Non-rigid shape correspondence by matching spectral features and global geodesic structures

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Non-rigid shape correspondence by matching spectral features and global geodesic structures

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Abstract

Finding a correspondence between two non-rigid shapes is one of the cornerstone problems in the field of three-dimensional shape processing. It plays an important role in multiple shape processing and analysis tasks, such as retrieval, morphing and deformation, symmetry and self-similarity detection. In this thesis we propose a framework for marker-less non-rigid shape correspondence, based on matching intrinsic invariant surface descriptors, and the metric structures of the shapes. We formulate the matching task as an optimization problem that can be used with any type of descriptors and metric, and solved using an integer optimization tool. We demonstrate our framework with a specific method for constructing isometry invariant descriptors using the Laplace-Beltrami operator, and with metric given by geodesic distances. We also explore the correspondence ambiguity problem arising when matching intrinsically symmetric shapes using only intrinsic surface properties. We show that when using the proposed invariant surface descriptors, it is possible to construct distinctive sets of surface descriptors for different possible correspondences. When used in a proper minimization problem, those descriptors allow us to explore all possible correspondences between two given shapes. To the best of our knowledge, this is the first attempt to deal with this correspondence ambiguity problem.
Glossary

$X, Y, M$ Two dimensional Riemannian manifolds

$x, \tilde{x}$ Points on manifold $X$

$y, \tilde{y}$ Points on manifold $Y$

$p, q$ Points on manifold $M$

$\varphi$ A correspondence function

$f^X, f^Y, f^M$ Surface descriptors defined on $X, Y$ and $M$, respectively

$d_F$ Distance measure defined in the descriptors space

$d_X, d_Y$ Distance measures defined on $X$ and $Y$, respectively

$Dist$ A correspondence distortion measure

$x_i, x_m$ Points sampled from the manifold $X$

$y_j, y_n$ Points sampled from the manifold $Y$

$p_i, p_j$ Points sampled from the manifold $M$

$P$ A discrete correspondence function

$b$ A vector of pairwise dissimilarities between the descriptors $f^X$ and $f^Y$, in the quadratic optimization problem

$Q$ A matrix of pairwise dissimilarities between the distances measures $d_X$ and $d_Y$, in the quadratic optimization problem
Abstract

A function defined on the manifold $M$.

Gradient and divergence operators, respectively, defined on the manifold $M$.

The Laplace-Beltrami operator defined on $M$.

$i^{th}$ eigenvalue, and corresponding eigenfunction, of the Laplace-Beltrami operator.

Number of the eigenfunctions of the Laplace-Beltrami operator used to construct a descriptor.

Optimal coarse correspondence function.

A $K$-dimensional sign sequence that aligns the eigenfunctions of $\Delta_X$ and $\Delta_Y$.

A matrix of the geodesic distance errors between pairs of corresponding points on $X$ and $Y$.

$j^{th}$ cluster of coarsely matched correspondence points.

Discretized Laplace-Beltrami operator.

A first ring neighborhood of a point on a triangular mesh.

A weight in the discretized Laplace-Beltrami operator.

Eigenfunction of the discretized Laplace-Beltrami operator.

Angles belonging to triangles sharing an edge $(p_i, p_j)$ on a mesh.

Voronoi areas of vertices on a mesh.

A matrix consisting of cotangent weights.

A diagonal matrix with diagonal elements equal to $A_i$. 
Chapter 1

Introduction

Correspondence detection is an important part of many three dimensional shape processing applications, such as shape retrieval, registration - either as a stand alone task, or as an initialization for morphing and deformation algorithms, symmetry and self-similarity detection. Unlike rigid shape correspondence, where the transformation connecting the two shapes can be modeled using six parameters, detecting non-rigid shape correspondence is a far more challenging task. Here, we assume that the two shapes we try to match are approximately isometric in terms of corresponding geodesic distances measured between corresponding surface points. Figure 1.1 presents an example of a human shape at different poses that differ by approximately isometric transformations.

In order to perform the matching we propose a framework based on comparing the intrinsic invariant surface descriptors, and the metric structures of the shapes. We formulate a measure of the correspondence distortion in terms of those surface descriptors and metric structures, and find the optimal correspondence by minimizing the distortion. The minimization can be performed using a standard integer
The above framework is general and can be used with different descriptors and metrics, depending on the specific matching problem. We demonstrate it with invariant surface descriptors based on the Laplace-Beltrami operator [26, 27, 44], and with metric given by geodesic distances. Furthermore, we show that when using the above descriptors, multiple correspondences may exist between two shapes that are intrinsically symmetric. The above correspondence ambiguity problem exists not only for the proposed framework, but for any algorithm that performs the matching using only intrinsic properties of the shapes. Figure 1.2 shows an example of two such correspondences between two symmetric shapes, shown by Voronoi cells associated with the matched points. We show that when using surface descriptors based on the Laplace-Beltrami operator, it is possible to construct distinctive sets of such descriptors for different possible correspondences. Using those sets of descriptors within the proposed framework we find several possible matches of the two shapes.

The two main contributions of this work are listed below.
We propose a new framework for shape matching, that incorporates both pointwise and pairwise surface descriptors. We formulate the matching as a quadratic optimization problem, and solve it using an integer quadratic programming solver.

We analyze the correspondence ambiguity problem arising when matching intrinsically symmetric shapes. We suggest using surface descriptors based on eigendecomposition of the Laplace-Beltrami operator in order to perform the matching, and show how they can be used to find all possible correspondences between the shapes.

The rest of this thesis is arranged as follows: we review the related efforts in the field of non-rigid shape matching in Chapter 2. In Chapter 3 we describe the correspondence detection problem, and show how it can be formulated as a quadratic
optimization problem. In Chapter 4 we describe isometry invariant surface descriptors, and an algorithm for constructing distinct sets of descriptors corresponding to each possible alignment. In Chapter 5 we present the matching results obtained with the proposed algorithm, and discuss its complexity. We summarize the results of the proposed algorithm and describe future research directions in Chapter 6.
Chapter 2

Review of previous work

The previous efforts in the field of non-rigid shape matching can be broadly divided into several main groups that we present in Section 2.1. It is interesting to note that algorithms from different groups share somewhat similar approach to the matching problem, as described in Section 2.2. It consists of using intrinsic properties of the shapes to define a dissimilarity measure over a set of possible matches and detecting the correct match by minimizing that measure. Finally, in Section 2.3 we describe previous applications of the Laplace-Beltrami operator in the field of non-rigid shape processing.

2.1 Non-rigid shape matching

A common approach to non-rigid shape matching consists in finding transformation invariant representations of the shapes, and performing the matching in the representation domain. Zigelman et al. [58] and Elad and Kimmel [14] embedded the shapes into a (flat) Euclidean domain using the Multidimensional Scaling (MDS) method
such that the Euclidian distances between points in the flat space approximate
the geodesic distances on the surface, and performed rigid matching. The drawback
of this method is the inherited embedding error that affects the matching accuracy.
Different embedding domains spanned by eigenfunctions of either affinity matrix or
a graph Laplacian operator defined on triangulated shapes were suggested by Jain et
al. [21], Mateus et al. [32] and Knossow et al. [24]. In order to perform the matching
in the embedding domain they used a non-rigid variant of the ICP (Iterative Closest
Point) algorithm, and a probabilistic framework, respectively. Rustamov [47] sug-
gested using the eigendecomposition of the Laplace-Beltrami operator to construct
an isometric invariant surface representation, though aiming rather to measure simi-
larity between non-rigid shapes, than for correspondence detection. Recently, Lipman
and Funkhouser [28] suggested conformal surface flattening to a complex plane, and
matched the shapes based on their corresponding conformal factors, thereby simpli-
fying the set of non-rigid isometric deformations to the Möbius group. The above
methods produce good correspondence results, but they do not deal with possible
correspondence ambiguity due to intrinsic symmetries.

A different approach to shape matching consists in calculating a set of pointwise
surface descriptors and employ them to perform the matching. Zhang et al. [57]
found feature points by examining extremities of the geodesic distance field defined
on the mesh. Zaharescu et al. [56] proposed an extension of an existing method for
salient feature detection in two-dimensional images. Sun et al. [49] introduced surface
descriptors based on the heat diffusion process on a shape. The correspondence based
on pointwise descriptors only is usually fast to calculate, but is not guaranteed to be
consistent in terms of pairwise point relationships.

Algorithms in the next group perform non-rigid shape comparison by minimizing
2.1. NON-RIGID SHAPE MATCHING

the dissimilarity between their the metric spaces [8, 33, 34]. Memoli and Sapiro [34], and Memoli [33] suggested using the Gromov-Hausdorf distance [17] to compare the metric structures of the shapes. Bronstein et al. [8] formulated the metric comparison as a problem of embedding one shape into another with minimal geodesic distance distortion by introducing the generalized MDS (GMDS). The GMDS algorithm also produces correspondence between two given shapes. Thorstensen and Keriven [51] extended the GMDS framework to surfaces with textures. Being mathematically sound, the above methods require significant computation efforts and good initialization to find the correct matching.

Other related approaches include work by Anguelov et al. [1], who proposed matching shapes by minimizing a probabilistic model based on geodesic distances between all pairs of corresponding points. Later, Tevs et al. suggested randomized geodesic distance preserving matching algorithm [50]. It produces dense correspondence set at cost of high computational complexity. Leordeanu and Hebert [25] employed both local descriptors and global pairwise similarity in their algorithm that tries to successively approximate the correspondence. Along the same line, Hu and Hua [19] used feature points calculated using the Laplace-Beltrami operator and distances between those feature points to perform the matching. Chang and Zwicker [12], and Huang et al. [20] approximated non-rigid transformations that align the two shapes by a finite set of rigid transformations, which were claimed to be simple to calculate. This constraint is too strong for general non-rigid transformations.

Intrinsic symmetry detection can be viewed as an extension of correspondence detection, where one needs to find the best possible mapping of a shape to itself. Raviv et al. [41, 42] formulated the symmetry detection as a problem of embedding a shape into itself, and used GMDS [8] to solve it. Ovsjanikov et al. [37] showed
that intrinsic symmetry detection can be reduced to extrinsic symmetry detection in
the domain of the Global Point Signature (GPS) embedding [47] of the surface. A
method for intrinsic regularity detection was presented by Mitra et al. in [36].

Looking for correspondence between two intrinsically symmetric non-rigid shapes
can be even more challenging, as there exists the correspondence ambiguity problem
which we will discuss in more detail in Chapter 4, and which has been addressed for
the first time in this thesis.

2.2 Non-rigid matching - a common approach

Many of the above algorithms for non-rigid matching share a common denominator,
namely the correspondence detection is performed by minimization of a certain shape
dissimilarity measure. The definition of the dissimilarity is usually based on surfaces’
properties that remain approximately invariant under the possible transformations of
the shapes. Roughly speaking, descriptor based methods [56, 49] measure the dis-
similarity between some local signatures (or, descriptors) associated with the shapes,
while metric based approaches [34, 33, 8, 10, 50] find correspondences by minimizing
the difference between the metric structures of the two shapes. There is also a family
of methods that measure dissimilarity using a mixture of several common quantities,
for example [19, 25, 51].

The approach we suggest in this thesis also belongs to the latter group of hybrid
methods, with dissimilarity measure based on two invariant surface properties: in-
vARIANT surface descriptors and the metric structures of the shapes. By employing
the both properties we attempt to achieve a geometrically consistent matching, while
the surface descriptors replace the need for good initialization required by metric
2.3. APPLICATIONS OF THE LAPLACE-BELTRAMI OPERATOR

structure-based methods. Thus, the proposed approach to the correspondence detection problem can be considered an expansion of the metric structure-based methods [8, 34, 33]. Moreover, the proposed framework can be combined with different surface descriptors and metric structures, and thus it generalizes the idea of hybrid dissimilarity measure presented in [19, 25, 51]

2.3 Applications of the Laplace-Beltrami operator

In order to perform the shape matching we suggest to employ the Laplace-Beltrami operator [2, 26, 27, 44]. It is defined solely by the internal geometry of the shape, and thus is invariant to isometric transformations the shape may undergo. Moreover, eigenfunctions of the Laplace-Beltrami operator constitute an orthogonal basis of energy bounded functions defined on the shape. This fact has been employed to create spectral embeddings of shapes [2], perform parameterization and re-meshing [26], detect similarity between shapes and perform shape matching [19, 44, 47], detect intrinsic symmetries [37], perform segmentation [43], etc. The discrete counterpart of the Laplace-Beltrami operator, the graph Laplacian, is based on mesh connectivity or, by other definition, on the connectivity plus the distances between neighboring points. It treats the shape as graph and is less descriptive of its intrinsic geometry than the Laplace-Beltrami operator. Non the less, it was successfully employed for various tasks, such as shape correspondence and retrieval [21, 24, 32, 57], geometry compression [22], etc.
Chapter 3

Problem formulation

Let us denote by $X$ and $Y$ the two shapes we would like to match. We represent the correspondence between $X$ and $Y$ by a bijective mapping $\varphi : X \rightarrow Y$, such that for each point $x \in X$ its corresponding point is $\varphi(x) \in Y$. We seek for correspondence that preserves both pointwise surface properties, and global pairwise relationships between corresponding points - those that remain approximately invariant under a given class of transformations. In order to measure the pointwise dissimilarity between $X$ and $Y$ we associate with each point $x \in X$ a surface descriptor $f^X(x)$, and, correspondingly, with each point $y \in Y$ - a descriptor $f^Y(y)$. The pointwise dissimilarity measure is defined as a sum of distances between the descriptors of all pairs of corresponding points

$$\sum_{x \in X} d_F \left( f^X(x), f^Y(\varphi(x)) \right). \quad (3.1)$$

The distance measure $d_F$ is defined in the descriptor space. It is chosen according to the type of the descriptors $f^X(x)$, $f^Y(y)$.

In order to compare the pairwise relationships between the corresponding points we adopt the metric space shape representation approach, along the line of [14, 34,
According to it, the shape is represented by a pair \((X, d_X)\), where \(X\) is a smooth compact connected Riemannian manifold, with associated distance measure \(d_X : X \times X \to \mathbb{R}_+ \cup \{0\}\).

Next, following [34, 33, 8, 9], we measure the metric dissimilarity induced by the correspondence given by \(\varphi\). Specifically, given two pairs of matched points \((x, \varphi(x))\) and \((\tilde{x}, \varphi(\tilde{x}))\), we can compare the corresponding distances measured on \(X\) and \(Y\) by \(|d_X(x, \tilde{x}) - d_Y(\varphi(x), \varphi(\tilde{x}))|\). Thereby, the overall pairwise dissimilarity induced by the correspondence set \(\varphi\), can be given by

\[
\sum_{x, \tilde{x} \in X} |d_X(x, \tilde{x}) - d_Y(\varphi(x), \varphi(\tilde{x}))|.
\] (3.2)

The overall distortion induced by the correspondence is a combination of the dissimilarity measures (3.1) and (3.2)

\[
Dist(\varphi) = \sum_{x \in X} d_F \left( f^X(x), f^Y(\varphi(x)) \right) + \lambda \cdot \sum_{x, \tilde{x} \in X} |d_X(x, \tilde{x}) - d_Y(\varphi(x), \varphi(\tilde{x}))|.
\] (3.3)

The scalar parameter \(\lambda\) determines the relative weight of the second term in the overall dissimilarity measure. The optimal correspondence \(\varphi^*\) would minimize \(Dist(\varphi)\). Similar formulations of the matching task were proposed by Berg et al. [4] and Torresani et al. [52] with respect to image registration, and by Hu and Hua [19] with respect to shape matching.

The above formulation is general and can be used with any choice of descriptors and distance measures, and with different minimization techniques. Here, we would like to match shapes that differ by approximately isometric transformations, that include all rigid transformations plus bending and twisting, but without significant stretching or tearing of the surface. Figure 1.1 shows several instances of a human body differing by approximately isometric transformations.
3.1. QUADRATIC OPTIMIZATION PROBLEM FORMULATION

The descriptors and the metric chosen for this application have to be as isometry invariant as possible. In Section 4 we describe such a descriptor based on the eigendecomposition of the Laplace-Beltrami operator. It is related to the Global Point Signature (GPS) proposed by Rustamov [47]. Other descriptors that can be employed in the proposed framework include the Gaussian curvature of the surface, histograms of geodesic distances like those used by [42] and [46], Gaussian curvature of the surface, or the heat kernel based descriptors proposed in [11].

A good choice of a metric for non-rigid transformation that does not tear or stretch the surface is based on geodesic distance. Given the points \(x, \tilde{x}\) on \(X\), the geodesic distance between them, denoted by \(d_X(x, \tilde{x})\), is equal to the length of the shortest path on the surface \(X\) connecting the points \(x\) and \(\tilde{x}\). In presence of topological changes, a better choice for the metric would be the diffusion distance [13], or the commute time distance [40], that are less sensitive to these kind of transformations.

3.1 Quadratic optimization problem formulation

In order to express the correspondence detection as an optimization problem we need to re-define the correspondence between \(X\) and \(Y\). Consider the set of all possible mappings \(P\) over the space of all pairs \((x_i, y_j)\), where \(x_i \in X\), and \(y_j \in Y\), such that

\[
P(x_i, y_j) = \begin{cases} 
1, & x_i \text{ corresponds to } y_j, \\
0, & \text{otherwise}
\end{cases}
\] (3.4)

The correspondence cost presented in Eq. (3.3) can be written equivalently in
terms of \( P \) as

\[
\text{Dist}(P) = \sum_{x \in X, y \in Y} d_F(f^X(x), f^Y(y)) P(x, y) + \\
\lambda \cdot \sum_{x, \tilde{x} \in X, y, \tilde{y} \in Y} \|d_X(x, \tilde{x}) - d_Y(y, \tilde{y})\| P(x, y) P(\tilde{x}, \tilde{y}), \tag{3.5}
\]

where \( P^* \) that minimizes \( \text{Dist}(P) \) represents the optimal correspondence between the shapes \( X \) and \( Y \).

To avoid a trivial solution when minimizing \( \text{Dist}(P) \) we must add constraints on \( P \) when minimizing the distortion (Equation (3.5)). Those constraints are subject to the type of correspondence we look for, and will be discussed in more detail for the discrete problem formulation shown next.

### 3.2 Discrete formulation

Next, we would like to define the optimization problem for the discrete setting, \( i.e. \) when \( X \) and \( Y \) are both discretized and are given either as triangulated meshes, or clouds of points. The mapping \( P \) could be approximated as a binary matrix. The correspondence between some \( x_i \in X \) and \( y_j \in Y \) is given by the \( (i, j) \) entry of \( P \), namely \( P_{ij} = P(x_i, y_j) \in \{0, 1\} \).

The cost (3.5) can then be discretized as follows

\[
\text{Dist}(P) = \sum_{i, j} d_F(f^X(x_i), f^Y(y_j)) P_{ij} + \\
\lambda \cdot \sum_{i, j, m, n} \|d_X(x_i, x_m) - d_Y(y_j, y_n)\| P_{ij} P_{mn}. \tag{3.6}
\]

To avoid a trivial solution we add constraints on \( P \) to Equation (3.6). Those constraints are subject to the type of correspondence we look for. We briefly review
several of them here. For an in-depth discussion on related constraint optimization
the reader is referred to [29].

When a bijective (one-to-one and onto) correspondence is required, the constraints
are given by
\[ \sum_i P_{ij} = 1, \sum_j P_{ij} = 1, \ \forall i, j \]  
(3.7)
In this case, the solution \( P \) is a permutation matrix.

This constraint may be too restrictive. If the shapes have different number of
points, or significantly different triangulations, the optimal correspondence is not
necessarily a bijection. In this case, we may fix the points on \( X \), and for each \( x \in X \)
look for a correspondence \( y \in Y \) that minimizes the distortion
\[ \sum_i P_{ij} = 1, \ \forall i. \]  
(3.8)
Each \( x \in X \) may have one or more corresponding points according to this constraint.
Finally, we re-write the dissimilarity measure (3.6) in matrix form, as a quadratic
function of the correspondence \( P \). We denote each double index \((i, j)\), as in \((x_i, y_j)\)
(or \(ij\) in \(P_{ij}\)) by a single index \( k \). Thus, we convert the correspondence matrix \( P \) into
a correspondence vector \( p \), with entries \( p_k = P_{i_k j_k} \). The pairwise dissimilarity term is
converted into a vector with entries
\[ b_k = d_F (f^X(x_{i_k}), f^Y(y_{j_k})) , \]  
(3.9)
and the metric dissimilarity term - into a matrix with entries
\[ Q_{kl} = \|d_X(x_{i_k}, x_{m_k}) - d_Y(y_{j_l}, y_{n_l})\| . \]  
(3.10)
We readily obtain the following quadratic optimization problem
\[ p^* = \arg \min_p \lambda_p^T Q p + b^T p \quad \text{s.t.} \quad S p = 1, \]  
(3.11)
where the sparse matrix $S$ represents the matrix form of the chosen constraints (Equations 3.7, 3.8); $\mathbf{1}$ is a vector with all entries equal to 1, of an appropriate size. The problem described by Equation (3.11) is called Integer Quadratic Programming (IQP), as the correspondence vector $p$ is a binary vector.
Chapter 4

Isometry invariant surface descriptors

In this section we describe a method for construction of isometry invariant surface descriptors based on eigendecomposition of the Laplace-Beltrami operator. We start with a brief review of the Laplace-Beltrami operator, and the associated eigendecomposition theory, and then describe the proposed descriptor.

4.1 Laplace-Beltrami operator

The Laplace-Beltrami operator is a generalization of the Laplacian operator from flat domain to compact Riemannian manifolds. Given a manifold $M$, its Laplace-Beltrami operator $\Delta_M$ is given by

$$\Delta_M f = -\text{div}_M (\nabla_M f),$$

(4.1)

for any function $f : M \to \mathbb{R}$. The divergence and the gradient operators, $\text{div}_M$ and $\nabla_M$, respectively, are defined by the intrinsic geometry of the manifold $M$. Explicitly,
the Laplace-Beltrami operator of a function $f : M \to \mathbb{R}$ defined on the manifold $M$ equipped with a Riemannian metric $g$, is given by
\[
\Delta_M f = -\frac{1}{\sqrt{\det g}} \sum_{j,k} \frac{\partial}{\partial x_j} \left( g^{jk} \sqrt{\det g} \frac{\partial f}{\partial x_k} \right).
\] (4.2)

In the above equation, $\det g = \det(g_{ij})$ and the $g^{jk}$ are the elements of $g^{-1}$. For more details see [45]. Hence, since isometric transformations of the manifold $M$ preserve the metric $g$, the operator $\Delta_M$ is invariant to these transformations.

Consider the Laplace-Beltrami operator eigenvalue problem given by
\[
\Delta_M \phi^M = \lambda^M \phi^M.
\] (4.3)

$\phi^M$ is the eigenfunction of $\Delta_M$, corresponding to the eigenvalue $\lambda^M$. The spectrum of the Laplace-Beltrami operator consists of positive eigenvalues (see, for example, [45]). When $M$ is a connected manifold without boundary, then $\Delta_M$ has additional eigenvalue equal to zero, with corresponding constant eigenfunction. We can order the eigenvalues as follows
\[
0 = \lambda_0^M < \lambda_1^M \leq \lambda_2^M \leq \lambda_3^M \leq ... \] (4.4)

The set of corresponding eigenfunctions given by
\[
\{ \phi_1^M, \phi_2^M, \phi_3^M, ... \}
\] (4.5)

forms an orthonormal basis defined on $M$ (see [45]).

## 4.2 Surface descriptors

Like the Laplace-Beltrami operator, the eigenvalues and the eigenfunctions are defined by the intrinsic geometry of the manifold, and thus remain invariant under its
isometric transformations. This fact has been exploited for non-rigid shape recognition [44, 47] and registration [57, 32, 24].

Let us consider a candidate surface descriptor constructed from the values of the eigenfunctions of $\Delta_M$

$$f^M(q) = \{\phi_1^M(q), \phi_2^M(q), ..., \phi_K^M(q)\}, \quad q \in M.$$  \hspace{1cm} (4.6)

$\phi_k^M(q)$ is the value of the $k$-th eigenfunction at point $q \in M$.

Here, we choose the dimension $K$ of the descriptor to be small. As we have already mentioned, the eigenfunction corresponding to the zero eigenvalue is a constant function. As we increase the value of the eigenvalue, the corresponding eigenfunction (or eigenfunctions) vary more rapidly over the manifold. Eigenfunctions corresponding to large eigenvalues are therefore more sensitive to the discretization. On the other hand, too small $K$ would reduce the discriminative power of the descriptor. In our experiments we used $K$ in the range of $5-15$.

Obviously, $f^M(q)$ is defined only by the intrinsic properties of $M$ and is thus suitable for isometry invariant matching. The descriptor $f^M(q)$ can be viewed as an embedding of the point $q$ into a $K$-dimensional Euclidean space spanned by the eigenfunctions $\{\phi_1^M, \phi_2^M, ..., \phi_K^M\}$. Hence, we can measure the dissimilarity between the descriptors in this space using $L_p$-norm (in our experiments we used $p = 2$).

Now, given two isometric shapes $X$ and $Y$, we denote the sets of eigenvalues and eigenfunctions of their Laplace-Beltrami operators by

$$\{\lambda_k^X\}_{k \geq 1}^K, \quad \{\phi_k^X\}_{k \geq 1}^K$$ \hspace{1cm} (4.7)

and

$$\{\lambda_k^Y\}_{k \geq 1}^K, \quad \{\phi_k^Y\}_{k \geq 1}^K.$$ \hspace{1cm} (4.8)
respectively. Despite the isometry invariance of the Laplace-Beltrami operator, the sets \( \{ \phi_X^k \}_{k \geq 1} \) and \( \{ \phi_Y^k \}_{k \geq 1} \) are not necessarily identical. Hence, we must preprocess the candidate descriptor \( f_M(p) \) before we can actually use it for matching. There are several factors that explain this loss of identity.

1. The eigenvalues of the Laplace-Beltrami operator may have multiplicity greater than one, with several eigenfunctions corresponding to each such eigenvalue. Each set of eigenfunctions corresponding to the eigenvalues that have the same values spans a subspace of the \( K \)-dimensional Euclidean space. Thus, in order to measure the distance between two such sets calculated for \( X \) and \( Y \) we need some distance measure between two subspaces, other than an \( L_p \)-norm. Additionally, since we work with shapes represented by discrete triangular meshes, the calculation of the Laplace-Beltrami operator and its eigendecomposition suffers from approximation and numerical errors. This sometimes leads to switching of eigenfunctions corresponding to eigenvalues with similar values.

2. The eigenfunctions of the Laplace-Beltrami operator are defined up to a sign. Therefore, the sets \( \{ \phi_X^k \}_{k \geq 1} \) and \( \{ \phi_Y^k \}_{k \geq 1} \) are connected by some arbitrary sign sequence, that has to be estimated prior to the matching. Figure 4.1 presents an example of this sign ambiguity. It shows two articulations of a human body, colored according to the values of the first four eigenfunctions of their corresponding Laplace-Beltrami operators. In order to obtain the same eigenfunctions for the two shapes, we need to multiply the second eigenfunction of the lower shape by -1.

3. An additional problem of eigenfunction sign ambiguity arises when matching intrinsically symmetric shapes, for example, the two shapes of the human body in
4.2. SURFACE DESCRIPTORS

Figure 4.1: Two articulations of a human shape, colored according to the values of
the first four eigenfunctions of their Laplace-Beltrami operators, from left to right. The two possible sign sequence relating the two groups of the eigenfunctions are
\([+,-,+,+]\) and \([+,+,−,+]\).

Figure 4.1. It was observed by Ovsjanikov [37] that eigenfunctions of the Laplace-
Beltrami operator of an intrinsically symmetric shape \(M\) are symmetric or antisym-
metric functions as well, with respect to the symmetry of \(M\) (Theorem 3.1 in [37]).
Eigenfunctions corresponding to eigenvalues without multiplicity exhibit reflection
symmetries, whereas eigenfunctions corresponding to eigenvalues with multiplicity
greater than one may also exhibit rotation symmetries. That is, for two intrinsically
symmetric points \(p, q \in M\), the eigenfunctions corresponding to eigenvalues with unit
multiplicities are related by

\[
\phi^M_k(p) = \phi^M_k(q) \quad \text{or} \quad \phi^M_k(p) = -\phi^M_k(q). \tag{4.9}
\]

We would like to display the effect of intrinsic symmetry on the matching problem
using the example in Figure 4.1. As we have already mentioned, we need to multiply the eigenfunctions of the lower shape by the sign sequence $[+,-,+,+]$ in order to align the two sets of eigenfunctions. We call the resulting alignment a \textit{primary alignment}, or a \textit{primary correspondence}. But, since the shapes are intrinsically symmetric, there exists another sign sequence that produces matching with the same dissimilarity value (calculated according to Eq. (3.1)). It is $[+,+,-,+]$, and results in what we call a \textit{symmetric correspondence}. That is, the signs corresponding to symmetric eigenfunctions $\phi_1^M$ and $\phi_4^M$ remain unchanged, and the signs corresponding to anti-symmetric eigenfunctions $\phi_2^M$ and $\phi_3^M$ change ("-" to "+" and "+" to "-"). Roughly speaking, the last sign sequence matches each point on the first shape not to its corresponding point on the second shape, but to a point intrinsically symmetric to it. When the shape has more than one intrinsic symmetry, the number of the possible \textit{symmetric correspondences} equals to the number of the symmetries. Moreover, it is impossible to distinguish between the different correspondences, since they result in equal dissimilarity value, according to Eq. (3.1).

Two of the above problems - eigenfunction ordering and sign ambiguity, were previously addressed with respect to the spectral decomposition-based shape matching, but not the problem of sign sequence ambiguity due to intrinsic symmetry. Several authors, amongst them Shapiro and Brady [48], and Jain \textit{et al.} [21], proposed using either exhaustive search or greedy approach for the eigenvalue ordering and sign detection. Umeyama [53] proposed using a combination of the absolute values of the eigenfunctions and an exhaustive search. Mateus \textit{et al.} [31] expressed the connection between the eigenfunctions of the two shapes by an orthogonal matrix. They formulated the matching as a global optimization problem, optimizing over the space
of orthogonal matrices, and solved it using the Expectation Minimization approach. Later, Mateus et al. [32] and Knossow et al. [24] suggested using histograms of eigenfunction values to detect their ordering and signs. On the other hand, Ovsjanikov et al. [37] suggested an exhaustive search-based algorithm for multiple sign-sequence estimation with respect to symmetry detection, thus having to deal with eigenfunctions of the Laplace-Beltrami operator of only one shape, instead of two shapes we deal with in the present work.

In this thesis, we suggest a novel algorithm that estimates several possible sign sequences connecting the two sets of eigenfunctions, thus accounting for sign ambiguity resulting from both eigenfunction sign ambiguity and intrinsic symmetry, and not only the sign sequence corresponding to the primary correspondence. To simplify the estimation, and solve the eigenfunction ordering problem, we discard eigenfunctions corresponding to eigenvalues with multiplicity greater than one. Therefore, we construct the descriptor $f^M(p)$ of the values of $K$ eigenfunctions corresponding to the first $K$ eigenvalues with no multiplicity. In practice, we discard eigenfunctions corresponding to eigenvalues with values closer than some predefined threshold, thus accounting for possible eigenfunction order flipping.

### 4.3 Multiple sign sequence estimation

In order to estimate all sign sequences relating the two sets of eigenfunctions we suggest using a coarse matching based on absolute values of eigenfunctions together with geodesic distances measured on the two shapes.

It follows from Equation (4.9) that the absolute values of the eigenfunction $\phi^M_k$ at
two symmetric points \( p, q \in M \) are equal, for all \( k \)

\[
|\phi^M_k(p)| = |\phi^M_k(q)|. 
\]

Equivalently, the absolute values of the descriptors \( f^M(p) \), \( f^M(q) \) are equal

\[
|f^M(p)| = |f^M(q)|. 
\]

In order to find the coarse matching between \( X \) and \( Y \) we minimize the pointwise dissimilarity cost, similar to (3.1), but using the absolute values of the descriptors \( f^X \), \( f^Y \)

\[
\tilde{\varphi}^* = \min_{\varphi} \sum_{x \in X} \left| |f^X(x)| - |f^Y(\varphi(x))| \right|,
\]

\( \tilde{\varphi}^* \) is the optimal coarse correspondence, and is less accurate than the correct correspondence. In order to simplify the problem, we fix the points on the first shape, \( X \), and search for their corresponding points on \( Y \)

\[
\tilde{\varphi}^*(x) = \arg \min_{\tilde{y} \in Y} \left| |f^X(x)| - |f^Y(\tilde{y})| \right|, \ \forall x \in X
\]

The sign sequence relating the two sets of descriptors can be calculated using the above correspondence set as follows

\[
S_k = \arg \min_{\{+,-\}} \sum_{x \in X} \left| f^X_k(x) - S_k f^Y_k(\tilde{\varphi}^*(x)) \right|, \ 1 \leq k \leq K,
\]

where by \( S_k \) we denote the \( k \)-th entry of the sign sequence. Similar approach to the sign sequence estimation was used by Ovsjanikov et al. [37].

If the shapes are intrinsically symmetric, for each \( x \in X \) there may exist several (symmetrical) points on \( Y \), with descriptors equal up to signs, that minimize the expression

\[
\left| |f^X(x)| - |f^Y(y)| \right|.
\]
Thus, the set $\tilde{\phi}^*$ would include both primary and symmetrical correspondences. In practice, as we work with sampled surfaces, the minimizer of (4.15) is usually unique, and may correspond to either primary or symmetrical alignment. We would like to estimate the sign sequences induced by all these alignments using (4.14). In order to do that we have to cluster the correspondences into groups according to the alignment they correspond to, and perform the sign sequence estimation for each group separately. The clustering procedure we suggest is based on comparing geodesic distances between pairs of corresponding points. Suppose we are given two such pairs, $(x, y)$ and $(\tilde{x}, \tilde{y})$. If $(x, y)$ and $(\tilde{x}, \tilde{y})$ represent the same alignment of $X$ and $Y$, the corresponding geodesic distances $d_X(x, \tilde{x})$ and $d_Y(y, \tilde{y})$ must be similar. Otherwise, the geodesic distance error $|d_X(x, \tilde{x}) - d_Y(y, \tilde{y})|$ could be large. Figure 4.2(a) shows an example of two human shapes with all possible correspondences between the four pairs of points marked on them. Primary and symmetrical correspondences are shown in blue and yellow, respectively. Shortest paths between three pairs of corresponding points are shown with dotted lines. From this example it follows that there indeed exists a significant difference in the geodesic distances measured between two points representing different correspondences (shown by the red dotted lines).

We construct a matrix of the geodesic distance errors of all pairs of corresponding points: for all $(x_m, \tilde{\phi}^*(x_m)), (x_n, \tilde{\phi}^*(x_n))$

$$A_{mn} = |d_X(x_m, x_n) - d_Y(\tilde{\phi}^*(x_m), \tilde{\phi}^*(x_n))|. \quad (4.16)$$

Figure 4.2(b) shows the values of the matrix $A$ calculated for the eight pairs of corresponding points shown in Figure 4.2(a), where the correspondences are ordered according to the alignment they represent - the first four correspondences represent the primary alignment, and the next four - the symmetrical alignment. We use the
matrix $A$ to cluster the correspondences, so that the sum of geodesic distance errors of all pair of corresponding points belonging to the same cluster is maintained low. In order to find the clusters we apply a dimensionality reduction algorithm, namely multidimensional scaling [5], to $A$ to obtain a set of points in Euclidean space, and cluster them using the K-means algorithm [30]. Given the clusters, we detect the corresponding sign sequences using (4.14). We suggest clustering the correspondence into a large number of clusters, greater than the expected number of intrinsic symmetries. Thus, each sign sequence is supported by several clusters, improving the robustness to imprecise coarse correspondences and clustering errors. If the number of intrinsic symmetries of the shapes is known a-priori, we choose a suitable number

Figure 4.2: (a) Mixed set of eight correspondences $(x_m, \tilde{\varphi}^*(x_m))$ between two human shapes. (b) Matrix of geodesic distances errors $A$ (b), with darker squares corresponding to entries with lower errors.
of sign sequences amongst sequences that were induced by the highest number of clusters. Alternatively, if we do not know the number of the intrinsic symmetries, we can choose the sign sequences supported by sufficiently high number of clusters. In our experiments, we clustered a set of 1000 initial correspondences into 50 clusters. Figure 4.3 presents an example of two correspondence clusters obtained for a cat model. Note that, knowing the exact number of intrinsic symmetries, it is also possible to cluster the correspondences into the number of the symmetries plus one clusters. The sign sequences induced by those clusters will reflect the possible alignments of the two shapes.

The sign sequence estimation algorithm is summarized below.

- **Coarse correspondence detection:** for each \( x \in X \) find its corresponding point using
  \[
  y = \arg \min_{\tilde{y} \in Y} \| f^X(x) - f^Y(\tilde{y}) \|.
  \]  
  \( (4.17) \)

- **Clustering:** construct a matrix
  \[
  A_{mn} = |d_X(x_m, x_n) - d_Y(\tilde{\varphi}^*(x_m), \tilde{\varphi}^*(x_n))|, \quad \forall x_m, x_n \in X.
  \]  
  \( (4.18) \)

  Apply to \( A \) MDS and K-means clustering to obtain \( J \) clusters of correspondences. We denote the set of clusters by \( \{C_j\}_{j=1}^J \).

- **Sign sequence estimation:** for each cluster, estimate the sign sequence \( S^j \) it induces by
  \[
  S^j_k = \arg \min_{\{+,-\}} \sum_{(x,y) \in C_j} \| f^X_k(x) - S_k f^Y_k(y) \|, \quad 1 \leq k \leq K.
  \]  
  \( (4.19) \)

Select the sign sequences induced by the highest number of clusters, according to the number of the intrinsic symmetries of the shape.
4.4 Discretization of the Laplace-Beltrami operator

Next, we describe the discretization of the Laplace-Beltrami operator for discrete representation of the manifold $M$ by a triangular mesh. Usually, the discretization is of a form

$$\Delta_M f(p_i) = \sum_{j \in N_1(i)} w_{ij} (f(p_i) - f(p_j)), \ \forall p_i \in M,$$

(4.20)

where $N_1(i)$ is a set of vertices belonging to the first ring neighborhood of the vertex $p_i$. That is, the Laplace-Beltrami operator $\Delta_M$ is now described by a matrix consisting of the weights $\{w_{ij}\}_{i,j}$

$$L = \begin{cases} 
-w_{ij}, & j \in N_1(i) \\
\sum_{k \in N_1(i)} w_{ik}, & j = i \\
0, & \text{else}
\end{cases}, \quad (4.21)$$

Figure 4.3: Correspondence clustering example.
4.4. DISCRETIZATION OF THE LAPLACE-BELTRAMI OPERATOR

Figure 4.4: Illustration of cotangent weight scheme for the discretization of the Laplace-Beltrami operator.

Consequently, the eigenvalues and eigenvectors of the Laplace-Beltrami operator are now given by the eigenvalues and the eigenvectors of the matrix $L$

$$Lv = \lambda v. \quad (4.22)$$

The choice of weights $\{w_{ij}\}_{i,j}$ varies between different discretization schemes. Here we used the cotangent weight scheme proposed by Pinkall and Polthier [39] and Meyer et al. [35]. For other discretizations of the Laplace-Beltrami operator the reader is referred to [55], and for Finite Element Methods - to [44]. According to the cotangent weight scheme, the weights are calculated as

$$w_{ij} = \frac{1}{2A_i}(\cot \alpha_{ij} + \cot \beta_{ij}), \quad j \in N_1(i), \quad (4.23)$$

where $\alpha_{ij}$ and $\beta_{ij}$ are the two angles belonging to the triangle sharing an edge $(p_i, p_j)$, and opposite to it, and $A_i$ is the Voronoi area of the vertex $p_i$, as shown in Figure ?? (see [35] for details).

The matrix $L$ defined by these weights according to (4.21) is not symmetric. To find its eigendecomposition we follow [47], and solve the generalized eigendecomposition problem

$$Wv = \lambda Sv \quad (4.24)$$
The matrix \( W \) consists of the cotangent weights
\[
W = \begin{cases} 
-\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}), & j \in N_1(i) \\
\sum_{k \in N_1(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}), & j = i \\
0 & \text{else}
\end{cases}
\tag{4.25}
\]
and \( S \) is a diagonal matrix with diagonal entries \( S_{ii} = A_i \). Thus, we obtain real spectrum \( \{ \lambda_i \} \) with corresponding orthonormal set of eigenvectors, with respect to the following inner product
\[
\langle v_i, v_j \rangle_S = v_i^T S v_j 
\tag{4.26}
\]
At the limit, when the surface is evenly sampled such that \( S_{ii} = S_{jj}, \forall i \neq j \), Equations (4.22) and (4.24) have identical solutions.
Chapter 5

Complexity analysis and results

The proposed algorithm was implemented in C++ and MATLAB®, and tested on various non-rigid shapes from the TOSCA high resolution database [9].

5.1 Complexity analysis

The algorithm consists of the following stages:

1. Descriptors calculation. This part requires the calculation of the Laplace-Beltrami operator with complexity $O(N)$, where $N$ is the number of vertices of the shape. The high resolution shapes we tested the algorithm on had 27K to 52K vertices. The eigendecomposition of the Laplace-Beltrami operator was efficiently performed using the ARPACK package (available within MATLAB). The worst performance complexity is $O(N^3)$.

2. Sign sequence estimation. First, we sub-sampled the shapes at $N_1 = 1000$ vertices, using the Farthest Point Sampling algorithm [16, 18], and estimated the geodesic distances between them using the Fast Marching method [23]. The
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CHAPTER 5. COMPLEXITY ANALYSIS AND RESULTS

overall complexity of the sub-sampling is therefore $O(N_1 N \log N)$. In order to reduce the calculation time, we could simplify the meshes prior to the sub-sampling [15], or approximate the distances using a fast approximation method that exploits parallel architectures [54]. Since the sub-sampling is not a part of the proposed matching algorithm, we did not include it in the calculation times presented in Table 5.1.

Other parts of the sign sequence estimation stage - coarse correspondence detection, the clustering using MDS and K-means, and the sigh sequence calculation can be performed in $O(JN_1^2)$, where $J$ is the number of clusters.

3. The largest complexity stage of the algorithm is solving the Quadratic Problem.

It has been shown in [38] that the Quadratic Assignment Problem, or Integer Quadratic Problem, is $NP$-complete. To approximate the solution we used the Mixed Integer Quadratic Programming (MIQP) solver, distributed as a part of the Hybrid Toolbox by Bemporad [3]. Its calculation time is exponential in the number of variables, due to branch-and-bound method it employs. Therefore, the number of correspondences we were able to find was relatively small. We sub-sampled 20 points from one shape, and 40 candidate corresponding points form the second shape, thus obtaining total of 20 correspondences.

Table 5.1 presents typical algorithm calculation times. The column named 'Total' presents the total computation time, and the column 'IQP' - computation time of the Integer Quadratic Programming solver. The rightmost column, named 'IQP (%)', presents the computation time of the IQP solver, as a fraction of the total calculation time. Note that the first three results in Table 5.1 correspond to shapes with two intrinsic symmetries, therefore two quadratic problems were solved for each. Future


5.2. CORRESPONDENCE DETECTION

<table>
<thead>
<tr>
<th>Model</th>
<th>#vertices</th>
<th>Total</th>
<th>IQP</th>
<th>IQP (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>human</td>
<td>53K</td>
<td>256.7</td>
<td>222.3</td>
<td>86.6</td>
</tr>
<tr>
<td>horse</td>
<td>19K</td>
<td>241.9</td>
<td>222.5</td>
<td>92.0</td>
</tr>
<tr>
<td>cat</td>
<td>28K</td>
<td>241.3</td>
<td>222.0</td>
<td>92.0</td>
</tr>
<tr>
<td>hand</td>
<td>7K</td>
<td>142.3</td>
<td>110.9</td>
<td>78.0</td>
</tr>
</tbody>
</table>

Table 5.1: Calculation time, in seconds, for different models from the TOSCA database, on a laptop with Intel Core 2 Duo T7500 processor and 2GBytes memory.

attempts to improve the algorithm must include a thoughtful analysis of the quadratic optimization problem, and suggestions for reducing its complexity.

5.2 Correspondence detection

We tested the proposed algorithm on shapes that underwent different types of transformations.

Different sampling and triangulation: the algorithm succeeded to match models with different number of vertices and different triangulations. It follows from the fact that both geodesic distances and eigenfunctions of the Laplace-Beltrami operator are intrinsic properties of the shapes, and remain approximately constant for different shape representations. Figures 5.1(a) and 5.1(b) show the correspondence detected between the shapes having vertex number ratio of 1 : 2 and 1 : 10, respectively.

The correspondence obtained for pairs of isometric shapes from the TOSCA database are shown in Figures 5.2, 5.3, 5.4, 5.5. Figures 5.2, 5.3, 5.4, 5.6 show both primary and symmetrical correspondence results, as all these shapes have exactly one intrinsic symmetry. The results presented in Figure 5.6 were calculated using the
Figure 5.1: Correspondence results obtained with the proposed method for a human body with different number of vertices and triangulations. (a) Original model (53K vertices) vs. a model with 25K vertices. (b) Original model vs. a model with 5K vertices.

commute time distances [40], and are similar to those obtained with geodesic distances (Figure 5.3). Figure 5.5 shows two examples of correspondences obtained for hand shapes.

In order to be able to match shapes that differ by scaling, we normalized the vertices of the two shapes to obtain maximal geodesic distance equal to one, on each shape. A more elegant way consists in using the GPS [47] instead of eigenfunctions to construct the descriptors, as they are invariant to scale. For the scale invariant metric one can use diffusion scalespace distance introduced in [6], instead of the geodesic distance. Figure 5.7 presents the correspondence detected between shapes that differ by both isometrical transformation and scaling.
5.3 Algorithm evaluation on the SHREC’10 benchmark

Additional results of the algorithm performance can be found in [7], presenting the algorithm evaluation on the SHREC’10 robust correspondence benchmark. The benchmark included shapes that underwent various types of transformations: isometric and topological transformations, insertion of micro holes and big holes, scaling, local scaling, additive Gaussian noise, shot noise, and down sampling. Each one of the transformations was presented at five different strengths. The algorithm performance was evaluated based on the quality of the correspondence it provided. That is, for each pair of shapes, denoted here by $X$ and $Y$, the algorithm provided a set of $M < |Y|$ correspondences $C(X,Y) = \{(x_k, y_k)\}_{k=1}^M$. Those correspondence were compared to the groundtruth correspondence set $C_0(X,Y) = \{(x'_k, y_k)\}_{k=1}^{|Y|}$, in the following manner

$$D(C, C_0) = \frac{1}{M} \sum_{k=1}^M d_X(x_k, x'_k), \quad (5.1)$$
where $d_X$ is the geodesic distances measured on the shape $X$.

The proposed method was evaluated with two different discretizations of the Laplace-Beltrami operator: the graph Laplacian, and the cotangent weight scheme discussed in Section 4.4. The results (see Tables 3 and 4 in [7]) indicate that the proposed method is robust to isometric transformations, holes, global scaling, sampling

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Transformation strength</th>
<th>1</th>
<th>$\leq 2$</th>
<th>$\leq 3$</th>
<th>$\leq 4$</th>
<th>$\leq 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMDS</td>
<td></td>
<td>39.92</td>
<td>36.77</td>
<td>35.24</td>
<td>37.40</td>
<td>39.10</td>
</tr>
<tr>
<td>The proposed method, cotangent weight scheme</td>
<td></td>
<td>15.51</td>
<td>18.21</td>
<td>22.99</td>
<td>25.26</td>
<td>28.69</td>
</tr>
<tr>
<td>The proposed method, graph Laplacian</td>
<td></td>
<td>10.61</td>
<td>15.48</td>
<td>19.01</td>
<td>23.22</td>
<td>23.88</td>
</tr>
</tbody>
</table>

Table 5.2: SHREC’10, algorithm performance comparison: average geodesic distances from the groundtruth correspondence.
5.3. ALGORITHM EVALUATION ON THE SHREC’10 BENCHMARK

Figure 5.4: Correspondence results obtained with the proposed method for a horse shape.

Figure 5.5: Two correspondence results obtained with the proposed method for hand shapes.

As expected, since neither the Laplace-Beltrami operator, nor the geodesic distance measure are invariant to changes of topology or local scale, the algorithm performs poorly for those transformations.

To compare the performance of the proposed method with the GMDS algorithm, we summarized the averaged values of the correspondence errors $D(X,Y)$ in Table 5.2. The results show that in both settings the average performance of the proposed method is better than of the GMDS algorithm, for all transformation strengths. When comparing the algorithm performances for each type of transformation separately (see Tables 3, 4 and 5 in [7]), the proposed method combined with cotangent weight
scheme outperforms the GMDS, except for the case of additive noise which affects the discretization of the Laplace-Betrami operator. In the case of graph-Laplacian, the algorithm performance deteriorates when the surface sampling changes.

For a more detailed description of the SHREC’10 correspondence benchmark, the compared algorithms and the comparison results the reader is referred to [7].

5.4 Combined distortion vs. other distortion measures

It order to demonstrate the improvement the combined distortion measure introduces over the distortion measures based solely on the metric structures of the shapes or the pointwise surface descriptors, we conducted the following experiment. We compared the correspondence obtained with the proposed method, which we will denote by $C_0$,
5.4. COMBINED DISTORTION VS. OTHER DISTORTION MEASURES

Figure 5.7: Two correspondence result obtained for shapes at different scales.

with correspondences obtained by

1. Minimizing the linear part of the distortion measure $Dist(P)$ (Equation (3.6)), thus only minimizing the difference between the pointwise surface descriptors;

2. Minimizing the quadratic part of the distortion measure $Dist(P)$ (Equation (3.6)), thus only minimizing the difference between the metric structures of the shapes;

3. Constructing the surface descriptors using the absolute values of the eigenvectors of the Laplace-Beltrami operator, and using those descriptors in the quadratic optimization problem.

Figure 5.8 shows the primary correspondences obtained as described above, for two shapes of a cat. In order to evaluate the correspondence results we calculated
the combined distortion $\text{Dist}(P)$ given by Equation (3.6), for each one of the correspondences (see Figure 5.8). The correspondence obtained by minimizing the linear part of $\text{Dist}(P)$ is very similar to $C_0$, but is less accurate in terms of pairwise point relationships: its distortion value is 10.91, greater than the distortion of $C_0$. The correspondence obtained by minimizing the quadratic part alone is clearly incorrect. We think it is the result of the minimization algorithm getting stuck at a local minima, with high distortion value. The fourth correspondence is the closest to $C_0$ in terms of the distortion $\text{Dist}(P)$. The main disadvantage of using the absolute values of the descriptors is the nonability to find more than one correspondence. Thus, the descriptors calculated using the absolute values of the eigenvectors should be considered when one is interested in a single correspondence, either primary or symmetrical one.
5.4. COMBINED DISTORTION VS. OTHER DISTORTION MEASURES

Figure 5.8: Correspondences obtained by minimizing different distortion measures: (a) Combined distortion measure, denoted by $C_0$; (b) Descriptor based distortion; (c) Metric structure based distortion; (d) Combined distortion with descriptors constructed of absolute values of eigenvectors of $\Delta_M$. 

(a) $\text{Dist}(P) = 8.53$ 
(b) $\text{Dist}(P) = 10.91$ 
(c) $\text{Dist}(P) = 19.81$ 
(d) $\text{Dist}(P) = 8.88$
Chapter 6

Conclusions

In this thesis we addressed the problem of finding correspondence between non-rigid shapes. We presented a general framework for finding correspondences between non-rigid shapes, which can be seen as a generalization of the metric-based matching methods proposed in [8, 34, 33]. We incorporated both pointwise surface properties and pairwise metric structures defined on the two shapes in one dissimilarity measure, and showed that it can be written as a quadratic function of a correspondence between the shapes. We then formulated correspondence detection as a minimization of this dissimilarity measure over the set of all possible correspondences. This correspondence detection scheme is general and does not rely on a particular choice of surface descriptors or metric structures, and thus can be adjusted to solve different types of correspondence problems.

We applied the proposed framework to find correspondence between pairs of approximately isometric shapes. We used geodesic distances and isometry invariant surface descriptors based on eigendecomposition of the Laplace-Beltrami operator to calculate the dissimilarity between the shapes. We showed that the optimization
problem can be solved using an integer programming solver, to find the best match between the shapes.

We also introduced the problem of correspondence ambiguity that occurs when matching intrinsically symmetric shapes: in case when the shapes admit one or more intrinsic symmetries, there exist multiple correspondences that minimize the above dissimilarity measure. We showed that using the proposed descriptors we could solve the correspondence ambiguity problem, and find all possible correspondences between the shapes. We demonstrated the performance of the proposed algorithm on multiple pairs of non-rigid shapes, showing that it is able to find correspondence between shapes that differ not only by approximately isometric transformations, but also scaling and re-meshing.

Various directions for future research include extending the proposed algorithm to handle partial shape matching and matching in presence of topological and local scale changes. The algorithm can also benefit from a more efficient solution of the minimization problem that currently constitutes the bottle-neck of the algorithm’s computational complexity.
References


REFERENCES


The algorithm's performance was tested on a number of isometric surfaces within the TOSCA dataset. Moreover, the surfaces were distinguished from each other by the number of points and the number of facets. It turned out that the algorithm is capable of accurately matching surfaces (up to 22 points) in a very good way, including finding symmetrical matches.

The constraint on the number of points we can achieve leads to a lower accuracy in turn.

Furthermore, we show that it is possible to use the algorithm to find a match between surfaces that are different by a scale transformation.

One of the future directions of research is to find an efficient way to perform minimization, such as working with continuous variables instead of discrete variables, finding approximations, and similar. Additionally, an open problem is to adapt the algorithm to find partial matches or matches between surfaces that have undergone topological transformations or non-uniform size changes (on the surface).
איור 2: דוגמה לשתי התאמות אפšíוניות של שני משטחים איזומטריים על ידי מחסנית איזומטרים בודק סימטריה פנימית החתמות מוצגות על ידי תאי Voronoi של הקודuctor החתמות, אשר התאים בצבע בצבע צבוע בצבע זהה עבר בין כל שני הקודuarios החתמות.

אופטימיזציה נפרדת עבור כל אחד מהחתמות האפשים,فو актуальнות את הלגת החתמות.

המעון של האלגוריתמיםۇ נשעט במשבר שבלם:

- מציאת הפירוק הספקטרלי אפיטוריי לפלס-בלטרמי המונדרידיך על תאי השטחים. פעולות זו ניתן לבצע באופן יעיל באמצעות חבילת האלגוריתמים לחישוב וקטורים וערכים עצמיים של מטריצות דלילות שמותורת ARPACK.

- חישוב מרחקים גיאודזיים ומיצאת קבוצות נפרדות של מאפיינים נקודתיים עבור כל החתמות האפישה, אשר נכנס את הביצויים של מאפיינים נקודתיים ועקרונות עבור כל החתמות האפישה. ניתן לכנס את מספר הפרטים של האלגוריתמים של הקדומים עם מספר הנקודות במשטחים המאוחים, ו-1, אם מספר הקדומים שאותו

- משלב על סיבוכיות החישוב המגולה בטירחו ועוד פרמטר בודק השטחמיה הלגיטימיים למדיות החתמה המסהח בודיק. מכיוון שמדירות בעיינון ברעפועה ידיע, ואת פרמטרים שמקנה שמקסימלים של הקדומים של החיתון הקדומים, אם הוא אקסטראקטלי במשבר השטח של הביצויים. כרבע החיתוב

- במשל זה מהווה את צוואר הבקוב של או החיתון של האלגוריתמים כולם.
של המשטח \( M \) אנו משתמשים באופרטור לפלס-בלטרמי (Laplace-Beltrami operator), אשר מסומן על ידי \( \Delta_M \).

אופרטור לפלס-בלטרי המוגדר על גבי המשטח \( M \) מודר בפיים בלתי על ידי הגיאומטריה פנימית \( M \) של הגל \( \Delta \), אלא עלiei הגלים של הגלרים התחלתיים–מיומדיי. ובכן, האופרטורyny הינו \( \Delta_M \) מושג נייח אם מ땅ה חלש המשטח \( M \) \( \Delta_M \).

אופרטור לפלס-בלטרי המוגדר על פני המשטח \( M \) מוגדר באופן בלעדי על ידי הגיאומטריה פנימית \( M \), ללא תלות בשיכוך של במאורות התחלתי–מיומדיי. שוב, האופרטורyny הינו \( \Delta_M \) מושג נייח אם מנטה חלש המשטח \( M \) \( \Delta_M \).

אופרטור לפלס בלטרי המוגדר על גבי המשטח \( M \) \( \Delta_M \) \( \Delta_M \) \( \Delta_M \).

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איור 1: נקודות מבוות משותפת, הנבדלות זו מזו על ידי תרנספורמציה כזו שאיומטרית.

אם מנתחים את בעיית התאמה של שני משטחים כבעית מינימיזציה בידידות עם פונקציה מטרית ריבועית, הממשת את הפונקציה של פונקציה מטרית וניי הפוקטר ביצייד המשטחים反省(morph) של המשטחים של שני המשטחים be,ัวו במישור הקופרנציאליים בין המשטחים, החולק הריבועי — את השוני בין המבננים המטריים של שני המשטחים (המזוגר על ידי פונקציה המטרית). לעモンולגיאית התאמה והאנשה פיתורית ואיצטדיון הפונקציה המטרית בין התאמה והנישת מזריף太阳能 בין המבנה המטרי של שני המשטחים (הנורמל על ידי פונקציה המטרית). יישוב דומים של פונקציה המטרית. ניסוח דומה של בעיית התאמה הופך לעבר יושר תומכות ד-מיטוריות. וניי הפוקטר, הנישת מזריף בין התאמה ומסתבניים קודגיים.

המעין מתוירו שונים,לב יי יחללהلاحניאס לוינים שלבעית התאמה.

הبعثה אוnant עסוקב ביה התאמה ביניב מיישנים הדיסלוזה והיווה על ידי תרנספורמציה ניצונית ביבמגראות ינדידים(inode) לדשום של נווכות הידידות בין נווכות התאמה על מדם שני המיישנים. דנונם למשתנים הדיסלוזה לעידי תרנספורמציה ניצונית ביבמגראות ינדידים על ילאמות הידידות ובאיאור 1. לע몬 שטקלבלע比亚 התאמה ביבמגראות ינדידים נקודאשיות מובנים מותרו אלאות והתווכזים בשיט שיאיר קביעות יכאר המשטוח מעל תרנספורמציה יאומורית יבמגראות. במאף טיב, הא限りים משלימים ביבמגראות ינדידים כי הידידות את המבנה המטרי של המשטוח. לעמון הלגידה משלימים מקודגיים.
תקציר

מציאת התאמה בין משטחים לא קשיחים היא אחת מבעיות הברזל בתחום עיבוד וניתוח משטחים בתלת-מימד. זו בעיה מעניינת וקשה בפני עצמה, והיא גם אחד מבעיות העיבוד והניתוח המשטחים המרכזיות של במסגרת האזורים המוספים, האזורים הגישיים של המשטחים המוספים, והעיבוד של התאמה בין משטחים קשיחים או רחבים במשטחים נוספים, או התאמה משטחים שונים קשישים.

הבעיה של התאמת בין משטחים קשיחים ננודת רבות בספרות המקצועית, אך התאמת משטחים שאליהם קשישים נותרת עדיין כבעיה פתוחה.

ניתן לשייך את האלגוריתמים הקיים למעט משטחים ואשר מציאת התאמה לא קשיחים למטרות עקרונות נוספים. קבוצת אחת כוללת אלגוריתמים אשר מבצעים את התאמה על ידי מציאת מאפיינים (descriptors) נקודתיים של המשטחים, והמתאמה של המ gratuiteים של המשטחים. קבוצת שנייה כוללת אלגוריתמים אשר מבצעים את התאמה על ידי השוואה של המבנים המטריים של המשטחים. קבוצת الثالثית מבוססת על התמרת המשטחים לצורה קנהונית שאינה משתנה תחת טרנספורמציה איזומטרית, ומציאת התאמה בין הצורות הקנוניות של המשטחים. אלגוריתמים נוספים בסיסים את התאמה לא קשיחים באמצעות ממحما תקופות של התאמה או באמצעות אלגוריתמים נוספים כמו תקופות הריצוף והפחתת התאמה לא קשיחים המשלים את האלגוריתמים הקודימים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים הקודולים ומאפשרים תקופות של התאמה לא קשיחים המשלים את האלגוריתמים לקיים.
 GRATITUDE

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מאמני סpettoראלים ושברונים ניאודידיים
galbilem

תניה על מחקק

לשמי מלי חלקי של הדרישות לקבלת תואר
מוניטור למדעי
בנדסת שימוש

אמנטיה דוברנית

허וג שסנט הแดนוני—מכן טכונולוגיה לישראל
ניחבה
נובמבר 2010
הופה
האמה בין misdemeanים לא קשים בزهرת
מאפיינים ספקראליים ומ):-ים גראורדים
גלוכליים

אנטסיה דוברובינה