A BRIEF SUMMARY OF OTAL’S PROOF OF MARKED LENGTH SPECTRUM RIGIDITY

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Abstract. We outline Otal’s proof of marked length spectrum rigidity for negatively curved surfaces. We omit all technical details, and refer the interested reader to the original [Ota90] or the course notes [Wil] for details, and to [Cro90] for different approach. (Actually the course notes [Wil] combine the approaches in [Ota90, Cro90].)

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Consider two negatively curved closed surfaces $S$ and $S'$. Fix a homeomorphism from $S$ to $S'$, or alternatively consider $S$ and $S'$ to be two Riemannian structures on the same topological surface. Due to negative curvature, every closed curve is homotopic to a unique closed geodesic, called the geodesic representative of the homotopy class. Let $\mathcal{C}$ denote the set of homotopy classes of closed curves. The marked length spectrum of $S$ is defined as the function $\ell_S : \mathcal{C} \to \mathbb{R}_{>0}$ which assigns to each homotopy class of curve the length of its geodesic representative.

Theorem 1 (Otal, Annals 1990). Let $S$ and $S'$ be two negatively curved closed marked surfaces. If $S$ and $S'$ have the same marked length spectrum, they are isometric.

Step 1: Coarse geometry gives a correspondence of geodesics. Let $\tilde{S}$ and $\tilde{S'}$ denote the universal covers of $S$ and $S'$. The homeomorphism $\text{Id} : S \to S'$ lifts to a homeomorphism

$$\tilde{\text{Id}} : \tilde{S} \to \tilde{S'},$$

which is in fact a quasi-isometry. Again due to negative curvature, both $\tilde{S}$ and $\tilde{S'}$ have boundaries, which are homeomorphic to a circle. The
quasi-isometry $\hat{\text{Id}} : \tilde{S} \to \tilde{S}'$ induced a homeomorphism on boundaries

$$\hat{\text{Id}} : \partial \tilde{S} \to \partial \tilde{S}'.$$

The image of any geodesic under a quasi-isometry is a quasi-geodesic, and in negative curvature every quasi-geodesic is a bounded distance from a genuine geodesic. Hence we get a correspondence $\phi$ between geodesics in $\tilde{S}$ and geodesics in $\tilde{S}'$: given a geodesic $\gamma$ in $\tilde{S}$, we define $\phi(\gamma)$ to be the unique geodesic which lies within bounded distance from the quasi-geodesic $\hat{\text{Id}}(\gamma)$.

The space $\mathcal{G}$ of geodesics in $\tilde{S}$ is identified naturally with

$$\partial \tilde{S} \times \partial \tilde{S} \setminus \Delta,$$

where $\Delta$ is the diagonal. The identification sends a geodesic to the ordered pair of its forward and backward endpoints and infinity. In these coordinates, we have that

$$\phi(\xi, \eta) = (\hat{\text{Id}}(\xi), \hat{\text{Id}}(\eta)).$$

In other words, given a geodesic $\gamma$ in $\tilde{S}$ with endpoints $\xi, \eta \in \partial \tilde{S}$ at infinity, we may map the endpoints to $\hat{\text{Id}}(\xi), \hat{\text{Id}}(\eta) \in \partial \tilde{S}'$ and the geodesic $\phi(\gamma)$ in $\tilde{S}'$ is simply the unique geodesic from $\hat{\text{Id}}(\xi)$ to $\hat{\text{Id}}(\eta)$.

**Lemma 2.** The correspondence of geodesics $\phi$ sends intersecting geodesics to intersecting geodesics.

**Proof.** Suppose we have geodesics $\gamma_1, \gamma_2$ in $\tilde{S}$, which may be described in terms of their endpoints at infinity as

$$\gamma_1 = (\xi_1, \eta_1), \quad \gamma_2 = (\xi_2, \eta_2).$$

Suppose that $\gamma_1$ and $\gamma_2$ intersect. We will treat the case where $(\xi_1, \xi_2, \eta_1, \eta_2)$ are cyclically ordered on the circle. (There is one other case which is identical: the endpoints at infinity may intertwine in a different order.)

The correspondence

$$\phi : \partial \tilde{S} \times \partial \tilde{S} \setminus \Delta \to \partial \tilde{S}' \times \partial \tilde{S}' \setminus \Delta'$$
is induced by the homeomorphism $\hat{\text{Id}} : \partial \tilde{S} \to \partial \tilde{S}'$. The boundary is homeomorphic to a circle, and any homeomorphism of a circle preserves the cyclic order of quadruples of points. Hence

$$(\hat{\text{Id}}(\xi_1), \hat{\text{Id}}(\xi_2), \hat{\text{Id}}(\eta_1), \hat{\text{Id}}(\eta_2))$$

are cyclically ordered on the circle $\partial \tilde{S}'$. It follows that the geodesics $\phi(\gamma_1)$ and $\phi(\gamma_2)$ intersect.

**Step 2: Marked length spectrum determines the Liouville current.** The Liouville current $\lambda$ of $\tilde{S}$ is a measure on the space $G$ of geodesics with the following property. If $\alpha$ is a bounded geodesic arc in $\tilde{S}$, then the measure of the set $G_\alpha$ of geodesics intersecting $\alpha$ is exactly the length of $\alpha$, that is

$$\lambda(G_\alpha) = \text{length}(\alpha).$$

Otal proves that marked length spectrum completely determines the Liouville current (“Crofton’s Formula”). As a result, we get that $\phi$ preserves the Liouville current:

**Lemma 3.** If $Q \subset G$ is a set of geodesics, then $\lambda(Q) = \lambda'(\phi(Q))$.

**Step 3: Understanding change in angle.** Two intersecting geodesics in $\tilde{S}$ are mapped via the correspondence $\phi$ to intersecting geodesics in $\tilde{S}'$ (Lemma 2). We will see that negative curvature restricts the change in angle.

To make this precise, let us define a function

$$\theta' : T^1 \tilde{S} \times [0, \pi] \to [0, \pi],$$

where $T^1 \tilde{S}$ is the unit tangent bundle to $\tilde{S}$. Given $v \in T^1 \tilde{S}$, denote by $\gamma_v$ the geodesic through $v$, and given $\theta \in [0, 2\pi]$ let $\theta v$ denote the tangent vector $v$ rotated by $\theta$. 
We consider the intersecting geodesics $\gamma_v$ and $\gamma_{\theta v}$, and the define $\theta'(v, \theta)$ to be the angle between $\phi(\gamma_v)$ and $\phi(\gamma_{\theta v})$. The notation is intuitive because $\theta$ is the angle between $\gamma_v$ and $\gamma_{\theta v}$ in $\tilde{S}$, and $\theta'$ is the angle between the corresponding geodesics in $\tilde{S}'$.

**Lemma 4.** Let $\theta(v)$ denote the rotation of $v$ by angle $\theta$. Then we have the following superadditivity relation:

$$\theta'(v, \theta_1 + \theta_2) \geq \theta'(v, \theta_1) + \theta'(\theta_1 v, \theta_2).$$

Equality is only possible if the corresponding geodesics in $\tilde{S}'$ of the three geodesics through $v, \theta_1 v$ and $(\theta_1 + \theta_2)v$ all intersect in a point.

**Proof.** The proof is by picture. Since the angles in a triangle add up to strictly less than $\pi$ in negative curvature, we obtain

$$\theta'(v, \theta_1) + \theta'(\theta_1 v, \theta_2) + (\pi - \theta'(v, \theta_1 + \theta_2)) < \pi,$$

which gives the superadditivity. Equality is only possible in the degenerate case when the triangle above is actually a point.

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**Step 4:** The correspondence $\phi$ of geodesics sends triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point. Roughly speaking, we wish to show that we are always in the equality case of the super-additivity relation Lemma 4. To do so, we will have to average $\theta'$.

Indeed, using the Liouville current, we can further constrain the average change in angle. We must first note that $\theta'$ descends to a well defined function on $T^1 S \times [0, \pi]$. The unit tangent bundle $T^1 S$ can
be equipped with a natural volume measure $\text{vol}$, called the Liouville measure. It is with respect to this measure that we average.

We define

$$\Theta'(\theta) = \frac{1}{\text{vol}(T^1S)} \int_{T^1S} \theta'(v, \theta) \text{d} \text{vol}.$$ 

Lemma 4 implies a corresponding superadditivity relation for $\Theta'$.

**Lemma 5.**

$$\Theta'(\theta_1 + \theta_2) \geq \Theta'(\theta_1) + \Theta'(\theta_2),$$

with equality if and only if $\phi$ sends triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point.

What is harder is to show the following.

**Proposition 6.** For all continuous convex functions $F : [0, \pi] \to \mathbb{R}$ we have

$$\int_0^\pi F(\Theta'(\theta)) \sin \theta \text{d} \theta \leq \int_0^\pi F(\theta) \sin \theta \text{d} \theta.$$

We omit the proof, but make a few remarks:

- The $\sin \theta$ term appears naturally in the expression of the Liouville current on $\mathcal{G}$, in certain natural local coordinates.
- The proof begins with Jensen’s inequality.
- The key step uses that the Liouville measure is preserved.
- Using this, Otal computes the average of

$$\int_0^\pi F(\theta'(v, \theta)) \sin \theta \text{d} \theta$$

over every closed orbit.

- The average of a continuous function on $T^1S$ over all closed geodesics determines its average with respect to the Liouville measure $\text{vol}$.

Otal deduces that $\Theta'$ is constant by applying the following elementary result on functions.

**Lemma 7.** Let $\Theta$ be an increasing homeomorphisms from $[0, \pi]$ to itself satisfying

1. $\Theta$ is super-additive and symmetric ($\Theta(\pi - \theta) = \pi - \Theta(\theta)$), and
2. for all continuous convex functions $F : [0, \pi] \to \mathbb{R}$ we have

$$\int_0^\pi F(\Theta'(\theta)) \sin \theta \text{d} \theta \leq \int_0^\pi F(\theta) \sin \theta \text{d} \theta.$$

Then $\Theta$ is the identity.
Since $\Theta'$ is the identity, it follows in particular that equality in Lemma 5 is achieved.

**Step 5: Constructing an isometry $\tilde{S} \to \tilde{S}'$.** To establish Theorem 1, it suffices to construct an isometry $f : \tilde{S} \to \tilde{S}'$ which is equivariant with respect to deck transformations (the action of the fundamental group).

The map $f$ is defined as follows. Given $p \in \tilde{S}$, we pick any two geodesics $\gamma_1, \gamma_2$ intersecting at $p$, and set $f(p)$ to be the unique point of intersection of the geodesics $\phi(\gamma_1), \phi(\gamma_2)$.

By Step 4, the result does not depend on which two geodesics through $p$ are chosen; $f$ is well defined.

**Lemma 8.** Let $p, q \in \tilde{S}$, and let $\alpha = [p, q]$ be the geodesic arc from $p$ to $q$. Furthermore let $\alpha' = [f(p), f(q)]$ denote the geodesic arc in $\tilde{S}'$ from $f(p)$ to $f(q)$.

Then if $\gamma$ is any geodesic in $\tilde{S}$ intersecting $\alpha$, then the corresponding geodesic $\phi(\gamma)$ in $\tilde{S}'$ intersects $\alpha'$.

**Proof.** Let $\gamma_p$ and $\gamma_q$ be geodesics through $p$ and $q$ respectively, each not intersecting $\gamma$. The geodesic $\gamma$ is thus “in between” $\gamma_p$ and $\gamma_q$. As in Lemma 2, since the correspondence $\phi$ of geodesics is induced by a
homeomorphism on boundaries at infinity, \( \phi(\gamma) \) lies in between \( \phi(\gamma_p) \) and \( \phi(\gamma_q) \). However, since \( \phi(\gamma_p) \) contains \( f(p) \) and \( \phi(\gamma_q) \) contains \( f(q) \), it follows that \( \phi(\gamma) \) must intersect \( \alpha' \).

The proof of the Theorem 1 is completed using the Liouville current again.

**Proof that \( f \) is an isometry.** Let \( p, q, \alpha, \alpha' \) be as above. Let \( G_\alpha \) be the set of all geodesics in \( \tilde{S} \) intersecting \( \alpha \); similarly \( G_{\alpha'} \) is the set of all geodesics in \( \tilde{S}' \) intersecting \( \alpha' \). Lemma 8 gives that

\[
\phi(G_\alpha) = G_{\alpha'}.
\]

We know that the Liouville current measure of \( G_\alpha \) is the length of \( G_\alpha \):

\[
\lambda(G_\alpha) = \text{length}(\alpha),
\]
and similarly for \( G_{\alpha'} \).

Now, using that the Liouville current is preserved (Lemma 3), we get

\[
\text{dist}_{\tilde{S}}(p, q) = \text{length}(\alpha) = \lambda(G_\alpha) = \lambda'(\phi(G_\alpha)) = \lambda'(G_{\alpha'}) = \text{length}(\alpha') = \text{dist}_{\tilde{S}'}(f(p), f(q)).
\]

That is, we have shown that \( f : \tilde{S} \to \tilde{S}' \) preserves distances. The isometry \( f \) descends to an isometry \( S \to S' \), completing the proof.

**References**


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