The Unruh Effect

The Unruh effect is the following strange phenomenon: an accelerating observer traveling through the Minkowski vacuum will observe a *thermal spectrum of particle excitations.* The goal of this problem is to derive some evidence for this strange result. We will do so in the simplest possible QFT: a real scalar field in 1+1D.

First, we need a bit of formalism that allows us to talk about fields in curved coordinates. In curved coordinates, the notion of distance can depend on the coordinates themselves, and the Lagrangian for our scalar field takes the form of

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

$g_{\mu\nu}$ is called the metric of the spacetime. In flat space, $-g = 1$ and $g^{\mu\nu} = \eta^{\mu\nu}$, and so this reduces to the typical Lagrangian.

For us, we will want a strange set of coordinates that is very useful for describing an accelerating observer. This set of coordinates is called **Rindler coordinates**. Using the variable η for Rindler time, and ζ for Rindler space, we may express our typical space and time, x and t, in terms of the Rindler coordinates as follows:

$$t = \pm\frac{\mathrm{e}^{a\zeta}}{a}\sinh(a\eta),$$
$$x = \pm\frac{\mathrm{e}^{a\zeta}}{a}\cosh(a\eta).$$

The reason for the signs is as follows. Note from the x expression that if we take the + sign, x is always ≥ 0. Thus, to describe all of Minkowski space, we will need to have two coordinate patches, which we will denote with I ($x > 0$) and II ($x < 0$).

(a) Show that in both region I and II, $|x| > |t|$. Thus, we are still not describing the full region of spacetime. However, for this problem, it will turn out that this is not an issue.

(b) Show that an observer with acceleration a (in a locally comoving reference frame) has a world line which is described by ζ constant and $\eta = \tau$, where τ is the proper time of the accelerating observer.

One can check that in Rindler coordinates, the metric is a diagonal matrix with the following components:

$$\frac{1}{\sqrt{-g}} = -g^{\eta\eta} = g^{\zeta\zeta} = \mathrm{e}^{-2a\zeta}.$$

(c) Verify that the solutions to the equation of motion in Rindler coordinates may be written as $\mathrm{e}^{\pm\mathrm{i}E(\zeta\pm\eta)}$.

We will quantize the theory as usual. Letting

$$\pi \equiv \frac{1}{\sqrt{-g}}\frac{\partial\phi}{\partial\eta},$$

we can choose the canonical commutation relations:

$$[\phi(\zeta,\eta),\phi(\zeta',\eta)] = [\pi(\zeta,\eta),\pi(\zeta',\eta)] = 0,$$
$$[\phi(\zeta,\eta),\pi(\zeta',\eta)] = \frac{\mathrm{i}}{\sqrt{-g}}\delta(\zeta-\zeta').$$

Then we define operators $b_{\rm I}(k)$, $b_{\rm II}(k)$ and their Hermitian conjugates, such that

$$\phi(\zeta,\eta) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{4\pi|k|} \left[b_{\rm I}(k)\mathrm{e}^{\mathrm{i}k\zeta - |k|\eta)} + b_{\rm II}(k)\mathrm{e}^{\mathrm{i}k\zeta + \mathrm{i}|k|\eta)} + \text{h.c.} \right]$$

(d) Verify that the $b_{\rm I}$ and $b_{\rm II}$ are indeed the annihilation operators for a particle with Rindler momentum k in regions I/II.

We have shown how to quantize our theory in Rindler coordinates. However, what we really want to do is relate the quantization in Rindler coordinates to our more typical quantization in Minkowski coordinates. Right away, we expect this is a bit subtle, for the following reason: a Rindler mode that is excited in region I somehow corresponds to a purely "$x > 0$" mode, whereas our normal harmonic oscillator modes extended through the full space. Thus, we expect that somehow the region I and II annihilation operators must come together to make the annihilation operator for Minkowski space. In fact, we will not go quite this far. It turns out that all we will care about is to find an operator which annihilates the Minkowski vacuum, and to do that what we need to do is find an operator which can be expressed purely in terms of a positive frequency component. Let's see how this can be done.

(e) Begin by showing that

$$\mathrm{e}^{a(\zeta-\eta)} = \begin{cases} a(x-t) & \text{I} \\ -a(x-t) & \text{II} \end{cases},$$

$$\mathrm{e}^{a(\zeta+\eta)} = \begin{cases} a(x+t) & \text{I} \\ -a(x+t) & \text{II} \end{cases}.$$

(f) Now use the identity that $(-x)^{\mathrm{i}b} = x^{\mathrm{i}b}\mathrm{e}^{-\pi b}$ to argue that the combination of $b_{\rm I}(k) + \mathrm{e}^{-\pi\omega/a} b_{\rm II}(-k)^\dagger$ is a positive frequency mode. From there, conclude that

$$c_{\rm I}(k) = \left[2\sinh\frac{\pi|k|}{a}\right]^{-1/2} \left(\mathrm{e}^{\pi|k|/2a} b_{\rm I}(k) + \mathrm{e}^{-\pi|k|/2a} b_{\rm II}(-k)^\dagger\right)$$

describes a positive frequency annihilation operator on the Minkowski vacuum, with its Hermitian conjugate a creation operator.

(g) Conclude that, if $|0_{\rm M}\rangle$ denotes the Minkowski vacuum,

$$\langle 0_{\rm M}|b_{\rm I}(k)^\dagger b_{\rm I}(k)|0_{\rm M}\rangle \sim \frac{1}{\mathrm{e}^{|k|/T} - 1}$$

where T is the Unruh temperature defined as

$$T = \frac{a}{2\pi}.$$

This shows that the accelerating observer will see a spectrum which appears to be at temperature T.

(h) Use dimensional analysis to convert T to SI units. Then, evaluate T for $a = 9.8$ m/s^2 – is the effect negligible for small accelerations?

The Unruh effect shows that the vacuum in quantum field theory is essentially thermal. It should also make you think about what do we really mean by the vacuum. If we interpret the vacuum as "the nothing state", then the Unruh effect seems very odd indeed, and there seems to be no physical explanation for it. The proper explanation for the vacuum state is thus "the state of lowest energy" – from this angle, the Unruh effect does not seem so strange anymore, since accelerating observes feel "forces" which will make them interpret the state of lowest energy differently.