CME 345: MODEL REDUCTION

Proper Orthogonal Decomposition (POD)

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Outline

1. Time-continuous Formulation
2. Method of Snapshots for a Single Parametric Configuration
3. The POD Method in the Frequency Domain
4. Connection with SVD
5. Error Analysis
6. Extension to Multiple Parametric Configurations
7. Applications
Nonlinear High-Dimensional Model

\[
\frac{d}{dt} \mathbf{w}(t) = f(\mathbf{w}(t), t) \\
y(t) = g(\mathbf{w}(t), t) \\
\mathbf{w}(0) = \mathbf{w}_0
\]

- \( \mathbf{w} \in \mathbb{R}^N \): vector of state variables
- \( \mathbf{y} \in \mathbb{R}^q \): vector of output variables (typically \( q \ll N \))
- \( f(\cdot, \cdot) \in \mathbb{R}^N \): together with \( \frac{d}{dt} \mathbf{w}(t) \), defines the high-dimensional system of equations
Consider a fixed initial condition $w_0 \in \mathbb{R}^N$

Associated state trajectory for the time-interval $[0, T]$

$$\mathcal{T}_w = \{w(t)\}_{0 \leq t \leq T}$$

The POD method seeks an orthogonal projector $\Pi_{V,V}$ of fixed rank $k$ that minimizes the integrated projection error

$$\int_0^T \|w(t) - \Pi_{V,V} w(t)\|^2 dt = \int_0^T \|\mathcal{E}_V(t)\|^2 dt = \|\mathcal{E}_V\|^2 = J(\Pi_{V,V})$$
Theorem

Let \( \hat{K} \in \mathbb{R}^{N \times N} \) be the real, symmetric, positive semi-definite matrix defined as

\[
\hat{K} = \int_0^T w(t)w(t)^T dt
\]

Let \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_N \geq 0 \) denote the ordered eigenvalues of \( \hat{K} \) and \( \hat{\phi}_i \in \mathbb{R}^N, \ i = 1, \cdots, N \) their associated eigenvectors

\[
\hat{K}\hat{\phi}_i = \hat{\lambda}_i\hat{\phi}_i, \ i = 1, \cdots, N
\]

Assume that \( \hat{\lambda}_k > \hat{\lambda}_{k+1} \)

The subspace \( \hat{\mathcal{V}} = \text{range}(\hat{\mathcal{V}}) \) minimizing \( J(\Pi_{\mathcal{V}}) \) is the invariant subspace of \( \hat{K} \) associated with the eigenvalues \( \hat{\lambda}_1, \cdots, \hat{\lambda}_k \)
Solving the eigenvalue problem $\hat{K}\hat{\phi}_i = \hat{\lambda}_i\hat{\phi}_i$ is in general computationally intractable because: (1) the dimension $N$ of the matrix $\hat{K}$ is usually large, (2) this matrix is usually dense.

However, the state data is typically available under the form of discrete “snapshot” vectors

$$\{w(t_i)\}_{i=1}^{N_{\text{snap}}}$$

In this case, $\int_0^T w(t)w(t)^T dt$ can be approximated using a quadrature rule as follows

$$K = \sum_{i=1}^{N_{\text{snap}}} \alpha_i w(t_i)w(t_i)^T$$

where $\alpha_i$, $i = 1, \ldots, N_{\text{snap}}$ are the quadrature weights.
Let \( S \in \mathbb{R}^{N \times N_{\text{snap}}} \) denote the snapshot matrix defined as follows:

\[
S = \begin{bmatrix}
\sqrt{\alpha_1}w(t_1) & \ldots & \sqrt{\alpha_{N_{\text{snap}}}}w(t_{N_{\text{snap}}})
\end{bmatrix}
\]

It follows that

\[
K = SS^T
\]

Note that \( K \) is still a large-scale \( N \)-by-\( N \) matrix.
Note also that the non-zero eigenvalues of the matrix
\[ K = SS^T \in \mathbb{R}^{N \times N} \] are the same as those of the matrix
\[ R = S^T S \in \mathbb{R}^{N_{\text{snap}} \times N_{\text{snap}}} \]

Since usually \( N_{\text{snap}} \ll N \), it is more economical to solve instead the symmetric eigenvalue problem
\[
R \psi_i = \lambda_i \psi_i, \quad i = 1, \cdots, N_{\text{snap}}
\]

Note that if \( S \) is ill-conditioned, then \( R \) is worse conditioned
\[
\kappa_2(S) = \sqrt{\kappa_2(S^T S)}
\]
\[
\kappa_2(R) = \kappa_2(S)^2
\]
If \( \text{rank}(R) = r \), then the first \( r \) POD modes \( \phi_i \) are given by

\[
\phi_i = \frac{1}{\sqrt{\lambda_i}} S \psi_i, \quad i = 1, \ldots, r
\]

Let \( \Phi = [\phi_1 \ldots \phi_r] \) and \( \Psi = [\psi_1 \ldots \psi_r] \) with

\[
\Psi^T \Psi = I_r \implies \Phi = S \Psi \Lambda^{-\frac{1}{2}}
\]

where

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 \\
& \ddots \\
0 & & \lambda_r
\end{bmatrix}
\]

\[
R \psi_i = \lambda_i \psi_i, \quad i = 1, \ldots, N_{\text{snap}} \implies \Psi^T R \Psi = \Psi^T S^T S \Psi = \Lambda
\]

Hence

\[
\Phi^T K \Phi = \Lambda^{-\frac{1}{2}} \Psi^T S^T S S^T S \Psi \Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}} \Lambda \Psi^T \Psi \Lambda \Lambda^{-\frac{1}{2}} = \Lambda
\]

Since the columns of \( \Phi \) are the eigenvectors of \( K \) ordered by decreasing eigenvalues, the optimal orthogonal basis of size \( k \leq r \) is

\[
V = [\Phi_k \Phi_{r-k}] [I_k \ 0] = \Phi_k
\]
Parseval's theorem\(^1\) (the Fourier transform is unitary)

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \| V^T w(t) \|_2^2 \, dt = \lim_{T, \Omega \to \infty} \frac{1}{2\pi T} \int_{-\Omega}^{\Omega} \| \mathcal{F} [ V^T w(t) ] \|_2^2 \, d\omega
\]

where \( \mathcal{F}[w(t)] = \mathcal{W}(\omega) \) is the Fourier transform of \( w(t) \)

Consequence

\[
V^T \left( \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w(t)w(t)^T \, dt \right) V
\]

\[
= V^T \left( \lim_{T, \Omega \to \infty} \frac{1}{2\pi T} \int_{-\Omega}^{\Omega} \mathcal{W}(\omega)\mathcal{W}(\omega)^* \, d\omega \right) V
\]

(Proof: see Homework assignment #2)

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\(^1\)Rayleigh’s energy theorem, Plancherel’s theorem
Let $\tilde{K}$ denote the analog to $K$ in the frequency domain

$$\tilde{K} = \int_{-\Omega}^{\Omega} \mathcal{W}(\omega)\mathcal{W}(\omega)^* d\omega \approx \sum_{i=-N_{\text{snap}}^{\text{C}}}^{N_{\text{snap}}^{\text{C}}} \alpha_i \mathcal{W}(\omega_i)\mathcal{W}(\omega_i)^*$$

The corresponding snapshot matrix for $\omega_{-i} = -\omega_i$ is

$$\tilde{S} = \begin{bmatrix} \sqrt{\alpha_0} \mathcal{W}(\omega_0) & \sqrt{2\alpha_1} \text{Re}\left(\mathcal{W}(\omega_1)\right) & \ldots & \sqrt{2\alpha_{N_{\text{snap}}^{\text{C}}}^{\text{C}}} \text{Re}\left(\mathcal{W}(\omega_{N_{\text{snap}}^{\text{C}}}^{\text{C}})\right) \\ \sqrt{2\alpha_1} \text{Im}\left(\mathcal{W}(\omega_1)\right) & \ldots & \sqrt{2\alpha_{N_{\text{snap}}^{\text{C}}}^{\text{C}}} \text{Im}\left(\mathcal{W}(\omega_{N_{\text{snap}}^{\text{C}}}^{\text{C}})\right) \end{bmatrix}$$

It follows that

$$\tilde{K} = \tilde{S}\tilde{S}^T \quad \tilde{R} = \tilde{S}^T\tilde{S} = \tilde{\Psi}\tilde{\Lambda}\tilde{\Psi}^T \quad \tilde{\Phi} = \tilde{S}\tilde{\Psi}\tilde{\Lambda}^{-\frac{1}{2}} \quad \tilde{V} = \begin{bmatrix} \tilde{\Phi}_k & \tilde{\Phi}_{N-r} \end{bmatrix} \begin{bmatrix} 1_k \\ 0 \end{bmatrix} = \tilde{\Phi}_k$$
The POD Method in the Frequency Domain

Case of Linear-Time Invariant Systems

\[ f(w(t), t) = Aw(t) + Bu(t) \]
\[ g(w(t), t) = Cw(t) + Du(t) \]

- Single input case: \( p = 1 \) \( \Rightarrow B \in \mathbb{R}^N \)
- Time trajectory

\[ w(t) = e^{At}w_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \]

- Snapshots in the time-domain for an impulse input \( u(t) = \delta(t) \) and zero initial condition

\[ w(t_i) = e^{At_i}B, \quad t_i \geq 0 \]

- In the frequency domain, the LTI system can be written as

\[ j\omega l \mathcal{W} = A\mathcal{W} + B, \quad \omega_l \geq 0 \]

and the associated snapshots are \( \mathcal{W}(\omega_l) = (j\omega_l l - A)^{-1}B \)
How to sample the frequency domain?

- approximate time trajectory for a zero initial condition

\[ \Pi_{\tilde{V}, \tilde{V}} w(t) = \tilde{V} \tilde{V}^T \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \]

- low-dimensional solution is accurate if the corresponding error is small — that is

\[ \| w(t) - \Pi_{\tilde{V}, \tilde{V}} w(t) \| = \| (I - \tilde{V} \tilde{V}^T) \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \| \]

is small, which depends on the frequency content of \( u(\tau) \)

\[ \implies \] the sampled frequency band should contain the dominant frequencies of \( u(\tau) \)
Given \( A \in \mathbb{R}^{N \times M} \), there exist 2 orthogonal matrices \( U \in \mathbb{R}^{N \times N} \) (\( U^T U = I_N \)) and \( Z \in \mathbb{R}^{M \times M} \) (\( Z^T Z = I_M \)) such that

\[
A = U \Sigma Z^T
\]

where \( \Sigma \in \mathbb{R}^{N \times M} \) has diagonal entries

\[
\Sigma_{ii} = \sigma_i
\]

satisfying

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(N,M)} \geq 0
\]

and zero entries everywhere else

\[
\{\sigma_i\}_{i=1}^{\min(N,M)}
\]

are the singular values of \( A \), and the columns of \( U \) and \( Z \) are the left and right singular vectors of \( A \), respectively

\[
U = [u_1 \cdots u_N], \quad Z = [z_1 \cdots z_M]
\]
The SVD of a matrix provides many useful information about it (rank, range, null space, norm,...)

- \(\{\sigma_i^2\}_{i=1}^{\min(N,M)}\) are the eigenvalues of the symmetric positive semi-definite matrices \(AA^T\) and \(A^TA\)
- \(Az_i = \sigma_i u_i, \ i = 1, \ldots, \min(N, M)\)
- \(\text{rank}(A) = r\), where \(r\) is the index of the smallest non-zero singular value
- If \(U_r = [u_1 \cdots u_r]\) and \(Z_r = [z_1 \cdots z_r]\) denote the singular vectors associated with the non-zero singular values and \(U_{N-r} = [u_{r+1} \cdots u_N]\) and \(Z_{M-r} = [z_{r+1} \cdots z_M]\), then
  \[
  A = \sigma_1 u_1 z_1^T + \cdots + \sigma_r u_r z_r^T = \sum_{i=1}^{r} \sigma_i u_i z_i^T
  \]
  - \(\text{range}(A) = \text{range}(U_r)\) \quad \(\text{range}(A^T) = \text{range}(Z_r)\)
  - \(\text{null}(A) = \text{range}(Z_{M-r})\) \quad \(\text{null}(A^T) = \text{range}(U_{N-r})\)
Given \( A \in \mathbb{R}^{N \times M} \) with \( N \geq M \), which matrix \( X \in \mathbb{R}^{N \times M} \) with \( \text{rank}(X) = k < r = \text{rank}(A) \) minimizes \( \|A - X\|_2 \)?

**Theorem (Schmidt-Eckart-Young-Mirsky)**

\[
\min_{X, \text{rank}(X) = k} \|A - X\|_2 = \sigma_{k+1}(A), \quad \text{if} \ \sigma_k(A) > \sigma_{k+1}(A)
\]

Hence, \( X = \sum_{i=1}^{k} \sigma_i u_i z_i^T \), where \( A = U \Sigma Z^T \), minimizes \( \|A - X\|_2 \)

This minimizer is also the unique solution of the related problem (Eckart-Young theorem)

\[
\min_{X, \text{rank}(X) = k} \|A - X\|_F
\]
Consider a color image in RGB representation made of $M$-by-$N$ pixels (assume here that $M < N$, i.e. landscape image)

- this image can be represented by an $M$-by-$N$-by-3 real matrix $A_1$
- $A_1$ is converted to a $3N$-by-$M$ matrix $A_3$ as follows

\[
A_1 \in \mathbb{R}^{M \times N \times 3} \quad \Rightarrow \quad A_2 \in \mathbb{R}^{M \times (N \times 3)} \quad \Rightarrow \quad A_3 = A_2^T \in \mathbb{R}^{(3N) \times M}
\]

- finally, $A_3$ is approximated using the SVD as follows

\[
A_3 = \sigma_1 u_1 z_1^T + \cdots + \sigma_r u_r z_r^T = \sum_{i=1}^{r} \sigma_i u_i z_i^T
\]
Connection with SVD

Application to Image Compression

Example: \( A_3 \in \mathbb{R}^{1497 \times 285} \)

(a) rank 1  
(b) rank 2  
(c) rank 3  

(d) rank 4  
(e) rank 5  
(f) rank 6
Connection with SVD

Application to Image Compression

(g) rank 10  (h) rank 20  (i) rank 50

(j) rank 75  (k) rank 100  (l) rank 285

⇒ The SVD can be used for data compression
The discretization of the POD by the method of snapshots requires computing the eigenspectrum of $K = SS^T$

$$\Phi^T K \Phi = \Phi^T SS^T \Phi = \Lambda$$

Corresponding to its non-zero eigenvalues.

Link with the SVD of $S$

$$S = U \Sigma Z^T = [U_r \ U_{N-r}] \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} Z^T$$

$$\Phi = U_r$$

$$\Lambda^{1/2} = \Sigma_r$$

Computing the SVD of the snapshot matrix $S$ is usually preferred to computing the eigendecomposition of $R = S^T S$ because, as noted earlier

$$\kappa_2(R) = \kappa_2(S)^2$$
How to choose the size $k$ of the reduced-order basis $V$ obtained using the POD method

- start from the property of the Frobenius norm of $S$

$$\|S\|_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2(S)}$$

- consider the error measured with the Frobenius norm induced by the truncation of the POD basis

$$\|(I_N - VV^T)S\|_F = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2(S)}$$

- the square of the relative error gives an indication of the magnitude of the “missing” information

$$\mathcal{E}_{POD}(k) = \frac{\sum_{i=1}^{k} \sigma_i^2(S)}{\sum_{i=1}^{r} \sigma_i^2(S)} \Rightarrow 1 - \mathcal{E}_{POD}(k) = \frac{\sum_{i=k+1}^{r} \sigma_i^2(S)}{\sum_{i=1}^{r} \sigma_i^2(S)}$$
How to choose the size $k$ of the reduced-order basis $V$ obtained using the POD method (continue)

$$E_{POD}(k) = \frac{\sum_{i=1}^{k} \sigma_i^2(S)}{\sum_{i=1}^{r} \sigma_i^2(S)}$$

- $E_{POD}(k)$ represents the energy of the snapshots captured by the $k$ first POD basis vectors
- $k$ is usually chosen as the minimum integer for which
  $$1 - E_{POD}(k) \leq \epsilon$$
  for a given tolerance $0 < \epsilon < 1$ (for instance $\epsilon = 0.1\%$)
- this criterion originates from turbulence applications
Recall the model reduction error components

\[
\mathcal{E}_{\text{ROM}}(t) = \mathcal{E}_{V^\perp}(t) + \mathcal{E}_V(t) = (I_N - \Pi_{V,V})w(t) + V(V^Tw(t) - q(t))
\]

- denote \(\mathcal{E}_{\text{ROM}}^{\text{snap}} = [\mathcal{E}_{\text{ROM}}(t_1) \cdots \mathcal{E}_{\text{ROM}}(t_{N_{\text{snap}}})]\)
- \(\|[\mathcal{E}_{V^\perp}(t_1) \cdots \mathcal{E}_{V^\perp}(t_{N_{\text{snap}}})]\|_F = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2(S)}\)
- hence

\[
1 - \mathcal{E}_{\text{POD}}(k) = \frac{\|[\mathcal{E}_{V^\perp}(t_1) \cdots \mathcal{E}_{V^\perp}(t_{N_{\text{snap}}})]\|_F^2}{\sum_{i=1}^{r} \sigma_i^2(S)}
\]

and

\[
1 - \mathcal{E}_{\text{POD}}(k) \leq \frac{\|\mathcal{E}_{\text{ROM}}^{\text{snap}}\|_F^2}{\sum_{i=1}^{r} \sigma_i^2(S)}
\]

- note that the energy criterion is valid only for the sampled snapshots
Consider the **parametrized steady** system of equations

\[ f(w; \mu) = 0, \quad \mu \in \mathcal{D} \subset \mathbb{R}^d \]

The goal is to build a reduced-order model for the solution

\[ w(\mu) \approx Vq(\mu), \quad \mu \in \mathcal{D} \]

How do we build a **global reduced-order basis** \( V \) that can capture the solution in the entire parameter domain \( \mathcal{D} \)?
Lagrange basis

\[ V \subset \text{span} \left\{ w(\mu^{(1)}), \ldots, w(\mu^{(s)}) \right\} \Rightarrow N_{\text{snap}} = s \]

Hermite basis

\[ V \subset \text{span} \left\{ w(\mu^{(1)}), \frac{\partial w}{\partial \mu_1}(\mu^{(1)}), \ldots, w(\mu^{(2)}), \ldots, \frac{\partial^d w}{\partial \mu_d}(\mu^{(s)}) \right\} \]

\[ \Rightarrow N_{\text{snap}} = s \times (d + 1) \]

Taylor basis

\[ V \subset \text{span} \left\{ w(\mu^{(1)}), \frac{\partial w}{\partial \mu_1}(\mu^{(1)}), \ldots, \frac{\partial^2 w}{\partial \mu_1^2}(\mu^{(1)}), \ldots, \frac{\partial^d w}{\partial \mu_d^q}(\mu^{(1)}) \right\} \]

\[ \Rightarrow N_{\text{snap}} = 1 + d + \frac{d(d + 1)}{2} + \ldots + \frac{(d + q - 1)!}{(d - 1)!q!} = 1 + \sum_{i=1}^{q} \frac{(d + i - 1)!}{(d - 1)!i!} \]
How do we choose the $s$ samples $\{w(\mu^{(1)}), \ldots, w(\mu^{(s)})\}$?

The samples location will determine the accuracy of the resulting ROM in the entire parameter domain $\mathcal{D} \subset \mathbb{R}^d$.

Possible approaches
- Uniform sampling for moderate dimensional spaces ($d \leq 5$)
- Latin Hypercube sampling for higher dimensional spaces
- Goal-oriented greedy sampling that exploits the accuracy of the ROM
Ideally, for a given ROM, one would like to add additional samples $\mu^{(i)}$ at the locations of the parameter space where the ROM is the most inaccurate:

$$
\mu^{(i)} = \arg\max_{\mu \in \mathcal{D}} ||\mathcal{E}_{\text{ROM}}(\mu)|| = \arg\max_{\mu \in \mathcal{D}} ||w(\mu) - Vq(\mu)||
$$

$q(\mu)$ can be efficiently computed

$w(\mu)$ is however expensive $\Rightarrow$ intractable approach

Idea: use instead a cheap a posteriori error estimator

- Option 1: error bound

$$
||\mathcal{E}_{\text{ROM}}(\mu)|| \leq \Delta(\mu)
$$

- Option 2: error indicator based on the residual norm (when it can be evaluated cheaply)

$$
||r(\mu)|| = ||f(Vq(\mu); \mu)||
$$

The set $\mathcal{D}$ is usually replaced by a large discrete set of candidate parameters

$$
\{\mu_1, \cdots, \mu_c\} \subset \mathcal{D}
$$
A greedy approach

Greedy procedure based on the residual norm as an error indicator

Algorithm

1. Select randomly a first sample $\mu^{(1)}$
2. Solve the HDM $f\left(w(\mu^{(1)}); \mu^{(1)}\right) = 0$
3. Build a corresponding ROB $V$
4. For $i = 2, \ldots, s$
5. Solve

$$\mu^{(i)} = \operatorname*{argmax}_{\mu \in \{\mu_1, \ldots, \mu_c\}} \|r(\mu)\|$$

6. Solve the HDM $f\left(w(\mu^{(i)}); \mu^{(i)}\right) = 0$
7. Build a ROB $V$ based on the samples $\{w(\mu^{(1)}), \ldots, w(\mu^{(i)})\}$
Parameterized HDM:

\[
\frac{d}{dt} w(t; \mu) = f(w(t; \mu), t; \mu)
\]

Lagrange basis

\[
V \subset \text{span} \left\{ w(t_1; \mu^{(1)}), \ldots, w(t_{N_t}; \mu^{(s)}) \right\} \Rightarrow N_{\text{snap}} = s \times N_t
\]

A posteriori error estimators

- Option 1: error bound

\[
\|E_{\text{ROM}}(\mu)\| = \left( \int_0^T \|E_{\text{ROM}}(t; \mu)\|^2 dt \right)^{1/2} \leq \Delta(\mu)
\]

- Option 2: error indicator based on the residual norm (when it can be evaluated cheaply)

\[
\|r(\mu)\| = \left( \int_0^T \|r(t; \mu)\|^2 dt \right)^{1/2} = \sqrt{\int_0^T \left\| \frac{d}{dt} w(t; \mu) - f(Vq(t; \mu), t; \mu) \right\|^2 dt}
\]
Greedy procedure based on the residual norm as an error indicator

Algorithm

1. Select randomly a first sample $\mu^{(1)}$
2. Solve the HDM
   \[ \frac{d}{dt} w(t; \mu^{(1)}) = f\left( w(t; \mu^{(1)}), t; \mu^{(1)} \right) \]
3. Build a ROB $\mathbf{V}$ based on the snapshots
   \[ \{ w(t_1; \mu^{(1)}), \ldots, w(t_{N_t}; \mu^{(1)}) \} \]
4. For $i = 2, \ldots, s$
5. Solve
   \[ \mu^{(i)} = \arg\max_{\mu \in \{\mu_1, \ldots, \mu_c\}} \| r(\mu) \| \]
6. Solve the HDM
   \[ \frac{d}{dt} w(t; \mu^{(i)}) = f\left( w(t; \mu^{(i)}), t; \mu^{(i)} \right) \]
7. Build a ROB $\mathbf{V}$ based on the snapshot
   \[ \{ w(t_1; \mu^{(1)}), \ldots, w(t_{N_t}; \mu^{(i)}) \} \]
Stanford picture example: $\epsilon = 1 - \epsilon_{\text{POD}}$

(m) $\epsilon < 10^{-1} \Rightarrow \text{rank 2}$
(n) $\epsilon < 10^{-2} \Rightarrow \text{rank 47}$
(o) $\epsilon < 10^{-3} \Rightarrow \text{rank 138}$

(p) $\epsilon < 10^{-4} \Rightarrow \text{rank 210}$
(q) $\epsilon < 10^{-5} \Rightarrow \text{rank 249}$
(r) $\epsilon < 10^{-6} \Rightarrow \text{rank 269}$
Applications

Image Compression

\[ 1 - E_{\text{POD}}(k) \]
Applications

Structural dynamical system

- $N_u = 48$ masses $\Rightarrow N = 96$ degrees of freedom in state space form
- Model reduction by the POD method in the frequency domain
Applications

Structural dynamical system

- Nyquist diagrams

This leads to a choice of a ROM of size $k = 18$
Applications

Structural dynamical system

- Bode diagram for a ROM of size $k = 18$
Applications

Fluid System - Advection-diffusion

- High-dimensional model \((N = 5, 402)\)
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Applications

Fluid System - Advection-diffusion

- POD modes

POD vector #1

POD vector #2

POD vector #3

POD vector #4

POD vector #5

POD vector #6
- Applications

- Fluid System - Advection-diffusion

- Projection error (singular values decay)
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Applications

Fluid System - Advection-diffusion

- POD-based ROM ($k = 1$ and $k = 2$)
Applications

Fluid System - Advection-diffusion

- POD-based ROM ($k = 3$ and $k = 4$)
POD-based ROM ($k = 5$ and $k = 6$)
Model reduction error $\mathcal{E}_{ROM}(t)$
Applications

Fluid System - Advection-diffusion

- Model reduction error $\varepsilon_{\text{ROM}}(t)$ and projection error $\varepsilon_{\mathbb{V}}(t)$

- For this problem $\varepsilon_{\mathbb{V}}(t)$ dominates $\varepsilon_{\mathbb{V}}(t)$
aeroelasticity analysis of an F-16 Block 40