STRUCTURE OF THE RESOLVENT FOR THREE-BODY POTENTIALS

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1. Introduction

In this paper we construct a parametrix for the resolvent of the three-body Hamiltonian $H_V = \Delta + \sum_i V_i$ when the two-body potentials are real valued and Schwartz: $V_i \in \mathcal{S}(X^i)$. Here Δ is the positive Laplacian on $\mathbb{R}^N$, and X^i are linear subspaces of $\mathbb{R}^N$. The (reduced) three-body problem has the important property that $X_i \cap X_j = \{0\}$ for $i \neq j$ where $X_i = (X^i)^\perp$. The construction is performed via a finite iteration of the two-body resolvents which is in many ways similar to Faddeev's original approach [10]. This is facilitated by the use of the scattering calculus introduced by Melrose in [29] and motivated by the parametrix construction of Melrose and Zworski for the Poisson operator in [31].

Thus, we identify $\mathbb{R}^N$ with the interior of its radial compactification $\mathbb{S}^N_+$, the upper hemisphere, via the stereographic projection $\mathrm{SP} : \mathbb{R}^N \to \mathbb{S}^N_+$. Let $C_i = \bar{X}_i \cap \mathbb{S}^{N-1}$ where $\mathbb{S}^{N-1} = \partial \mathbb{S}^N_+$, $\bar{X}_i = \mathrm{cl}(\mathrm{SP}(X_i))$. Then $V_i \in \mathcal{C}^\infty(\mathbb{S}^N_+ \setminus C_i)$ and vanishes to infinite order on $\mathbb{S}^{N-1} \setminus C_i$. Note that the condition $X_i \cap X_j = \{0\}$ for $i \neq j$ becomes $C_i \cap C_j = \emptyset$. Our approach will enable us to describe the asymptotic behavior of the resolvent applied to Schwartz functions and to obtain the structure of the three-cluster to three-cluster part of the scattering matrix. Namely, we prove the following theorem which was conjectured by Melrose (motivated by [31]):

Theorem. *The three-cluster to three-cluster part of the (absolute) scattering matrix is a sum of Fourier integral operators on $\mathbb{S}^{N-1} \setminus \cup_i C_i$ associated to the 'broken' geodesic flow, broken at points in $\cup_i C_i$, at distance π.*

Andrew Hassell had previously proved the corresponding theorem for the scattering matrix of the unreduced two-body Hamiltonian (i.e. the case when we only have one two-body potential) [16].

We denote the (modified) resolvent of H_V by $R_V(\lambda)$. Thus, our normalization is that of [30], so $R_V(\lambda) = (H_V - \lambda^2)^{-1}$ in the 'physical half plane' where $\operatorname{Im} \lambda < 0$, and for $\sigma > 0$ $(H_V - (\sigma \pm i0))^{-1} = R_V(\mp \sigma^{1/2})$. We also show in this paper that away from $\cup_i C_i$, $R_V(\lambda) f$ (here $f \in \mathcal{S}(\mathbb{R}^N)$) has the same behavior as in the free problem, i.e. for $\lambda > 0$

$$R_V(\lambda) f(r\theta) \sim e^{-i\lambda r} r^{-(N-1)/2} \sum_{j \geq 0} r^{-j} g_j(\theta)$$

with $g_j \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \setminus \cup_i C_i)$, and that the precise asymptotics can be described after blowing up $\cup_i C_i$. In fact, let $\beta : [\mathbb{S}^N_+; \cup_i C_i] \to \mathbb{S}^N_+$ be the blow down map, $\mathrm{ff}_i = \beta^{-1}(C_i)$, mf the lift of $\mathbb{S}^{N-1}$ under β. Then, assuming that the two-body subsystems have no bound states, $R_V(\lambda) f = e^{-i\lambda r} r^{-(N-1)/2} g$ where

Date: February 29, 1996, revised October 22, 1996.

$\tilde{g} = \beta^*(\mathrm{SP}^{-1})^* g \in \mathcal{C}^{-\infty}([\mathbb{S}^N_+; \cup_i C_i])$ is conormal to the boundary of $[\mathbb{S}^N_+; \cup_i C_i]$ in the sense of iterative regularity (as in [28]); in the general case there are additional terms corresponding to the bound states. We also prove a polyhomogeneous expansion at the boundary of the blown up space $[\mathbb{S}^N_+; \cup_i C_i; \cup_i(\mathrm{mf} \cap \mathrm{ff}_i)]$ when $\dim X^i \geq 3$ is odd. Isozaki had previously obtained the top term of the expansion off $\cup_i C_i$ in his paper on scattering with initial state in a two-cluster [22, Theorem 1.1], and he had also proved a theorem there in which the boundary behavior of $R_V(\lambda) f$ near $\cup_i C_i$ is described in an averaged sense.

The first progress for the 3-body problem was made by Faddeev [10] who used a finite iteration involving the two-body resolvents to construct the limit of the resolvent on the real axis (away from the thresholds and eigenvalues) modulo a compact error on spaces of Hölder continuous functions. Ginibre and Moulin [14] modified his approach so that only Hilbert space techniques were necessary. Later Mourre [32] introduced a new method: instead of constructing a parametrix for the resolvent, he obtained positive commutator estimates for the three-body Hamiltonian. Given that σ_0 is not an eigenvalue of H_V and it is not a threshold of H_V either, that is $\sigma_0 \neq 0$ and σ_0 is not an eigenvalue of any of the two-body subsystems, this proves that the limits $(H_V - (\sigma \pm i0))^{-1}$ exist near σ_0 as continuous functions of bounded operators between appropriate weighted Sobolev spaces. A major advantage of Mourre's method is that it is applicable to long-range two-body potentials while Faddeev's construction relies on the fact that the unreduced two-body resolvents are locally good approximations for (part of) the three-body resolvent which certainly limits it to short-range potentials. This phenomenon is similar to the fact that the free resolvent can be perturbed into the resolvent of the two-body problem with a short range potential (using the Lippman-Schwinger equation), while long-range potentials require other methods. Mourre's original result was extended by Perry, Sigal and Simon to N-body Hamiltonians [36]; Froese and Herbst later simplified their proof [12]. Froese and Herbst also proved the absence of positive eigenvalues for N-body Hamiltonians [11] thereby showing the existence of $(H_V - (\sigma \pm i0))^{-1}$ for $\sigma > 0$.

These global results were microlocalized by Gérard, Isozaki and Skibsted [13] and Wang [42] in the sense that the global operator of the commutator estimate of [32] was replaced by microlocal ones. This allowed the characterization of the resolvent by Isozaki [24], and Isozaki and Skibsted obtained the structure of certain parts of the S-matrices as well some of the asymptotic properties of generalized eigenfunctions [21, 22, 40]. Bommier's finer estimates allowed him to describe the two-cluster to two-cluster S-matrices for long-range potentials [2, 3]. It should be noted that there have been many related results in the physics literature; see for example Newton's work [33, 34] and Nuttal's [35].

There is another issue which arises in the study of N-body Hamiltonians, namely the asymptotic completeness of the wave operators. This problem is not discussed in this paper, but we briefly summarize the main steps of progress: Enss [8] proved the asymptotic completeness for 3-body short range potentials. The corresponding result was obtained in the N-body case by Sigal and Soffer [39]. Dereziński [4] and Graf [15] provided new proofs of this result. Dereziński [5] also proved asymptotic completeness in the long-range case.

Since two-body scattering is so well understood, it is natural to try to use it in the three-body problem. In fact, if we have a single potential V_i, then the unreduced two-body problem (i.e. where the center of mass coordinates are not removed) can be considered as a prototype of the (reduced) three-body problem. To see the similarity note that the Hamiltonian in the former case is $\Delta_{\bar{w}} + V_i(z^i)$ where $\bar{w} = (w^i, z^i)$, w^i is the center of mass coordinate and z^i is the separation of the particles given by orthogonal projection to X_i and X^i respectively. In the compactified picture V_i is still singular at C_i, so the geometry of the unreduced problem is rather close to the 3-body case.

The basic idea of the proofs is simple. One can construct a parametrix for $H_V - \lambda^2$ by iterating the two-body resolvents $R_{V_i}(\lambda) = (\Delta_{\bar{w}} + V_i(z^i) - (\lambda - i0)^2)^{-1}$. As $(H_V - \lambda^2)R_0(\lambda) = \mathrm{Id} + VR_0(\lambda)$, $V = \sum_i V_i$, we can try to remove the error $VR_0(\lambda)$ term by term using $R_{V_i}(\lambda)$, that is by subtracting an extra term $R_{V_i}(\lambda)V_iR_0(\lambda)$. Then we are left with an error

$$(H_V - \lambda^2)(R_0(\lambda) - \sum_i R_{V_i}(\lambda)V_iR_0(\lambda)) - \mathrm{Id} = \sum_{i\neq j} V_jR_{V_i}(\lambda)V_iR_0(\lambda). \tag{1.1}$$

In general we iterate this method, so in the next step we add

$$\sum_{i\neq j} R_{V_j}(\lambda)V_jR_{V_i}(\lambda)V_iR_0(\lambda)$$

to the approximation of the three-body resolvent on the left hand side. Of course, one must check that the compositions are defined when acting on appropriate spaces, e.g. on sufficiently highly weighted Sobolev spaces or on spaces of distributions with no 'incoming' scattering wave front set, and one must understand the structure of the resulting operators. This is the crucial part of the analysis which can be carried out directly using Legendre distributions (cf. [31]), or, as done in this paper, employing the Fourier transform to work in phase space. The second approach then reduces the task to a traditional wave front set and conormal analysis, except that the scattering calculus, $\Psi_{\mathrm{sc}}^{*,*}(\mathbb{S}^N_+)$, must be utilized to take care of the behavior at infinity.

Away from $\cup_i C_i$ the propagation by $R_{V_i}(\lambda)$ is very similar to propagation by the free resolvent, $R_0(\lambda)$. Thus, over $\mathbb{S}^{N-1} \setminus \cup_i C_i$ the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$, propagates along bicharacteristics of $\Delta - \lambda^2$ in the scattering cotangent bundle, ${}^{\mathrm{sc}}T^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}}$ (see [29]). Note that for $p = \mathrm{SP}(x^{-1}\theta)$, $\theta \in \mathbb{S}^{N-1}$, $v \in {}^{\mathrm{sc}}T^*_p\mathbb{S}^N_+$ can be written as

$$v = \tau x^{-2}\,dx + x^{-1}\mu$$

where $\mu \in T^*_\theta\mathbb{S}^{N-1}$, and correspondingly we obtain coordinates (x, θ, τ, μ) on ${}^{\mathrm{sc}}T^*\mathbb{S}^N_+$ near $\mathbb{S}^{N-1}$. The characteristic set of $\Delta - \lambda^2$ is $\Sigma_{\Delta-\lambda^2} = \{(\theta, \tau, \mu) : \tau^2 + |\mu|^2 = \lambda^2\}$, so on the bicharacteristics $\tau \in [-\lambda, \lambda]$. The important property of propagation is that it increases the τ coordinate of covectors in the scattering wave front set. Thus, propagating from C_i to C_j, $i \neq j$, $\arccos(\tau/\lambda)$ increases by $\mathrm{dist}(C_i, C_j)$. Although $V_i \notin \Psi_{\mathrm{sc}}^{*,*}(\mathbb{S}^N_+)$, and in particular it does not preserve $\mathrm{WF}_{\mathrm{sc}}$, it *does* preserve the τ coordinate of $\mathrm{WF}_{\mathrm{sc}}$. Hence, this analysis shows that in a finite number of iterative steps we reach $\tau = \lambda$. This final error can be reduced to an element of $\mathcal{S}(\mathbb{R}^N)$ by a slight modification of the last step in the construction, hence giving a right parametrix for $H_V - \lambda^2$. We can similarly construct a left parametrix (in an appropriate sense) which allows us to describe the structure of $R_V(\lambda)$. In fact, we shall remove the Schwartz error by showing the connection between our approach and that of Gérard, Isozaki and Skibsted [13, 20]. Here we only mention that their microlocal estimates in terms of the operator B correspond to estimates in terms of τ since the normal symbol of B is exactly τ; see Section 4.

Note that by applying these results we indirectly use the Mourre estimate and the absence of positive eigenvalues.

The meaning of the terms in (1.1) becomes more transparent if we consider the geometric picture. The first term, $R_0(\lambda)$, corresponds to a free wave propagating along geodesics (which are just projections of the bicharacteristics of $\Delta - \lambda^2$ to the base) on $\mathbb{S}^{N-1}$, the sphere at 'infinity'. When it hits a component of the singular set, say C_i, part of the wave is transmitted and part of it is reflected. Upon reflection the direction of the propagation can change so that the usual law of reflection holds (the precise statement is that of Theorem 6.3); this gives the first 'break' of the geodesics. The new propagating wave corresponds to the second term, $-R_{V_i}(\lambda)V_iR_0(\lambda)$. If this hits another component of the singular set, C_j, the same phenomenon occurs, and the reflected wave is represented by $R_{V_j}(\lambda)V_jR_{V_i}(\lambda)V_iR_0(\lambda)$. During the propagation $\arccos(\tau/\lambda)$ increases by the distance travelled along the broken geodesics. A free wave starting off away from $\cup_i C_i$ has $\tau = -\lambda$ at the initial point. This increases to $\tau = \lambda$ after distance π has been covered along the broken geodesic, hence the statement of the main Theorem.

In Section 2 we shall analyze the unreduced two-body resolvents and their wave front set properties. The results will be applied in Section 3 to obtain approximate eigenfunctions u of H_V, i.e. u such that $(H_V - \lambda^2)u$ is Schwartz, and in Section 4 to construct the three-body resolvent on weighted Sobolev spaces. In Section 5 the Poisson operator will be examined. In Section 6 we shall analyze the scattering matrix. Finally, in Section 7 the asymptotic behavior of $R_V(\lambda)f$ will be described for $f \in \mathcal{S}(\mathbb{R}^N)$.

I would like to thank A. Hassell, M. Joshi and J. Wunsch for the helpful discussions. I am particularly grateful to R. B. Melrose who suggested the problem and with whom I discussed it on numerous occasions. Melrose's ideas had a great influence on this paper.

2. The unreduced two-body resolvent

In this section we analyze the unreduced two-body Hamiltonian $\Delta_{w,z} + V(z)$. Thus, we shall consider $\mathbb{R}^N$ as a product $\mathbb{R}^m \times \mathbb{R}^n$ with coordinates $\bar{w} = (w, z)$ and dual coordinates $\bar{\xi} = (\xi, \zeta)$. Let $H^{r,s}(\mathbb{R}^N_{\bar{w}})$ be the weighted Sobolev space $\langle \bar{w} \rangle^{-s} H^r(\mathbb{R}^N)$. As indicated in the introduction, we shall be working in the dual space most of the time. Let $\mathcal{F}$ be the Fourier transformation from $\bar{w}$ coordinates to $\bar{\xi}$ coordinates, $\mathbb{S}^{N-1}_\lambda = \{\bar{\xi} : |\bar{\xi}| = \lambda\}$,

$$N^\pm_\lambda = \{(\bar{\xi}, \pm\bar{\xi}/\lambda) : |\bar{\xi}| = \lambda\} \subset S^*\mathbb{R}^N \tag{2.1}$$

be the outgoing (N^+_λ) and the incoming (N^-_λ) half of the conormal bundle of $\mathbb{S}^{N-1}_\lambda$ respectively, and we also let

$$\rho_0(\lambda) = (|\bar{\xi}|^2 - (\lambda - i0)^2)^{-1}. \tag{2.2}$$

Thus, the (modified) resolvent of the Laplacian $\Delta_{\bar{w}}$ for $\lambda > 0$ is given by $R_0(\lambda) = \mathcal{F}^{-1}\rho_0(\lambda)\mathcal{F}$. Throughout the paper we shall think of the wave front set both as a conic subset of $T^*\mathbb{R}^N \setminus 0$ and as a subset of the cosphere bundle $S^*\mathbb{R}^N$. We shall use a slightly different notation for conormal distributions than that of [17, p. 115] mainly to avoid confusion with the notation for paired Lagrangian distributions as used in [26]; this will be helpful in Section 7. Thus, we write $I^k(Y)$ for $I^k(\mathbb{R}^N, Y)$ when Y is a submanifold of $\mathbb{R}^N$. We have the following basic result (cf. [17, 18]).

Lemma 2.1. *Suppose* $\lambda \in \mathbb{R}^+$. *Then*

$$\rho_0(\lambda) \in I^{-(N-2)/4}(\mathbb{S}_\lambda^{N-1}), \qquad \mathrm{WF}(\rho_0(\lambda)) \subset N_\lambda^+$$

and for all $\epsilon > 0$ $\rho_0(\lambda) \in H_{loc}^{-1/2-\epsilon}(\mathbb{R}^N)$.

Proof. The distribution is $\mathcal{C}^\infty$ away from $\mathbb{S}_\lambda^{N-1}$. Moreover, $(|\bar\xi|^2 - (\lambda - i0)^2)^{-1} = (|\bar\xi|^2 - \lambda^2 + i0)^{-1}$. But $(t+i0)^{-1}$ is conormal to $t = 0$, its symbol is a constant multiple of $\chi_+^0(\tau)$ (χ_+^0 being the characteristic function of $(0,\infty)$), so

$$\mathrm{WF}((t+i0)^{-1}) \subset \{(0,\tau), \tau > 0\},$$

and $f : \bar\xi \mapsto |\bar\xi|^2 - \lambda^2$ is a $\mathcal{C}^\infty$ map with $f_{\bar\xi}^*(0,\tau) = (\bar\xi, 2\tau\bar\xi)$ if $|\bar\xi| = \lambda$, so the behavior of conormal distributions and wave front sets under pull-backs shows the lemma. □

Now we shall start studying the potential. Let

$$Y_1 = \{(\xi,\zeta,\xi,\zeta') : \xi \in \mathbb{R}^m,\ \zeta,\zeta' \in \mathbb{R}^n\}. \tag{2.3}$$

Lemma 2.2. *If* $V \in \mathcal{S}(\mathbb{R}_z^n)$ *then* $\tilde V = \mathcal{F}V\mathcal{F}^{-1}$ *is a Fourier integral operator with Schwartz kernel* $K_{\tilde V} \in I^{-n/2}(Y_1)$. *If* $u \in H^{k,l}(\mathbb{R}^N)$ *then* $\tilde V u \in H^{k,l}(\mathbb{R}^N)$. *In particular if* $u \in \mathcal{S}(\mathbb{R}^N)$ *then* $\tilde V u \in \mathcal{S}(\mathbb{R}^N)$.

Proof. The assumptions imply that

$$\mathcal{F}V\mathcal{F}^{-1}u = (2\pi)^{-n}\hat V *_\zeta u \tag{2.4}$$

where $\hat V = \mathcal{F}_z V$ since this is true when $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and both sides are continuous as maps $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$. Thus, the Schwartz kernel of $\tilde V$ is

$$K_{\tilde V}(\xi,\zeta,\xi',\zeta') = (2\pi)^{-n}\hat V(\zeta-\zeta')\delta(\xi-\xi') \tag{2.5}$$

which is the product of a $\mathcal{C}^\infty$ function and a distribution in $I^{-n/2}(Y_1)$ (in fact, the tensor product of a smooth function and a conormal distribution), proving the lemma. □

We next analyze the behavior of the unreduced two-body resolvent $R_V(\lambda)$ for $V \in \mathcal{S}(\mathbb{R}^n)$. This is the resolvent of the unreduced two-body Laplacian $\Delta_w + \Delta_z + V(z) - \lambda^2$ (which is an unbounded self-adjoint operator on $L^2(\mathbb{R}^N)$), i.e. for $\operatorname{Im}\lambda < 0$

$$R_V(\lambda) = (\Delta_{w,z} + V(z) - \lambda^2)^{-1}, \tag{2.6}$$

and for $\lambda \in \mathbb{R}$ it is the limit (in a suitable sense) of $(\Delta + V - (\lambda')^2)^{-1}$ as $\lambda' \to \lambda$ from the lower half plane, $\mathcal{P}$. The basic idea in dealing with $R_V(\lambda)$ is that partial Fourier transformation in w changes Δ_w into $|\xi|^2$, so we can obtain $R_V(\lambda)$ from the reduced two-body resolvent, $(\Delta_z + V(z) - \sigma)^{-1}$. In fact, for $\lambda \in \mathcal{P}$

$$R_V(\lambda) = \mathcal{F}_\xi^{-1}(\Delta_z + V + |\xi|^2 - \lambda^2)^{-1}\mathcal{F}_w. \tag{2.7}$$

It is actually more convenient to consider

$$B_V = \mathcal{F}V(\Delta + V - (\sigma - i0))^{-1}V\mathcal{F}^{-1} \tag{2.8}$$

to describe $R_V(\lambda)$ for $\lambda > 0$. For $n \geq 3$ let Λ' be the set of $\sigma \in (-\infty, 0]$ for which

$$\mathrm{Id} + \operatorname{sign} V|V|^{1/2}(\Delta - (\sigma - i0))^{-1}|V|^{1/2} \in \mathcal{B}(L^2(\mathbb{R}^n)) \tag{2.9}$$

is not invertible; elements of $\Lambda' \setminus \{0\}$ are eigenvalues of the reduced two-body Hamiltonian $\Delta_z + V(z)$. Let $\Lambda = \mathrm{spec}_p(\Delta + V) \cup \{0\}$ be the set of thresholds, so for $n \geq 3$ $\Lambda = \Lambda' \cup \{0\}$, and let

$$Y_0 = \{(\sigma, \zeta, \zeta') : \sigma \in \Lambda\}. \tag{2.10}$$

Note that Λ is finite. We shall only use the following lemma about B_V in this paper; it is essentially a special case of Jensen's and Kato's results [25]. Its proof is included in the Appendix for the convenience of the readers.

Lemma 2.3. *Suppose $V \in \mathcal{S}(\mathbb{R}^n)$. Then B_V is a continuous map from $H^{k,l}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}(\mathbb{R}_\sigma; H^{k',l+2}(\mathbb{R}^n))$ for all k, k', $l \in \mathbb{R}$ which is $\mathcal{C}^\infty$ for $\sigma \notin \Lambda$ and in $\mathcal{C}^\infty_\infty$ for $\sigma < \inf \Lambda - 1$. Its Schwartz kernel, K_0, is conormal to Y_0, and it satisfies*

$$\mathrm{WF}(K_0) \subset \{(\sigma, \zeta, \zeta', \eta, 0, 0) : \eta < 0, \sigma \in \Lambda\}. \tag{2.11}$$

Moreover, if $n \geq 3$ is odd, then it has a 'classical' expansion

$$B_V = B_e + (\sigma - i0)^{(n-2)/2} B_o + \sum_{\tau \in \Lambda'} (\sigma - i0 - \tau)^{-1} B_\tau + \sum_{j=1,2} (\sigma - i0)^{-j/2} B_0^{(j)} \tag{2.12}$$

with B_e, B_o continuous maps from $H^{k,l}(\mathbb{R}^n)$ to $\mathcal{C}^\infty(\mathbb{R}_\sigma; H^{k',l+2}(\mathbb{R}^n))$ for any k, k', l, and B_τ, $B_0^{(j)}$ map $H^{k,l}$ to $\mathcal{C}^\infty_c(\mathbb{R}_\sigma; \mathcal{S}(\mathbb{R}^n))$. In particular, the Schwartz kernels of B_e, B_o, B_τ and $B_0^{(j)}$ are in $\mathcal{C}^\infty(\mathbb{R}_\sigma \times \mathbb{R}^n \times \mathbb{R}^n)$.

Remark 2.4. For other values of n we still have an expansion, but it involves logarithmic terms in σ. $\mathcal{C}^\infty_\infty$ stands for bounded with all derivatives.

Lemma 2.3 allows us to define the operator $\tilde{B}_V(\lambda) = \mathcal{F} V R_V(\lambda) V \mathcal{F}^{-1}$ for $u \in \mathcal{S}(\mathbb{R}^N)$ by

$$\tilde{B}_V(\lambda) u(\xi, \zeta) = (B_V u(\xi, .))(\lambda^2 - |\xi|^2, \zeta). \tag{2.13}$$

In fact, the Schwartz kernel of $\tilde{B}_V(\lambda)$ is given by

$$K_{\tilde{B}_V(\lambda)} = \delta(\xi - \xi') \iota^* K_0 \tag{2.14}$$

where $\iota : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is the map

$$\iota(\xi, \zeta, \xi', \zeta') = (\lambda^2 - |\xi|^2, \zeta, \zeta'). \tag{2.15}$$

Here the distribution we are pulling back is conormal to Y_0 and ι is transverse to Y_0 so the pull back is defined. We then take the product of two distributions conormal to transverse submanifolds which makes sense. Now (2.12) shows that for $n \geq 3$ odd

$$\tilde{B}_V(\lambda) = B_1 + (\lambda^2 - |\xi|^2)^{(n-2)/2} B_2 + \sum_{\tau \in \Lambda',\ j=1,2} (\lambda^2 - |\xi|^2 - \tau)^{-j/2} B_3(\tau, j) \tag{2.16}$$

where all of B_1 and B_2, $B_3(\tau, j)$ have Schwartz kernels conormal to $\xi = \xi'$, and which are thus Fourier integral operators. Here the notation $(\lambda^2 - |\xi|^2 - \tau)^{-1}$, etc. refers to $\iota^*((\sigma - i0) - \tau)^{-1}$, etc. The absence of $-i0$ in this notation should not cause any problems, for we shall only consider this case, not the case of $+i0$. If $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^m)$ is identically 1 when $\lambda^2 - |\xi|^2 \geq \inf \Lambda - 1$ then

$$(1 - \phi) \tilde{B}_V(\lambda) \in \mathcal{B}(H^{r,s}(\mathbb{R}^N), H^{r+2,s}(\mathbb{R}^N)) \tag{2.17}$$

as follows from the expansion

$$(1-\phi)\tilde{B}_V(\lambda) = (1-\phi)\tilde{V}\rho_0(\lambda)\tilde{V} - \tilde{V}\rho_0(\lambda)(1-\phi)\tilde{B}_V(\lambda)$$

and Lemma 2.3. Again, these results also hold for other values of n, but (2.16) will have logarithmic terms.

Let

$$E_\lambda(\tau) = \{(\xi,\zeta) : \lambda^2 - |\xi|^2 = \tau\}, \qquad E_\lambda = \cup_{\tau\in\Lambda} E_\lambda(\tau). \tag{2.18}$$

We introduce the incoming and outgoing halves of their conormal bundles

$$N^\pm E_\lambda(\tau) = \{(\xi,\zeta,c\xi,0) : \lambda^2 - |\xi|^2 = \tau, \pm c > 0\}, \qquad N^\pm E_\lambda = \cup_{\tau\in\Lambda} N^\pm E_\lambda(\tau). \tag{2.19}$$

Restricted to $E_\lambda(0)\cap\mathbb{S}^{N-1}_\lambda$ $N^\pm E_\lambda(0) = N^\pm_\lambda$, and for $\tau\in\Lambda\setminus\{0\}$ $E_\lambda(\tau)\cap\mathbb{S}^{N-1}_\lambda = \emptyset$. We also define $0_{\mathbb{R}^N}$ to be the zero section of $T^*\mathbb{R}^N$, and let

$$Y_1' = \{(\xi,\zeta,\xi,\zeta') :\ (\xi,\zeta)\notin E_\lambda\} = Y_1\setminus(E_\lambda\times\mathbb{R}^N). \tag{2.20}$$

We have thus shown:

Lemma 2.5. *For $\lambda > 0$, $V\in\mathcal{S}(\mathbb{R}^n)$, the Schwartz kernel $K_{\tilde{B}_V(\lambda)}$ of $\tilde{B}_V(\lambda)$ is given by* (2.14)*, is in $I^{-n/2}(Y_1')$, and*

$$\mathrm{WF}(K_{\tilde{B}_V(\lambda)}) \subset N^*Y_1 \cup ((N^+E_\lambda\times 0_{\mathbb{R}^N})|_{Y_1} + N^*Y_1|_{E_\lambda\times\mathbb{R}^N}). \tag{2.21}$$

For $n\geq 3$ odd $\tilde{B}_V(\lambda)$ can be written as (2.16) *where B_1, B_2, $B_3(\tau,j)$ are Fourier integral operators with Schwartz kernel in $I^{-n/2}(Y_1)$.*

It will be convenient to identify $\mathbb{R}^N$ with the interior of its radial compactification $\mathbb{S}^N_+$ via the stereographic projection $\mathrm{SP}:\mathbb{R}^N\to\mathbb{S}^N_+$ (as in [29]); $\mathbb{S}^N_+$ is just the closed upper hemisphere of $\mathbb{S}^N$. The boundary of $\mathbb{S}^N_+$ is $\mathbb{S}^{N-1}$; it is thought of as the sphere at infinity. Imbedding $\mathbb{S}^N_+$ in $\mathbb{R}^{N+1}_y$ (so $\mathbb{S}^N_+ = \{y :\ |y| = 1,\ y_{N+1}\geq 0\}$), $\partial\mathbb{S}^N_+ = \mathbb{S}^{N-1}$ becomes imbedded in $\mathbb{R}^N \simeq \{y\in\mathbb{R}^{N+1} :\ y_{N+1} = 0\}$ as the unit sphere $\mathbb{S}^{N-1}_1$; in the following we often use this identification of $\partial\mathbb{S}^N_+$ with $\mathbb{S}^{N-1}_1$. If $\bar\theta\in\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$ is written as $(\theta_1,\theta_2)\in\mathbb{S}^{N-1}_1\subset\mathbb{R}^m\times\mathbb{R}^n = \mathbb{R}^N$ then the singular set is defined as

$$C = \{\bar\theta : \theta_2 = 0\}. \tag{2.22}$$

Thus, V is smooth on $\mathbb{S}^N_+\setminus C$ and vanishes to infinite order on $\mathbb{S}^{N-1}\setminus C$. We shall say that $\psi\in\mathcal{C}^\infty(\mathbb{S}^N_+)$ is a conic cutoff in a direction $\bar\theta\in\mathbb{S}^{N-1}$ if $\operatorname{supp}\psi$ intersects $\partial\mathbb{S}^N_+ = \mathbb{S}^{N-1}$ in a (small) neighborhood of $\bar\theta$. Note that $\mathrm{SP}^*(\dot{\mathcal{C}}^\infty(\mathbb{S}^N_+)) = \mathcal{S}(\mathbb{R}^N)$, and $\mathrm{SP}^*(\mathcal{C}^{-\infty}(\mathbb{S}^N_+)) = \mathcal{S}'(\mathbb{R}^N)$.

To facilitate our Fourier transformed approach we introduce

$$T(\lambda) = R_V(\lambda)V, \qquad \tilde{T}(\lambda) = \mathcal{F}T(\lambda)\mathcal{F}^{-1} = \rho_0(\lambda)(\tilde{V} - \tilde{B}_V(\lambda)). \tag{2.23}$$

We can now give a rather precise description of $\tilde{T}(\lambda)$, which will be transposed into a description of $T(\lambda)$. It will be more convenient to think of $\mathrm{WF}(u)$ as a subset of the cosphere bundle $S^*\mathbb{R}^N$ instead of being a conic subset of $T^*\mathbb{R}^N\setminus 0$. In fact, using radial compactification $\mathbb{S}^N_+$ we shall start working with the scattering calculus $\Psi^{*,*}_{\mathrm{sc}}(\mathbb{S}^N_+)$. This was introduced by Shubin [38], and it was generalized by Melrose in [29] to cover scattering metrics on manifolds with boundary, see also [31]. This is the calculus of pseudo-differential operators obtained by the usual (left, right or Weyl) quantization of (polyhomogeneous) symbols $a \in S^{k,l}$ which for all multi-indices $\alpha, \beta \in \mathbb{N}^N$ satisfy estimates

$$|D^\alpha_{\bar{w}} D^\beta_{\bar{\xi}} a(\bar{w}, \bar{\xi})| \le C_{\alpha,\beta} \langle\bar{\xi}\rangle^{k-|\beta|} \langle\bar{w}\rangle^{-l-|\alpha|}; \tag{2.24}$$

$S^{k,l}$ gives rise to operators in $\Psi^{\mathrm{k,l}}_{\mathrm{sc}}(\mathbb{S}^N_+)$. Alternatively, these operators can be described intrinsically on arbitrary manifolds with boundary; see [29]. These operators define the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$, which is a closed subset of the boundary $C_{\mathrm{sc}}\mathbb{S}^N_+$ of the compactified scattering cotangent bundle ${}^{\mathrm{sc}}\bar{T}^*\mathbb{S}^N_+$. Of course, over the interior of $\mathbb{S}^N_+$ these just correspond to the usual WF statements in the cosphere bundle, i.e. $\mathrm{WF}_{\mathrm{sc}}(u) \cap S^*\mathbb{R}^N = \mathrm{WF}(u)$.

Let $\psi_0 \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ be identically 1 near $\mathbb{S}^{N-1}_\lambda$, and let

$$\rho_1(\lambda) = \psi_0\rho_0(\lambda), \qquad \rho_2(\lambda) = (1-\psi_0)\rho_0(\lambda). \tag{2.25}$$

Then $\rho_1(\lambda)$ is compactly supported and so multiplication by it can be understood in terms of its wave front set only, while $\rho_2(\lambda) \in S^{-2,0}$, so

$$R_2(\lambda) = \mathcal{F}^{-1}\rho_2(\lambda)\mathcal{F} \in \Psi^{-2,0}_{\mathrm{sc}}(\mathbb{S}^N_+). \tag{2.26}$$

One of the most important properties of (scattering) pseudodifferential operators is that they are microlocal: $\mathrm{WF}_{\mathrm{sc}}(Au) \subset \mathrm{WF}_{\mathrm{sc}}(u)$ if $A \in \Psi^{*,*}_{\mathrm{sc}}(\mathbb{S}^N_+)$, $u \in \mathcal{S}'(\mathbb{R}^N)$ ([29], Proposition 6). We also define

$$\Psi^{\mathrm{k},\infty}_{\mathrm{sc}}(\mathbb{S}^N_+) = \cap_{l\in\mathbb{R}} \Psi^{\mathrm{k,l}}_{\mathrm{sc}}(\mathbb{S}^N_+); \tag{2.27}$$

this set contains the operators with 'trivial' behavior at infinity. Thus, if $u \in \mathcal{S}'(\mathbb{R}^N)$ with $\mathrm{WF}_{\mathrm{sc}}(u)$ a compact subset of ${}^{\mathrm{sc}}T^*_{\mathbb{S}^{N-1}}\mathbb{S}^N_+$, and $A \in \Psi^{\mathrm{k},\infty}_{\mathrm{sc}}(\mathbb{S}^N_+)$, then $Au \in \mathcal{S}(\mathbb{R}^N)$.

As expected from its definition, the scattering calculus is invariant under conjugation by the Fourier transform. To describe how the scattering wave front set transforms we introduce polar coordinates on $\mathbb{R}^N$: $\bar{w} = x^{-1}\bar{\theta}$, so $x = (\mathrm{SP}^{-1})^*|\bar{w}|^{-1}$ is a defining function of the boundary of $\mathbb{S}^N_+$. In these coordinates we can write a covector $v \in {}^{\mathrm{sc}}T^*_p\mathbb{S}^N_+$, $p = \mathrm{SP}(\bar{w})$, as

$$v = \tau x^{-2}\,dx + x^{-1}\mu \tag{2.28}$$

where $\mu \in T^*_{\bar{\theta}}\mathbb{S}^{N-1}$ (see [29]). The relationship between $\mathrm{WF}(\mathcal{F}u)$ and $\mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}T^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}}$ is then given by the Legendre diffeomorphism $L : {}^{\mathrm{sc}}T^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}} \to S^*\mathbb{R}^N$,

$$L(\bar{\theta}, \tau, \mu) = (\mu - \tau\bar{\theta}, -\bar{\theta}). \tag{2.29}$$

Namely, as shown in [31],

$$\mathrm{WF}(\mathcal{F}u) = L(\mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}T^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}}). \tag{2.30}$$

The characteristic set of the Laplacian in these coordinates is

$$\Sigma_{\Delta-\lambda^2} = \{(\bar{\theta}, \tau, \mu) : \bar{\theta} \in \mathbb{S}^{N-1},\ \tau^2 + |\mu|^2 = \lambda^2\}, \tag{2.31}$$

so $L(\Sigma_{\Delta-\lambda^2}) = S^*\mathbb{R}^N|_{\mathbb{S}^{N-1}_\lambda}$. Corresponding to $N^\pm_\lambda$

$$R^\pm_\lambda = \{(\bar{\theta}, \pm\lambda, 0) : \bar{\theta} \in \mathbb{S}^{N-1}\} \tag{2.32}$$

are the incoming and outgoing radial surfaces respectively, and we let

$$F^-(\lambda) = L^{-1}(N^- E_\lambda). \tag{2.33}$$

Our immediate goal is to understand the propagation of (scattering) singularities under the action of the operator $T(\lambda)$ as outlined in the Introduction. We now introduce a useful piece of notation which we shall employ until Corollary 2.20.

Definition 2.6. Let $\psi \in \mathcal{C}^\infty(\mathbb{S}^N_+)$ vanish to infinite order on a neighborhood $U \subset \mathbb{S}^{N-1}$ of C, and let $K = \mathbb{S}^{N-1} \setminus U$. Let $\tilde{\psi} = \mathcal{F}\psi\mathcal{F}^{-1} \in \Psi^{0,0}_{\mathrm{sc}}(\mathbb{S}^N_+)$ where we consider ψ as a multiplication operator.

Hence the subsets of $\mathbb{S}^{N-1}$ on which ψ and V do *not* vanish to infinite order, i.e. K and C respectively, are disjoint. This models the iterative scheme of the Introduction which shall be performed in detail in the next section. Propagation will be measured by the τ coordinate of elements of the scattering wave front set. For $u \in \mathcal{S}'(\mathbb{R}^N)$ let

$$\tau(u) = \min\{\lambda, \inf\{\tau : \exists \bar{\theta}, \mu \text{ s.t.} (\bar{\theta}, \tau, \mu) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap \Sigma_{\Delta-\lambda^2}\}\}, \tag{2.34}$$

$$\gamma(u) = \arccos(\tau(u)/\lambda). \tag{2.35}$$

Thus, $\gamma(u)$ is the supremum of the geodesic distance between $(1,0) \in \mathbb{S}^{N-1}$ and $\lambda^{-1}(\tau,\mu) \in \mathbb{S}^{N-1}$ for $(\bar{\theta},\tau,\mu) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap \Sigma_{\Delta-\lambda^2}$, since for ω, $\omega' \in \mathbb{S}^{N-1}$ the geodesic distance between ω and ω' is exactly

$$\mathrm{dist}(\omega,\omega') = \arccos(\omega \cdot \omega'). \tag{2.36}$$

For arbitrary $U_i \subset \mathbb{S}^{N-1}$ $(i = 1,2)$ let

$$\mathrm{dist}(U_1,U_2) = \inf\{\mathrm{dist}(\omega,\omega') : \omega \in U_1,\ \omega' \in U_2\}. \tag{2.37}$$

Our basic result will be that $\psi T(\lambda) = \psi R_V(\lambda) V$ decreases γ by $\mathrm{dist}(C,K)$ at least. In fact, we shall prove the following proposition (see Corollary 2.17), and we shall also obtain a more precise version in terms of continuity on weighted Sobolev spaces.

Proposition 2.7. *If $u \in \mathcal{S}'(\mathbb{R}^N)$, $\mathrm{WF}_{\mathrm{sc}}(u) \subset \Sigma_{\Delta-\lambda^2}$, $\mathrm{WF}_{\mathrm{sc}}(u) \cap F^-(\lambda) = \emptyset$ then $T(\lambda)u \in \mathcal{S}'(\mathbb{R}^N)$ is defined by continuity from $\mathcal{S}(\mathbb{R}^N)$, and for some k $T(\lambda)u \in H^{\infty,k}(\mathbb{R}^N)$. Here we use the notation of* (2.22)- (2.37). *Moreover,*

$$\mathrm{WF}_{\mathrm{sc}}(\psi T(\lambda) u) \subset \Sigma_{\Delta-\lambda^2} \setminus R^-_\lambda, \tag{2.38}$$

$$\gamma(\psi T(\lambda) u) \le \max(0, \gamma(u) - \mathrm{dist}(C,K)). \tag{2.39}$$

As the first step towards the proof, note that propagation of singularities (at infinity) for the free Laplacian must take place within the characteristic set, so there is no propagation due to $R_2(\lambda)$, the non-singular part of the free resolvent defined in (2.26). This is expressed in the following result which will allow us to concentrate on a compact region of phase space (thereby simplifying the necessary machinery).

Lemma 2.8. *If $\psi_i \in \mathcal{C}^{-\infty}(\mathbb{S}^N_+) \cap \mathcal{C}^\infty(\mathrm{int}(\mathbb{S}^N_+))$ $(i = 1,2)$ are such that for two disjoint closed subsets K_i of $\mathbb{S}^{N-1}$, $\psi_i \in \dot{\mathcal{C}}^\infty(\mathbb{S}^N_+ \setminus K_i)$, then $\psi_1 R_2(\lambda)\psi_2 \in \Psi^{-2,\infty}_{\mathrm{sc}}(\mathbb{S}^N_+)$.*

Proof. We shall use the notation of the Appendix of [29]. The Schwartz kernel K of $R_2(\lambda) \in \Psi^{-2,0}_{\mathrm{sc}}(\mathbb{S}^N_+)$ is conormal to sf and Δ_{sc} in $(\mathbb{S}^N_+)^2_{\mathrm{sc}}$. If $\pi^L_{\mathrm{sc}}, \pi^R_{\mathrm{sc}} : (\mathbb{S}^N_+)^2_{\mathrm{sc}} \to \mathbb{S}^N_+$ are the left and right projections then the kernel of $\psi_1 R_2(\lambda)\psi_2$ is just $K_\psi = ((\pi^L_{\mathrm{sc}})^*\psi_1)((\pi^R_{\mathrm{sc}})^*\psi_2)K$. But our assumptions imply that $((\pi^L_{\mathrm{sc}})^*\psi_1)((\pi^R_{\mathrm{sc}})^*\psi_2)$ is smooth in a neighborhood of sf (of course, it is also smooth in the interior of $(\mathbb{S}^N_+)^2_{\mathrm{sc}}$) and it vanishes at sf to infinite order, so K_ψ is the kernel of an operator in $\Psi^{-2,\infty}_{\mathrm{sc}}(\mathbb{S}^N_+)$. □

To simplify the notation let

$$\tilde{T}'(\lambda) = \rho_1(\lambda)(\tilde{V} - \tilde{B}_V(\lambda)), \qquad \tilde{T}''(\lambda) = \rho_2(\lambda)(\tilde{V} - \tilde{B}_V(\lambda)). \tag{2.40}$$

The kernel K_1 of $\tilde{T}'(\lambda)$ is simply $K_1 = (\rho_1(\lambda) \boxtimes 1_{\bar{\xi}'})K_{\tilde{V}-\tilde{B}_V(\lambda)}$, so Lemmas 2.2 and 2.5 show:

Lemma 2.9. *The Schwartz kernel K_1 of $\tilde{T}'(\lambda)$ satisfies*

$$\mathrm{WF}(K_1) \subset N^*Y_1 \cup (((N^+E_\lambda \cup N^+_\lambda) \times 0_{\mathbb{R}^N})|_{Y_1} + N^*Y_1|_{(\mathbb{S}^{N-1}_\lambda \cup E_\lambda)\times\mathbb{R}^N}). \tag{2.41}$$

In particular,

$$\mathrm{WF}(K_1) \cap ((N^-E_\lambda \cup N^-_\lambda) \times (T^*\mathbb{R}^N \setminus N^+E_\lambda)) = \emptyset, \tag{2.42}$$

$$\mathrm{WF}(K_1) \cap (0_{\mathbb{R}^N} \times T^*\mathbb{R}^N) \subset 0_{\mathbb{R}^N} \times N^+E_\lambda, \tag{2.43}$$

$$\mathrm{WF}(K_1) \cap (T^*\mathbb{R}^N \times 0_{\mathbb{R}^N}) \subset (N^+E_\lambda \cup N^+_\lambda) \times 0_{\mathbb{R}^N}. \tag{2.44}$$

Corollary 2.10. *Suppose that $(\bar{\xi}, \bar{\eta}, \bar{\xi}', \bar{\eta}') \in \mathrm{WF}'(K_1)$, $\omega' = \bar{\xi}'/\lambda \in \mathbb{S}^{N-1}$, $\bar{\eta}$, $\bar{\eta}' \neq 0$, $(\bar{\xi}', \bar{\eta}') \notin N^-E_\lambda$ and $\theta = \bar{\eta}/|\bar{\eta}| \notin C$. Let $\theta' = \bar{\eta}'/|\bar{\eta}'|$. Then $\omega = \bar{\xi}/\lambda \in \mathbb{S}^{N-1}$ and*

$$\mathrm{dist}(\omega, \theta) = \mathrm{dist}(\omega', \theta') - \mathrm{dist}(\theta, \theta'). \tag{2.45}$$

Proof. If $\bar{\xi} \notin \mathbb{S}^{N-1}_\lambda$ then $\bar{\eta} = \bar{\eta}' + c(\xi, 0)$ by the previous lemma where $c > 0$. Since $\bar{\eta}' = (\tilde{\xi}, 0)$, this implies that $\bar{\eta}/|\bar{\eta}| \in C$. So assume that $\bar{\xi} \in \mathbb{S}^{N-1}_\lambda$. Now, $\bar{\xi} = (\xi, \zeta)$, $\bar{\xi}' = (\xi, \zeta')$, $\theta' \in C$ and $\theta = \alpha\theta' + \beta\omega$ where $\alpha \geq 0$, $\beta > 0$. Thus, θ lies on the geodesic arc of length less than π between θ' and ω, so

$$\mathrm{dist}(\omega, \theta) = \mathrm{dist}(\omega, \theta') - \mathrm{dist}(\theta, \theta').$$

But $\omega - \omega' = (0, (\zeta - \zeta')/\lambda)$ and $\theta' \in C$, so $(\omega - \omega') \cdot \theta' = 0$, hence $\mathrm{dist}(\omega', \theta') = \mathrm{dist}(\omega, \theta')$ from which (2.45) follows. $\square$

The propagation result, Proposition 2.7, is essentially based on this corollary and on (2.30), but we still need to show that we have control over the action of $\tilde{T}(\lambda)$ on distributions which are not compactly supported, and over the behavior of the result at infinity in either case. First we prove that $\tilde{\psi}\tilde{T}''(\lambda)$ ($\tilde{\psi}$ defined in Definition 2.6) maps compactly supported distributions satisfying appropriate wave front set conditions to Schwartz functions. Now, Lemma 2.8 shows that for $\tilde{\psi}\rho_2(\lambda)\tilde{V}$, $\tilde{\psi}\rho_2(\lambda)B_j$, are all continuous from $\mathcal{C}_c^{-\infty}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$, while with $A \in \Psi^0_c(\mathbb{R}^N)$, $\mathrm{WF}'(A) \cap N^-E_\lambda = \emptyset$

$$(\lambda^2 - |\xi|^2)^{1/2}A : \mathcal{C}_c^{-\infty}(\mathbb{R}^N) \to \mathcal{C}_c^{-\infty}(\mathbb{R}^N)$$

is continuous. Similar results hold for the other singular terms so we obtain:

Lemma 2.11. *If $A \in \Psi^0_c(\mathbb{R}^N)$, $\mathrm{WF}'(A) \cap N^-E_\lambda = \emptyset$, $\tilde{\psi}$ as in Definition 2.6 then*

$$\tilde{T}''(\lambda)A : \mathcal{C}_c^{-\infty}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N),$$

$$\tilde{\psi}\tilde{T}''(\lambda)A : \mathcal{C}_c^{-\infty}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$$

are continuous.

Since $\rho_1(\lambda)$, $(\lambda^2-|\xi|^2)^{1/2}$, $(\lambda^2-|\xi|^2-\sigma)^{-j/2}$ $(j=1,2)$ have singularities, we need to obtain a microlocal understanding of the behavior of the corresponding multiplication operators on Sobolev spaces. For the sake of simplicity we restrict ourselves to $n\geq 3$ odd. The general case is similar, we only need to adjust some of the conormal factors. The proof of the following lemma is analogous to that of the usual statement on the wave front set of the product of two distributions.

Lemma 2.12. *Suppose $u\in H^{-l}_{loc}(\mathbb{R}^s)$ $(l\geq 0,\ s\in\mathbb{N})$, and $A\in\Psi^0_p(\mathbb{R}^N)$ (p stands for properly supported) with $\mathrm{WF}'(A)\cap\mathrm{WF}(u\boxtimes 1_{\mathbb{R}^{N-s}})=\emptyset$. Then, as a multiplication operator,*

$$u\boxtimes 1_{\mathbb{R}^{N-s}}:H^k_{loc}(\mathbb{R}^N)\to H^{-k}_{loc}(\mathbb{R}^N)\tag{2.46}$$

is continuous for $k>l+s/2$, as is

$$A(u\boxtimes 1_{\mathbb{R}^{N-s}}):H^k_{loc}(\mathbb{R}^N)\to H^{k-l-s/2}_{loc}(\mathbb{R}^N).\tag{2.47}$$

Moreover, if $A'\in\Psi^0_p(\mathbb{R}^N)$ with the property that $(x,\eta)\in\mathrm{WF}'(A')$, $(x,\eta')\in\mathrm{WF}(u\boxtimes 1_{\mathbb{R}^{N-s}})$ imply that $(x,\eta+\eta')\notin\mathrm{WF}'(A)$ then

$$A(u\boxtimes 1_{\mathbb{R}^{N-s}})A':H^k_{loc}(\mathbb{R}^N)\to\mathcal{C}^\infty(\mathbb{R}^N)\tag{2.48}$$

is continuous.

Since $\rho_0(\lambda)=F^*(t+i0)^{-1}$ where $F(\bar\xi)=|\bar\xi|^2-\lambda^2$, and the Sobolev spaces are invariant under diffeomorphisms, by introducing local coordinates (in small subsets of $\mathbb{R}^N$), one of which is F, we deduce:

Corollary 2.13. *If $A\in\Psi^0_c(\mathbb{R}^N)$, $\mathrm{WF}'(A)\cap N^+_\lambda=\emptyset$ then*

$$\rho_1(\lambda):H^k_{loc}(\mathbb{R}^N)\to H^{-1-\epsilon}_c(\mathbb{R}^N),\tag{2.49}$$

$$A\rho_1(\lambda):H^k_{loc}(\mathbb{R}^N)\to H^{k-1-\epsilon}_c(\mathbb{R}^N)\tag{2.50}$$

are continuous when $k>1$, $\epsilon>0$.

Remark 2.14. The regularity estimates are not optimal since we only use that $(t+i0)^{-1}\in H^{-1/2-\epsilon}(\mathbb{R})$, and not the more precise conormal regularity statement. In fact, $\rho_1(\lambda):H^{1/2+\epsilon}_{loc}(\mathbb{R}^N)\to H^{-1/2-\epsilon}_c(\mathbb{R}^N)$ is continuous, and (2.50) holds when we merely assume that $k>1/2$. However, it would be more complicated to keep the corresponding precision in the following lemmas, and we are primarily interested in the wave front set results of Lemma 2.9, so we opted for this weaker statement. Note also that the Besov spaces give a natural setting for dividing by functions with simple zeros such as $|\bar\xi|^2-\lambda^2$ (see Agmon's and Hörmander's paper [1], or [18]), but it seems to be difficult to work with these spaces in what follows.

For $A\in\Psi^{*,*}_{\mathrm{sc}}(\mathbb{S}^N_+)$ let

$$\tilde\gamma_\pm(A)=\sup\{\mathrm{dist}(\pm\omega,\theta):\ \omega,\theta\in\mathbb{S}^{N-1},\ (\lambda\omega,\theta)\in\mathrm{WF}'_{\mathrm{sc}}(A)\}.\tag{2.51}$$

Here we again identify the unit sphere $\mathbb{S}^{N-1}=\mathbb{S}^{N-1}_1\subset\mathbb{R}^N$ with $\mathbb{S}^{N-1}=\partial\mathbb{S}^N_+$. Thus, $\tilde\gamma_\pm(A)$ measure how far the operator wave front set of A can be from N^+_λ and N^-_λ respectively. First we obtain a Sobolev space version of our wave front relation estimate (2.41); in the proof $\pi_2:S^*\mathbb{R}^N\to\mathbb{S}^{N-1}$ will denote the projection to the fiber.

Lemma 2.15. *Suppose that $A \in \Psi_c^0(\mathbb{R}^N)$ and $\tilde{\gamma}_-(A) < \operatorname{dist}(\operatorname{supp}\psi \cap \mathbb{S}^{N-1}, C)/3$. Then for $r > 2$*

$$\tilde{T}'(\lambda) : H^{r,s}(\mathbb{R}^N) \to \mathcal{C}_c^{-\infty}(\mathbb{R}^N), \tag{2.52}$$

$$\tilde{\psi} A\tilde{T}'(\lambda) : H^{r,s}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N) \tag{2.53}$$

are continuous.

Proof. First of all, $\tilde{V}, B_j \in \mathcal{B}(H^{r,s}, H^{r,s})$, so it suffices to consider $\rho_1(\lambda)$, $\rho_1(\lambda)(\lambda^2 - |\xi|^2)^{-1}$, etc. as multiplication operators on $H^{r,s}$. In fact, we shall only consider the second operator; the others are easier to deal with. Let $A' \in \Psi_c^0(\mathbb{R}^N)$ be such that $\mathrm{WF}'(A') \cap N^+E_\lambda = \emptyset$ and $\mathrm{WF}'(\mathrm{Id} - A') \cap N_\lambda^- = \emptyset$. By Lemma 2.12

$$(\lambda^2 - |\xi|^2)^{-1} : H^{r,s}(\mathbb{R}^N) \to \mathcal{C}^{-\infty}(\mathbb{R}^N),$$

$$A'(\lambda^2 - |\xi|^2)^{-1} : H^{r,s}(\mathbb{R}^N) \to H^{r-1-\epsilon,s}(\mathbb{R}^N)$$

are continuous. As $\rho_1(\lambda)(\mathrm{Id} - A') : \mathcal{C}^{-\infty}(\mathbb{R}^N) \to \mathcal{C}_c^{-\infty}(\mathbb{R}^N)$ is continuous, these facts together with Corollary 2.13 prove (2.52).

Since (2.53) requires better wave front set estimates, we shall only check the terms $\rho_1(\lambda)\tilde{V}$ and $\rho_1(\lambda)B_1$. To obtain the other ones we must apply (2.48) twice. So let $A'' \in \Psi_c^0(\mathbb{R}^N)$ be such that

$$\operatorname{dist}(-\pi_2(\mathrm{WF}'(A'')), \operatorname{supp}\psi \cap \mathbb{S}^{N-1}) > \operatorname{dist}(C, \operatorname{supp}\psi \cap \mathbb{S}^{N-1})/2, \tag{2.54}$$

$\mathrm{WF}'(\mathrm{Id} - A'') \cap (\operatorname{supp}\rho_1(\lambda) \times C) = \emptyset$. Then $\phi(\mathrm{Id} - A'')\tilde{V} : H^{r,s} \to \mathcal{C}_c^\infty(\mathbb{R}^N)$ for $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ supported sufficiently close to $\operatorname{supp}\rho_1(\lambda)$, identically 1 on $\operatorname{supp}\rho_1(\lambda)$, and similarly for B_j, so it suffices to deal with the terms involving A''. Now, if $(\bar{\xi}', \bar{\eta}') \in \mathrm{WF}'(A'')$, $\bar{\eta} = \bar{\eta}' + c\bar{\xi}$, $c > 0$, $\theta = \bar{\eta}/|\bar{\eta}|$, $\theta' = \bar{\eta}'/|\bar{\eta}'|$ then similarly to Corollary 2.10

$$\operatorname{dist}(-\bar{\xi}/\lambda, \theta) = \operatorname{dist}(-\bar{\xi}'/\lambda, \theta') + \operatorname{dist}(\theta, \theta').$$

Since $(\bar{\xi}, \bar{\eta}) \in \mathrm{WF}'(\tilde{\psi}A)$ implies $-\theta \in \operatorname{supp}\psi \cap \mathbb{S}^{N-1}$, by (2.54)

$$\operatorname{dist}(\theta, \theta') > \operatorname{dist}(C, \operatorname{supp}\psi \cap \mathbb{S}^{N-1})/2,$$

so $\operatorname{dist}(-\bar{\xi}/\lambda, \theta) > \operatorname{dist}(C, \operatorname{supp}\psi \cap \mathbb{S}^{N-1})/2$. Hence $(\bar{\xi}, \theta) \notin \mathrm{WF}'(\tilde{\psi}A)$, so

$$\tilde{\psi}A\rho_1(\lambda)A'' : H^{1+\epsilon,s} \to \mathcal{S}(\mathbb{R}^N)$$

is continuous by (2.48) which then proves that $\tilde{\psi}A\rho_1(\lambda)\tilde{V}$ has the same mapping property. Similar, though somewhat more complicated, estimates work also for the terms involving $(\lambda^2 - |\xi|^2)^{-1}$, thus proving the lemma. $\square$

Lemma 2.16. *For $r > 2$, $r', s \in \mathbb{R}$*

$$\tilde{\psi}\tilde{T}''(\lambda) \in \mathcal{B}(H^{r,s}, H^{r',s+2}). \tag{2.55}$$

For $r' < -1/2$, $r, s \in \mathbb{R}$

$$\tilde{T}''(\lambda)\tilde{\psi} \in \mathcal{B}(H^{r,s}, H^{r',s+2}). \tag{2.56}$$

Proof. By Lemma 2.8 $\tilde\psi\rho_2(\lambda)\tilde V \in \Psi_{\mathrm{sc}}^{-\infty,0}(\mathbb{S}^N_+)$. Since $\tilde B_V(\lambda) = \tilde V\rho_0(\lambda)(\tilde V - \tilde B_V(\lambda))$, it suffices to prove that $\rho_0(\lambda)(\tilde V - \tilde B_V(\lambda)) \in \mathcal{B}(H^{r,s}, H^{-2-\epsilon,s})$, $\epsilon > 0$, to obtain (2.55). This holds since $\tilde V, B_j \in \mathcal{B}(H^{r,s}, H^{r,s})$, and, as a multiplication operator,

$$(\lambda^2 - |\xi|^2)^{-1}\rho_0(\lambda) \in \mathcal{B}(H^{r,s}, H^{-2-\epsilon,s}).$$

Since $\tilde V\tilde\psi \in \Psi_{\mathrm{sc}}^{-\infty,0}(\mathbb{S}^N_+)$, (2.56) can be proved similarly. □

Now we can describe the action of $\tilde T(\lambda)$ on distributions with nice scattering wave front set properties using these lemmas. For $u \in \mathcal{S}'(\mathbb{R}^N)$ let

$$\tilde\tau(u) = \min\{\lambda, \inf\{\bar\xi\cdot\bar\theta : \bar\xi \in \mathbb{S}^{N-1}_\lambda, (\bar\xi,\bar\theta) \in \mathrm{WF}(u)\}\}, \tag{2.57}$$

$$\tilde\gamma(u) = \arccos(\tilde\tau(u)/\lambda). \tag{2.58}$$

Thus, $\tilde\tau(u) \in [-\lambda,\lambda]$ is (λ times) the infimum of the 'radial' components of the covectors in $\mathrm{WF}(u)$ which lie above points on $\mathbb{S}^{N-1}_\lambda$, and $\tilde\gamma(u) \in [0,\pi]$ is the supremum of the angles between such covectors and the outgoing radial direction. Note that $\tau(u) = \tilde\tau(\mathcal{F}u)$, and $\gamma(u) = \tilde\gamma(\mathcal{F}u)$, so the following Corollary proves Proposition 2.7.

Corollary 2.17. *Suppose $u \in \mathcal{S}'(\mathbb{R}^N)$ satisfies $\mathrm{WF}_{\mathrm{sc}}(u) \subset S^*\mathbb{R}^N|_{\mathbb{S}^{N-1}_\lambda}$, and in addition $\mathrm{WF}(u)\cap N^-E_\lambda(0) = \emptyset$. Then $v = \tilde T(\lambda)u \in \mathcal{S}'(\mathbb{R}^N)$ is defined by continuity from $\mathcal{C}^\infty_c(\mathbb{R}^N)$, $v \in H^{k,\infty}(\mathbb{R}^N)$ for some k, and*

$$\mathrm{WF}(v) \cap (N^-_\lambda \cup N^-E_\lambda) = \emptyset. \tag{2.59}$$

Moreover, with $\tilde\psi$ as in Definition 2.6, $\mathrm{WF}_{\mathrm{sc}}(\tilde\psi v) \subset S^*\mathbb{R}^N|_{\mathbb{S}^{N-1}_\lambda}$ and

$$\tilde\gamma(\tilde\psi v) \le \max(0, \tilde\gamma(u) - \mathrm{dist}(C,K)). \tag{2.60}$$

Proof. Write

$$\tilde T(\lambda) = \tilde T'(\lambda)\psi_0 + \tilde T'(\lambda)(1-\psi_0) + \tilde T''(\lambda)\psi_0 + \tilde T''(\lambda)(1-\psi_0),$$

and apply Lemma 2.9 and Corollary 2.10 to $\tilde T'(\lambda)\psi_0$, Lemma 2.15 to $\tilde T'(\lambda)(1-\psi_0)$, Lemma 2.16 to $\tilde T''(\lambda)(1-\psi_0)$, and Lemma 2.11 to $\tilde T''(\lambda)\psi_0$. □

We need to make a modification of the unreduced two-body resolvents in the case when $\gamma(u) = 0$. In this case Proposition 2.7 shows that $T(\lambda)u$ is defined, but it does not give any improvement for $\gamma(\psi T(\lambda)u)$. The improvement is important for we would like to find v with $\mathrm{WF}_{\mathrm{sc}}(v) \subset R^+_\lambda \cup \mathrm{cl}(F^+(\lambda))$, $(\Delta + V - \lambda^2)v - Vu \in \mathcal{S}(\mathbb{R}^N)$ and $\psi v \in \mathcal{S}(\mathbb{R}^N)$ when $\operatorname{supp}\psi$ is disjoint from the singular set. We can achieve this by using a cut off function in phase space. The main reason why this approach works is that there is no propagation at R^+_λ, and such cut-offs essentially commute with the Hamiltonian away from $\cup_i C_i$ (in an appropriate sense).

Proposition 2.18. *Suppose that $\rho \in \mathcal{C}^\infty(\mathbb{R}^N)$ is such that $\operatorname{supp}(1-\rho)$ is compact, $\operatorname{supp}(1-\rho) \subset \{(\xi,\zeta) : |\xi| < \lambda\}$, and $\theta \in \operatorname{supp}\psi \cap \mathbb{S}^{N-1}$ implies $-\lambda\theta \notin \operatorname{supp}\rho$; here ψ is as in Definition 2.6. Suppose also that $A \in \Psi^0_c(\mathbb{R}^N)$,*

$$\tilde\tau_+(A) > \sup\{\bar\xi\cdot\theta :\ \bar\xi \in \operatorname{supp}(1-\rho),\ \theta \in C\}, \tag{2.61}$$

$$\tilde\tau_+(A) > \sup\{\bar\xi\cdot\theta :\ \bar\xi \in \operatorname{supp}\rho \cap \mathbb{S}^{N-1}_\lambda,\ -\theta \in \operatorname{supp}\psi \cap \mathbb{S}^{N-1}\}, \tag{2.62}$$

where $\tilde{\tau}_+(A) = \inf\{\bar{\xi}\cdot\theta : \ (\bar{\xi},\theta) \in \mathrm{WF}'(A)\}$. *Then* $\rho\tilde{T}(\lambda)A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ *is continuous, as are*

$$(|\bar{\xi}|^2 + \tilde{V} - \lambda^2)(1-\rho)\tilde{T}(\lambda)A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N), \tag{2.63}$$

$$\tilde{\psi}\rho\tilde{T}(\lambda)A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N). \tag{2.64}$$

Proof. Continuity of $\rho\tilde{T}(\lambda)A$ as stated follows from our earlier results. Next,

$$(1-\rho)(\tilde{V} - \tilde{B}_V(\lambda))A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N). \tag{2.65}$$

In fact, if $((\bar{\xi},\bar{\eta}),(\bar{\xi}',\bar{\eta}'))$ is in the wave front relation of $(1-\rho)(\tilde{V} - \tilde{B}_V(\lambda))A$ and $\bar{\eta}' = 0$ then $(\bar{\xi},\bar{\eta}) \in N^+E_\lambda$ (by Lemma 2.5) and $\bar{\xi} \in \mathrm{supp}(1-\rho)$ which is impossible. If $\bar{\eta}' \neq 0$ then by Lemma 2.5 (cf. the proof of Corollary 2.10)

$$\bar{\xi}\cdot\frac{\bar{\eta}}{|\bar{\eta}|} \geq \bar{\xi}'\cdot\frac{\bar{\eta}'}{|\bar{\eta}'|} \geq \tilde{\tau}_+(A) \tag{2.66}$$

which contradicts (2.61) since $\frac{\bar{\eta}}{|\bar{\eta}|} \in C$ and $\bar{\xi} \in \mathrm{supp}(1-\rho)$. Hence the wave front relation of $(1-\rho)(\tilde{V} - \tilde{B}_V(\lambda))A$ is empty, i.e. this operator is smoothing. Now, $\rho_0(\lambda)$ can be commuted through $1-\rho$, so the term of (2.63) arising from $|\bar{\xi}|^2 - \lambda^2$ is exactly (2.65). On the other hand, $\mathrm{supp}(1-\rho) \cap E_\lambda(0) = \emptyset$, and applied to smooth functions supported away from $E_\lambda(0)$, $\rho_0(\lambda)$ does not give any wave front set in the direction of C, so

$$\tilde{V}\rho_0(\lambda)(1-\rho)(\tilde{V} - \tilde{B}_V(\lambda))A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$$

continuously. Finally, to show (2.64) suppose that $((\bar{\xi},\bar{\eta}),(\bar{\xi}',\bar{\eta}'))$ is in the wave front relation of $\tilde{\psi}\rho\tilde{T}(\lambda)A$. If $\bar{\eta}' = 0$ then $(\bar{\xi},\bar{\eta}) \in N^+_\lambda \cup N^+E_\lambda$ by Lemma 2.9, $-\frac{\bar{\eta}}{|\bar{\eta}|} \in \mathrm{supp}\,\psi \cap \mathbb{S}^{N-1}$, $\bar{\xi} \in \mathrm{supp}\,\rho$ contradicting the assumptions on ρ. If $\bar{\eta}' \neq 0$ then by Lemma 2.9 (2.66) still holds. On the other hand, $-\frac{\bar{\eta}}{|\bar{\eta}|} \in \mathrm{supp}\,\psi \cap \mathbb{S}^{N-1}$, $\bar{\xi} \in \mathrm{supp}\,\rho$ which is impossible by (2.62), so $\tilde{\psi}\rho\tilde{T}(\lambda)A$ is indeed smoothing. □

Remark 2.19. Note that the expressions on the right hand side of (2.61) and (2.62) are strictly less than λ by the assumptions on ρ and ψ.

Corollary 2.20. *Let* ψ, ρ *be as in the previous proposition, and let* $\tilde{\rho} = \mathcal{F}^{-1}\rho\mathcal{F}$. *If* $u \in \mathcal{S}'(\mathbb{R}^N)$, $\mathrm{WF}_{\mathrm{sc}}(u) \subset R^+_\lambda$ *then* $v = \tilde{\rho}R_V(\lambda)Vu \in \mathcal{S}'(\mathbb{R}^N)$ *is in* $H^{\infty,k}(\mathbb{R}^N)$ *for some* k, *and it satisfies*

$$(\Delta + V - \lambda^2)(v - T(\lambda)u) \in \mathcal{S}(\mathbb{R}^N), \tag{2.67}$$

$\psi v \in \mathcal{S}(\mathbb{R}^N)$, $\mathrm{WF}_{\mathrm{sc}}(v) \subset R^+_\lambda \cup \mathrm{cl}(F^+(\lambda))$.

Finally, we note a result on the product of conormal distributions and some of its consequences which will be useful in Section 6.

Lemma 2.21. *Suppose* $u \in I^r(Y)$, $v \in I^s(Y')$ *where* Y *and* Y' *are transverse submanifolds of* $\mathbb{R}^M$. *If* $A \in \Psi^0_p(\mathbb{R}^M)$ *with*

$$\mathrm{WF}'(A) \cap (N^*Y \cup N^*Y') = \emptyset \tag{2.68}$$

then $A(uv) \in I^{r+s+M/4}(Y \cap Y')$.

Proof. This lemma is local so we may assume that Y is defined by $x = 0$, Y' by $y = 0$ where (x, y, z) are coordinates on $\mathbb{R}^M$. If

$$u = \int e^{ix\cdot\eta} a(\eta, x, y, z)\, d\eta, \qquad v = \int e^{iy\cdot\zeta} b(\zeta, x, y, z)\, d\zeta \tag{2.69}$$

where

$$a \in S^{r+(M-2n_x)/4}(\mathbb{R}^{n_x}; \mathbb{R}^M), \qquad b \in S^{s+(M-2n_y)/4}(\mathbb{R}^{n_y}; \mathbb{R}^M) \tag{2.70}$$

then

$$\psi ab \in S^{r+s+(2M-2n_x-2n_y)/4}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}; \mathbb{R}^M) \tag{2.71}$$

when $\psi \in \mathcal{C}^\infty(\mathbb{S}_+^{n_x+n_y})$ is supported away from the directions $(\eta, 0)$ and $(0, \zeta)$. Thus,

$$\mathcal{F}^{-1}_{\eta,\zeta} \psi \mathcal{F}_{x,y} uv = \int e^{i(x\cdot\eta + y\cdot\zeta)} \psi(\eta, \zeta) a(\eta, x, y, z) b(\zeta, x, y, z)\, d\eta\, d\zeta \tag{2.72}$$

is in $I^{r+s+M/4}(Y \cap Y')$. This proves the proposition if we choose ψ so that it is identically 1 on the projection of $\mathrm{WF}'(A)$ to the (η, ζ) variables. (Note that we can assume that A has compact support.) $\square$

Let Y_1 be as in (2.3), and let $Y_2 = \mathbb{S}_\lambda^{N-1} \times \mathbb{R}^N$. Then Y_1 and Y_2 are transverse and

$$Y_1 \cap Y_2 = \{(\xi, \zeta, \xi, \zeta') : |\xi|^2 + |\zeta|^2 = \lambda^2\}. \tag{2.73}$$

Let

$$M = \{(\xi, \zeta, \tilde{\xi}, 0) :\ \xi, \tilde{\xi} \in \mathbb{R}^m,\ \zeta \in \mathbb{R}^n,\ \tilde{\xi} \neq 0\} = \pi_L(N^*Y_1 \cap (T^*\mathbb{R}^{2N} \setminus 0))$$

where $\pi_L : T^*\mathbb{R}^N \times T^*\mathbb{R}^N \to T^*\mathbb{R}^N$ is projection to the left factor.

Corollary 2.22. *If $A, A' \in \Psi_c^0(\mathbb{R}^N)$, $\mathrm{WF}'(A') \cap N^- E_\lambda = \emptyset$, $\mathrm{WF}'(A) \cap (N_\lambda^+ \cup M) = \emptyset$, then $A\rho_0(\lambda)\tilde{V}A'$ is a Fourier integral operator with Schwartz kernel $K \in I^{-(n-1)/2}(Y_1 \cap Y_2)$. The same result holds when $\tilde{V}$ is replaced by $\tilde{B}_V(\lambda)$.*

Proof. By Lemma 2.2 the Schwartz kernel of $\tilde{V}$ is conormal to Y_1. Since $A' \in \Psi_c^0(\mathbb{R}^N)$ the composition $\tilde{V}A'$ is still an FIO with kernel conormal to Y_1, but now

$$\mathrm{WF}(K_{\tilde{V}A'}) \subset N^*Y_1 \setminus (T^*\mathbb{R}^N \times N^+E_\lambda(0)). \tag{2.74}$$

By Lemma 2.1 $\rho_0(\lambda)_{\bar{\xi}} \boxtimes 1_{\bar{\xi}'}$ is conormal to Y_2. Thus, the Schwartz kernel of $\rho_0(\lambda)\tilde{V}A'$ is $K_0 = (\rho_0(\lambda) \boxtimes 1)K_{\tilde{V}A'}$, and by (2.74)

$$\mathrm{WF}(K_0) \subset (N^*Y_1 \cup N^*Y_2 \cup N^*(Y_1 \cap Y_2)) \cap ((T^*\mathbb{R}^N \setminus 0) \times T^*\mathbb{R}^N. \tag{2.75}$$

Let $B \in \Psi_p^0(\mathbb{R}^{2N})$ be such that $\mathrm{WF}'(B) \cap ((M \cup N_\lambda^+) \times T^*\mathbb{R}^N) = \emptyset$ (so in particular $\mathrm{WF}'(B) \cap (N^*Y_1 \cup N^*Y_2) = \emptyset$), and $\mathrm{WF}'(\mathrm{Id} - B)$ is sufficiently close to $(M \cup N_\lambda^+) \times T^*\mathbb{R}^N$. The previous lemma shows that $BK_0 \in I^{-(n-1)/2}(Y_1 \cap Y_2)$. Let T_1 be the operator with kernel BK_0, T_2 the operator with kernel $(\mathrm{Id} - B)K_0$. Now, the kernel of AT_1 is in $I^{-(n-1)/2}(Y_1 \cap Y_2)$ by the composition theorem of FIOs. On the other hand, by our assumption on A

$$\mathrm{WF}((\mathrm{Id} - B)K_0) \cap \{(\bar{\xi}, \bar{\xi}', \bar{\eta}, \bar{\eta}') : (\bar{\xi}, -\bar{\eta}) \in \mathrm{WF}'(A)\} = \emptyset$$

if $\mathrm{WF}'(\mathrm{Id} - B)$ is sufficiently close to $(M \cup N_\lambda^+) \times T^*\mathbb{R}^N$. Hence, with π_C denoting projection off the $\bar{\xi}$ variable,

$$K_{AT_2} = (\pi_C)_*((K_A \boxtimes 1_{\bar{\xi}'})(1_{\bar{\xi}''} \boxtimes (\mathrm{Id} - B)K_0)) \in \mathcal{C}_c^\infty(\mathbb{R}^{2N}), \tag{2.76}$$

i.e. AT_2 is smoothing. Therefore, $A\rho_0(\lambda)\tilde{V}A' = AT_1 + AT_2$ is indeed an FIO. The proof is very similar for $A\rho_0(\lambda)\tilde{B}_V(\lambda)A'$, since the additional wave front set of the kernel of $\rho_0(\lambda)\tilde{B}_V(\lambda)A'$ is in $M \times T^*\mathbb{R}^N$, and hence (2.76) still holds. □

3. The parametrix construction

For the three-body problem our basic space will be $\mathbb{R}^N$, and we shall study perturbations of the Laplacian Δ on it. More precisely, let X_i, $i = 1, ..., M$ be linear subspaces of $\mathbb{R}^N$, and assume that for $i \neq j$ $X_i \cap X_j = \{0\}$. Let $m_i = \dim X_i$, $n_i = \operatorname{codim} X_i$, and introduce the orthogonal projections $\pi_i : \mathbb{R}^N \to X_i$ and $\pi^i : \mathbb{R}^N \to X^i$ where $X^i = X_i^\perp$. (This set-up is described, for example, in [4, 13, 20, 24, 39, 44].) In accordance with the notation of the unreduced two-body problem we shall write $w^i = \pi_i$, $z^i = \pi^i$ when (w^i, z^i) are thought of as coordinates on $\mathbb{R}^N$, and correspondingly (ξ^i, ζ^i) for the dual coordinates. Let $V_i \in \mathcal{S}(X^i)$ be real valued, $\Lambda_i = \operatorname{spec}_p(\Delta_{z^i} + V_i) \cup \{0\}$ be the set of thresholds, and define Λ_i' analogously to (2.9). We also introduce

$$N^\pm E^i_\lambda = \{(\xi^i, \zeta^i, c\xi^i, 0) : \pm c > 0,\ \lambda^2 - |\xi^i|^2 \in \Lambda_i\}. \tag{3.1}$$

Note that in Sigal's and Soffer's notation, [39], $\cup_i N^* E^i_\lambda$ corresponds to the set PS_{λ^2} in whose complement there is 'no propagation' in the sense of Kato-smoothness. Recall also that $\Sigma_{\Delta-\lambda^2}$ and $R^\pm_\lambda$ have been defined in (2.31)-(2.32) and let $F_i^-(\lambda)$ be the image of $N^- E^i_\lambda$ under the inverse of the Legendre diffeomorphism (given by (2.29)). We shall consider the perturbed Laplacian

$$H_V = \Delta + V, \qquad V = \sum_{i=1}^M (\pi^i)^* V_i; \tag{3.2}$$

of course, we shall drop $(\pi^i)^*$ from now on and consider V_i as a function on $\mathbb{R}^N$ as well as a function on X^i. This is the reduced three-body Laplacian if we take $M = 3$, but this particular value of M plays no role in the analysis. In fact, the only important condition is that $X_i \cap X_j = \{0\}$. To see the effect of this assumption consider the radial compactification $\mathbb{S}^N_+$ of $\mathbb{R}^N$. Then the singular sets C_i are defined as

$$C_i = \bar{X}_i \cap \mathbb{S}^{N-1} \tag{3.3}$$

where $\mathbb{S}^{N-1} = \partial \mathbb{S}^N_+$, and $\bar{X}_i$ is the closure of the image of X_i under SP. The condition on the X_i is then equivalent to the statement that for $i \neq j$ $C_i \cap C_j = \emptyset$, i.e. the singular sets of the different potentials are separated. This will allow us to use the unreduced two-body resolvents to construct a parametrix for the Poisson operator and the resolvent in the (generalized) three-body case.

The construction proceeds as in the Introduction, though we construct approximate eigenfunctions (e.g. perturbed plane waves) instead. So suppose $u \in \mathcal{S}'(\mathbb{R}^N)$, $(\Delta - \lambda^2)u = 0$. This implies that $\mathrm{WF}_{\mathrm{sc}}(u) \subset \Sigma_{\Delta-\lambda^2}$. We assume in addition that, with R^-_λ as in (2.32),

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap R^-_\lambda \cap {}^{\mathrm{sc}}\bar{T}^* \mathbb{S}^N_+|_{C_i} = \emptyset \tag{3.4}$$

for each i, i.e. u has no incoming scattering wave front set over the singular set. This is satisfied, for example, if $u = e^{i\lambda\omega\cdot\bar{w}}$ with $\omega \notin \cup_i C_i$, i.e. for plane waves starting away from the singular set. If we apply the perturbed Laplacian $\Delta + V - \lambda^2$ to u, the error is Vu. To remove this we would like to take $v = u - R_V(\lambda)Vu$ where $R_V(\lambda)$

would be the three-body resolvent, since then we would have $(\Delta+V-\lambda^2)v=0$. As a first approximation for $R_V(\lambda)V$ we could take $\sum_i T_i(\lambda)$ where $T_i(\lambda)=R_{V_i}(\lambda)V_i$; in general we iterate this.

To simplify the bookkeeping we introduce the following notation. Let I_k be the set of k-tuples α of integers in $\{1,...,M\}$ such that $\alpha_i\neq\alpha_{i+1}$ for $i=1,...,k-1$. For $\alpha\in I_k$ we shall write $\alpha'=(\alpha_2,\alpha_3,...,\alpha_k)\in I_{k-1}$ and $\alpha''=(\alpha_3,...,\alpha_k)\in I_{k-2}$, and let

$$T_\alpha(\lambda)=T_{\alpha_1}(\lambda)T_{\alpha_2}(\lambda)...T_{\alpha_k}(\lambda),\qquad T^{(k)}(\lambda)=\sum_{j=1}^{k}(-1)^j\sum_{\alpha\in I_j}T_\alpha(\lambda). \tag{3.5}$$

Let $\psi_i\in\mathcal{C}^\infty(\mathbb{S}^N_+)$ be identically 1 near C_i (so $(1-\psi_i)V_i\in\mathcal{S}(\mathbb{R}^N)$), $\operatorname{supp}\psi_i\cap C_j=\emptyset$ for $i\neq j$. Inserting factors of ψ_{α_j} and $1-\psi_{\alpha_j}$ between each $T_{\alpha_j}(\lambda)$ and $T_{\alpha_{j+1}}(\lambda)$ and applying Proposition 2.7 and Lemma 2.16 respectively $T^{(k)}(\lambda)u$ is defined for $u\in\mathcal{S}'(\mathbb{R}^N)$ satisfying (3.4). We claim that

$$(\Delta+V-\lambda^2)(\mathrm{Id}+T^{(k)}(\lambda))u=(\Delta-\lambda^2)u+\sum_{\alpha\in I_{k+1}}(-1)^kV_{\alpha_1}T_{\alpha'}(\lambda)u. \tag{3.6}$$

In fact, this follows by induction since it holds for $k=0$ and if it holds for $k-1$ then

$$\begin{aligned}(\Delta+V-\lambda^2)(\mathrm{Id}+T^{(k)}(\lambda))u=&(\Delta-\lambda^2)u+(-1)^{k-1}\sum_{\alpha\in I_k}V_{\alpha_1}T_{\alpha'}(\lambda)u\\&+(-1)^k\sum_{\alpha\in I_k}(\Delta+V_{\alpha_1}-\lambda^2)T_{\alpha_1}(\lambda)T_{\alpha'}(\lambda)u\\&+(-1)^k\sum_{\alpha\in I_k}\sum_{j\neq\alpha_1}V_jT_\alpha(\lambda)u\end{aligned}$$

proving the result since the first two terms cancel. Moreover, for $\alpha\in I_{k+1}$

$$\gamma(\psi_{\alpha_1}T_{\alpha'}(\lambda)u)\leq\max(0,\gamma(\psi_{\alpha_2}T_{\alpha''}(\lambda)u)-\operatorname{dist}(\operatorname{supp}\psi_{\alpha_1}\cap\mathbb{S}^{N-1},C_{\alpha_2})). \tag{3.7}$$

Then induction shows that

$$\gamma(\psi_{\alpha_1}T_{\alpha'}(\lambda)u)\leq\max(0,\gamma(\psi_{\alpha_{k+1}}u)-(k-1)C)\leq\max(0,\pi-(k-1)C) \tag{3.8}$$

where

$$C=\min\{\operatorname{dist}(C_i,C_j):\ i\neq j,\ i,j\in\{1,...,M\}\} \tag{3.9}$$

since the cutoffs ψ_{α_j} do not appear on the left hand side of (3.8) for $j\neq1$, $j\neq k+1$, and their support can be chosen to lie arbitrarily close to C_{α_j}. Thus, for k large enough (and independent of u) $\gamma(\psi_{\alpha_1}T_{\alpha'}(\lambda)u)=0$, so

$$\mathrm{WF}_{\mathrm{sc}}(\psi_{\alpha_1}T_{\alpha'}(\lambda)u)\subset R^+_\lambda.$$

We can actually improve the error $(-1)^k\sum_{\alpha\in I_{k+1}}V_{\alpha_1}T_{\alpha'}(\lambda)$ to a Schwartz function by using that there is no propagation at R^+_λ. Thus, we take $\tilde\rho_{\alpha_1}T_{\alpha_1}(\lambda)T_{\alpha'}(\lambda)u$ instead of $T_{\alpha_1}(\lambda)T_{\alpha'}(\lambda)u$, $\alpha\in I_{k+1}$, in the next step of the iteration. Here we choose ρ_i as in Proposition 2.18, that is $\rho_i\in\mathcal{C}^\infty(\mathbb{R}^N)$, $\operatorname{supp}(1-\rho_i)$ is compact,

$\operatorname{supp}(1-\rho_i) \subset \{(\xi^i,\zeta^i) : |\xi^i| < \lambda\}$, for $i \neq j$ $\theta \in \operatorname{supp}\psi_j$ implies $-\lambda\theta \notin \operatorname{supp}\rho_i$, and we let $\tilde{\rho}_i = \mathcal{F}^{-1}\rho_i\mathcal{F}$. Thus let

$$T^\flat(\lambda) = T^{(k)}(\lambda) + (-1)^{k+1} \sum_{\alpha \in I_{k+1}} \tilde{\rho}_{\alpha_1} T_\alpha(\lambda). \tag{3.10}$$

Corollary 2.20 shows that

$$(\Delta + V_{\alpha_1} - \lambda^2)(1 - \tilde{\rho}_{\alpha_1}) T_\alpha(\lambda) u \in \mathcal{S}(\mathbb{R}^N),$$

and for $j \neq \alpha_1$, $V_j \tilde{\rho}_{\alpha_1} T_\alpha(\lambda) u \in \mathcal{S}(\mathbb{R}^N)$. Then

(3.11)

$$\begin{aligned}(\Delta + V - \lambda^2)(\mathrm{Id} + T^\flat(\lambda))u = (\Delta - \lambda^2)u &+ (-1)^k \sum_{\alpha \in I_{k+1}} V_{\alpha_1} T_{\alpha'}(\lambda) u \\ &+ (-1)^{k+1} \sum_{\alpha \in I_{k+1}} (\Delta + V_{\alpha_1} - \lambda^2) \tilde{\rho}_{\alpha_1} T_{\alpha_1}(\lambda) T_{\alpha'}(\lambda) u \\ &+ (-1)^{k+1} \sum_{\alpha \in I_{k+1}} \sum_{j \neq \alpha_1} V_j \tilde{\rho}_{\alpha_1} T_\alpha(\lambda) u.\end{aligned}$$

The third sum is in $\mathcal{S}(\mathbb{R}^N)$, and the second differs from

$$(-1)^{k+1} \sum_{\alpha \in I_{k+1}} (\Delta + V_{\alpha_1} - \lambda^2) T_{\alpha_1}(\lambda) T_{\alpha'}(\lambda) u$$

by an element of $\mathcal{S}(\mathbb{R}^N)$, so the sum of the second and third terms on the right hand side of (3.11) is in $\mathcal{S}(\mathbb{R}^N)$. Hence,

$$(\Delta + V - \lambda^2)(\mathrm{Id} + T^\flat(\lambda))u = (\Delta - \lambda^2)u + F(\lambda)u \tag{3.12}$$

where $F(\lambda)$ maps u satisfying (3.4) into $\mathcal{S}(\mathbb{R}^N)$. Thus, we have shown the following theorem.

Theorem 3.1. *Suppose $u \in \mathcal{S}'(\mathbb{R}^N)$, $(\Delta - \lambda^2)u = 0$ and for all i*

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap R_\lambda^- \cap {}^{sc}\bar{T}^*\mathbb{S}^N_+|_{C_i} = \emptyset. \tag{3.13}$$

Then, with $T^\flat(\lambda)$ defined in (3.10), *$u_V = (\mathrm{Id} + T^\flat(\lambda))u \in \mathcal{S}'(\mathbb{R}^N)$ satisfies*

$$(\Delta + V - \lambda^2)u_V \in \mathcal{S}(\mathbb{R}^N), \qquad \mathrm{WF}_{\mathrm{sc}}(u_V - u) \cap R_\lambda^- = \emptyset. \tag{3.14}$$

4. The three-body resolvent

The construction leading to Theorem 3.1 still has an error in $\mathcal{S}(\mathbb{R}^N)$ which has to be removed. As usual in microlocal analysis, this is done by performing the parametrix construction (for $R_V(\lambda)$) of the previous section on weighted Sobolev spaces. We shall be left with $\mathrm{Id} + E_R(\lambda)$ to invert where $E_R(\lambda)$ maps $H^{k,l}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$ for sufficiently large l. Hilbert space techniques allow us to do this modulo a finite rank error. To show that the resolvent exists everywhere on the positive real axis, however, we shall use the results of Gérard, Isozaki and Skibsted [13] by relating their microlocal estimates to ours.

We shall continue using the notation of Section 3. As $\Delta + V - \lambda^2 \in \Psi^2_\infty(\mathbb{R}^N)$ is elliptic for $\lambda \in \mathbb{C}$ (this is the standard calculus of pseudo-differential operators on $\mathbb{R}^N$, as in [18], corresponding to symbol classes with uniform estimates in $\bar{w}$), it has a parametrix $P \in \Psi^{-2}_\infty(\mathbb{R}^N)$. Thus,

$$(\Delta + V - \lambda^2)P = \mathrm{Id} + F_R,\ \ P(\Delta + V - \lambda^2) = \mathrm{Id} + F_L \tag{4.1}$$

where $F_R,\ F_L \in \Psi_\infty^{-\infty}(\mathbb{R}^N)$. Note that operators in $\Psi_\infty^k(\mathbb{R}^N)$ map $H^{r,s}(\mathbb{R}^N)$ to $H^{r-k,s}(\mathbb{R}^N)$, so the smoothing elements only map $H^{r,s}(\mathbb{R}^N)$ to $H^{\infty,s}(\mathbb{R}^N)$ (and not to $\mathcal{S}(\mathbb{R}^N)$). By (3.12) $T^\flat(\lambda)R_0(\lambda)$ is a right parametrix for $\Delta+V-\lambda^2$, $\lambda>0$, if it acts on smooth functions satisfying appropriate $\mathrm{WF}_{\mathrm{sc}}$ conditions. To deal with the lack of smoothness we break up $T_i(\lambda)$ into two pieces as in (2.40), that is we let

$$T_i'(\lambda)=\mathcal{F}^{-1}\psi_0\mathcal{F}T_i(\lambda),\qquad T_i''(\lambda)=\mathcal{F}^{-1}(1-\psi_0)\mathcal{F}T_i(\lambda) \tag{4.2}$$

with $\psi_0\in\mathcal{C}_c^\infty(\mathbb{R}^N_\xi)$ identically 1 near $\mathbb{S}^{N-1}_\lambda$. Now let

(4.3)
$$R_V^R(\lambda)=R_0(\lambda)-\sum_i T_i(\lambda)R_0(\lambda)+\sum_{i\neq j}PV_jT_i''(\lambda)R_0(\lambda)$$
$$+\sum_{j=2}^k(-1)^j\sum_{\alpha\in I_j}T_{\alpha'}(\lambda)T'_{\alpha_j}(\lambda)R_0(\lambda)+(-1)^{k+1}\sum_{\alpha\in I_{k+1}}\tilde\rho_{\alpha_1}T_{\alpha'}(\lambda)T'_{\alpha_{k+1}}(\lambda)R_0(\lambda)$$

where we use the notation $\alpha=(\alpha',\alpha_j)$ for $\alpha\in I_j$. This construction separates the error $\sum_{i\neq j}V_jT_i(\lambda)R_0(\lambda)$ of the first two terms (as a parametrix) into two pieces. The term $V_jT_i'(\lambda)R_0(\lambda)$ satisfies the conditions for the construction of Section 3, while P will be adequate for removing the error $V_jT_i''(\lambda)R_0(\lambda)$, since this lies in $H^{r,\infty}(\mathbb{R}^N)$ for some r. Of course, $R_V^R(\lambda)$ depends on k even though this is suppressed in the notation; however, as long as we choose k large enough (as the construction of the following proposition indicates), this dependence is not important. (Technically we need one more iteration than in the previous section to make the proof work.) We proceed to demonstrate that $R_V^R(\lambda)u$ is indeed defined for $u\in H^{r,s}$ for sufficiently large s and to show that it is a parametrix for $\Delta+V-\lambda^2$.

Proposition 4.1. *For k sufficiently large, $s>3$, $R_V^R(\lambda)$ defines a continuous linear map from $H^{r,s}(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$ with*

$$(\Delta+V-\lambda^2)R_V^R(\lambda)=\mathrm{Id}+E_R,\qquad E_R\in\mathcal{B}(H^{r,s}(\mathbb{R}^N),H^{r',s'}(\mathbb{R}^N)) \tag{4.4}$$

for all r', s'. Moreover, for $u\in\mathcal{S}(\mathbb{R}^N)$

$$\mathrm{WF}_{\mathrm{sc}}(R_V^R(\lambda)u)\subset R_\lambda^+\cap\mathrm{cl}(F^+(\lambda)). \tag{4.5}$$

Proof. Let $A\in\Psi_c^0(\mathbb{R}^N)$ be such that

$$\mathrm{WF}'(A)\cap(N_\lambda^-\cup\cup_iN^-E_\lambda^i)=\emptyset,\qquad \mathrm{WF}'(\mathrm{Id}-A)\cap N_\lambda^+=\emptyset.$$

Write

$$\rho_0(\lambda)=A\rho_1(\lambda)+(\mathrm{Id}-A)\rho_1(\lambda)+\rho_2(\lambda).$$

Of course, $\rho_2(\lambda)\in\Psi_{\mathrm{sc}}^{0,2}(\mathbb{S}^N_+)$, while $(\mathrm{Id}-A)\rho_1(\lambda)\in\mathcal{B}(H^{r,s}(\mathbb{R}^N),H_c^{r-1-\epsilon}(\mathbb{R}^N))$ by Corollary 2.13. Moreover, for all i

$$\mathrm{WF}'(K_{\tilde T_i(\lambda)A})\cap(0_{\mathbb{R}^N}\times(T^*\mathbb{R}^N\setminus 0))=\emptyset$$

by Lemma 2.9. Hence, the results of Section 2 show that $\tilde T_i(\lambda)\rho_2(\lambda)$, $\tilde T_i(\lambda)(\mathrm{Id}-A)$ are continuous operators from $H^{s,r}$ to $\mathcal{S}'(\mathbb{R}^N)$, so this also holds for $\tilde T_i(\lambda)\rho_0(\lambda)$. In addition, by Lemma 2.16

$$V_jT_i''(\lambda)R_0(\lambda)\in\mathcal{B}(H^{r,s},H^{r+4,s'}) \tag{4.6}$$

for all $s'\in\mathbb{R}$, while by Lemmas 2.9 and 2.15 $T_i'(\lambda)R_0(\lambda)$ maps to compactly supported distributions with wave front set disjoint from $N^-E^j_\lambda$ for $j\neq i$. Although we did not state the argument of Section 3 in terms of continuous operators, this can be remedied easily (again, using the results of Section 2), and hence we deduce that all terms of (4.3) are continuous operators from $H^{r,s}(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$.

A slight modification of the argument leading to (3.12) shows that

$$(\Delta + V - \lambda^2)R_V^R(\lambda) = \mathrm{Id} + \sum_i F_i T_i'(\lambda)R_0(\lambda) + \sum_{i\neq j} F_R V_j T_i''(\lambda)R_0(\lambda) \tag{4.7}$$

where, for sufficiently large k, $F_i T_i'(\lambda)R_0(\lambda) : H^{r,s}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is continuous. Since $F_R \in \Psi_\infty^{-\infty}(\mathbb{R}^N)$, (4.6) shows that the last term has the same property. Thus,

$$E_R = (\Delta + V - \lambda^2)R_V^R(\lambda) - \mathrm{Id} : H^{r,s}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N) \tag{4.8}$$

is continuous. Finally, if $u \in \mathcal{S}(\mathbb{R}^N)$ then the results of Section 2 show (4.5). $\square$

Unfortunately $R_V^R(\lambda)$ is not a left parametrix for $\Delta + V - \lambda^2$ (in the sense that its error does not map sufficiently highly weighted Sobolev spaces to Schwartz functions). However, it comes reasonably close to being one. Namely, analogously to (3.6) we have

$$(\mathrm{Id} + T^{(k)}(\lambda))R_0(\lambda)(\Delta + V - \lambda^2) = \mathrm{Id} + E_k \tag{4.9}$$

with

$$E_k = (-1)^k \sum_{\alpha \in I_{k+1}} T_{\alpha'}(\lambda)R_0(\lambda)V_{\alpha_{k+1}}. \tag{4.10}$$

Similarly to (4.3) let

$$R_V^L(\lambda) = R_0(\lambda) + T^{(k)}(\lambda)R_0(\lambda) + (-1)^{k+1} \sum_{\alpha \in I_{k+1}} T_{\alpha'}(\lambda)R_2(\lambda)V_{\alpha_{k+1}}P. \tag{4.11}$$

Since $V_i R_2(\lambda)V_j \in \Psi_{\mathrm{sc}}^{-2,\infty}(\mathbb{S}_+^N)$ for $i \neq j$, $V_i R_2(\lambda)V_j P \in \Psi_{\mathrm{sc}}^{-2,\infty}(\mathbb{S}_+^N)$ too. In addition, $F_L \in \Psi_\infty^{-\infty}(\mathbb{R}^N)$, so

$$V_i R_2(\lambda)V_j F_L \in \Psi_{\mathrm{sc}}^{-\infty,\infty}(\mathbb{S}_+^N). \tag{4.12}$$

We thus deduce that

$$R_V^L(\lambda)(\Delta + V - \lambda^2) = \mathrm{Id} + E_L, \tag{4.13}$$

$$E_L = (-1)^k \sum_{\alpha \in I_{k+1}} T_{\alpha'}(\lambda)R_1(\lambda)V_{\alpha_{k+1}} + (-1)^{k+1} \sum_{\alpha \in I_{k+1}} T_{\alpha'}(\lambda)R_2(\lambda)V_{\alpha_{k+1}}F_L. \tag{4.14}$$

Thus,

$$E_L = \sum_i R_{V_i}(\lambda)S_i + R_0(\lambda)S_0 \tag{4.15}$$

where $S_i : H^{r,s}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is continuous for $s > 3$. The usual argument shows that difference of the two parametrices is not great:

$$R_V^L(\lambda) = R_V^R(\lambda) + E_L R_V^R(\lambda) - R_V^L(\lambda)E_R, \tag{4.16}$$

so it is of the form (4.15). Thus, $R_V^R(\lambda)$ is also a left parametrix with error term of the same form.

Since the error term $E_R \in \mathcal{B}(H^{r,s}, H^{r,s})$ of the proposition is compact, we can invert $\mathrm{Id} + E_R$ modulo a finite rank operator, i.e. E_R can be replaced by a finite rank operator K which is continuous from $H^{r,s}$ to $\mathcal{S}(\mathbb{R}^N)$. Notice that if $\operatorname{Id} + E_R$ is actually invertible then writing

$$(\operatorname{Id} + E_R)^{-1} = \operatorname{Id} - E_R(\operatorname{Id} + E_R)^{-1} \tag{4.17}$$

shows that $S = (\operatorname{Id} + E_R)^{-1} - \operatorname{Id}$ is a continuous map from $H^{r,s}$ to $\mathcal{S}(\mathbb{R}^N)$. In that case the actual resolvent

$$R_V(\lambda) = R_V^R(\lambda)(\operatorname{Id} + E_R)^{-1} \tag{4.18}$$

exists, and for $u \in H^{r,s}$

$$R_V(\lambda)u - R_V^R(\lambda)u = R_V^R(\lambda)Su \tag{4.19}$$

which is in the image of $R_V^R(\lambda)$ acting on Schwartz functions, hence its behavior is easily describable (this will be done in the next sections).

We can actually show that $\operatorname{Id} + E_R(\lambda)$ is invertible for 'most' $\lambda \in \mathbb{R} \setminus \{0\}$. In fact, there is no reason to restrict ourselves to $\lambda > 0$; the construction of the proposition also works for $\lambda \in \mathcal{P}$ with $\operatorname{Re}\lambda \neq 0$ (where $\mathcal{P}$ is the open lower half plane). Indeed, for such values of λ there is no propagation due to $R_0(\lambda)$, so already $V_j R_{V_i}(\lambda) V_i : H^{k,l}(\mathbb{R}^N) \to H^{k+2,\infty}(\mathbb{R}^N)$ (continuously) for $i \neq j$. We can also see then that $R_V^R(\lambda)$ is a two-sided parametrix for $\Delta + V - \lambda^2$ in the sense that

$$(\Delta + V - \lambda^2) R_V^R(\lambda) = \operatorname{Id} + E_R, \qquad R_V^R(\lambda)(\Delta + V - \lambda^2) = \operatorname{Id} + E_L \tag{4.20}$$

with both E_R and E_L being continuous maps from $\mathcal{S}'(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$. Note also that for $\lambda \in \mathcal{P}$, $\operatorname{Re}\lambda \neq 0$ $\Delta + V - \lambda^2$ is invertible as an operator on $L^2(\mathbb{R}^N)$ (since $\Delta + V$ is self-adjoint). We can then use the standard identity

$$(\Delta + V - \lambda^2)^{-1} = R_V^R(\lambda) - R_V^L(\lambda) E_R + E_L (\Delta + V - \lambda^2)^{-1} E_R \tag{4.21}$$

with $R_V^L(\lambda) = R_V^R(\lambda)$. Thus, $(\Delta + V - \lambda^2)^{-1}$ differs from $R_V^R(\lambda)$ by a smoothing operator if we keep λ^2 away from the real axis.

Now for $\lambda \in \mathcal{P}$, $\operatorname{Re}\lambda \neq 0$ the construction of $R_V^R(\lambda)$ is holomorphic in λ (at least if we keep $\operatorname{Re}\lambda$ bounded and bounded away from 0), and it is $\mathcal{C}^\infty$ up to $\mathbb{R} \setminus \{0\}$, so the error term $E_R(\lambda)$ is a holomorphic family of trace class operators in $\mathcal{B}(H^{r,s})$ for $\lambda \in \mathcal{P}$ with a $\mathcal{C}^j$ extension to $\overline{\mathcal{P}} \setminus \Lambda$ if $j < r - 3$. Hence the operator determinant $h(\lambda) = \det(\operatorname{Id} + E_R(\lambda))$ is a holomorphic function in $\mathcal{P}$ with a $\mathcal{C}^j$ extension to $\overline{\mathcal{P}} \setminus \Lambda$ (see [37]). For $\lambda \in \overline{\mathcal{P}} \setminus \Lambda$ with $h(\lambda) \neq 0$ $\operatorname{Id} + E_R(\lambda)$ has an inverse in $\mathcal{B}(H^{r,s})$. When $\operatorname{Im}\lambda$ is large and negative, $R_{V_i}(\lambda)$ has a small norm for each i, so $E_R(\lambda)$ also has a small norm, so h is not identically 0. But $h(\lambda)$ being holomorphic in $\mathcal{P} \setminus \Lambda$ it follows that it cannot vanish identically on any open interval in $\mathbb{R} \setminus \{0\}$ ([18]), so the set where $h(\lambda) \neq 0$ is an open subset of $\mathbb{R} \setminus \{0\}$ whose complement contains no open interval.

We shall now use the results of Gérard, Isozaki and Skibsted [13, 20, 23] (see also Wang's paper [42]) about the existence of $R_V(\lambda)$ (which we could not show so far if $\operatorname{Id} + E_R$ was not invertible) to describe it rather nicely. In [20, 23] Isozaki defines $\mathcal{R}^k_\pm(a)$ to be the set of $\mathcal{C}^\infty$ functions p on $T^*\mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N$ satisfying

$$|\partial_{\bar{w}}^r \partial_{\bar{\xi}}^s p(\bar{w}, \bar{\xi})| \leq C_{r,s} \langle \bar{w} \rangle^{-r} \langle \bar{\xi} \rangle^{-k} \tag{4.22}$$

for $0 \leq r \leq k$, $0 \leq s \leq k$ and on $\operatorname{supp} p$

$$\inf_{\bar{w}, \bar{\xi}} \pm \frac{\bar{w} \cdot \bar{\xi}}{\langle \bar{w} \rangle} > a \tag{4.23}$$

with the $+$ sign corresponding to $\mathcal{R}^k_+(a)$ and the $-$ sign to $\mathcal{R}^k_-(a)$. Note that if $p \in \mathcal{C}^\infty(\mathbb{S}^N_+ \times \mathbb{S}^N_+)$ has compact support in the $\bar{\xi}$ variable, then (4.22) is automatically satisfied. Introducing the usual coordinates on ${}^{\mathrm{sc}}T^*\mathbb{S}^N_+$ near ${}^{\mathrm{sc}}T^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}}$ we have

$$\tau = -\frac{\bar{w} \cdot \bar{\xi}}{|\bar{w}|}. \tag{4.24}$$

Thus, (4.23) states in this case that there exists $\epsilon > 0$ such that $\pm\tau < -a - \epsilon$ on $\operatorname{supp} p$.

Although τ arises naturally from the scattering calculus, not all symbols in $\mathcal{R}^k_\pm(a)$ are scattering symbols. The problem is that τ is not a smooth function at the corner of ${}^{\mathrm{sc}}\bar{T}^*\mathbb{S}^N_+$, although it is smooth in the interior of ${}^{\mathrm{sc}}\bar{T}^*\mathbb{S}^N_+|_{\mathbb{S}^{N-1}}$. In particular, the symbol

$$p(\bar{w}, \bar{\xi}) = \rho(-\frac{\bar{w} \cdot \bar{\xi}}{\langle \bar{w} \rangle})\langle \bar{\xi} \rangle^{-2k} \tag{4.25}$$

with $\rho \in \mathcal{C}^\infty(\mathbb{R})$, $\operatorname{supp} \rho \subset (-\infty, 0)$, $\rho \equiv 1$ on $(-\infty, -\lambda/2)$, used in [13, 23] is not a scattering symbol; in fact, it satisfies the estimates

$$|\partial^r_{\bar{w}} \partial^s_{\bar{\xi}} p(\bar{w}, \bar{\xi})| \leq C_{r,s} \langle \bar{w} \rangle^{-r} \langle \bar{\xi} \rangle^{-2k+r-s}. \tag{4.26}$$

Of course, such a discrepancy can be expected, since our potentials V_i are not in $\Psi^{*,*}_{\mathrm{sc}}(\mathbb{S}^N_+)$ either.

It is shown in [13], see also [23, Theorem 1.2], that for all $s > -1/2$, $t > 1$ there exists $k = k(s)$ such that

$$QR_V(\lambda) \in \mathcal{B}(H^{0,s+t}, H^{0,s}) \tag{4.27}$$

for all $Q \in \mathcal{R}^k_+(-\lambda)$. Now let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N_{\bar{\xi}})$, $\phi \equiv 1$ for $|\bar{\xi}| \leq 2\lambda$, $\rho \in \mathcal{C}^\infty(\mathbb{R})$, $\operatorname{supp} \rho \subset (-\infty, 0)$, $\rho \equiv 1$ on $(-\infty, -\lambda/2)$, and $\psi \in \mathcal{C}^\infty(\mathbb{S}^N_+)$ supported away from 0, identically 1 near $\mathbb{S}^{N-1} = \partial \mathbb{S}^N_+$. Then

$$p(\bar{w}, \bar{\xi}) = \phi(\bar{\xi})\psi(\bar{w})\rho(\tau) \in \mathcal{R}^k_+(-\lambda) \tag{4.28}$$

for all k so the corresponding operator $Q \in \Psi^{-\infty,0}_{\mathrm{sc}}(\mathbb{S}^N_+)$ satisfies (4.27). In fact, as $Q \in \Psi^{-\infty,0}_{\mathrm{sc}}(\mathbb{S}^N_+)$ (4.27) implies that

$$QR_V(\lambda) \in \mathcal{B}(H^{0,s+t}, H^{r,s}) \tag{4.29}$$

for all r. Also,

$$R_V(\lambda) \in \mathcal{B}(H^{0,1/2+\epsilon}, H^{0,-1/2-\epsilon}) \tag{4.30}$$

(see [36]); this is obtained using positive commutator estimates. Since analogously to (4.21) we have

$$R_V(\lambda) = P - PF_R + F_L R_V(\lambda) F_R, \tag{4.31}$$

we see that $R_V(\lambda) \in \mathcal{B}(H^{r,1/2+\epsilon}, H^{r+2,-1/2-\epsilon})$ for all r. Now,

$$VR_V(\lambda) = \mathrm{Id} - (\Delta - \lambda^2) Q R_V(\lambda) - (\Delta - \lambda^2)(\mathrm{Id} - Q) R_V(\lambda). \tag{4.32}$$

With ψ_i as in Section 3 $(1 - \psi_i)V_i \in \mathcal{S}(\mathbb{R}^N)$ and $\psi_i V_j \in \mathcal{S}(\mathbb{R}^N)$ for $i \neq j$, so

$$V_i R_V(\lambda) - \psi_i V R_V(\lambda) \in \mathcal{B}(H^{r,1/2+\epsilon}, H^{r+2,s}) \tag{4.33}$$

for all r and s. Since $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q) \cap R^-_\lambda = \emptyset$, we have that for $\alpha \in I_k$

$$T_{\alpha'}(\lambda) R_1(\lambda) \psi_{\alpha_k} (\Delta - \lambda^2)(\mathrm{Id} - Q) = \sum R_{V_i}(\lambda) S'_i + R_0(\lambda) S'_0, \tag{4.34}$$

with S_i', $S_0' \in \Psi_{\mathrm{sc}}^{-\infty,\infty}(\mathbb{S}_+^N)$. We have also seen that

$$T_{\alpha'}(\lambda)R_1(\lambda)\psi_{\alpha_k} = \sum R_{V_i}(\lambda)S_i'' + R_0(\lambda)S_0'', \tag{4.35}$$

where S_i'', $S_0'' \in \mathcal{B}(H^{r,s}, H^{r',s'})$ for $s > 3$, $s', r, r' \in \mathbb{R}$. Thus, by (4.14), (4.32) and (4.33)

$$E_L R_V(\lambda) = \sum R_{V_i}(\lambda)S_i^\flat + R_0(\lambda)S_0^\flat \tag{4.36}$$

with $S_i^\flat$, $S_0^\flat \in \mathcal{B}(H^{0,s}, H^{r',s'})$ for $s > 4$ and for all r' and s'. Combining this with Proposition 4.1 we obtain the following lemma:

Lemma 4.2. *$E_L(\Delta + V - (\lambda - i0)^2)^{-1}E_R = \sum_i R_{V_i}(\lambda)S^i + R_0(\lambda)S^0$ where S^i : $H^{r,s} \to \mathcal{S}(\mathbb{R}^N)$ is continuous if $s > 3$.*

An immediate consequence of this lemma and (4.21) is the following theorem on the structure of the resolvent.

Theorem 4.3. *For $\lambda > 0$, $s > 3$, $s' < -1/2$, $r \in \mathbb{R}$, $R_V(\lambda)$ defines a continuous linear map from $H^{r,s}(\mathbb{R}^N)$ to $H^{r+2,s'}(\mathbb{R}^N)$ which can be written as*

$$R_V(\lambda) = R_V^R(\lambda) + \sum_i R_{V_i}(\lambda)S^i + R_0(\lambda)S^0 \tag{4.37}$$

where $S^i : H^{r,s}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is continuous for each i. Moreover, for $u \in \mathcal{S}(\mathbb{R}^N)$

$$\mathrm{WF}_{\mathrm{sc}}(R_V(\lambda)u) \subset R_\lambda^+ \cup \cup_i \,\mathrm{cl}(F_i^+(\lambda));$$

in particular $\mathrm{WF}_{\mathrm{sc}}(R_V(\lambda)u) \cap R_\lambda^- = \emptyset$.

Remark 4.4. The assumption $s > 3$ is certainly too strong; one expects that the same result holds if we merely assume $s > 1/2$. However, that would require a more careful examination of the unreduced resolvents than performed in this section. In any case, we shall only need to apply $R_V(\lambda)$ to functions with an appropriate $\mathrm{WF}_{\mathrm{sc}}$.

5. The Poisson operator

The Poisson operator of the unperturbed Laplacian is the map

$$P_0(\lambda) : \mathcal{C}^{-\infty}(\mathbb{S}^{N-1}) \to \mathcal{S}'(\mathbb{R}^N) \tag{5.1}$$

with kernel

$$P_0(\lambda, \bar\theta, \bar w) = \lambda^{(N-1)/2} e^{\pi(N-1)i/4} (2\pi)^{-(N-1)/2} e^{i\lambda\bar\theta\cdot\bar w}. \tag{5.2}$$

It gives a parametrization of the null space of $\Delta - \lambda^2$ acting on $\mathcal{S}'(\mathbb{R}^N)$, i.e. it maps into the null space and it is bijective as a map into the null space. We can rewrite $P_0(\lambda)$ in a more convenient form if we introduce the projections $\pi : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$, $p : \mathbb{R}^N \setminus \{0\} \to \mathbb{S}^{N-1}$ defined by $\pi(r\theta) = r$, $p(r\theta) = \theta$ with (r, θ) being polar coordinates. Then

$$P_0(\lambda)g = \lambda^{-(N-1)/2} e^{\pi(N-1)i/4} (2\pi)^{(N+1)/2} \mathcal{F}^{-1}((\pi^*\delta_\lambda)(p^*g)) \tag{5.3}$$

for $g \in \mathcal{C}^{-\infty}(\mathbb{S}^{N-1})$. We can now use the method of Section 3 to correct $P_0(\lambda)$ to obtain a parametrix for the 3-body Poisson operator. Namely we let

$$P_V^\flat(\lambda) = P_0(\lambda) - R_V^R(\lambda) V P_0(\lambda). \tag{5.4}$$

Unfortunately, the second operator is not defined for every $g \in \mathcal{C}^{-\infty}(\mathbb{S}^{N-1})$. However, it is well defined when $g \in \mathcal{C}_c^{-\infty}(U)$ with $U = \mathbb{S}^{N-1} \setminus \cup_i C_i$, and then

$$(\Delta + V - \lambda^2) P_V^\flat(\lambda) g \in \mathcal{S}(\mathbb{R}^N). \tag{5.5}$$

It is now easy to obtain the true Poisson operator. We let

$$P_V(\lambda) = P_V^\flat(\lambda) - R_V(\lambda)(\Delta + V - \lambda^2) P_V^\flat(\lambda). \tag{5.6}$$

Of course, if we let $R_V(\lambda)$ as constructed in Section 4 act on distributions with appropriate wave front sets (which is essentially what we are doing right now), we can write this as

$$P_V(\lambda) = P_0(\lambda) - R_V(\lambda) V P_0(\lambda). \tag{5.7}$$

We have the following result:

Proposition 5.1. *$P_V(\lambda) : \mathcal{C}_c^{-\infty}(U) \to \mathcal{S}'(\mathbb{R}^N)$ maps injectively into*

$$\mathcal{N} = \{u \in \mathcal{S}'(\mathbb{R}^N) : (\Delta + V - \lambda^2)u = 0,\ \mathrm{WF}_{\mathrm{sc}}(u) \cap R_\lambda^- \cap \cup_i F_i^-(\lambda) = \emptyset\}. \tag{5.8}$$

For $g \in \mathcal{C}_c^{-\infty}(U)$

$$\mathrm{WF}_{\mathrm{sc}}(P_V(\lambda) g - P_0(\lambda) g) \cap R_\lambda^- = \emptyset. \tag{5.9}$$

In addition, if $0 < \epsilon < \lambda$, k is sufficiently large, $g \in \mathcal{C}_c^\infty(U)$ then $P_V(\lambda) g$ is the only solution of $(\Delta + V - \lambda^2)u = 0$ which satisfies

$$P_+(u - P_0(\lambda) g) \in L^2 \tag{5.10}$$

for all $P_+ \in \mathcal{R}_+^k(-\epsilon)$.

Proof. The fact that $P_V(\lambda)$ maps into $\mathcal{N}$ and (5.9) follow from Theorem 4.3. Due to (5.9) $P_V(\lambda) g = 0$ implies

$$\mathrm{WF}_{\mathrm{sc}}(P_0(\lambda) g) \cap R_\lambda^- = \emptyset$$

which is impossible unless $g = 0$, so $P_V(\lambda)$ is indeed injective. If $g \in \mathcal{C}_c^\infty(U)$, then $V P_0(\lambda) g \in \mathcal{S}(\mathbb{R}^N)$, so by (4.27) and (5.7) $P_+(P_V(\lambda) g - P_0(\lambda) g) \in L^2$ for all $P_+ \in \mathcal{R}_+^k(-\epsilon)$. Now, if u_j, $j = 1, 2$, satisfy

$$(\Delta + V - \lambda^2) u_j = 0, \qquad P_+(u_j - P_0(\lambda) g) \in L^2$$

for all $P_+ \in \mathcal{R}_+^k(-\epsilon)$ then $v = u_1 - u_2$ satisfies $(\Delta + V - \lambda^2) v = 0$ and $P_+ v \in L^2$, so by Isozaki's uniqueness theorem [23, Theorem 1.3] $v = 0$, proving the proposition. □

Remark 5.2. To obtain a complete characterization of the range of $P_V(\lambda)$ we would have to use a different notion of wave front set (from $\mathrm{WF}_{\mathrm{sc}}$), since in 3-body scattering $\Delta + V - \lambda^2 \notin \Psi_{\mathrm{sc}}^{*,*}(\mathbb{S}_+^N)$.

6. The scattering matrix

The scattering matrix is defined by the asymptotic behavior of $R_V(\lambda) f(\bar{w})$ for $\lambda \in \mathbb{R} \setminus \{0\}$, $f \in \mathcal{S}(\mathbb{R}^N)$ as $|\bar{w}| \to \infty$, so first of all we shall obtain these asymptotics. We shall use the compactification $\mathbb{S}_+^N$ of $\mathbb{R}^N$ to do this more conveniently. Thus, for $f \in \mathcal{S}(\mathbb{R}^N)$ $R_0(\lambda) f = e^{-i\lambda/x} x^{(N-1)/2} g$ where $x = |\bar{w}|^{-1}$ is the defining function of $\partial\mathbb{S}_+^N = \mathbb{S}^{N-1}$ and $g \in \mathcal{C}^\infty(\mathbb{S}_+^N)$. The same result holds for the reduced two-body resolvents if the two-body potential is in $x^2 \mathcal{C}^\infty(\mathbb{S}_+^N)$ (cf. [29]). In the three-body problem we assume that $V_i \in \mathcal{S}(\mathbb{R}^{n_i})$ and we consider $(\pi^i)^* V_i \in \mathcal{C}^\infty(\mathbb{R}^N)$ (where we drop the pull back notation). Thus, V_i is rapidly decreasing in the z_i variable, but it does not decay as $|\bar{w}| \to \infty$ if we keep z_i fixed. Hence, (unless V_i is identically 0)

$V_i \notin \mathcal{C}^\infty(\mathbb{S}^N_+)$. However, all singularities of V_i are located at C_i. Correspondingly it is much easier to describe $R_V(\lambda)f$ away from $\cup_i C_i$; in fact, it has the usual asymptotic expansion there.

Proposition 6.1. *For $f \in \mathcal{S}(\mathbb{R}^N)$, $\lambda \in \mathbb{R} \setminus \{0\}$*

$$R_V(\lambda)f = e^{-i\lambda/x}x^{(N-1)/2}g \tag{6.1}$$

where $g \in \mathcal{C}^\infty(\mathbb{S}^N_+ \setminus \cup_i C_i)$.

Proof. We shall take $\lambda > 0$; $\lambda < 0$ is similar. For $f \in \mathcal{S}(\mathbb{R}^N)$ $R_V(\lambda)f$ is given by (4.37). First of all note that the term of $R^R_V(\lambda)f$ which involves P is Schwartz, so it can be ignored. The other terms are all smooth and of the type $R_0(\lambda)u$ where $\mathrm{WF}_{\mathrm{sc}}(u) \subset \mathrm{cl}(F^+_i(\lambda))$, so $\mathrm{WF}(\mathcal{F}u) \subset N^+E^i_\lambda$. Let $K \subset \mathbb{S}^{N-1}$ be compact and such that $K \cap \cup_i C_i = \emptyset$. Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ such that ϕ vanishes in a neighborhood of each E^i_λ and it is identically 1 on a neighborhood of $\{\lambda\bar\theta : -\bar\theta \in K\}$. Then

$$\mathrm{WF}(\rho_0(\lambda)(1-\phi)\mathcal{F}u) \subset N^+E^i_\lambda \cup (\mathrm{supp}(1-\phi) \cap N^+_\lambda),$$

so its inverse Fourier transform, $R_0(\lambda)\mathcal{F}^{-1}(1-\phi)\mathcal{F}u$ vanishes to infinite order near $K \subset \partial\mathbb{S}^N_+$. On the other hand, $\phi\mathcal{F}u \in \mathcal{S}(\mathbb{R}^N)$, so the same holds for its inverse Fourier transform, and hence $R_0(\lambda)\mathcal{F}^{-1}\phi\mathcal{F}u$ has an asymptotic expansion as stated in the lemma with $g \in \mathcal{C}^\infty(\mathbb{S}^N_+)$. Thus, $R_V(\lambda)f$ has the same asymptotic behavior away from the singular set. □

This proposition and (5.7) show that for $g \in \mathcal{C}^\infty_c(U)$

$$P_V(\lambda)g = e^{i\lambda/x}x^{(N-1)/2}f_+ + e^{-i\lambda/x}x^{(N-1)/2}f_- \tag{6.2}$$

with $f_\pm \in \mathcal{C}^\infty(\mathbb{S}^N_+ \setminus \cup_i C_i)$. In fact, $(\pi^*\delta_\lambda)(p^*g)$ is conormal to $\mathbb{S}^{N-1}_\lambda$ and it is supported away from $\cup_i E^i_\lambda$, so $\tilde V(\pi^*\delta_\lambda)(p^*g)$ is smooth, and now we can apply the proposition. Furthermore, f_+ is rapidly decreasing in a neighborhood of the singular set $\cup_i C_i$, so actually $f_+ \in \mathcal{C}^\infty(\mathbb{S}^N_+)$. The boundary values of f_+ are simply $f_+(\theta) = g(\theta)$ due to our normalization of the Poisson operator. The three-cluster to three-cluster scattering matrix $S_V(\lambda) : \mathcal{C}^\infty_c(U) \to \mathcal{C}^\infty(U)$ is defined as the map $S_V(\lambda)f_+ = f_-$, i.e. $S_V(\lambda)g = f_-$. As a first step we wish to extend $S_V(\lambda)$ to an operator from $\mathcal{C}^{-\infty}_c(U)$ to $\mathcal{C}^{-\infty}(U)$.

Lemma 6.2. *The scattering matrix extends to an operator $\mathcal{C}^{-\infty}_c(U) \to \mathcal{C}^{-\infty}(U)$ by continuity from $\mathcal{C}^\infty_c(U)$.*

Proof. For $g \in \mathcal{C}^\infty_c(U)$

$$P_V(\lambda)g = P_0(\lambda)g - R_0(\lambda)(V - VR_V(\lambda)V)P_0(\lambda)g, \tag{6.3}$$

so taking the Fourier transform

$$c\mathcal{F}P_V(\lambda)g = \rho_0(-\lambda)p^*g - \rho_0(\lambda)(p^*g + h) \tag{6.4}$$

where (with p, π as in Section 5)

$$h = \pi i\lambda^{-1}\mathcal{F}(V - VR_V(\lambda)V)\mathcal{F}^{-1}((\pi^*\delta_\lambda)(p^*g)) \tag{6.5}$$

is rapidly decreasing, $\mathrm{WF}(h) \subset \cup_i N^+E^i_\lambda$, and c is a constant coming from (5.3). Thus,

$$f_+|_{\mathbb{S}^{N-1}} = g, \qquad f_-|_{\mathbb{S}^{N-1}} = i^{N-1}j^*(g + \iota^*_\lambda h), \tag{6.6}$$

with j being the antipodal map $j(\theta) = -\theta$ and $\iota_\lambda : \mathbb{S}^{N-1} \to \mathbb{R}^N$ the inclusion $\iota_\lambda(\theta) = \lambda\theta$. But defining h this way for $g \in \mathcal{C}_c^{-\infty}(U)$

$$\mathrm{WF}(h) \cap (N_\lambda^+ \cup N_\lambda^-) \subset \cup_i N^+ E_\lambda^i, \tag{6.7}$$

so the pull back by ι_λ is even defined when $g \in \mathcal{C}_c^{-\infty}(U)$ (by continuity from $\mathcal{C}_c^\infty(U)$), at least as a distribution in $\mathcal{C}^{-\infty}(U)$. Thus, for $g \in \mathcal{C}_c^{-\infty}(U)$

$$S_V(\lambda)g = i^{N-1} j^*(g + \pi i \lambda^{-1} \iota_\lambda^* \mathcal{F}(V - VR_V(\lambda)V)\mathcal{F}^{-1}(\pi^*\delta_\lambda)p^*g), \tag{6.8}$$

proving the lemma. □

Of course, we have very precise information on the structure of $R_V(\lambda)$, so it is easy to describe the singularities of the Schwartz kernel of $S_V(\lambda)$. First of all, if we are only interested in $S_V(\lambda)$ modulo smoothing operators, we can replace $R_V(\lambda)$ in (6.8) by $R_V^R(\lambda)$ due to Theorem 4.3. Moreover, as $(\pi^*\delta_\lambda)(p^*g) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^N)$, we can replace $R_V^R(\lambda)V$ by $\mathrm{Id} + T^\flat(\lambda)$, and in fact by $\mathrm{Id} + T^{(k)}(\lambda)$ for sufficiently large k due to the additional factor of V. Thus, as a map $\mathcal{C}_c^{-\infty}(U) \to \mathcal{C}^{-\infty}(U)$

$$\begin{aligned} S_V(\lambda) = &i^{N-1} j^*(\mathrm{Id} + \\ &\frac{\pi i}{\lambda}\iota_\lambda^* \sum_{j=1}^{k} (-1)^{j-1} \sum_{\alpha \in I_j} \mathcal{F}(V_{\alpha_1} - V_{\alpha_1} T_{\alpha_1}(\lambda)) T_{\alpha'}(\lambda) \mathcal{F}^{-1}(\pi^*\delta_\lambda)p^*) + S \end{aligned} \tag{6.9}$$

where S is a smoothing operator, i.e. its kernel $K_S \in \mathcal{C}^\infty(U \times U)$. We shall see in a moment that the singularities of $S_V(\lambda)$ are related to the 'broken' geodesic flow on $\mathbb{S}^{N-1}$. Since $\mathbb{S}^{N-1}$ is a submanifold of $\mathbb{R}^N$, its tangent vectors can be thought of as elements of $T\mathbb{R}^N$, which in turn can be identified with $\mathbb{R}^N$. We can identify $T_\theta^*\mathbb{S}^{N-1}$ and $T_\theta\mathbb{S}^{N-1}$ using the standard metric on $\mathbb{S}^{N-1}$, which thus allows us to identify $T_\theta^*\mathbb{S}^{N-1}$ (and $T_\theta\mathbb{S}^{N-1}$) with $\mathrm{span}\{\theta\}^\perp \subset \mathbb{R}^N$. For $\theta \in \mathbb{S}^{N-1}$ let $P_\theta : T_\theta\mathbb{R}^N \to T_\theta\mathbb{S}^{N-1}$ be the orthogonal projection. If θ, $\omega \in \mathbb{S}^{N-1}$, $\theta \neq \pm\omega$, γ is the unique geodesic of length less than π from θ to ω, and $\theta' \in \gamma$ then an (unnormalized) tangent vector at θ' to γ is given by $P_{\theta'}(\omega - \theta)$.

Theorem 6.3. *$S_V(\lambda)$ is a sum of Fourier integral operators associated to the broken geodesic flow at time $-\pi$ on $\mathbb{S}^{N-1}$. That is, if $(\omega, \theta, \omega', \theta')$ is in the canonical relation of $S_V(\lambda)$ then there exists a broken geodesic of length π from ω' starting in the direction of $-\theta'$, ending at ω in direction $-\theta$, breaking only at points in $\cup_i C_i$ such that the unit tangent vectors to the segments at any break point $\tilde{\theta} \in C_i$ differ by a vector perpendicular to $T_{\tilde{\theta}} C_i$.*

Proof. By (6.9) it suffices to prove that

$$\iota_\lambda^* \tilde{V}_{\alpha_1} \tilde{T}_{\alpha'}(\lambda)(\pi^*\delta_\lambda)p^*,\ \iota_\lambda^* B_{V_{\alpha_1}}(\lambda)\tilde{T}_{\alpha'}(\lambda)(\pi^*\delta_\lambda)p^* \tag{6.10}$$

are FIOs for $\alpha \in I_k$, $k \geq 1$ where $\tilde{T}_{\alpha'}(\lambda) = \mathcal{F}T_{\alpha'}(\lambda)\mathcal{F}^{-1}$. Here $B_{V_{\alpha_1}}(\lambda)$ does not require a special treatment since we are only interested in what happens away from the singular set, hence away from $E_\lambda^i \cap \mathbb{S}_\lambda^{N-1}$. We can also replace $\tilde{T}_{\alpha_j}(\lambda)$ by $\tilde{T}'_{\alpha_j}(\lambda)$ (i.e. by $\psi_0 \tilde{T}_{\alpha_j}(\lambda)$ where $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ is identically 1 near $\mathbb{S}_\lambda^{N-1}$) due to the propagation results of Section 2 (although we will not show this in the notation), so the non-compactness of the supports cannot cause any problems. Let W_{pr} be the product of the wave front relations of these operators

$$W_{pr} = W_{\iota_\lambda^*} \times W_{\tilde{V}_{\alpha_1}} \times \ldots \times W_{\tilde{T}_{\alpha_k}(\lambda)} \times W_{(\pi^*\delta_\lambda)p^*}, \tag{6.11}$$

and let $\Delta \subset M = T^*U \times (T^*\mathbb{R}^N)^{2k+2} \times T^*U$ be the diagonal relation. The wave front relation of the composite operator is just $\pi_c(W_{pr} \cap \Delta)$ with $\pi_c : W_{pr} \cap \Delta \to (T^*U)^2$ the projection. Suppose that

$$p = (\xi_0, \xi_0', ..., \xi_{k+1}, \xi_{k+1}', \eta_0, ..., \eta_{k+1}') \in W_{pr} \cap \Delta; \tag{6.12}$$

here the η variables are the fiber coordinates. Thus, ξ_j, $\xi_j' \in \mathbb{S}_\lambda^{N-1}$, $\eta_j' \in X_{\alpha_j}$ for $j = 1, ..., k$. We now show that $\xi_j \notin X_{\alpha_{j-1}}$ for $j = 2, ..., k$. In fact, if $\xi_j \in X_{\alpha_{j-1}}$ then $\xi_{j-1} \in X_{\alpha_{j-1}}$ too: their difference is in $X^{\alpha_{j-1}}$ and they are both in $\mathbb{S}_\lambda^{N-1}$. Since $\xi_0 \in U$, $\xi_1 = \lambda \xi_0$, this implies $\xi_2 \notin X_{\alpha_1}$. On the other hand for $j \geq 3$, as $\eta_{j-1}' \in X_{\alpha_{j-1}}$, this implies $\eta_{j-2}' \in X_{\alpha_{j-1}}$ which cannot happen. Since $\eta_j \in X_{\alpha_{j-1}}$ this implies that $(\xi_j, \eta_j) \notin N_\lambda^+$, $j = 2, ..., k$. This shows that with A, A' as in Corollary 2.22 the composite operator becomes smoothing if we replace $\tilde{T}_{\alpha_j}(\lambda)$ by $\tilde{T}_{\alpha_j}(\lambda) - A\tilde{T}_{\alpha_j}(\lambda)A'$, so we can replace $\tilde{T}_{\alpha_j}(\lambda)$ by the FIO $A\tilde{T}_{\alpha_j}(\lambda)A'$ in (6.10) (again, we shall not indicate this in the notation; A, A' depend on α_j, of course). Thus, we only need to check that the product of their canonical relation

$$C_{pr} = C_{\iota_\lambda^*} \times C_{\tilde{V}_{\alpha_1}} \times ... \times C_{\tilde{T}_{\alpha_k}(\lambda)} \times C_{(\pi^*\delta_\lambda)p^*} \tag{6.13}$$

(which is the restriction of W_{pr} to the conormal part) intersects Δ transversally in $T^*U \times (T^*\mathbb{R}^N)^{2k+2} \times T^*U$, and then the composition theorem in [17], or rather its generalization for more than two factors, proves that $S_V(\lambda)$ is an FIO. (It is easy to see that the projection $\pi_c : C_{pr} \cap \Delta \to (T^*U)^2$ is proper.) To prove transversality suppose that $p \in C_{pr} \cap \Delta$. Our propagation results in Section 2 show that for $j = 2, ..., k$ there exist $c_j > 0$ such that $\eta_j = \eta_{j+1} + c_j \xi_j$ and $\eta_{j+1} = \eta_j' \in X_{\alpha_j}$. One can show by an inductive argument starting from $j = k$ that for $j = 2, ..., k$ there exists $A_j \in \mathrm{End}(T_{\xi_j}\mathbb{S}^{N-1})$ positive, invertible, such that for $v \in T_{\xi_j}\mathbb{S}^{N-1}$

$$v\frac{\partial}{\partial \eta_j} + (A_j v)\frac{\partial}{\partial \xi_j} \in T_p C_{pr} + T_p \Delta. \tag{6.14}$$

In fact, let A_k be multiplication by c_k^{-1}, and given A_{j+1} define $B_j \in \mathrm{End}(P_{\xi_j} X_{\alpha_j})$ by

$$B_j P_{\xi_j} v = \pi_2 A_{j+1} P_{\xi_{j+1}} v, \qquad v \in X_{\alpha_j}$$

where π_2 is projection onto the second factor of the direct sum decomposition $\mathbb{R}^N = X^{\alpha_j} \oplus P_{\xi_j} X_{\alpha_j}$. It is easy to check that B_j is positive and invertible. Now, using the orthogonal decomposition

$$T_{\xi_j}\mathbb{S}^{N-1} = P_{\xi_j} X_{\alpha_j} \oplus (X^{\alpha_j} \cap T_{\xi_j}\mathbb{S}^{N-1})$$

define $A_j = B_j(\mathrm{Id} + c_j B_j)^{-1} \oplus c_j^{-1}$; this provides the inductive step. Now an inductive argument starting at $j = 1$ and (6.14) show that $v\frac{\partial}{\partial \eta_j}$, $v\frac{\partial}{\partial \xi_j} \in T_p C_{pr} + T_p \Delta$ for $j = 1, ..., k$, $v \in \mathbb{R}^N$, so $T_p C_{pr} + T_p \Delta = T_p M$ indeed.

We still need to check that the canonical relation of the terms of $S_V(\lambda)$ is given by the broken geodesic flow as described above. Suppose that

$$(\omega, \eta, \omega', \eta') = \pi_c(\xi_0, ..., \xi_{k+1}', \eta_0, ..., \eta_{k+1}')$$

with $(\xi_0, ..., \eta_{k+1}') \in C_{pr} \cap \Delta$. Hence $\xi_j' = \xi_{j+1}$, $\eta_j' = \eta_{j+1}$ for $j = 0, ..., k$, $\omega_j = \xi_j/\lambda \in \mathbb{S}^{N-1}$, $\theta_j = \eta_j/|\eta_j| \in C_{\alpha_{j-1}}$ for $j = 2, ..., k+1$ and $\theta_1 = \theta_2$. Let $\theta = \eta/|\eta|$, $\theta' = \eta'/|\eta'|$. Let γ_j be the broken geodesic consisting of segments of length less than π, starting at $\omega' = \omega_{k+1}$, breaking at $-\theta_{k+1}$, $-\theta_k$, ..., $-\theta_{j+1}$, ending at $-\omega_j$, and let $\gamma = \gamma_1$. Thus, γ ends at $-\omega = -\omega_1$, its tangent vector at ω' is $P_{\omega'}(-\theta_{k+1}) = -c'\theta'$

and at $-\omega$ it is $P_{-\omega}(\theta_2) = c\theta$ where $c, c' > 0$. Here we need $(-\omega, \theta)$ rather than $(\omega, -\theta)$ as in the statement of the theorem due to the presence of j^* in the definition of $S_V(\lambda)$. Thus, it suffices to check that γ has length π and the segments of γ satisfy the stated law of reflection.

To see the former note that γ_k has length π since

$$\operatorname{dist}(-\theta_{k+1}, -\omega_k) = \operatorname{dist}(-\theta_{k+1}, -\omega_{k+1})$$

as $\omega_k - \omega_{k+1} \in X^{\alpha_k}$ and $\theta_{k+1} \in X_{\alpha_k}$, and

$$\operatorname{dist}(\omega_{k+1}, -\theta_{k+1}) + \operatorname{dist}(-\theta_{k+1}, -\omega_{k+1}) = \pi.$$

We now proceed by induction. Since $\theta_j = \alpha_j\omega_j + \beta_j\theta_{j+1}$ with $\alpha_j, \beta_j > 0$ we see that $-\theta_j$ lies on the last segment of γ_j. Hence, the difference between γ_j and γ_{j-1} is that the geodesic between $-\theta_j$ and $-\omega_j$ is replaced by the geodesic between $-\theta_j$ and $-\omega_{j-1}$. But the same argument as above shows that these have the same length, so the length of γ is indeed π.

We have seen that a tangent vector V_j to the segment of γ from $-\theta_{j+1}$ to $-\theta_j$ at $-\theta_j$ is also a tangent vector to the geodesic between $-\theta_j$ and $-\omega_j$, so we can take $V_j = P_{-\theta_j}(-\omega_j)$. Similarly, a tangent vector to this segment at $-\theta_{j+1}$ is $W_j = P_{-\theta_{j+1}}(-\omega_j)$, so we only need to prove that $W_{j-1} - V_j \in X^{\alpha_{j-1}}$ provided that we also prove that $|W_{j-1}| = |V_j|$. But $W_{j-1} - V_j = P_{-\theta_j}(\omega_j - \omega_{j-1})$, and $\omega_j - \omega_{j-1} \in X^{\alpha_{j-1}}$, $\theta_j \in X_{\alpha_{j-1}}$, so $W_{j-1} - V_j = \omega_j - \omega_{j-1} \in X^{\alpha_{j-1}}$ indeed. In addition

$$|V_j|^2 = 1 - (\omega_j \cdot \theta_j)^2 = 1 - (\omega_{j-1} \cdot \theta_j)^2 = |W_{j-1}|^2.$$

As indicated, this proves the theorem. □

The composition theorem for FIOs also gives the conormal orders of the terms in (6.9). For the sake of simplicity assume that $n_i = m_i = N/2$ for all i (as in the real 3-body problem). Then the term

$$S_\alpha = \frac{\pi i}{\lambda}\iota_\lambda^*\mathcal{F}(V_{\alpha_1} - V_{\alpha_1}T_{\alpha_1}(\lambda))T_{\alpha'}(\lambda)\mathcal{F}^{-1}(\pi^*\delta_\lambda)p^* \in I^{-|\alpha|(N-2)/4}(\pi_c(C_{pr} \cap \Delta)');$$

here the prime denotes the Lagrangian corresponding to the canonical relation. Thus, for $N > 2$ (i.e. $n_i \geq 2$) all terms S_α with $|\alpha| \geq 2$ have higher Sobolev regularity than the others (see [7]), hence the principal symbol of $\sum S_j$ is determined by $S_V(\lambda)$. Let

$$Y^j = \{(\omega_1, \omega_2, \omega_1, \omega_2') : \ (\omega_1, \omega_2), (\omega_1, \omega_2') \in \mathbb{S}^{N-1} \setminus C_j\}$$

where we wrote $\omega = (\omega_1, \omega_2) \in X_j \times X^j$, so the kernel of S_j is conormal to Y^j. Since $Y^j \cap (U \times U)$ are (pairwise) transversal, the principal symbol of each S_j is also determined on $N^*(Y^j \cap (U \times U))$, hence on N^*Y^j. Here we need to use that $Y^j \cap (U \times U)$ is dense in Y^j for $n \geq 2$ since $U \times C_i$ and Y^j $(i \neq j)$ are submanifolds of $\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$ of dimension $(3/2)N - 2$ whose tangent spaces do not coincide.

Let $S_j^\natural(\sigma)$ be the (reduced) 2-body scattering matrix of the potential V_j at energy σ^2. The kernel of $S_j^\natural(\sigma) - i^{n-1}j^*$ is

$$S_j^\natural(\sigma, \theta, \theta') = c(\sigma)(\hat{V}_j(\sigma(-\theta - \theta')) - B_{V_j}(\sigma, -\sigma\theta, \sigma\theta'));$$

$c(\sigma) = \sigma^{n-2}i^n\pi(2\pi)^{-n}$ (see [30, Equation (2.30)]). Let

$$K = \delta(\xi^j - (\xi^j)') \in \mathcal{C}^{-\infty}(\mathbb{R}^N \times \mathbb{R}^N),$$

$\iota_{\lambda,L,R} : (\mathbb{S}_\lambda^{N-1} \setminus X_j)^2 \to \mathbb{R}^N$ the inclusion in both factors. Then the kernel of S_j is

$$S_j(\omega,\omega') = \frac{\pi i}{\lambda}(\hat{V}_j(\lambda(\omega_2 - \omega_2')) - B_{V_j}(\lambda|\omega_2|, \lambda\omega_2, \lambda\omega_2'))\iota^*_{\lambda,L,R}K,$$

so S_j determines $S_j^\natural(\sigma, -\theta, \theta')$ when $\sigma \in (0,\lambda]$. If V_j are exponentially decaying (as in (A.2)) then $S_j^\natural(\sigma)$ is real analytic on $(0,\infty)$, so their asymptotic behavior as $\sigma \to \infty$ is also determined. But V_j can be reconstructed from the high energy asymptotics. Therefore, we have shown:

Proposition 6.4. *If for all i $n_i = N/2 \geq 2$ then $S_V(\lambda)$ determines the two-body scattering matrices $S_j^\natural(\sigma)$ for $\sigma \in (0,\lambda]$. If V_j decay exponentially as in* (A.2)*, then these in turn determine V_j.*

Remark 6.5. Wang [43], Enss and Weder [9] have shown that the two-body potentials can be reconstructed from the high energy asymptotics of $S_V(\lambda)$ for a much wider class of potentials.

7. Weak asymptotics of the resolvent

In Section 6 we obtained the asymptotic behavior of $u = R_V(\lambda)f$ away from the singular set where $f \in \mathcal{S}(\mathbb{R}^N)$. Now we describe the complete behavior of u for $n_i \geq 3$ odd by examining what happens near points on the singular set. When some n_i are even, the conormality statements still hold, but the additional logarithmic terms would make an asymptotic expansion more complicated.

In the proof of Corollary 7.5 we show that it suffices to describe

$$R_{V_i}(\lambda)f = R_0(\lambda)f - R_0(\lambda)V_iR_0(\lambda)f + R_0(\lambda)(V_iR_{V_i}(\lambda)V_i)R_0(\lambda)f; \tag{7.1}$$

so we start with this simpler task. We drop the index i from now on since we are in the setting of the unreduced two-body problem as in Section 2. It is convenient to analyze $R_V(\lambda)$ on the radial compactification $\mathbb{S}^N_+$ of $\mathbb{R}^N$. Proposition 6.1 shows that

$$R_V(\lambda)f = e^{-i\lambda/x}x^{(N-1)/2}g$$

with $g \in \mathcal{C}^\infty(\mathbb{S}^N_+ \setminus C)$, $x = |\bar{w}|^{-1}$. Since $V \notin \mathcal{C}^\infty(\mathbb{S}^N_+)$ (if V is not identically 0), we do not expect that $g \in \mathcal{C}^\infty(\mathbb{S}^N_+)$. However, since all the singularities of V are located on C we might expect that the situation improves if we blow up the p-submanifold C of $\mathbb{S}^N_+$ to obtain the new space $X = [\mathbb{S}^N_+; C]$ (see [27, 28, 29] for a description of blow-ups). Let $\beta : X \to \mathbb{S}^N_+$ be the blow down map, ff the front face of the blow up (so ff $= \beta^{-1}(C)$), and mf the main face, i.e. the lift of $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$ under β. In the following lemma we construct local coordinates on X near ff and show that V pulls back to a smooth function on X which vanishes to infinite order on mf (i.e. it is in $\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X)$).

Lemma 7.1. *If $V \in \mathcal{S}(\mathbb{R}^n)$ then $\beta^*V \in \dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X)$.*

Proof. Away from ff the blow-down map is a diffeomorphism, so β^*V is $\mathcal{C}^\infty$ there. To see how it behaves near ff note that near $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$ $\mathbb{S}^N_+$ is a product $[0,1)_x \times \mathbb{S}^{N-1}_{\bar{\theta}}$ where we write $\bar{w} \in \mathbb{R}^N$ as $\bar{w} = x^{-1}\bar{\theta}$. Furthermore, writing $\bar{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^m \times \mathbb{R}^n$ we have

$$C = \{(x,\bar{\theta}) :\ x = 0, \theta_2 = 0\}, \tag{7.2}$$

so near $C\,\mathbb{S}^N_+$ has the product decomposition $[0,1)_x \times B(1/2)_{\theta_2} \times \mathbb{S}^{m-1}_{\tilde\theta_1}$ with $\tilde\theta_1 = \theta_1/|\theta_1|$. Thus, near the corner $\mathrm{ff} \cap \mathrm{mf}$ X is diffeomorphic to $[0,1)_\nu \times [0,1/2)_{s'} \times \mathbb{S}^{n-1}_{\tilde\theta_2} \times \mathbb{S}^{m-1}_{\tilde\theta_1}$ with $s' = |\theta_2|$, $\tilde\theta_2 = (s')^{-1}\theta_2$, $\nu = x(s')^{-1}$. In terms of coordinates on $\mathbb{R}^N$ this means that

$$s' = |z|/|\bar w|,\ \nu = |z|^{-1},\ \tilde\theta_1 = w/|w|,\ \tilde\theta_2 = z/|z|. \tag{7.3}$$

Here mf is defined by $\nu = 0$, and ff by $s' = 0$ locally. This proves that $\beta^* V$ is smooth near the corner and vanishes to infinite order in ν (hence on mf). Near interior points of ff $(x, \theta_2/x, \tilde\theta_1) \in [0,1) \times \mathbb{R}^n \times \mathbb{S}^{m-1}$ give coordinates; now $x = 0$ defines ff locally. In terms of coordinates on $\mathbb{R}^N$ these are just

$$x = |\bar w|^{-1}, \theta_2/x = z, \tilde\theta_1 = w/|w|, \tag{7.4}$$

so $\beta^* V$ is smooth in the region of validity of these coordinates too. □

We actually use slightly different coordinates in the following. Near the singular set C $(\mathrm{SP}^{-1})^*(|w|/|\bar w|)$ is smooth and does not vanish, so we can use the following coordinates near the corner:

$$s = |z|/|w|,\ \nu = |z|^{-1},\ \tilde\theta_1 = w/|w|, \tilde\theta_2 = z/|z|, \tag{7.5}$$

while near ff but away from the corner

$$\rho = |w|^{-1},\ z,\ \tilde\theta_1 = w/|w|. \tag{7.6}$$

Now that space on which we shall be working has been constructed, we examine the distributions involved in (7.1). We start by analyzing their Fourier transforms. Note that we still use the notation $I^k(Y)$ for distributions on $\mathbb{R}^N$ conormal to a submanifold Y os $\mathbb{R}^N$.

Lemma 7.2. *For $f \in \mathcal{S}(\mathbb{R}^N)$*

$$\mathcal{F} V R_0(\lambda) f \in I^{-(N-2)/4-n/2}(E_\lambda(0)), \tag{7.7}$$

$$\begin{aligned}\tilde B_V(\lambda)\mathcal{F} R_0(\lambda) f \in I^{-(N-2)/4-n/2}(E_\lambda(0)) &+ \sum_{\sigma\in\Lambda\setminus\{0\}} I^{-(N-2)/4}(E_\lambda(\sigma))\\ &+ \sum_{j=0}^{1} I^{-(N-2)/4-j/2}(E_\lambda(0))\end{aligned} \tag{7.8}$$

where the last term arises only if $0 \in \Lambda'$, and the wave front set is in the outgoing half of the conormal bundles.

Proof. Since $f \in \mathcal{S}(\mathbb{R}^N)$, the same holds for $\mathcal{F} f$, so $\rho_0(\lambda)\mathcal{F} f \in I^{-(N-2)/4}(\mathbb{S}^{N-1}_\lambda)$. Since $\tilde V$ is an FIO we see at once that $\tilde V \rho_0(\lambda)\mathcal{F} f \in I^{-(N-2)/4-n/2}(E_\lambda(0))$. The second statement can be proved similarly since $\tilde B_V(\lambda)$ is an FIO followed by multiplication by conormal distributions. The FIO part gives a conormal distribution in $I^{-(N-2)/4-n/2}(E_\lambda(0))$ with one-sided wave front sets, and we multiply it by elements of $I^{-(N-2)/4-n/2}(E_\lambda(0))$ (if $0 \notin \Lambda'$) or $I^{-(N-2)/4-j/2}(E_\lambda(0))$ ($j = 0, 1$) (if $0 \in \Lambda'$) with same-sided wave front sets (hence conormality follows in this case), or by elements of $I^{-(N-2)/4}(E_\lambda(\sigma))$, $\sigma \neq 0$, in which case disjoint singular supports give the result. □

The last two distributions in (7.1) are somewhat more complicated. In fact, being the product of two conormal distributions (conormal to submanifolds of codimension one and with appropriate wave front sets) whose conormal bundles intersect cleanly they are paired Lagrangian distributions. (For a detailed treatment of such distributions see [26].) We do not use this in the following since we shall obtain the asymptotics more explicitly, but for the sake of completeness we state the result. Since the Lagrangians are conormal bundles, we employ the notation $I^{k,l}(Y_1, Y_2)$ instead of $I^{k,l}(N^*Y_1, N^*Y_2)$ if Y_1, Y_2 are submanifolds of $\mathbb{R}^N$ whose conormal bundles give an intersecting pair of Lagrangians.

Lemma 7.3. *For $f \in \mathcal{S}(\mathbb{R}^N)$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ with support close to $\mathbb{S}_\lambda^{N-1} \cap E_\lambda(0)$*

$$\phi \mathcal{F} R_0(\lambda) V R_0(\lambda) f \in I(\mathbb{S}_\lambda^{N-1}, E_\lambda(0)), \tag{7.9}$$

$$\phi \mathcal{F} R_0(\lambda) V R_V(\lambda) V R_0(\lambda) f \in I(\mathbb{S}_\lambda^{N-1}, E_\lambda(0)). \tag{7.10}$$

Proof. This follows from the iterative regularity of conormal distributions, i.e. the stability of their regularity under applications of first order pseudo-differential operators characteristic to the conormal bundle of the submanifold. Since we shall use a more explicit approach, we write down now what this means in terms of symbols. On $\operatorname{supp} \phi$ $\mathbb{S}_\lambda^{N-1}$ is defined by $|\xi| + F(\zeta) - \lambda = 0$ with F smooth, $F(0) = 0$, $dF(0) = 0$, but the Hessian $F''(0)$ is non-degenerate. In fact,

$$F(\zeta) = \lambda - (\lambda^2 - |\zeta|^2)^{1/2}. \tag{7.11}$$

Since $\phi \rho_0(\lambda) \in I^{-(N-2)/4}(\mathbb{S}_\lambda^{N-1})$ with wave front set in N_λ^+, we can write it (modulo $\mathcal{C}_c^\infty(\mathbb{R}^N)$) as

$$v_1 = \int e^{i(|\xi| + F(\zeta) - \lambda)\eta} b(\eta, \xi, \zeta)\, d\eta \tag{7.12}$$

where $b \in S^0(\mathbb{R}; \mathbb{R}^N)$ has support in $[1, \infty) \times K$, $K \subset \mathbb{R}^N$ compact. Similarly, modulo $\mathcal{C}_c^\infty(\mathbb{R}^N)$ we can write $\phi \tilde{V} \rho_0(\lambda) \mathcal{F} f \in I^{-(N-2)/4 - n/2}(E_\lambda(0))$ as

$$v_2 = \int e^{i(|\xi| - \lambda)\tau} a(\tau, \xi, \zeta)\, d\tau \tag{7.13}$$

with $a \in S^{-n/2}(\mathbb{R}; \mathbb{R}^N)$ with the same support conditions as we had for b. Thus, modulo conormal terms $\phi^2 \mathcal{F} R_0(\lambda) V R_0(\lambda) f$ is just

$$v_1 v_2 = \int\int e^{i((|\xi| - \lambda)\mu + F(\zeta)\eta)} c(\mu - \eta, \eta, \xi, \zeta)\, d\eta\, d\mu \tag{7.14}$$

with the notation $\mu = \tau + \eta$. Here $c(\tau, \eta, \xi, \zeta) = a(\tau, \xi, \zeta) b(\eta, \xi, \zeta)$ so c is a product-type symbol in the first two variables. We introduce a bump function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\psi(\alpha) = 1$ for $\alpha > 2/3$, $\psi(\alpha) = 0$ for $\alpha < 1/3$, and let

$$c_1 = \psi(\eta/(\eta + \tau)) c, \;\; c_2 = (1 - \psi(\eta/(\eta + \tau))) c \tag{7.15}$$

so that c_j are also product-type symbols. It is now straightforward to show that

$$\tilde{c}_1(\mu, \xi, \zeta) = \int e^{iF(\zeta)(\eta - \mu)} c_1(\mu - \eta, \eta, \xi, \zeta)\, d\eta \tag{7.16}$$

$$\tilde{c}_2(\mu, \xi, \zeta) = \int e^{iF(\zeta)\eta} c_2(\mu - \eta, \eta, \xi, \zeta)\, d\eta, \tag{7.17}$$

are marked symbols (marked at $\zeta = 0$). Note that the integrals are over compact sets due to the support condition on a and b. Thus,

$$\int e^{i(|\xi|+F(\zeta)-\lambda)\mu}\tilde{c}_1(\mu,\xi,\zeta)\,d\mu, \qquad \int e^{i(|\xi|-\lambda)\mu}\tilde{c}_2(\mu,\xi,\zeta)\,d\mu \tag{7.18}$$

are marked Lagrangian distributions associated to the Lagrangians $\mathbb{S}^{N-1}_\lambda$ and $E_\lambda(0)$ respectively, marked at $\zeta = 0$, so their sum $v_1 v_2$ is a paired Lagrangian distribution in $I(\mathbb{S}^{N-1}_\lambda, E_\lambda(0))$ with outgoing wave front set. Since we only needed the conormality and the wave front set condition on $\tilde{V}\rho_0(\lambda)\mathcal{F}f$, the proof for $\tilde{B}_V(\lambda)\rho_0(\lambda)\mathcal{F}f$ works exactly the same way, though the orders must be adjusted in accordance with (7.8). □

In this lemma we already had a product type symbol (which was in fact the product of two symbols). It will be useful to define a more general class of such symbols. So assume that $k > 0$, M is a manifold without boundary, and for $i = 1, ..., k$, $U_i = \mathbb{R}$ or $U_i = \mathbb{R}^+$. We first define the class $S^{0,\dots,0}_c(\mathbb{R}^k; M) \subset \mathcal{C}^\infty(\mathbb{R}^k \times M)$ using the stereographic projection $\mathrm{SP} : \mathbb{R} \to \mathbb{S}^1_+$. Here $\mathbb{S}^1_+$ can be identified with $[0, \pi]$ in the usual way so that as $p \to \infty$ $\mathrm{SP}(p) \to 0$, while as $p \to -\infty$ $\mathrm{SP}(p) \to \pi$. Now consider $Y = [0, \pi]^k \times M$; it is a manifold with corners. Its boundary faces of codimension 1 are

$$Y_{j,+} = [0,\pi]^{j-1} \times \{0\} \times [0,\pi]^{k-j}, \qquad Y_{j,-} = [0,\pi]^{j-1} \times \{\pi\} \times [0,\pi]^{k-j} \tag{7.19}$$

for $j = 1, ..., k$. Let C be the set of compactly supported smooth functions on Y which vanish to infinite order at $Y_{j,-}$ for all j, i.e.

$$C = \dot{\mathcal{C}}^\infty_{c,Y_{1,+},\dots,Y_{k,+}}(Y),$$

and let $S^{0,\dots,0}_c(\mathbb{R}^k; M) = (\mathrm{SP}^k \times \mathrm{id})^* C$. More generally, if $\alpha \in \mathbb{R}^k$, let

$$S^\alpha_c(\mathbb{R}_{\eta_1} \times \dots \times \mathbb{R}_{\eta_k}; M) = (\prod_{j=1}^k \langle \eta_j\rangle^{\alpha_j}) S^{0,\dots,0}_c(\mathbb{R}^k; M).$$

Finally let

$$S^\alpha_c(U_1 \times \dots \times U_k; M) \subset \mathcal{C}^\infty(U_1 \times \dots \times U_k \times M)$$

be the set of symbols a whose support lies in $U_1 \times \dots \times U_k \times K$ for some $K \subset M$ compact and for which for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is identically 1 near 0

$$(\prod_{j\in J}(1 - \phi(\eta_j)))a \in S^\alpha_c(\mathbb{R}^k; M)$$

where J is the set of indices j with $U_j = \mathbb{R}^+$. With this notation $S^\alpha_c(\mathbb{R}_\eta; M)$ is the usual space of one-step polyhomogeneous symbols of order $\alpha \in \mathbb{R}$ which are supported in $\mathbb{R} \times K$, K compact, and which are rapidly decreasing with all derivatives as $\eta \to -\infty$, and in (7.14) $c \in S^{-n/2,0}_c(\mathbb{R}^2; \mathbb{R}^N)$. From now on we shall assume that $c \in S^{l_1,l_2}_c(\mathbb{R}^2; \mathbb{R}^N)$ to deal with the more general orders in (7.8).

We wish to examine the inverse Fourier transform of the paired Lagrangian distributions of the previous lemma. Note that $v_1 v_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^N)$, so its inverse Fourier transform is certainly smooth. We can write down the Fourier transform explicitly (where c was changed by a constant factor):

$$u(w, z) = \mathcal{F}^{-1}(v_1 v_2)(w, z) = \int e^{i(\xi\cdot w+\zeta\cdot z+(|\xi|-\lambda)(\eta+\tau)+F(\zeta)\eta)} c(\tau, \eta, \xi, \zeta)\,d\tau\,d\eta\,d\xi\,d\zeta.$$

Here the integrals are defined by continuity from residual (rapidly decreasing) symbols, hence we can interchange them. We shall carry out the ξ integral first. Note that $|\xi|$ is away from 0 on $\operatorname{supp} c$, so we may introduce polar coordinates $\xi = r\omega$. We also write $w = |w|\tilde{\theta}_1$ since we are only interested in the behavior of u in a neighborhood of the singular set, and there w is bounded away from 0. The measure $d\xi$ becomes $r^{m-1}\,d\omega\,dr$. Now, by the stationary phase lemma

$$\int e^{ir|w|\omega\cdot\tilde{\theta}_1} c(\tau,\eta,r\omega,\zeta)\,d\omega = e^{ir|w|}c_+(\tau,\eta,|w|,r,\tilde{\theta}_1,\zeta) + e^{-ir|w|}c_-(\tau,\eta,|w|,r,\tilde{\theta}_1,\zeta)$$

where $c_\pm \in S_c^{l_1,l_2,-(m-1)/2}(\mathbb{R}^2\times\mathbb{R}^+;\mathbb{R}^+\times\mathbb{S}^{m-1}\times\mathbb{R}^n)$. Next we do the integral in r. Namely,

$$\int e^{ir(\pm|w|+\tau+\eta)}c_\pm(\tau,\eta,|w|,r,\tilde{\theta}_1,\zeta)r^{m-1}\,dr = c'_\pm(\tau,\eta,|w|,\pm|w|+\tau+\eta,\tilde{\theta}_1,\zeta)$$

with $c'_\pm \in S_c^{l_1,l_2,-(m-1)/2,-\infty}(\mathbb{R}^2\times\mathbb{R}^+\times\mathbb{R}_{\mu'};\mathbb{S}^{m-1}\times\mathbb{R}^n)$ where we write $\mu' = \pm|w|+\tau+\eta$. The rapid decay in μ' is due to the compactness of the projection of the support of c to the r variable. At this point we change variables using $d\tau\,d\eta = d\mu'\,d\eta$, and perform the integration with respect to μ':

$$\begin{aligned}\int e^{-i\lambda(\mu'\mp|w|)}c'_\pm(\mu'\mp|w|-\eta,\eta,|w|,\mu',\tilde{\theta}_1,\zeta)\,d\mu'\\ = e^{\pm i\lambda|w|}\tilde{c}_\pm(\mp|w|-\eta,\eta,|w|,\tilde{\theta}_1,\zeta)\end{aligned}$$

due to the rapid decay of $c'_\pm$ in μ'. Here $\tilde{c}_\pm \in S_c^{l_1,l_2,-(m-1)/2}(\mathbb{R}^2\times\mathbb{R}^+;\mathbb{S}^{m-1}\times\mathbb{R}^n)$. Moreover, $\tilde{c}_+$ is a Schwartz function in $|w|$ and η, since we have rapid decay in its first two variables for negative values of the arguments, and if $|(|w|,\eta)|\to\infty$ then at least one of $-|w|-\eta$ and η tends to $-\infty$ (at least as fast as the other goes to ∞, if it goes there at all). Thus, the contribution of the term involving $\tilde{c}_+$ to u is Schwartz, so it can be neglected. Note that $\tilde{c}_-$ satisfies estimates

$$|D_1^{k_1}D_2^{k_2}D_3^{k_3}D_{\tilde{\theta}_1}^{\gamma}D_\zeta^{\delta}\tilde{c}_-| \le C_{k_1,k_2,k_3,\gamma,\delta,r}\langle|w|-\eta\rangle^{-r}\langle\eta\rangle^{l_2-k_2}\langle|w|\rangle^{-(m-1)/2-k_3}$$

for any r when $|w|-\eta<0$, and

$$|D_1^{k_1}D_2^{k_2}D_3^{k_3}D_{\tilde{\theta}_1}^{\gamma}D_\zeta^{\delta}\tilde{c}_-| \le C_{k_1,k_2,k_3,\gamma,\delta,r}\langle|w|-\eta\rangle^{l_1-k_1}\langle\eta\rangle^{-r}\langle|w|\rangle^{-(m-1)/2-k_3}$$

for any r when $\eta<0$ since c satisfied similar estimates, so if we replace $\tilde{c}_-$ by

$$\tilde{c} = \psi_1(\eta/|w|)\tilde{c}_-, \tag{7.20}$$

$\psi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$, identically 1 in a neighborhood of $[0,1]$, we change the integral by a function which is rapidly decreasing in $|w|$. Thus, the interesting term is

$$u_- = e^{-i\lambda|w|}\int e^{i(\zeta\cdot z+F(\zeta)\eta)}\tilde{c}(|w|-\eta,\eta,|w|,\tilde{\theta}_1,\zeta)\,d\zeta\,d\eta. \tag{7.21}$$

We shall now show that the pull-back of $e^{i\lambda|(w,z)|}u_-$ to $[\mathbb{S}^N_+;C]$ is conormal to the boundary. Note that we have already constructed coordinates in (7.5) and (7.6) near ff. Since we are only interested in what happens near the singular set, one of these two coordinate systems can always be used in the region of interest.

It is reasonable to expect that we can also obtain asymptotic expansions (as opposed to just stability under repeated application of b-vectorfields). However, it seems that we need to blow up the corner $\mathrm{mf}\cap\mathrm{ff}$ to have such an expansion. Let cf be the front face of this blow up, and β_c the blow down map. In terms of our coordinates (7.5) the new coordinates near $\mathrm{cf}\cap\mathrm{ff}$ become

$$\nu,\ y=s/\nu,\ \tilde\theta_1,\ \tilde\theta_2 \tag{7.22}$$

with $\nu=0$ defining cf, $y=0$ defining ff locally, while near $\mathrm{cf}\cap\mathrm{mf}$ we have

$$s,\ y'=\nu/s,\ \tilde\theta_1,\tilde\theta_2 \tag{7.23}$$

where $s=0$ defines cf and $y'=0$ defines mf.

Proposition 7.4. *Let u_- be as in* (7.21)*. Then*

$$\beta^*(\mathrm{SP}^{-1})^*(e^{i\lambda|(w,z)|}u_-)\in\mathcal{A}([\mathbb{S}^N_+;C]), \tag{7.24}$$

$$\beta_c^*\beta^*(\mathrm{SP}^{-1})^*(e^{i\lambda|(w,z)|}u_-)\in\mathcal{A}_{phg}([\mathbb{S}^N_+;C;\mathrm{mf}\cap\mathrm{ff}]). \tag{7.25}$$

Proof. We shall prove the existence of the classical expansion on $[\mathbb{S}^N_+;C;\mathrm{mf}\cap\mathrm{ff}]$. The conormality on $[\mathbb{S}^N_+;C]$ automatically follows from this. It can also be shown directly using integration by parts.

We divide u_- into three pieces. Let ψ be as in (7.15), and let $\phi\in\mathcal{C}^\infty_c(\mathbb{R})$ be identically 1 on $[-2C,2C]$ where $C=\sup_{\zeta\in\pi(\operatorname{supp}\tilde c)}|F'(\zeta)|$; π is projection to the last factor. Let $\alpha=\eta/|w|$. Now define

$$\tilde c_1=\psi(\alpha)|w|\tilde c((1-\alpha)|w|,\alpha|w|,|w|,\tilde\theta_1,\zeta),$$

$$\tilde c_2=(1-\psi(\alpha))\phi(z/(\alpha|w|))|w|\tilde c((1-\alpha)|w|,\alpha|w|,|w|,\tilde\theta_1,\zeta),$$

$$\tilde c_3=(1-\psi(\alpha))(1-\phi(z/(\alpha|w|)))|w|\ \tilde c((1-\alpha)|w|,\alpha|w|,|w|,\tilde\theta_1,\zeta);$$

$c_j\in S_c^{l_1,l_2,-(m-3)/2}(\mathbb{R}^2\times\mathbb{R}^+;\mathbb{R}_\alpha\times\mathbb{S}^{m-1}\times\mathbb{R}^n)$. For $j=1,2,3$ we also let

$$I_j=\int e^{i(\zeta\cdot z+F(\zeta)\alpha|w|+\lambda(|(w,z)|-|w|))}\tilde c_j\,d\zeta\,d\alpha. \tag{7.26}$$

Since $\alpha\geq 1/3$ on $\operatorname{supp}\psi$ the symbolic dependence on $\alpha|w|$ in $\tilde c_1$ means a smooth dependence on α and a symbolic dependence on $|w|$, and similarly in $\tilde c_2$ and $\tilde c_3$ the symbolic dependence on $(1-\alpha)|w|$ means smooth dependence on α and symbolic dependence on $|w|$. In addition, $\tilde\phi(\sigma)=1-\phi(\sigma^{-1})$ is a smooth function of σ supported in $\sigma\leq(2C)^{-1}$. We now proceed to analyze these three pieces. We shall be working near the corner $\mathrm{mf}\cap\mathrm{ff}$; the proofs in the interior of ff are similar, and in fact easier.

Near the corner $|w|=s^{-1}\nu^{-1}$, so $|(w,z)|=s^{-1}\nu^{-1}(1+s^2)^{1/2}$. Thus, the exponent is

$$G=(\zeta\cdot\tilde\theta_2+\alpha s^{-1}F(\zeta)+g(s))\nu^{-1} \tag{7.27}$$

where

$$g(s)=\lambda((1+s^2)^{1/2}-1)s^{-1} \tag{7.28}$$

is smooth. Note that $d_\zeta G=0$ just means that $s\tilde\theta_2+\alpha F'(\zeta)=0$. Using the non-degeneracy of the Hessian of F we deduce that for $\alpha\neq 0$ this is satisfied if and only if $\zeta=h(s\tilde\theta_2/\alpha)$ where h is smooth and $h(0)=0$. Since we know F explicitly we see that

$$h(s\tilde\theta_2/\alpha)=-\lambda(1+(s/\alpha)^2)^{-1/2}(s/\alpha)\tilde\theta_2, \tag{7.29}$$

and correspondingly

$$G(h(s\tilde{\theta}_2/\alpha)) = (g(s) - g(s/\alpha))\nu^{-1}. \tag{7.30}$$

In particular, taking $\alpha = 1$ $G(h(s\tilde{\theta}_2)) = 0$. Since α near 1 corresponds to the marked Lagrangian distribution associated to $N^*\mathbb{S}^{N-1}_\lambda$ (as in (7.18)), the factor $e^{i\lambda|(w,z)|}$ comes from the stationary phase lemma which can be applied away from the marking as usual.

We first consider I_1. Here we can use the stationary phase lemma on the ζ integral in $x' = s\nu$ since α is away from 0 so we do not have degenerate critical points. We find this way that

$$I_1 = \int e^{i(g(s)-g(s/\alpha))\nu^{-1}} c_1'((1-\alpha)s^{-1}\nu^{-1}, s^{-1}\nu^{-1}, \alpha, \tilde{\theta}_1, s\tilde{\theta}_2)\, d\alpha \tag{7.31}$$

where $c_1' \in S_c^{l_1, l_2-(N-3)/2}(\mathbb{R}\times\mathbb{R}^+; \mathbb{R}\times\mathbb{S}^{m-1}\times\mathbb{R}^n)$. Since $g(s)$ vanishes at $s=0$, our exponent is of the form $\tilde{G}(s,\alpha)s\nu^{-1}$ where $\tilde{G}(s,1)\equiv 0$ and given $\alpha_0 > 0$ there exists $s_0 > 0$ such that $D_\alpha\tilde{G}(s,\alpha) \neq 0$ when $\alpha \geq \alpha_0$ and $s \leq s_0$. We blow up the corner $s=0$, $\nu = 0$ as indicated beforehand, so near ff the exponent is a smooth function, and hence it can be incorporated into c_1'. We can expand c_1' in its second variable and take a homogeneous term (which can be factored out of the integral). Thus, we only need to show that for $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$I(x,t) = \int \chi(\beta) b(\beta x^{-1}, t)\, d\beta \tag{7.32}$$

is a classical symbol when b is one-step classical of order l in the first variable and depends smoothly on the second (here t lies in a parameter set M). Here $\beta = 1-\alpha$, $x = s\nu$ and $l = l_1$. This follows immediately from the push-forward theorem in [28], but we shall need related, though slightly more complicated results momentarily, and a direct proof will be useful then.

If χ vanishes to order k at 0, direct differentiation shows that $a(x,\beta,t) = x^l b(\beta x^{-1}, t)$ satisfies

$$D_t^\gamma D_x^r D_\beta^s a \in L^\infty([0,1)_x \times (-1,1)_\beta \times M) \tag{7.33}$$

when $r + s \leq k + l$, so a is in $\mathcal{C}^{k+l-1}$. Performing the integral shows then that $x^l I(x,t) \in \mathcal{C}^{k+l-1}$. In particular, infinite order vanishing at 0 shows that $x^l I(x,t)$ is smooth. Therefore it suffices to consider $\chi(\beta) = \beta^k\tilde{\chi}(\beta)$ where $\tilde{\chi}$ is identically 1 in a neighborhood of 0. Let

$$\tilde{b}(\eta, t) = \int_0^\eta (\eta')^k b(\eta', t)\, d\eta'. \tag{7.34}$$

Then $\tilde{b} = b_1(\eta,t) + (\log\eta)b_2(t) + b_3(t)$ where b_1 is a one-step classical symbol of order $k+l+1$ with smooth dependence on t, and b_2, b_3 are smooth. The log term can arise only if $l+k+1$ is a non-negative integer. Writing the factor β^k as $x^k(\beta x^{-1})^k$ and integrating by parts in (7.32) gives

$$\tilde{I}(x,t) = -x^{k+1}\int \tilde{\chi}'(\beta)\tilde{b}(\beta x^{-1}, t)\, d\beta. \tag{7.35}$$

Since $\tilde{\chi}'$ vanishes near $\beta = 0$, the previous argument shows that the right hand side is the sum of two classical conormal distributions of symbolic order l and $-k-1$ respectively, and (possibly) of a term $x^{k+1}\log x$ where $k+l+1 \in \mathbb{N}$. Putting all the previous information together shows that $I(x,t)$ is the sum of two one-step classical conormal distributions (depending smoothly on t), one of order l, and one of order -1, and of the product of $\log x$ and a one-step classical conormal distribution of order l.

Returning to I_1 we thus conclude that near ff

$$I_1 = (y\nu^2)^{-l_2+(N-1)/2} I_{11} + (y\nu^2)^{(N-3)/2-l_1-l_2}(I_{12} + \log(y\nu^2) I_{13}) \tag{7.36}$$

with $I_{1j} \in \mathcal{C}^\infty([0,\infty)^2_{\nu,y} \times \mathbb{S}^{m-1}_{\tilde\theta_1} \times \mathbb{S}^{n-1}_{\tilde\theta_2})$. We must also look at the region near mf. There the situation is nicer, for the exponent gives infinite order vanishing in y' if c_1' vanishes to infinite order at $\alpha = 1$; of course, finite order vanishing gives correspondingly finite order decay. We can use the Taylor series expansion of c_1' to obtain classical conormal behavior in this region. Note that I_2 and I_3 vanish to infinite order on mf for the same reason.

We can also use the stationary phase lemma for the ζ integral in I_2 if we introduce $\mu = s\nu/\alpha$ and $\sigma = s/\alpha$. Then the integral in ζ becomes

$$\int e^{i(F(\zeta)+\sigma\zeta\cdot\tilde\theta_2+\sigma g(\sigma))\mu^{-1}} c_2''(\alpha(s\nu)^{-1}, (s\nu)^{-1}, \sigma, \alpha, \tilde\theta_1, \zeta)\, d\zeta.$$

The stationary phase lemma shows that this is

$$\begin{aligned} &c_2'''(\mu^{-1}, \alpha(s\nu)^{-1}, (s\nu)^{-1}, \sigma, \alpha, \tilde\theta_1, \tilde\theta_2) \\ &\qquad\qquad \in S_c^{-n/2, l_2, l_1-(m-3)/2}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+; \mathbb{R}^2 \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1}). \end{aligned}$$

Thus, after substituting back the original variables we are left with the integral

$$I_2 = \int e^{i(g(s)-g(s/\alpha))\nu^{-1}} c_2'(\alpha(s\nu)^{-1}, (s\nu)^{-1}, s/\alpha, \alpha, \tilde\theta_1, \tilde\theta_2)\, d\alpha. \tag{7.37}$$

Here $c_2' \in S_c^{l_2-n/2, l_1-(m-3)/2}(\mathbb{R}\times\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$; in particular has compact support in s/α. If we have infinite order vanishing in the α variable at 0 (i.e. $D_4^j c_2'$ vanishes for all j when $\alpha = 0$) then the argument given for I_1 applies, and $(y\nu^2)^{l_1+l_2-(N-3)/2} I_2$ is smooth. Hence it suffices to consider the Taylor expansion in α, i.e. we replace c_2' by

$$\alpha^k \chi_1(\alpha) b(\alpha(s\nu)^{-1}, s/\alpha, \tilde\theta_1, \tilde\theta_2)$$

with $\chi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near 0. Here we also expanded c_2' in $(s\nu)^{-1}$, and moved this factor outside the integral. It is convenient to take the factor $e^{ig(s)\nu^{-1}}$ outside the integral too; since $g(s)$ vanishes at $s = 0$ this is smooth on $\mathrm{ff} \cup (\mathrm{cf} \setminus \mathrm{mf})$. Then we are left with

$$J = \int e^{-ig(s/\alpha)\nu^{-1}} \alpha^k \chi_1(\alpha) b(\alpha(s\nu)^{-1}, s/\alpha, \tilde\theta_1, \tilde\theta_2)\, d\alpha. \tag{7.38}$$

Changing the variables back to σ gives

$$J = -\int e^{-ig(\sigma)\nu^{-1}} (y\nu/\sigma)^{k+1} \chi_1(y\nu/\sigma) \sigma^{-1} b(\sigma^{-1}\nu^{-1}, \sigma, \tilde\theta_1, \tilde\theta_2)\, d\sigma.$$

Multiplying through by $(y\nu)^{-(k+1)}$ and differentiating with respect to y yields

$$\begin{aligned} (y\nu)^{k+2} y\partial_y (y\nu)^{-(k+1)} J = -\int e^{-ig(\sigma)\nu^{-1}} &(y\nu/\sigma)^{k+3} \chi_1'(y\nu/\sigma) \\ &b(\sigma^{-1}\nu^{-1}, \sigma, \tilde\theta_1, \tilde\theta_2)\, d\sigma. \end{aligned}$$

Now the integrand vanishes near $\alpha = y\nu/\sigma = 0$, so the right hand side is a smooth multiple of $y\nu(y\nu^2)^{-l_2+n/2}$. Integration with respect to y then gives that

$$J = (y\nu^2)^{-l_2+n/2} a(y, \nu, \tilde{\theta}_1, \tilde{\theta}_2) + (y\nu)^{k+1} C(\nu, \tilde{\theta}_1, \tilde{\theta}_2) \tag{7.39}$$

where $a = a_1 + (\log y)a_2$, a_1, $a_2 \in \mathcal{C}^\infty([0,\infty)^2 \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Since J is smooth in the interior of $\mathbb{S}^N_+$, C is also smooth away from $\nu = 0$. Hence, we only need to examine how C behaves when $\nu \to 0$, i.e. how J behaves in the interior of cf (since C is independent of y).

As b is compactly supported in σ, σ^{-1} is bounded away from 0 on supp b. If we are given j and b has sufficiently low symbolic order then $\nu^{-j}J$ is bounded and similar estimates hold for its derivatives. Hence it suffices to examine the homogeneous terms of b; we shall take it to have order s. Moreover, if b vanishes in σ at 0 to a sufficiently high order, integration by parts shows again that $\nu^{-j}J$ is bounded and analogous estimates also hold for the derivatives. We can therefore assume that b is supported sufficiently close to 0 in σ so that we can change coordinates into $\tilde{\sigma} = g(\sigma)$ (since g vanishes at 0 but has a nonvanishing first derivative) and expand b in Taylor series in $\tilde{\sigma}$ around 0. Here we take the order p term. Hence, we only need to analyze

$$J_{s,p} = y^{k+1}\nu^{k+1-s} \int e^{-i\tilde{\sigma}\nu^{-1}} \tilde{\sigma}^r \chi_2(y\nu/\tilde{\sigma}, \tilde{\sigma}, \tilde{\theta}_1, \tilde{\theta}_2)\, d\tilde{\sigma} \tag{7.40}$$

where $r = -s - k - 2 + p$ and χ_2 is smooth, compactly supported, and identically 1 near $\tilde{\sigma} = 0$. Note that

$$J_{s,p} \in \mathcal{C}^\infty((0,\infty)^2_{y,\nu} \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1}). \tag{7.41}$$

Now,

$$\nu\partial_\nu \nu^{2s-p} J_{s,p} = \int e^{-i\tilde{\sigma}\nu^{-1}} \tilde{\sigma}^{r+1} y^{k+1} \nu^{-r-1} \partial_2\chi_2(y\nu/\tilde{\sigma}, \tilde{\sigma}, \tilde{\theta}_1, \tilde{\theta}_2)\, d\tilde{\sigma},$$

and $\partial_2\chi_2$ is supported away from $\tilde{\sigma} = 0$. Thus, as we have seen, the right hand side is a smooth function which vanishes to infinite order at $\nu = 0$. Therefore

$$J_{s,p} = \tilde{a}(y, \nu, \tilde{\theta}_1, \tilde{\theta}_2) + \nu^{p-2s} c(y, \tilde{\theta}_1, \tilde{\theta}_2)$$

with $\tilde{a} \in \mathcal{C}^\infty((0,\infty) \times [0,\infty) \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ vanishing to infinite order at $\nu = 0$ and $c \in \mathcal{C}^\infty((0,\infty) \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ by (7.41). This implies that

$$J = \nu^{-2(l_2-n/2)} \hat{c}(y, \nu, \tilde{\theta}_1, \tilde{\theta}_2)$$

with $\hat{c} \in \mathcal{C}^\infty((0,\infty) \times [0,\infty) \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Hence in (7.39) $\nu^{2(l_2-n/2)}\nu^{k+1}C \in \mathcal{C}^\infty([0,\infty) \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. We thus conclude that with $I_{2j} \in \mathcal{C}^\infty([0,\infty)^2 \times \mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$

$$I_2 = (y\nu^2)^{(m-1)/2-l_1} I_{21} + (y\nu^2)^{(N-3)/2-l_1-l_2}(I_{22} + (\log y) I_{23}). \tag{7.42}$$

Finally consider I_3. Note that

$$\tilde{\theta}_2 \cdot D_\zeta e^{iG} = (1 + \alpha s^{-1}\tilde{\theta}_2 \cdot F'(\zeta))\nu^{-1} e^{iG},$$

and on supp $\tilde{\phi}$ $|\alpha s^{-1}\tilde{\theta}_2 \cdot F'(\zeta)| \leq 1/2$, so integration by parts shows that the integral is rapidly decreasing in ν. Introducing $\hat{\sigma} = \alpha/s$ gives

$$I_3 = \int e^{i(F(\zeta)\hat{\sigma} + \zeta\cdot\tilde{\theta}_2)\nu^{-1}} c_3'(\hat{\sigma}\nu^{-1}, (s\nu)^{-1}, \hat{\sigma}, s, \tilde{\theta}_1, \zeta)\, d\zeta\, d\hat{\sigma}. \tag{7.43}$$

Since c_3' has compact support in $\hat{\sigma}$ and the integral decreases rapidly in ν^{-1}, the symbolic behavior in $\hat{\sigma}\nu^{-1}$ can be thought of as smooth behavior in ν and in $\hat{\sigma}$ with infinite order vanishing at $\nu = 0$. Then the fact that c_3' has a classical expansion of order $l_1 - (m-3)/2$ in $(s\nu)^{-1}$ together with the smooth behavior and first order vanishing in s shows that I_3 is classical conormal to the boundary of $[\mathbb{S}^N_+; C]$ with infinite order vanishing on mf and order $-l_1 + (m-1)/2$ on ff. $\square$

We now define an index family $\mathcal{K}$ for $[\mathbb{S}^N_+; \cup_i C_i; \cup_i(\mathrm{mf} \cap \mathrm{ff}_i)]$ (see [28]). We let $\frac{1}{2}\mathbb{N} + r = \{\frac{1}{2}k + r :\ k \in \mathbb{N}\}$ where $\mathbb{N}$ is the set of non-negative integers,

$$\mathcal{K}(\mathrm{mf}) = \mathbb{N} \times \{0\},\ \mathcal{K}(\mathrm{ff}_i) = \frac{1}{2}\mathbb{N} \times \{0\} \cup (\frac{1}{2}\mathbb{N} + (n_i - 2)/2) \times \{0, 1\}, \tag{7.44}$$

$$\mathcal{K}(\mathrm{cf}_i) = \mathbb{N} \times \{0\} \cup (\mathbb{N} + n_i - 2) \times \{0, 1\} \tag{7.45}$$

if $0 \notin \Lambda_i'$, and

$$\mathcal{K}(\mathrm{ff}_i) = \mathbb{N} \times \{0\} \cup (\frac{1}{2}\mathbb{N} - 1) \times \{0, 1\}, \tag{7.46}$$

$$\mathcal{K}(\mathrm{cf}_i) = \mathbb{N} \times \{0\} \cup (\mathbb{N} - 2) \times \{0, 1\} \tag{7.47}$$

if $0 \in \Lambda_i'$. If the indicial set at a boundary hypersurface H is $\mathbb{N} \times \{0\}$ and $u \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}$, then it has a $\mathcal{C}^\infty$ expansion at H. If it is $(\frac{1}{2}\mathbb{N} + r) \times \{0\}$, it simply means that ρ_H^{-r} multiplied by the conormal function is smooth in $\rho_H^{1/2}$ (ρ_H a defining function for H), while $(\frac{1}{2}\mathbb{N} + r) \times \{0, 1\}$ gives the sum of such a function and $\log \rho_H$ times such a function.

Corollary 7.5. *Suppose that $V_i \in \mathcal{S}(\mathbb{R}^{n_i})$. Then for $f \in \mathcal{S}(\mathbb{R}^N)$*

$$\beta^*(\mathrm{SP}^{-1})^* R_V(\lambda) f = e^{-i\lambda/x} g + \sum_j \sum_{\sigma \in \Lambda_j} e^{-i(\lambda^2 - \sigma)^{1/2}/x} g_{j,\sigma}$$

where g, $g_{j,\sigma} \in \mathcal{A}([\mathbb{S}^N_+; \cup_i C_i])$, $g_{j,\sigma}$ vanish to infinite order on mf. *Moreover, if $n_i \geq 3$ are odd then*

$$\beta_c^* \beta^*(\mathrm{SP}^{-1})^* R_V(\lambda) f = e^{-i\lambda/x} x^{(N-1)/2} g + \sum_j \sum_{\sigma \in \Lambda_j} e^{-i(\lambda^2 - \sigma)^{1/2}/x} x^{(m_i - 1)/2} g_{j,\sigma}$$

where

$$g \in \mathcal{A}^{\mathcal{K}}_{phg}([\mathbb{S}^N_+; \cup_i C_i; \cup_i(\mathrm{mf} \cap \mathrm{ff}_i)]),\ g_{j,\sigma} \in \dot{\mathcal{C}}^\infty_{\mathrm{ff}_j}([\mathbb{S}^N_+; \cup_i C_i]).$$

Proof. Just as in the proof of Proposition 6.1 we can use the expansion (4.37) of $R_V(\lambda) f$ and we can ignore the term involving P. All other terms (except the trivial one, $R_0(\lambda) f$) are of the form $R_{V_i}(\lambda) V_i R_0(\lambda) u$ where $\mathrm{WF}_{\mathrm{sc}}(u) \subset \mathrm{cl}(F_j^+(\lambda))$ for some $j \neq i$. Choosing a cutoff function $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ such that ϕ is identically 1 in a neighborhood of $\mathbb{S}^{N-1}_\lambda \cap E^i_\lambda(0)$ but vanishes in a neighborhood of $\cup_{\sigma \in \Lambda_j} E^j_\lambda(\sigma)$, we see that

$$V_i R_0(\lambda) \mathcal{F}^{-1}(1 - \phi)\mathcal{F} u \in \mathcal{S}(\mathbb{R}^N), \qquad \mathcal{F}^{-1}\phi\mathcal{F} u \in \mathcal{S}(\mathbb{R}^N).$$

Since $R_{V_i}(\lambda) V_i R_0(\lambda) = R_0(\lambda) - R_{V_i}(\lambda)$, we have

$$R_{V_i}(\lambda) V_i R_0(\lambda) u = R_{V_i}(\lambda)(V_i R_0(\lambda) \mathcal{F}^{-1}(1 - \phi)\mathcal{F} u) + (R_0(\lambda) - R_{V_i}(\lambda))\mathcal{F}^{-1}\phi\mathcal{F} u,$$

so it suffices to obtain the asymptotic behavior of $R_{V_i}(\lambda)u$ for $u \in \mathcal{S}(\mathbb{R}^N)$. But by (7.1) $\mathcal{F}R_{V_i}(\lambda)u$ is a sum of conormal distributions to $\mathbb{S}^{N-1}_\lambda$ and $E^i_\lambda(\sigma)$, $\sigma \in \Lambda_i$, and paired Lagrangian distributions in $I(\mathbb{S}^{N-1}_\lambda, E^i_\lambda(0))$. The inverse Fourier transform of a distribution conormal to $\mathbb{S}^{N-1}_\lambda$ with wave front set in N^+_λ is $e^{-i\lambda|\bar{w}|}g$ where $(\mathrm{SP}^{-1})^*g \in \mathcal{A}_{\mathrm{phg}}(\mathbb{S}^N_+)$ (see [31]). The inverse Fourier transform of a conormal distribution to $E^i_\lambda(\sigma)$ (which is rapidly decreasing at infinity) is even easier to calculate. In fact, the inverse Fourier transform in ζ^i gives a conormal distribution to $\lambda^2 - |\xi^i|^2 = \sigma$ which is rapidly decreasing in z^i. Then the inverse Fourier transform in ξ^i yields $e^{-i(\lambda^2-\sigma)^{1/2}|w^i|}g'(w^i, z^i)$ (assuming we had outgoing wave front set to start with; $\lambda^2 - \sigma > 0$, and we are taking its positive square root), where g' is a symbol in the w^i variable, and it is (uniformly) rapidly decreasing in z^i. Translated into the compactified picture this means that g' is the pull-back of an element of $\mathcal{A}_{\mathrm{phg}}([\mathbb{S}^N_+; C_i])$ which vanishes to all orders on mf. Thus, the same is true for $e^{i(\lambda^2-\sigma)^{1/2}(|\bar{w}|-|w^i|)}g'$ since in terms of our coordinates (7.5) near the corner the exponential is just $e^{i(\lambda^2-\sigma)^{1/2}((1+s^2)^{1/2}-1)s^{-1}\nu^{-1}}$ and $((1+s^2)^{1/2}-1)s^{-1}$ is smooth in s. Finally, we have already calculated the inverse Fourier transform of the paired Lagrangian distributions in Proposition 7.4. Note that the factor of $1/2$ in (7.44), etc. arises due to the presence of the square root in (2.16). Putting these results together proves the corollary. $\square$

Appendix A. The reduced two-body resolvent

For $V \in \mathcal{S}(\mathbb{R}^n)$ real valued the reduced two-body operator, $\Delta + V$, is an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$. Its resolvent is $(\Delta + V - \sigma)^{-1}$ which is a bounded operator for $\sigma \in \mathbb{C} \setminus [\inf V, \infty)$ with $\|(\Delta + V - \sigma)^{-1}\| \leq |\operatorname{Im}\sigma|^{-1}$. Thus, it defines the distributions (in $\sigma \in \mathbb{R}$) with values in $\mathcal{B}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ by $(\Delta + V - (\sigma \pm i0))^{-1}$. Often it is better to use the operator

$$R^\natural_V(\sigma) = (\Delta + V - \sigma^2)^{-1} \tag{A.1}$$

for $\sigma \in \mathcal{P}$ instead of $(\Delta + V - \sigma)^{-1}$; here $\mathcal{P}$ is the open lower half plane, also called the physical half plane. The reason is that for $n \geq 3$ odd and $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ exponentially decaying, i.e. for some $T > 0$ and for all multiindices $\alpha \in \mathbb{N}^n$

$$|D^\alpha V(z)| \leq C_\alpha e^{-T|z|}, \tag{A.2}$$

$R^\natural_V(\sigma)$ extends to a meromorphic family of operators in

$$\mathcal{B}(e^{-T\langle z\rangle/2}H^k(\mathbb{R}^n), e^{T\langle z\rangle/2}H^{k+2}(\mathbb{R}^n))$$

in $\Omega_T = \{\sigma \in \mathbb{C} : \operatorname{Im}\sigma < T/2\}$ (see, for example, [30, 37] for a treatment of the reduced two-body problem; for the analytic continuation of exponentially decaying potentials see [6, 41]). We shall also refer to $R^\natural_V(\sigma)$ as the reduced two-body resolvent. Let Λ_p be the set of its poles in $\overline{\mathcal{P}}$, so Λ_p is finite, $\Lambda_p \cap \mathbb{R} \subset \{0\}$. For other values of n $R^\natural_V(\sigma)$ has a meromorphic extension to a strip of the logarithmic plane, and its behavior near 0 can be described rather precisely (see [41]).

It is actually convenient to consider $VR^\natural_V(\sigma)V$ instead of $R^\natural_V(\sigma)$ since its properties are easier to describe. Let $B_\epsilon(0) = \{\sigma \in \overline{\mathcal{P}} : |\sigma| < \epsilon\}$. Then, for $n \geq 3$ odd, V as in (A.2) $VR^\natural_V(\sigma)V$ is a meromorphic family of operators in $\mathcal{P}$ with values in $\mathcal{B}(H^{r,s}(\mathbb{R}^n), H^{r+2,s'}(\mathbb{R}^n))$, with a $\mathcal{C}^\infty$ extension to $\overline{\mathcal{P}} \setminus \Lambda_p$, and for sufficiently small

$\epsilon > 0$ we can write

$$VR_V^\natural(\sigma)V = \sum_{j=1}^{2} \sigma^{-j} A_j + A(\sigma) \tag{A.3}$$

with $A_j \in \mathcal{B}(H^{r,s}(\mathbb{R}^n), H^{r',s'}(\mathbb{R}^n))$ and $A(\sigma)$ holomorphic in $\mathcal{P} \cap B_\epsilon(0)$ with a $\mathcal{C}^\infty$ extension to $B_\epsilon(0)$ with values in $\mathcal{B}(H^{r,s}, H^{r+2,s'})$ for all r, r', s, s'. This statement is, of course, much weaker than the possibility of a holomorphic extension to a strip, but this is all that we shall need. Again, as noted above, we have a very similar expansion for other values of n, but it includes logarithmic terms in σ.

If we merely assume that $V \in \mathcal{S}(\mathbb{R}^n)$, we cannot use the idea of analytic continuation through the real axis. However, for $n \geq 3$ odd it is still true that $VR_0^\natural(\sigma)$ is a holomorphic family of compact operators on $H^{r,s}(\mathbb{R}^N)$ in $\mathcal{P}$ with a $\mathcal{C}^j$ extension to $\overline{\mathcal{P}}$ for $j \leq r-1$. The absence of imbedded eigenvalues shows that $\mathrm{Id} + VR_0^\natural(\sigma)$ is invertible for $\sigma \in \mathbb{R} \setminus \{0\}$, and the inverse is $\mathcal{C}^j$. Of course, in $\mathcal{P}$ we can use analytic Fredholm theory as usual. So we only have to consider what happens near $\sigma = 0$. Now it is actually advantageous to use the operator $K(\sigma) = V_1 R_0^\natural(\sigma) V_2$ with $V = V_1 V_2$, where it suffices for our purposes to take V_j with $\langle z\rangle^s V_j \in L^\infty(\mathbb{R}^n)$ for all s. Then $K(\sigma)$ is a holomorphic family of compact operators on $L^2(\mathbb{R}^n)$ in $\mathcal{P}$ with a $\mathcal{C}^\infty$ extension to $\overline{\mathcal{P}}$. Moreover, as $\|R_V^\natural(\sigma)\| \leq 2|\operatorname{Im}\sigma|^{-1}|\operatorname{Re}\sigma|^{-1}$ for $\operatorname{Im}\sigma < 0$, $\operatorname{Re}\sigma \neq 0$, we see that

$$\|(\mathrm{Id} + K(\sigma))^{-1}\| \leq c_1 + c_2 |\operatorname{Im}\sigma|^{-1} |\operatorname{Re}\sigma|^{-1} \tag{A.4}$$

since

$$(\mathrm{Id} + K(\sigma))^{-1} = \mathrm{Id} - V_1 R_0^\natural(\sigma) V_2 + V_1 R_V^\natural(\sigma) V R_0^\natural(\sigma) V_2. \tag{A.5}$$

Now a slight modification of analytic Fredholm theory proves that for $\epsilon > 0$ sufficiently small there exist $k \in \mathbb{N}$, $S_j \in \mathcal{B}(L^2(\mathbb{R}^n))$ finite rank, $S(\sigma)$ a holomorphic family of compact operators on $L^2(\mathbb{R}^N)$ in $\mathcal{P} \cap B_\epsilon(0)$ with a $\mathcal{C}^\infty$ extension to $B_\epsilon(0)$ such that

$$(\mathrm{Id} + K(\sigma))^{-1} = \sum_{j=1}^{k} \sigma^{-j} S_j + S(\sigma). \tag{A.6}$$

In fact, following the proof of the analytic Fredholm theorem in [37], we can choose F finite rank such that $\|K(0) - F\| < 1/4$, and $r > 0$ such that $\|K(\sigma) - K(0)\| < 1/4$ for $\sigma \in B_\epsilon(0)$. Then $\mathrm{Id} + K(\sigma) - F$ is invertible on $B_\epsilon(0)$, and its norm is less than 2. In addition, the inverse is holomorphic in $\mathcal{P} \cap B_\epsilon(0)$ with a $\mathcal{C}^\infty$ extension to $B_\epsilon(0)$. Let

$$C(\sigma) = F(\mathrm{Id} + K(\sigma) - F)^{-1},$$

so $C(\sigma) = \sum_{j=1}^{l} (\phi_j(\sigma), .)\psi_j$ with $\phi_j(\sigma)$ still holomorphic with a smooth extension (with values in $L^2(\mathbb{R}^n)$), $\psi_j \in L^2(\mathbb{R}^n)$. Moreover,

$$(\mathrm{Id} + C(\sigma))^{-1} = (\mathrm{Id} + K(\sigma) - F)(\mathrm{Id} + K(\sigma))^{-1}, \tag{A.7}$$

so it has bounds as in (A.4). Let $W = \operatorname{span}\{\psi_j : j = 1, ..., l\}$, π the orthogonal projection to W, $\pi^\perp$ to $W^\perp$. Then

$$(\mathrm{Id} + C(\sigma)) = (\mathrm{Id} + C(\sigma)\pi^\perp)(\mathrm{Id} + C(\sigma)\pi),$$

and $\mathrm{Id} + C(\sigma)\pi^\perp$ is invertible, its inverse being $\mathrm{Id} - C(\sigma)\pi^\perp$, and this is holomorphic with a smooth extension. Hence, $(\mathrm{Id} + C(\sigma)\pi)^{-1}$ also has bounds as in (A.4). But

$\mathrm{Id} + C(\sigma)\pi$ is just the tensor product of $\mathrm{Id} + C(\sigma)\pi$ on W and Id on $W^\perp$, so the problem is reduced to inverting a matrix. Its determinant is

$$h(\sigma) = \det\{\delta_{ij} + (\phi_i(\sigma), \psi_j)\} \tag{A.8}$$

which is also holomorphic in $\mathcal{P} \cap B_\epsilon(0)$ with a smooth extension to $B_\epsilon(0)$ and by (A.4)

$$|h(\sigma)^{-1}| \leq C|\operatorname{Im}\sigma|^{-l}|\operatorname{Re}\sigma|^{-l}.$$

But this implies at once that $h(\sigma)$ can vanish at most to order $2l$ at 0; in particular, it is invertible in a punctured neighborhood of 0 in $\overline{\mathcal{P}}$. Hence, $(\mathrm{Id} + C(\sigma))^{-1}$, and therefore $(\mathrm{Id} + K(\sigma))^{-1}$ are indeed of the form (A.6) where we might have to choose a smaller $\epsilon > 0$. As

$$VR_V^\natural(\sigma)V = VR_0^\natural(\sigma)V - VR_0^\natural(\sigma)V_2(\mathrm{Id} + K(\sigma))^{-1}V_1R_0^\natural(\sigma)V, \tag{A.9}$$

$VR_V^\natural(\sigma)V$ has an expansion as in (A.3), where the operators are in $\mathcal{B}(H^{r,s}, H^{r+2,s'})$ for any s, s', and for $-2 \leq r \leq 0$. Further expansion shows that this holds for all r. Note that $k \leq 2$ in (A.6), hence in (A.3), due to (A.4).

In Lemma 2.3 we actually need $(\Delta + V - \sigma)^{-1}$ rather than $(\Delta + V - \sigma^2)^{-1}$, i.e. we must consider $VR_V^\natural(\sigma^{1/2})V$. Here for $\sigma \in \mathbb{C} \setminus [0,\infty)$ we let $\sigma^{1/2}$ be the branch of the square root which takes values in the lower half plane $\mathcal{P}$, so for $\sigma \in \mathcal{P}$ $\sigma = (\sigma^2)^{1/2}$. We shall restrict our attention to $(\Delta + V - (\sigma - i0))^{-1}$, $+i0$ would give similar results with some sign changes. But (A.3) allows us to deal with this easily, for away from 0 the square root does not cause any problems, while near 0 we can write $A(\sigma) = A_e(\sigma^2) + \sigma A_o(\sigma^2)$ with A_e, A_o smooth, and then

$$VR_V^\natural((\sigma - i0)^{1/2})V = A_e(\sigma) + (\sigma - i0)^{1/2}A_o(\sigma) + \sum_{j=1}^{2}(\sigma - i0)^{-j/2}A_j. \tag{A.10}$$

The odd part is actually of the form $\sigma^{n-2}A_o'(\sigma^2)$ due to the corresponding vanishing of the first few odd derivatives of $R_0^\natural(\sigma)$, and hence we can write $(\sigma - i0)^{(n-2)/2}A_o'$ above.

We have defined $B_V : H^{k,l}(\mathbb{R}^n) \to \mathcal{C}^{-\infty}(\mathbb{R}, H^{k',l+2}(\mathbb{R}^n))$ in (2.8):

$$B_V u(\sigma, \zeta) = (\mathcal{F}VR_V^\natural((\sigma - i0)^{1/2})V\mathcal{F}^{-1})u(\zeta). \tag{A.11}$$

Let $\Lambda' = \{\sigma \in \mathbb{R} :\ \sigma^{1/2} \in \Lambda_p\}$; so $\operatorname{spec}_p(\Delta + V) \subset \Lambda' \subset \operatorname{spec}_p(\Delta + V) \cup \{0\}$. By (A.10)

$$B_V = B_e + (\sigma - i0)^{(n-2)/2}B_o + \sum_{\tau \in \Lambda' \setminus \{0\}} (\sigma - i0 - \tau)^{-1}B_\tau + \sum_{j=1,2}(\sigma - i0)^{-j/2}B_0^{(j)} \tag{A.12}$$

with B_e, B_o continuous maps from $H^{k,l}(\mathbb{R}^n)$ to $\mathcal{C}^\infty(\mathbb{R}; H^{k',l+2}(\mathbb{R}^n))$ for any k, k', l, and B_τ, $B_0^{(j)}$ map $H^{k,l}$ to $\mathcal{C}_c^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$. Thus, they all map $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$, so the Schwartz kernels are in $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$. Hence, the Schwartz kernel K_0 of B_V is conormal to

$$Y_0 = \{(\sigma, \zeta, \zeta') : \sigma \in \Lambda\}, \tag{A.13}$$

where $\Lambda = \Lambda' \cup \{0\}$ is the set of thresholds (so Λ is finite), and

$$\mathrm{WF}(K_0) \subset \{(\sigma, \zeta, \zeta', \eta, 0, 0) : \eta < 0, \sigma \in \Lambda\}. \tag{A.14}$$

Moreover, the norms of $VR_V^\natural(\sigma)V$ and its derivatives (with respect to σ) stay bounded as $\operatorname{Im}\sigma \to -\infty$ due to the corresponding estimates on $R_0^\natural(\sigma)$, so B_V actually maps to $\mathcal{C}_\infty^\infty$ for σ sufficiently small where the subscript infinity stands for bounded with all derivatives. Although we have assumed that $n \geq 3$ is odd, a similar argument works for other values of n, but the 'classical' expansion of (A.12) is replaced by a more general one involving $\log\sigma$. This proves Lemma 2.3.

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