition 7.24**

We have

\[
\int_{-1}^{\ell-1} \{V(\phi_a(x)) - V(\phi(x))\} dx = \int_{-1}^{\ell-1} \{V(\phi_a(x)) - V(\phi(x))\} dx \\
+ \left( \int_{x_0+a}^{x_2} V(\phi_a(x)) dx - \int_{x_0}^{x_2} V(\phi(x)) dx \right) \\
+ \left( \int_{-1+a}^{-1} V(\phi_a(x)) dx - \int_{-1}^{-1} V(\phi(x)) dx \right) \\
+ \int_{-1}^{-1+a} V(\phi_a(x)) dx.
\]

(163)

Notice that the first and the third terms on the right hand side are zero. Also,

\[
\int_{x_0+a}^{x_2} V(\phi_a(x)) dx - \int_{x_0}^{x_2} V(\phi(x)) dx = -\frac{a}{x_2 - x_0} \int_{x_0}^{x_2} V(\phi(x)) dx, \\
\int_{-1}^{-1+a} V(\phi_a(x)) dx = aV(\phi^*),
\]

(164)

where the second equation holds because \( \phi_a(x) = \phi^* \) for \( x \in [-1, -1 + a] \) in view of Eq. (106).
Substituting Eq. (164) in Eq. (163), we get
\[
\int_{-1}^{\ell-1} \{ V(\phi_a(x)) - V(\phi(x)) \} dx = \frac{a}{x_2 - x_0} \int_{x_0}^{x_2} \{ V(\phi^*) - V(\phi(x)) \} dx.
\] (165)

Now we upper bound the right hand side of Eq. (165).

By Proposition 7.19, we have
\[
V(\phi^*) \leq \frac{\delta}{2} + \frac{\delta - d(p_0)}{4} \log(2\sigma^2),
\] (166)
for \( \sigma \in (0, \sigma_2] \). Also, since \( \phi(x) > \kappa/2 \) for \( x \in [x_0, x_2] \), we have \( V(\phi(x)) \geq (\delta/2) \log \phi > (\delta/2) \log(\kappa/2) \). Therefore,
\[
\frac{1}{2} \int_{-1}^{\ell-1} \{ V(\phi_a(x)) - V(\phi(x)) \} dx = \frac{a}{2(x_2 - x_0)} \int_{x_0}^{x_2} \{ V(\phi^*) - V(\phi(x)) \} dx
\leq \frac{a}{2} \left[ \frac{\delta}{2} + \frac{\delta - d(p_0)}{4} \log(2\sigma^2) - \frac{\delta}{2} \log(\frac{\kappa}{2}) \right].
\] (167)

It is now obvious that by choosing \( \sigma_0 > 0 \) small enough, we can ensure that for values \( \sigma \in (0, \sigma_0] \),
\[
\frac{a}{2} \left[ \frac{\delta}{2} + \frac{\delta - d(p_0)}{4} \log(2\sigma^2) - \frac{\delta}{2} \log(\frac{\kappa}{2}) \right] < -2C(\kappa, K)a.
\] (168)
(Notice that the right hand side of Eq. (168) does not depend on \( \sigma \)).

I Proof of Claim 7.26

Similar to the proof of Claim 7.21, the assumption \( \int_{-1}^{\ell-1} |\phi(x) - \phi^*| dx > C\sigma^2\ell \) implies \( \phi(\theta\ell - 1) > C\sigma^2(1 - \alpha) \), where
\[
0 < \theta = \frac{\Phi_M - C\sigma^2}{\Phi_M - C\sigma^2(1 - \alpha)} < 1.
\]

Choose \( \sigma \) small enough such that \( \phi^* < \phi_1 \). Let \( \kappa = (\phi_1 - \phi^*)(1 - \theta)/2 \). Applying Lemma 7.20, there exists \( \ell_0 \), and \( \sigma_0 \), such that, \( \int_{-1}^{\ell-1} |\phi(x) - \phi^*| dx \leq \kappa\ell \), for \( \ell > \ell_0 \) and \( \sigma \in (0, \sigma_0] \). We claim that \( \phi(\mu\ell - 1) < \phi_1 \), with
\[
\mu = 1 - \frac{\kappa}{\phi_1 - \phi^*} = \frac{1 + \theta}{2}.
\]
Otherwise, by monotonicity of \( \phi(x) \),
\[
(\phi_1 - \phi^*)(1 - \mu)\ell \leq \int_{\mu\ell - 1}^{\ell-1} |\phi(x) - \phi^*| dx < \int_{-1}^{\ell-1} |\phi(x) - \phi^*| dx \leq \kappa\ell.
\] (169)
Plugging in for \( \mu \) yields a contradiction.

Therefore, \( C\sigma^2(1 - \alpha) < \phi(x) < \phi_1 \), for \( x \in [\theta\ell - 1, \mu\ell - 1] \), and \( (\mu - \theta)\ell = (1 - \theta)\ell/2 \). Choosing \( \ell > \max\{\ell_0, 2K/(1 - \theta)\} \) gives the result.
J Proof of Proposition 7.27

To prove Eq. (118), we write

\[ \int_{\ell-1}^{x_2} V_{\text{rob}}(\phi_a(x)) \, dx = - \int_{x_1}^{x_2} \int_{\phi_a(x)} V'(s) \, ds \, dx \]

\[ \leq - \int_{x_1}^{x_2} \int_{\phi_a(x)} V'(s) \frac{\delta}{2s^2} (s - \sigma^2) \, ds \, dx \]

\[ = - \frac{\delta}{2} \int_{x_1}^{x_2} \left\{ \log \left( \frac{\phi(x)}{\phi_a(x)} \right) + \frac{\sigma^2}{\phi(x)} - \frac{\sigma^2}{\phi_a(x)} \right\} \, dx \]

\[ \leq \frac{\delta}{2} K \log(1-a) + K \frac{C}{2(1-\alpha)(1-a)}, \]  

where the second inequality follows from the fact \( C\sigma^2/2 < \phi(x) \), for \( x \in [x_1, x_2] \).

Next, we pass to prove Eq. (119).

\[ \int_{\ell-1}^{x_2} (\varsigma^2(x) - \sigma^2) \left( \frac{1}{\phi_a(x)} - \frac{1}{\phi(x)} \right) \, dx = \int_{x_1}^{x_2} \varsigma^2(x) - \sigma^2 \left( \frac{1}{\phi(x)} - 1 \right) \]

\[ \leq \frac{a}{1-a} \int_{x_1}^{x_2} \frac{\sigma^2}{\phi(x)} \, dx \leq K \frac{a}{C(1-\alpha)(1-a)}, \]

where the first inequality follows from Remark F.1.

Finally, we have

\[ \tilde{E}_{W,\text{rob}}(\phi_a) - \tilde{E}_{W,\text{rob}}(\phi) = \int_0^\ell \{ I(W * \phi_a(y)^{-1}) - I(W * \phi(y)^{-1}) \} \, dy \]

\[ = \int_0^\ell \int_{W * \phi(y)^{-1}} \frac{1}{2} \text{mmse}(s) \, ds \, dy \]

\[ \leq \frac{\overline{D}(p_X) + \varepsilon}{2} \int_0^\ell \int_{W * \phi(y)^{-1}} s^{-1} \, ds \, dy \]

\[ \leq \frac{\overline{D}(p_X) + \varepsilon}{2} \int_0^\ell \log \left( \frac{W * \phi_a(y)^{-1}}{W * \phi(y)^{-1}} \right) \, dy \]

\[ \leq - \frac{\overline{D}(p_X) + \varepsilon}{2} (K + 2) \log(1-a), \]

where the first inequality follows from Eq. (115) and Claim 7.26.

Acknowledgements

Andrea Montanari would like to thank Florent Krzakala, Marc Mézard, François Sausset, Yifan Sun and Lenka Zdeborová for a stimulating exchange about their results. The authors thank the anonymous reviewers for their valuable comments.
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