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SUBSOLUTIONS AND THE SUPERCORE OF COOPERATIVE GAMES*

ALVIN E. ROTH†
University of Illinois, Urbana-Champaign

A generalization of the von Neumann-Morgenstern solution, called a subsolution, is introduced. Subsolutions exist for all games (in a nontrival way for games with a nonempty core), and can be interpreted as "standards of behavior." A unique, distinguished subsolution called the supercore is also identified; it is the intersection of all subsolutions.

1. Solutions. When von Neumann and Morgenstern first defined the solution of a cooperative game, they did so in the context of characteristic function games with side payments. They defined a solution to be a set S of imputations such that: (a) "No y contained in S is dominated by an x contained in S". (b) "Every y not contained in S is dominated by some x contained in S". Condition (a) is called *internal stability*, while condition (b) is called *external stability*.

The above definition does not involve any of the special structure associated with characteristic function games with side payments, and has subsequently been used in precisely the same form to define solutions for more general kinds of games.[1] Here, we will require very little structure of any sort and, following von Neumann and Morgenstern,[2] we will direct our attention to *abstract games,* which we define to be pairs $(X, >)$, where X is an arbitrary set whose members are called the *outcomes* of the game, and $>$ is an arbitrary binary relation defined on X and called *domination*.

Following Gillies [1959], we will define the *dominion* of a point $x \in X$ to be the set $D(x) \equiv \{y \in X \mid x > y\}$. That is, the dominion of an outcome x consists of all those outcomes that are dominated by it. In the same way, we define the *dominion* of a set of outcomes $S \subset X$ to be $D(S) \equiv \bigcup_{x \in S} D(x)$, the set of outcomes dominated by some point in S. Finally, we denote the complement of $D(S)$, the set of outcomes undominated by any outcome in the set S, by $U(S) \equiv X - D(S)$, and denote $U^2(S) \equiv U(U(S))$.

We are now in a position to rewrite the conditions of internal and external stability in terms of the function U, and define a solution to be a set $S \subset X$ such that $S \subset U(S)$ and $S \supset U(S)$. These two conditions are precisely the same as conditions (a) and (b), and they make it apparent that S is a solution if and only if[3] (c) $S = U(S)$.

In a similarly compact way, the *core* of an abstract game is defined to be the set of outcomes undominated by any other outcome, i.e., the set $C = U(X)$.

von Neumann and Morgenstern interpreted a solution as a 'standard of behavior', which once generally accepted by the players of the game, would become self-enforcing. They justified this by arguing that outcomes outside of the solution would

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[1] See, for instance, Aumann [1967].

[2] [1953, pp. 587–603]. For a further discussion of solutions of abstract games, see Richardson [1953a, b].

[3] Cf. von Neumann and Morgenstern [1953, p. 40, (4:A:c)].

be considered 'unsound' and thus be 'overruled', by virtue of the fact that such outcomes are dominated by outcomes in the solution. By the same token, no outcome in the solution overrules any other outcome in the solution, and so no contradictions arise. Thus, once a solution comes to be looked on as 'sound', it creates expectations that reinforce this notion.

In this paper we define a generalization of the solution called a "subsolution", which may also be interpreted as a self-reinforcing standard of behavior, though not precisely in the same sense as that of von Neumann and Morgenstern.

2. Subsolutions. We define a *subsolution* of the abstract game $(X, >)$ to be a set $S \subset X$ such that:

(1) $S \subset U(S)$; and

(2) $S = U^2(S) = U(U(S))$.

Condition (1) needs little explanation; it says simply that the set S is internally stable—no element of S dominates any other element of S.

To understand the second condition, consider an arbitrary set $A \subset X$, and the set $U^2(A) = U(U(A))$. No point x in $U^2(A)$ is dominated by any point in the set $U(A)$. Therefore, if x is dominated by some point y, then y is not contained in $U(A)$, but lies instead in its complement, $D(A)$. This means that y is dominated by the set A; i.e., there is a point in A which dominates y. We may therefore regard the set $U^2(A)$ as the set of points "protected" by A, in the sense that any point which dominates some point in $U^2(A)$ is in turn dominated by some point in A. We say a set $A \subset X$ is *self-protecting* if $A \subset U^2(A)$.

We can now interpret a subsolution S as a standard of behavior, using the kind of reasoning originally proposed by von Neumann and Morgenstern to interpret solutions. We will show that once a subsolution S becomes "generally accepted" by the players of a game, it creates expectations that reinforce the notion that only the outcomes in S can be considered "sound".

Suppose then that a subsolution S becomes generally accepted. Any point in the set $D(S)$ is "overruled" by some point in S, and hence unacceptable. Because S is internally stable, no contradiction is implied.

Since, by condition (2), the set S is self-protecting, the set of points thus overruled includes any point which dominates some point in S. In particular, no point which is not immediately overruled by S—that is, no point in $U(S)$—can exert a destabilizing influence on S by means of domination.

Condition (2), in addition to saying that S is self-protecting ($S \subset U^2(S)$), says also that no point outside of S is protected by S ($S \supset U^2(S)$). In particular, no point in the set $U(S) - S$ is protected by S. This means that if x is a point in $U(S) - S$, then it is dominated by some point y in $U(S)$. (But since x is undominated by any point in S, it must be that y is in $U(S) - S$.) Thus every point in the set $U(S) - S$ is dominated by some other point in the same set, and the entire set thus "overrules" itself leaving only the set S as 'sound'.

Looking at condition (2) as a whole, we see that the set $S = U^2(S) = U(U(S))$ is stable with respect to those points it fails to dominate—the set $U(S)$—in precisely the same way that the core, $C = U(X)$, is stable with respect to the entire set of outcomes X.

It is clear that every solution is also a subsolution, since $S = U(S)$ implies $S = U^2(S)$. If S is a subsolution but not a solution, then it does not dominate every point outside of S, but it does dominate all points which dominate some point in S. Every subsolution S contains the core, since points outside of S are dominated either by S or by $U(S) - S$. Every solution is a maximal subsolution, since a solution is a maximal internally stable set.

Our basic theorem is:

THEOREM 1. *Every abstract game has a subsolution.*

A slightly stronger theorem can actually be stated, which says that for every abstract game a maximal subsolution exists [Roth, 1975].

A warning is in order here. For games in which the core is empty, the empty set satisfies the definition of a subsolution, since $U(\varnothing) = X$, and $U^2(\varnothing) = U(X) = C$. Indeed, it is a simple matter to construct an *abstract* game for which the empty set is the *only* subsolution. For games in characteristic function form, however, we conjecture that a nonempty subsolution must always exist. This conjecture is weaker than the presently unresolved question of whether a solution exists for every game in characteristic function form with side payments having an empty core, since every solution is also a subsolution.

For games in which the core is nonempty, no problems arise. Every subsolution is nonempty for such games, and so Theorem 1 assures the existence of a nonempty subsolution. In particular, for Lucas' famous example [1968a] of a game in characteristic function form with side payments for which no solution exists, a nonempty subsolution can be found.

The proof of Theorem 1 which appears in §4 is in a form suggested by R. J. Aumann. The original proof in Roth [1975] presented Theorem 1 as a corollary of a general lattice theorem.

3. The Supercore[4]. The study of von Neumann-Morgenstern solutions is complicated by the fact that solutions often occur in embarrassing profusion. In general, it is not possible to single out in a natural way a unique "distinguished" solution.[5] In addition, it has not proven possible to characterize the intersection of all solutions, even in the case when this intersection is known to be nonempty—e.g., when the core is nonempty (see Lucas [1967]).

In the context of subsolutions, these difficulties can be at least partially overcome. While uncountably many subsolutions may exist for a given game (indeed, every solution is a subsolution), it is possible to identify a "distinguished" subsolution in every game, which can be constructed from the core in a natural manner. Define the *supercore* of a game to be the intersection of all subsolutions. Then we have

THEOREM 2. *The supercore is a subsolution.*

The supercore is the empty set if and only if the core is empty. The supercore reflects all symmetries of the game (unlike general solutions and subsolutions).

4. Proofs[6].

LEMMA 1. (a) $A \subset B$ *implies* $U(B) \subset U(A)$. (b) $A \subset B$ *implies* $U^2(A) \subset U^2(B)$.

PROOF. $A \subset B$ yields $D(A) \subset D(B)$, and hence $U(A) \supset U(B)$; this proves (a). Part (b) follows by applying (a) to $U(B)$ and $U(A)$.

Next, define the set A_α for all (finite or transfinite) ordinals α as follows: $A_0 = \varnothing$; if A_β has been defined for all $\beta < \alpha$, define $A_\alpha = U^2(\bigcup_{\beta<\alpha} A_\beta)$. We show by induction that the sets A_α are nondecreasing in α.

[4] Sets similar to the supercore have been studied in other settings. See for instance the footnotes on pp. 58 and 60 of Shapley and Shubik [1973], or p. 299 of Berge [1970].

[5] See Lucas [1968b] for an example of a symmetric game with solutions but no symmetric solution.

[6] For a discussion of ordinal numbers and transfinite induction, see Hausdorff [1962, Chapter IV, pp. 65–89].

LEMMA 2. *If $\beta < \alpha$ then $A_\beta \subset A_\alpha$.*

PROOF. $A_0 \subset A_1$. Suppose the lemma has been established for all $\gamma < \alpha$. Then for $\beta < \alpha$, $\bigcup_{\gamma<\beta} A_\gamma \subset A_\beta \subset \bigcup_{\gamma<\alpha} A_\gamma$, and by Lemma 1(b), $A_\beta = U^2(\bigcup_{\gamma<\beta} A_\gamma) \subset U^2(\bigcup_{\gamma<\alpha} A_\gamma) = A_\alpha$.

It follows from Lemma 2 that there must be some γ such that $A_\gamma = A_{\gamma+1}$, because each A_α is a subset of X, so that there can be no more than $2^{|X|}$ distinct A_α. Thus there is a γ such that $A_\gamma = A_{\gamma+1} = U^2(\bigcup_{\beta<\gamma+1} A_\beta) = U^2(A_\gamma)$, where the last equality also follows from Lemma 2. We complete the proof of Theorem 1 by showing that such a set A_γ is in fact a subsolution. To do this, we must show that $A_\gamma \subset U(A_\gamma)$, i.e., that the set A_γ is internally stable. In fact, we show by induction that for all α, $A_\alpha \subset U(A_\alpha)$.

PROOF OF THEOREM 1. $A_0 \subset U(A_0)$. Suppose it has been established, for all $\beta < \alpha$, that $A_\beta \subset U(A_\beta)$. Consider A_β and A_δ for some β, $\delta < \alpha$: If $\beta \leqslant \delta$ then $A_\beta \subset A_\delta \subset U(A_\delta)$. If $\delta \leqslant \beta$ then $A_\delta \subset A_\beta$, which implies (by Lemma 1(a)) that $U(A_\delta) \supset U(A_\beta) \supset A_\beta$. Hence $A_\beta \subset U(A_\delta)$ for all β, $\delta < \alpha$. Thus, $\bigcup_{\beta<\alpha} A_\beta \subset \bigcap_{\beta<\alpha} U(A_\beta) = U(\bigcup_{\beta<\alpha} A_\beta)$. Hence, by Lemma 1(b), for all α we have

$$A_\alpha = U^2\Big(\bigcup_{\beta<\alpha} A_\beta\Big) \subset U^3\Big(\bigcup_{\beta<\alpha} A_\beta\Big) = U\Big(U^2\Big(\bigcup_{\beta<\alpha} A_\beta\Big)\Big) = U(A_\alpha).$$

In particular, $A_\gamma \subset U(A_\gamma)$, so the set A_γ is a subsolution, and Theorem 1 is proved.

PROOF OF THEOREM 2. Let S be any subsolution; we prove by induction that $S \supset A_\alpha$ for each α. Suppose that it has been demonstrated for all $\beta < \alpha$. Then $S \supset \bigcup_{\beta<\alpha} A_\beta$, and thus by Lemma 1(b), $S = U^2(S) \supset U^2(\bigcup_{\beta<\alpha} A_\beta) = A_\alpha$. In particular $S \supset A_\gamma$. So A_γ is a subsolution which is contained in every subsolution, and the theorem is proved.

We will find it convenient to denote by γ^* the first ordinal such that $A_{\gamma^*} = A_{\gamma^*+1}$, and to call A_{γ^*} the *supercore* of the game. (This name is meant to reflect the fact that the set A_1 is the core of the game $(X, >)$, the set A_2 is the core of the game $(X - D(A_1), >)$, and so forth.) Example 5.4 shows that for abstract games, γ^* need not be finite, and can in fact take arbitrary values. However, for games in characteristic function form with side payments, the question remains open whether γ^* can be infinite. The author knows of no examples of this type for which $\gamma^* > 2$.

5. Examples. We define a *microcosm* of the game to be a subset $W \subset X$ such that no element x outside of W dominates any element y in W; i.e., $W \subset U(X - W)$. Any nonempty, internally consistent subset $S \subset W$ with the property $S \subset U(U(S) \cap W)$ will have the property $S \subset U^2(S)$, and will be contained in some subsolution. In particular, if $S = U(S) \cap W$, that is, if S is a solution of the microcosm W, then S is contained in some subsolution. This heuristic has proved useful in analyzing some examples, and may prove to have some theoretical interest as well.

EXAMPLE 5.1. Market game with 1 seller and 2 buyers. Consider the three player game with characteristic function $v(\cdot)$, where $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = v(23) = 0$, while $v(12) = v(13) = v(123) = 1$. This game is studied in detail by Shapley [1959].[7] The set of imputations $X = \{(x_1, x_2, x_3) \mid x_i \geqslant 0,\ i = 1, 2, 3, \sum x_i = 1\}$ is depicted in barycentric coordinates in Figure 1. The core is the point $C = \{(1, 0, 0)\}$.

The set of solutions of this game is the set of monotone curves $z(p) = (p; f(p), g(p))$, $0 \leqslant p \leqslant 1$, where f, $g \geqslant 0$ are nonincreasing, continuous functions such that $p + f(p) + g(p) = 1$. A typical solution is depicted in Figure 1, which also shows the dominion of a typical point $x \in X$. For every $0 \leqslant c \leqslant 1$, the subset $M_c \subset X$

[7] See also von Neumann and Morgenstern [1953, pp. 550–554].

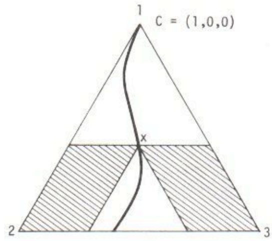

FIGURE 1

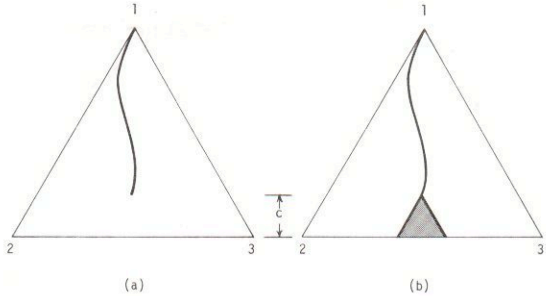

FIGURE 2

defined by $M_c = \{x \in X \mid x_1 \geqslant c\}$ is a microcosm. The set of solutions of the microcosms can be shown to be precisely the set of subsolutions of the game.

That is to say, each element of the family of monotone curves $z_c(p) = (p, f(p), g(p))$, $0 \leqslant c \leqslant p \leqslant 1$, where f and g are as before, is a subsolution. The core itself is thus a subsolution, and is therefore equal to the supercore (i.e., $\gamma^* = 1$).

A typical subsolution S is depicted in Figure 2(a), while Figure 2(b) depicts the set $U(S)$. It is easy to see from the diagram that any x which dominates a point on the curve $S = z_c(p)$ is in turn dominated by a point $z \in S$, and any point $x \in U(S) - S$ is dominated by a point $y \in U(S) - S$.

Note that for games in characteristic function form the dominion of an arbitrary set is open, and thus a subsolution, which is the complement of a dominion, is a closed set. Thus the curve $S' = z_c(p)$, $c < p \leqslant 1$, which is open at its lower end, is not a subsolution, although it is both internally consistent and self-protecting. That is, $S' \subset U(S')$ and $S' \subset U^2(S')$, but $S' \neq U^2(S')$.

EXAMPLE 5.2. Symmetric three person game. Consider the three person game in characteristic function form where $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = v(23) = v_2$, $v(123) = 1$ where $2/3 < v_2 \leqslant 1$. This game, which has no core, is studied in detail by von Neumann and Morgenstern [1953, pp. 550–554].

The set of imputations is $X = \{(x_1, x_2, x_3) \mid x_i \geqslant 0,\ i = 1, 2, 3\ \sum x_i = 1\}$. The region

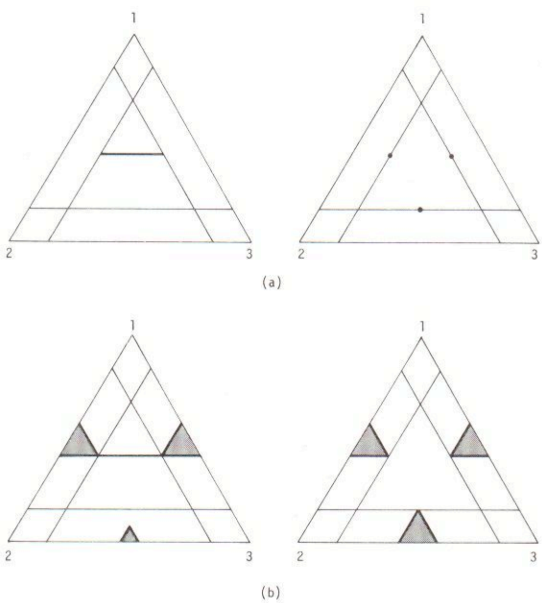

FIGURE 3

W defined by the relations $\sum x_i = 1$ and $x_i + x_j \leqslant v_2$ for all $i, j \in N$, is a microcosm, since any coalition $S \subset N$ which prefers a point $x \notin W$ to a point $y \in W$ is not effective for x. It can be shown that the set of solutions of the microcosm (i.e., of the game $(W, >)$) is identical to the set of nonempty subsolutions of $(W, >)$. In particular, if $v_2 = 1$, then $W = X$, and all of the nonempty subsolutions of the game are in fact solutions.

For the case $v_2 < 1$, Figure 3(a) shows two solutions of the microcosm W (the interior triangle) which are subsolutions of the game. In Figure 3(b) they are pictured with the region they do not dominate.

No point in any of the shaded triangles dominates any point in another shaded triangle, hence each triangle is exactly the same as the game in Example 1. A solution of the game would therefore consist of a solution of the microcosm W together with a solution of each of the shaded triangles, as described in Example 1. A subsolution of the game consists of a solution of W, together with a subsolution of each of the shaded triangles.

The available experimental evidence (see, for instance, Riker [1971]) seems consistent with the proposition that games resolve themselves along the lines suggested by subsolutions.

EXAMPLE 5.3. Game with no solution (Lucas [1968a]). Lucas constructs a ten person game in which the set of imputations X can be partitioned into regions: $X = \{X - B\} \cup \{B - (C \cup E \cup F)\} \cup C \cup E \cup F$, where C is the (nonempty) core. The domination relations is such that:

(1) $D(C) \supset \{X - B\} \cup \{B - (C \cup E \cup F)\}$;

(2) $E \cap D(C \cup F) = \varnothing$;

(3) $F \cap D(C \cup E \cup F) = \varnothing$;

(4) $D(E) \supset E$.

This information is sufficient to show that the set $C \cup F$ is a subsolution, since (1) $\{C \cup F\} \subset U(C \cup F) = \{C \cup F \cup E\}$; and (2) $U^2(C \cup F) = U(C \cup F \cup E) = \{C \cup F\}$. In fact, $C \cup F$ is the supercore, so for this game $\gamma^* = 2$.

EXAMPLE 5.4. Game with arbitrary γ^*. Let α be an ordinal number, let $X = \{x_\beta\}_{\beta<\alpha} \cup \{y_\beta\}_{\beta<\alpha}$, and define domination on X by $x_\beta > y_\beta$ for all β and $y_\beta > x_\gamma$ for all $\gamma > \beta$. Then the core—i.e., the set A_1—is $\{x_1\}$, the set $A_2 = U^2(A_1)$ is $\{x_1, x_2\}$, and so on; the supercore is the set $A_\alpha = U^2(\{x_\beta\}_{\beta<\alpha})$, and $\gamma^* = \alpha$.

6. Postscript. The theorems proved in this paper are of sufficient generality to have other applications for game theory. In particular, since abstract games were defined in terms of an arbitrary binary relation on the set of outcomes, we may interpret the theorems presented here in terms of relations other than domination.

For instance Harsanyi [1974] proposes a relation called *indirect dominance* which he uses to strengthen the requirement of internal consistency for stable sets. The theorems in this paper can be reinterpreted in terms of sets of outcomes that are stable with respect to indirect dominance (rather than the usual domination relation).

In a subsequent paper, we hope to explore this line of reasoning as it applies to the bargaining set of Aumann and Maschler [1964].

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