SPECTRAL AND SCATTERING THEORY FOR SYMBOLIC POTENTIALS OF ORDER ZERO

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Abstract. The spectral and scattering theory is investigated for a generalization, to scattering metrics on two-dimensional compact manifolds with boundary, of the class of smooth potentials on $\mathbb{R}^2$ which are homogeneous of degree zero near infinity. The most complete results require the additional assumption that the restriction of the potential to the circle(s) at infinity be Morse. Generalized eigenfunctions associated to the essential spectrum at non-critical energies are shown to originate both at minima and maxima, although the latter are not germane to the L^2 spectral theory. Asymptotic completeness is shown, both in the traditional L^2 sense and in the sense of tempered distributions. This leads to a definition of the scattering matrix, the structure of which will be described in a future publication.

Introduction

In [7], Herbst initiated the spectral analysis and scattering theory for the Schrödinger operator $P = \Delta + V$ on $\mathbb{R}^n$ where V is a smooth, real-valued potential which is homogeneous of degree zero near infinity. Further results for such potentials were obtained by Agmon, Cruz and Herbst [1] and Herbst and Skibsted [6]. Such a potential is so large that it deforms the geometry near spatial infinity and consequently the scattering theory of P is quite different from that of a Schrödinger operator with a short-, or even a long-, range potential. Herbst [7] and Herbst-Skibsted [6] showed that solutions of the time-dependent Schrödinger equation, with L^2 initial data, concentrate for large times near critical directions of the potential restricted to the sphere at infinity.

In this paper we study in some detail the tempered eigenspaces of P, consisting for each fixed λ of those functions u, of polynomial growth, satisfying $(P-\lambda)u = 0$. We restrict attention to the two- dimensional case but generalize from Euclidean space, $\mathbb{R}^2$, to scattering metrics (defined below) on arbitrary compact 2-manifolds X with boundary. Such metrics give the interior of X the structure of a complete Riemannian manifold with curvature vanishing at infinity. In this wider context the natural condition replacing, and generalizing, homogeneity of the potential near infinity is its smoothness up to the boundary of ∂X. For the Euclidean case, now appearing through the radial compactification to a ball, this allows potentials which are classical symbols of order 0. Generally we assume that V_0, the restriction of the potential to the boundary, is Morse, so V_0 has only nondegenerate critical points.

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The scattering wavefront set from [13] allows the eigenfunctions of P to be analyzed microlocally and hence to be related to the classical dynamics of P. It is the asymptotic behaviour which is relevant here and this is described through the contact geometry of the cotangent bundle over the boundary; the classical dynamics of P are then given by a Legendre vector field over the boundary. The critical points of this vector field correspond to the critical points of V_0. We give microlocal versions of the familiar notions of incoming and outgoing eigenfunctions, depending on which integral curves of the Legendre vector fields are permitted in the scattering wavefront set. By direct construction we show that there is a non-trivial space of outgoing microlocal eigenfunctions associated to each critical point q of V_0, at energies λ above $V_0(q)$. For a minimum this space is isomorphic to the space of Schwartz functions on a line, for a maximum it is isomorphic to the space of formal power series (i.e. Taylor series at a point) in one variable. These isomorphisms are realized directly in terms of asymptotic expansions near the corresponding critical point of V_0; the form of the expansion reflects the behaviour of the classical trajectories near these points.

For non-critical eigenvalues there is a well-defined space of 'smooth' eigenfunctions, which may be characterized either as the range of the spectral projection on Schwartz functions or else as the space of tempered eigenfunctions with no purely tangential oscillation at infinity (this latter condition is conveniently expressible in terms of the scattering wavefront set). We show this space to have a 'microlocal Morse decomposition' in terms of the spaces of microlocally outgoing eigenfunctions associated to the critical points. The space of all tempered eigenfunctions is identified by a boundary (or Green) pairing with the dual of the space of smooth eigenfunctions, so it has a similar decomposition in terms of the duals of the microlocal spaces. The latter are isomorphic either to Schwartz distributions on a line (for minima) or polynomials in one variable (for maxima). The map from generalized eigenfunctions to the duals of the microlocally outgoing eigenspaces thus correspond to 'generalized boundary data'; the collective boundary data fixes the eigenfunction via the inverse map, which is an analogue of the Poisson operator. There are two such families of boundary data (corresponding to an underlying choice of orientation of the Legendre vector field) and the isomorphism between them is the scattering matrix.

These results are new even in the Euclidean setting. Although for the most part the results here are limited to the two-dimensional case there are obvious generalizations to higher dimensions. These extensions will be discussed in future publications as will the related microlocal description of the Poisson operators and scattering matrix.

Superficially, the existence of smooth eigenfunctions associated to a maximum might appear to be in conflict with the results of Herbst and Skibsted on decay, near maxima, of solutions to the time-dependent Schrödinger equation [6]. This is not the case, since these eigenfunctions are, near the maximum point, in a weighted L^2 space which implies that they are too small to violate this decay estimate. In fact an eigenfunction which is microlocally outgoing near a maximum, and is non-trivial there, is necessarily non-trivial (but not outgoing) near the minimum. Thus there are no eigenfunctions *purely* associated to a maximum, rather these outgoing eigenfunctions can be thought of as corresponding to the completion of the Schwartz functions on a line to smooth functions on a circle by the addition of the Taylor series at the point at infinity. When completed in a norm equivalent to the L^2 norm on initial data for the Schrödinger operator, these extra terms are already in the closure of outgoing eigenfunctions at the minimum, thus they do not 'appear' in the L^2 theory.

Next we give a more precise description of our results. Thus, let X be a compact manifold with boundary where, for the moment, we do not restrict the dimension. The boundary $Y = \partial X$ consists of a finite union of compact manifolds without boundary, $Y = Y_1 \cup \cdots \cup Y_N$. It is always possible to find a boundary defining function on X, $x \in \mathcal{C}^\infty(X)$ such that $x \geq 0$, $Y = \{x = 0\}$ and $dx \neq 0$ on Y. A Riemannian metric on the interior of X is a scattering metric if, for some choice of defining function, it takes the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},\ h \in \mathcal{C}^\infty(X; S^2X),\ h_0 = h\big|_Y \text{ positive definite.} \tag{I.1}$$

That is, $h = x^2(g - x^{-4}dx^2)$ is a smooth 2-cotensor on X which restricts to a metric on Y. In this setting we consider a real potential $V \in \mathcal{C}^\infty(X)$ and examine the spectral and scattering theory of $\Delta + V$ where Δ is the Laplace operator of a scattering metric. The Euclidean case is included since the Euclidean metric is a scattering metric on the radial compactification of $\mathbb{R}^n$ to a ball (or half-sphere) and a smooth potential which is homogeneous of degree zero near infinity is a smooth function up to the boundary of the radial compactification. The lower order parts of the Taylor series of V allow additional terms decaying at Coulomb rate or faster.

Under these assumptions, for $\dim X \geq 2$, the Schrödinger operator $P = \Delta + V$ is self-adjoint and has continuous spectrum of infinite multiplicity occupying the interval $[\kappa, \infty)$ where

$$\kappa = \inf_Y V. \tag{I.2}$$

In addition there may be point spectrum in the interval $[m, K]$,

$$m = \inf_X V,\ K = \sup_Y V \tag{I.3}$$

which is discrete in the open set $(-\infty, K) \setminus \mathrm{Cv}(V)$,

$$\mathrm{Crit}(V) = \{p \in Y; d_Y V(p) = 0\},\ \mathrm{Cv}(V) = V(\mathrm{Crit}(V)). \tag{I.4}$$

The main interest lies in the continuous spectrum which we analyse in detail here under the assumption that $\dim X = 2$ and $V_0 = V\big|_Y$ is Morse. For simplicity in this Introduction we outline the results under the additional assumption that Y has only one component circle and V_0 is perfect Morse, so only has a global maximum, $z_{\max}$, and minimum $z_{\min}$, forming $\mathrm{Crit}(V)$. These restrictions are removed in the body of the paper.

Our central result is the parameterization of all tempered distributions associated to the continuous spectrum; this constitutes a distributional form of 'asymptotic completeness'. Thus we examine

$$E(\lambda) = \left\{u \in \mathcal{C}^{-\infty}(X); (\Delta + V - \lambda)u = 0\right\}. \tag{I.5}$$

The space of extendible distributions, $\mathcal{C}^{-\infty}(X)$, is the dual of the space $\dot{\mathcal{C}}^\infty(X)$ of smooth functions vanishing at ∂X with all derivatives. Thus, $\dot{\mathcal{C}}^\infty(X)$ reduces precisely to the space of Schwartz functions, and $\mathcal{C}^{-\infty}(X)$ to the space of tempered distributions in case X is the radial compactification of $\mathbb{R}^n$. For eigenfunctions this is equivalent to a polynomial bound. For $\lambda < \kappa$, $E(\lambda)$ is finite-dimensional and consists of square-integrable eigenfunctions. In fact, for any $\lambda \notin \mathrm{Cv}(V)$, we show in Proposition 4.9 that $E_{\mathrm{pp}}(\lambda) = E(\lambda) \cap L^2(X)$ is contained in $\dot{\mathcal{C}}^\infty(X)$ and finite dimensional (where $L^2(X)$ is computed with respect to the Riemannian volume form) and is trivial for $\lambda > K$. Since $\mathcal{C}^{-\infty}(X)$ is the dual of $\dot{\mathcal{C}}^\infty(X)$ this allows us to consider

$$\begin{aligned} E_{\mathrm{ess}}^{-\infty}(\lambda) &= \left\{u \in \mathcal{C}^{-\infty}(X); (\Delta + V - \lambda)u = 0,\ \langle u, v\rangle = 0 \ \forall\, v \in E_{\mathrm{pp}}(\lambda)\right\}, \\ E(\lambda) &= E_{\mathrm{ess}}^{-\infty}(\lambda) \oplus E_{\mathrm{pp}}(\lambda). \end{aligned} \tag{I.6}$$

The structure of the space $E_{\mathrm{ess}}^{-\infty}(\lambda)$ depends on λ; there are three distinct cases, corresponding to the values of κ, K and the additional transition point

$$\lambda_{\mathrm{Hess}} = \kappa + 2V''(z_{\min}),\ \ \kappa = V(z_{\min}) \tag{I.7}$$

where the derivatives are with respect to boundary arclength. Clearly $\lambda_{\mathrm{Hess}} > \kappa$ but the three possibilities $\lambda_{\mathrm{Hess}} < K$, $\lambda_{\mathrm{Hess}} = K$, $\lambda_{\mathrm{Hess}} > K$ may all occur for different V (or for the same V but different metrics). For any particular problem only two or three of the following four intervals can occur

$$\begin{cases} \kappa < \lambda < \min(\lambda_{\mathrm{Hess}}, K) & \text{“Near minimum”} \\ \lambda_{\mathrm{Hess}} < \lambda < K & \text{“Hessian range”} \\ K < \lambda < \lambda_{\mathrm{Hess}} & \text{“Mixed range”} \\ \max(\lambda_{\mathrm{Hess}}, K) < \lambda & \text{“Above thresholds.”} \end{cases} \tag{I.8}$$

The critical values κ and K of V_0 are thresholds corresponding to changes in the geometry of the fixed energy (i.e. characteristic) surface of the classical problem. Above these thresholds both the maximum and the minimum correspond to zeroes, which we also call radial points, of the classical flow (at infinity); the Hessian transition corresponds to an energy at which there is a change in the local geometry at the radial points corresponding to $z_{\min}$.

To analyse $E_{\mathrm{ess}}^{-\infty}(\lambda)$ we use the limit of the resolvent on the spectrum. This exists as an operator for $\lambda \in (\kappa, +\infty) \setminus (\mathrm{Cv}(V) \cup \sigma_{\mathrm{pp}}(\Delta + V))$, as a consequence of an appropriate version of the Mourre estimate, or of microlocal estimates closely related to it, see [7] for the proof in the Euclidean setting.

Theorem I.1 (Herbst [7], see also Theorem 3.3)**.** *The resolvent*

$$R(\lambda \pm it) = (\Delta + V - (\lambda \pm it))^{-1},\ t > 0,\ \lambda \notin \mathrm{Cv}(V) \tag{I.9}$$

extends continuously to the real axis, i.e. $R(\lambda \pm i0)$ *exist, as bounded operators*

$$x^{1/2+\delta} L^2(X) \ominus E_{\mathrm{pp}}(\lambda) \longrightarrow x^{-1/2-\delta} L^2(X) \ominus E_{\mathrm{pp}}(\lambda),\ \forall\, \delta > 0. \tag{I.10}$$

Note that

$$f \in x^{1/2+\delta} L^2(X) \ominus E_{\mathrm{pp}}(\lambda) \Longrightarrow [R(\lambda + i0) - R(\lambda - i0)] f \in E_{\mathrm{ess}}^{-\infty}(\lambda) \tag{I.11}$$

and conversely (essentially by Stone's theorem) the range is dense.

For $\lambda \notin \mathrm{Cv}(V)$ we can thus define spaces of 'smooth' eigenfunctions $E_{\mathrm{ess}}^{\infty}(\lambda)$ by

$$E_{\mathrm{ess}}^{\infty}(\lambda) = [R(\lambda + i0) - R(\lambda - i0)](\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda)) \tag{I.12}$$

and $E_{\mathrm{ess}}^{\infty}(\lambda)$ inherits a Fréchet topology from $\dot{\mathcal{C}}^\infty(X)$. We give a microlocal characterization of these spaces below. In all of the non-transition regions

$$E_{\mathrm{ess}}^{\infty}(\lambda) \subset E_{\mathrm{ess}}^{-\infty}(\lambda) \text{ is dense} \tag{I.13}$$

in the topology of $\mathcal{C}^{-\infty}(X)$. In fact, approximating sequences can be constructed rather explicitly by extending $R(\lambda \pm i0)$ to distributions satisfying a scattering wave front set condition; see Corollary 3.13.

One of the aims of scattering theory is to give explicit, and geometric, parameterizations of the continuous spectrum. The reality of the Laplacian implies that there are two equivalent such representations, which are interchanged by complex conjugation, and the scattering matrix gives the relation between them. The main task here is to describe the structure of the smooth eigenfunctions in the four non-transition regions in (I.8); our explicit parameterizations arise as the 'leading terms' in the complete asymptotic expansions that these eigenfunctions possess. This also identifies $E^{\infty}_{\mathrm{ess}}(\lambda)$ in terms of familiar Fréchet spaces. The expansions are intimately connected to the corresponding classical problem, which is described in detail in §1. For $\lambda \notin \mathrm{Cv}(V)$ the classical system consists of a Legendre vector field W (so defined up to a conformal factor) on a compact hypersurface, the characteristic variety $\Sigma(\lambda)$, in a contact manifold. The dual variable to the variable $1/x$ (which is the radial variable $r = |z|$ in the Euclidean case) is a function, ν on $\Sigma(\lambda)$ with $W\nu \geq 0$.

The microlocal structure of this problem, near each radial point, when transformed from the scattering to the traditional context (essentially by Fourier transformation) is the problem considered by Guillemin and Schaeffer in [5]. Although we do not use their work explicitly here, several of our results could be proved by their techniques, provided one made the additional assumption of non-resonance of the linearization of $\Delta + V - \lambda$ at the radial points.

The projections of the radial points (that is, the zeroes of W) are the critical points of V on the boundary. Each such critical point $z \in \mathrm{Crit}(V)$ corresponds to two radial points $q_\pm \in \Sigma(\lambda)$ with $\pm\nu(q_\pm) > 0$ if $\lambda > V(z)$; it corresponds to a singular point on $\Sigma(\lambda)$ if $\lambda = V(z)$. The two points correspond to opposite, non-zero, values of ν. We shall denote by $\mathrm{RP}_+(\lambda) \subset \Sigma(\lambda)$, resp. $\mathrm{RP}_-(\lambda) \subset \Sigma(\lambda)$, the set of radial points at which $\nu > 0$, resp. $\nu < 0$. This can also be identified with the subset $\{q \in \mathrm{Crit}(V); V(q) < \lambda\}$ and we let

$$\mathrm{RP}_+(\lambda) = \mathrm{Max}_+(\lambda) \cup \mathrm{Min}_+(\lambda) \tag{I.14}$$

be the decomposition into points associated to maxima and to minima of V_0. We shall describe points in $\mathrm{RP}_+(\lambda)$, resp. $\mathrm{RP}_-(\lambda)$, as outgoing, resp. incoming, radial points.

Our examination of the structure of the eigenfunctions is based on the description of the microlocally outgoing eigenfunctions associated to each of the outgoing radial points. The notion of microlocality here is with respect to the scattering wavefront set, which is the notion of wavefront set associated to the scattering calculus in [13]. By microlocal elliptic regularity, for any $u \in \mathcal{C}^{-\infty}(X)$, and any open set O

$$\mathrm{WF}_{\mathrm{sc}}\left((\Delta + V - \lambda)u\right) \cap O = \emptyset \Longrightarrow \mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \Sigma(\lambda).$$

If $q \in \mathrm{RP}_\pm(\lambda)$ and O is a sufficiently small open neighbourhood of q, which meets $\Sigma(\lambda)$ in a W-convex neighbourhood with each W curve meeting $\nu = \nu(q)$ we set

$$\tilde{E}_{\mathrm{mic},+}(O,\lambda) = \big\{u \in \mathcal{C}^{-\infty}(X); O \cap \mathrm{WF}_{\mathrm{sc}}\left((P-\lambda)u\right) = \emptyset \\ \text{and } \mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \{\nu \geq \nu(q)\}\big\}, \tag{I.15}$$

with $\tilde{E}_{\mathrm{mic},-}(O,\lambda)$ defined by reversing the inequality. We may consider this as a space of microfunctions, $E_{\mathrm{mic},+}(q,\lambda)$, by identifying elements when they differ by functions with wavefront set not meeting O. The result is then independent of the choice of O. That is, such microlocal solutions are determined by their behaviour in an arbitrarily small neighbourhood of q. Our convention is to call elements of either $\tilde{E}_{\mathrm{mic},+}(q,\lambda)$ or $E_{\mathrm{mic},+}(q,\lambda)$ 'microlocally outgoing eigenfuncions at $q \in \mathrm{RP}_+(\lambda)$.' The structure of the microlocal eigenfunctions at a radial point is essentially determined by the linear part of W, the normalized eigenvalues of which are given by (1.25) below. The character of the radial point is determined by the quantity

$$r(\lambda) = r(q,\lambda) = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{V_0''}{2\tilde{\nu}^2}},\ \tilde{\nu} = \nu(q), \tag{I.16}$$

where V_0'' is the Hessian with respect to boundary arclength at the critical point.

The case $q \in \mathrm{Max}_+(\lambda)$ is the most straightforward, with W having a saddle point at q at which $r(q,\lambda) < 0$. Elements $u \in E_{\mathrm{mic},+}(q,\lambda)$ are then (i.e. have representatives which are) functions of the form

$$u = e^{i\phi_\lambda/x} x^{\frac{1}{2}+\epsilon} v,\ v \in L^\infty(X) \cap \mathcal{C}^\infty(X^\circ),\ \epsilon = -\frac{1}{2} r(q,\lambda) > 0, \tag{I.17}$$

i.e. Legendre distributions (see [15]) where $\phi_\lambda \in \mathcal{C}^\infty(X)$ is a real function parameterizing the unstable manifold for W at q and v has a complete asymptotic expansion with $\mathcal{C}^\infty$ coefficients as $x \to 0$ in powers of x starting with a term in which the power is imaginary. Appropriately evaluating the terms in the expansion of v at $z = \pi(q)$ gives a parameterizing map,

$$M_+(q,\lambda) : E_{\mathrm{mic},+}(q,\lambda) \ni u \longrightarrow \mathbb{C}[[x]], \tag{I.18}$$

which is an isomorphism to the space of formal power series in one variable, or equivalently the space of arbitrary sequences $s : \mathbb{N} \longrightarrow \mathbb{C}$. This result is proved in Theorem 9.9. Notice that the factor x^ϵ in (I.17) means that these microlocal eigenfunctions are 'small' near the maximum, as they lie in the space $x^{-\frac{1}{2}} L^2(X)$ locally near z. Here, the exponent $-1/2$ is critical for eigenfunctions; any eigenfunction which is *globally* in $x^{-\frac{1}{2}} L^2(X)$ is actually rapidly decreasing.

When $q \in \mathrm{Min}_+(\lambda)$ the microlocal space $E_{\mathrm{mic},+}(q,\lambda)$ can be identified with a global space of eigenfunctions

$$E_{\mathrm{mic},+}(q,\lambda) \simeq \big\{u \in E^{-\infty}_{\mathrm{ess}}(\lambda); \mathrm{WF}_{\mathrm{sc}}(u) \cap \{\nu > 0\} \subset \{q\}\big\} \subset E^{\infty}_{\mathrm{ess}}(\lambda) \tag{I.19}$$

(see Corollary 5.7). In this case, the structure of $E_{\mathrm{mic},+}(q,\lambda)$ undergoes a distinct change as λ crosses the Hessian threhold. First consider the 'near minimal' range $\kappa < \lambda < \lambda_{\mathrm{Hess}}$. In this case W has a center at q with $\mathrm{Re}\, r(q,\lambda) = \frac{1}{2}$, and the square root in (I.16) is imaginary. It is convenient to introduce the space $[X;z]_{\frac{1}{2}}$ obtained by parabolic blow up, with respect to the boundary, of X at $z = \pi(q)$ (see section 2 for a discussion of blowups). Then the space $\mathcal{C}^\infty_{\mathrm{ff}}([X;z]_{\frac{1}{2}})$ consists of the smooth functions on this blown-up space which vanish rapidly at all boundary faces other than the 'front face' created by the blowup. Each microlocal eigenfunction has a representation in the form of an infinite sum

$$\begin{gathered} u = e^{i\tilde{\nu}/x} \sum_{j\geq 0} x^{\frac{1}{4}+i((2j+1)\alpha + V_1(z))} a_j v_j,\ a_j \in \mathcal{S}(\mathbb{N}),\ v_j \in \mathcal{C}^\infty_{\mathrm{ff}}([X;z]_{\frac{1}{2}}), \\ \tilde{\nu} = \sqrt{\lambda - V_0(z)},\ V_1(z) = (\partial_x V)(z), \end{gathered} \tag{I.20}$$

where the a_j, as indicated, form a Schwartz sequence in j, and the v_j restricted to $\mathrm{ff}([X;z]_{\frac{1}{2}})$ are L^2-normalized eigenfunctions for a self-adjoint globally elliptic operator on $\mathrm{ff}([X;z]_{\frac{1}{2}})$, namely a harmonic oscillator with eigenvalues $(j+1/2)\alpha$. The sequence of coefficients, a_j, is an arbitrary Schwartz sequence and the map obtained by normalization from (I.20) gives an isomorphism

(I.21)
$$M_+(q,\lambda): E_{\mathrm{mic},+}(q,\lambda) \longrightarrow \dot{\mathcal{C}}^\infty(\mathrm{ff}[X;z]_{\frac{1}{2}}),\ u \longmapsto \sum_{j\geq 0}(a_j v_j\big|_{\mathrm{ff}}), \kappa < \lambda < \lambda_{\mathrm{Hess}},$$

onto the Schwartz functions on a line. We prove this statement in Theorem 7.4. Note that the normalization of v_j is only well-defined up to a factor $e^{is(j)}$, where $s(j)$ is a real sequence. When it is analysed in a subsequent paper, it will convenient to introduce such terms so that $M_+(q,\lambda)$ (or more precisely its inverse) will be a Fourier integral operator of an appropriate type.

The case $\lambda_{\mathrm{Hess}} \leq \lambda$, at a minimum, is similar, except that the discrete spectrum of the model problem has collapsed, and q has become a sink for W. At $\lambda = \lambda_{\mathrm{Hess}}$ the 'local homogeneity' is still parabolic as it is for $\lambda < \lambda_{\mathrm{Hess}}$. As λ increases the local homogeneity is determined by $r = r(q,\lambda)$ given by (I.16). Corresponding to this homogeneity we introduce the space $[X;z]_r$ obtained by blow up with the correct scaling of the tangential variable relative to the normal variable. Now r is not necessarily rational, hence we need to replace $\mathcal{C}^\infty_{\mathrm{ff}}([X;z]_r)$ by a space of polyhomogeneous functions allowing both homogeneities x^n and x^{nr} where n is a non-negative integer. This space, $\mathcal{A}^I_{\mathrm{phg,ff}}([X;z]_r)$, is described in more detail in Section 8, but here we mention that its elements are in particular continuous on $[X;z]_r$, and vanish at the boundary away from ff . If $1/r(\lambda)$ is not an integer, then the microlocal eigenfunctions $u \in E_{\mathrm{mic},+}(q,\lambda)$ take the form

$$u = e^{i\phi_\lambda/x} x^{\frac{1-r}{2}+ic} a,\ a \in \mathcal{A}^I_{\mathrm{phg,ff}}([X;z]_r),\ r = r(\lambda), \tag{I.22}$$

where $\phi_\lambda \in \mathcal{C}^\infty(X)$ is now a real function parameterizing a smooth (minimal) Legendre submanifold through q and $c = c(\lambda)$ is a real constant defined from the subprincipal symbol of P. Again, restriction to the front face of the blown-up space gives an isomorphism

$$M_+(q,\lambda): E_{\mathrm{mic},+}(q,\lambda) \longrightarrow \dot{\mathcal{C}}^\infty(\mathrm{ff}[X;z]_r),\ q \in \mathrm{Min}_+(\lambda), \tag{I.23}$$

which, after selection of coordinates, maps isomorphically onto the Schwartz functions on an associated line. The passage of λ across the Hessian threshold results in a change in the underlying parameterizing space, which from then onwards depends on λ. This result is proved in Theorem 8.6, together with a statement of the minor changes needed when $1/r(\lambda)$ is an integer.

In each case the analysis of $E_{\mathrm{mic},+}(q,\lambda)$ proceeds in two steps. First, we construct a 'test module' $\mathcal{M}$ of pseudodifferential operators, and show that elements of $E_{\mathrm{mic},+}(q,\lambda)$ are stable under iterated application of elements of $\mathcal{M}$. Such a result is analogous to the description of conormal regularity in [12]. Its proof proceeds via a positive commutator argument, whose abstract version is given in § 6. We emphasize that just as the form of $u \in E_{\mathrm{mic},+}(q,\lambda)$ depends on the type of q, so does $\mathcal{M}$.

In the second step we multiply $u \in E_{\mathrm{mic},+}(q,\lambda)$ by an oscillatory factor given by the exponential in (I.17), (I.20) and (I.22), respectively. This conjugates $\mathcal{M}$ to another module $\widetilde{\mathcal{M}}$ of pseudodifferential operators. Iterated regularity of distributions under $\widetilde{\mathcal{M}}$ is equivalent to conormality on a geometric resolution of X, such as $[X;z]_r$ above. The conjugation also gives rise to a model operator which differs from the conjugate of the Hamiltonian, $\Delta+V-\lambda$, by a term that can be controlled using the already established $\widetilde{\mathcal{M}}$-regularity. The precise form of the asymptotic expansion then follows from the description of $\widetilde{\mathcal{M}}$-regular solutions of these model operators. We emphasize that the model operators are not complete microlocal models: they are microlocal models only *modulo* terms involving $\widetilde{\mathcal{M}}$, that become trivial when applied to $\widetilde{\mathcal{M}}$-regular solutions. In particular, we do *not* need to use the Sternberg linearization theorem or its consequences that figure prominently in [5].

The space of smooth eigenfunctions can be directly related to these microlocal spaces and this is especially simple in case the potential is perfect Morse on the one boundary component. There is then a restriction map 'to the maximum'

$$E^{\infty}_{\mathrm{ess}}(\lambda) \longrightarrow E_{\mathrm{mic},+}(q,\lambda),\ q \in \mathrm{Max}_+(\lambda) \text{ for } K<\lambda. \tag{I.24}$$

Theorem I.2. *[Microlocal Morse decomposition] If* $\dim X = 2$, ∂X *is connected,* V_0 *is perfect Morse and* $\lambda \notin \mathrm{Cv}(V)$, *the map* (I.19) *is an isomorphism*

$$E_{\mathrm{mic},+}(q^+_{\min},\lambda) \longrightarrow E^{\infty}_{\mathrm{ess}}(\lambda),\ \textit{for } \kappa<\lambda<K, \tag{I.25}$$

whereas, for $\lambda > K$, *it combines with the map* (I.24) *to give a short exact sequence*

$$0 \longrightarrow E_{\mathrm{mic},+}(q^+_{\min},\lambda) \longrightarrow E^{\infty}_{\mathrm{ess}}(\lambda) \longrightarrow E_{\mathrm{mic},+}(q^+_{\max},\lambda) \longrightarrow 0. \tag{I.26}$$

Whilst (I.26) does not split, there is a left inverse on any finite dimensional subspace. Combined with the isomorphisms (I.18) and (I.23) this result gives a quite complete description of the smooth eigenfunctions, which we could call 'smooth asymptotic completeness at energy λ' in view of (I.12).

There is a non-degenerate sesquilinear 'Green pairing' on $E^{\infty}_{\mathrm{ess}}(\lambda)$ which extends to a continuous bilinear map (see section 12)

$$B : E^{\infty}_{\mathrm{ess}}(\lambda) \times E^{-\infty}_{\mathrm{ess}}(\lambda) \longrightarrow \mathbb{C} \tag{I.27}$$

allowing $E^{-\infty}_{\mathrm{ess}}(\lambda)$ to be identified with the dual space of $E^{\infty}_{\mathrm{ess}}(\lambda)$. This pairing also restricts to a pairing between $E_{\mathrm{mic},+}(q,\lambda)$ and $E_{\mathrm{mic},-}(q,\lambda)$ for each radial point q and $\lambda > V(\pi(q))$. The adjoint of the map (I.18) becomes an isomorphism

$$\mathbb{C}[x] \longleftrightarrow E_{\mathrm{mic},-}(q,\lambda),\ q \in \mathrm{Max}_+(\lambda), \tag{I.28}$$

where $\mathbb{C}[x]$ is the space of finite power series (i.e. polynomials) — see Theorem 12.5. The range space in (I.28) again consists of functions which are Legendre distributions near q; only the trivial elements of $E_{\mathrm{mic},-}(q,\lambda)$ are in the space $x^{-\frac{1}{2}}L^2(X)$ near $\pi(q)$ (this is dual to the property (I.17)). Similarly the adjoint of (I.23) extends to an isomorphism

$$\mathcal{S}'(\mathbb{R}_r) \longleftrightarrow E_{\mathrm{mic},-}(q,\lambda),\ q \in \mathrm{Min}_+(\lambda). \tag{I.29}$$

In the perfect Morse case (with ∂X connected) microlocal decompositions of the distributional eigenspaces are given by the identification

$$E_{\mathrm{mic},-}(q^+_{\min},\lambda) \longrightarrow E^{-\infty}_{\mathrm{ess}}(\lambda),\quad \text{for } \kappa<\lambda<K \tag{I.30}$$

and for $\lambda > K$ the short exact sequence

$$0 \longrightarrow E_{\mathrm{mic},-}(q^+_{\max},\lambda) \longrightarrow E^{-\infty}_{\mathrm{ess}}(\lambda) \longrightarrow E_{\mathrm{mic},-}(q^+_{\min},\lambda) \longrightarrow 0. \tag{I.31}$$

Combined with the isomorphism (I.28) and (I.29) this gives a decomposition of the full distributional eigenspaces in terms of microlocal eigenspaces.

One can compare the sequence in (I.26) with the familiar short exact sequence

$$\mathcal{S}(\mathbb{R}) \hookrightarrow \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathbb{C}[[x]] \tag{I.32}$$

where the second map is the passage to Taylor series at a point and the first is by identification of the complement of this point with a line. Then (I.31) corresponds to the dual sequence for distributions

$$\mathbb{C}[x] \longrightarrow \mathcal{C}^{-\infty}(\mathbb{S}) \longrightarrow \mathcal{S}'(\mathbb{R}) \tag{I.33}$$

where the polynomials correspond to the (Dirac delta) distributions at the point.

In this way we may think of $E^\infty_{\mathrm{ess}}(\lambda)$ as a space of test functions and $E^{-\infty}_{\mathrm{ess}}(\lambda)$ as the dual space of distributions. It is then natural to think of the intermediate 'Sobolev spaces' of eigenfunctions. In this case they may be defined directly:

$$E^s_{\mathrm{ess}}(\lambda) = \left\{u \in E^{-\infty}_{\mathrm{ess}}(\lambda);\ \mathrm{WF}^{0,s-1/2}_{\mathrm{sc}}(u) \cap \{\nu = 0\} = \emptyset\right\},\ s \in \mathbb{R}. \tag{I.34}$$

The case $s = 0$ is particularly interesting since it is related to the notion of asymptotic completeness. In Section 12 we prove the following theorem.

Theorem I.3 (Asymptotic completeness at energy λ)**.** *If $\lambda \notin \mathrm{Cv}(V)$ then*

$$\begin{gathered} E_{\mathrm{ess},+}(q^+_{\min}, \lambda) \hookrightarrow E^0_{\mathrm{ess}}(\lambda) \text{ is dense and} \\ M_+(q^+_{\min}, \lambda) \text{ extends to a unitary operator } E^0_{\mathrm{ess}}(\lambda) \longrightarrow L^2(\mathbb{R}_r), \end{gathered} \tag{I.35}$$

with respect to the metric induced by B from (I.27) *on the one hand, and $L^2(\mathbb{R}_r)$ equipped with a translation-invariant measure induced by h_0 from* (I.1) *on the other.*

Integration with respect to λ gives the usual version of asymptotic completeness which we state in Theorem 12.13.

These two theorems, combined with (I.17), have immediate implications for the 'size' of $u_\pm = R(\lambda \pm i0)f$, $f \in \dot{\mathcal{C}}^\infty(X)$, away from the minima of V_0. Namely, for any pseudodifferential operator Q with $q^+_{\min} \notin \mathrm{WF}'_{\mathrm{sc}}(Q)$, in particular for multiplication operators supported away from $z_{\min}$, $Qu_+ \in x^s L^2(X)$ for all $s < (-r(q^+_{\min}, \lambda) - 1)/2$. Since $r(q^+_{\min}, \lambda) < 0$, this is an improvement over the statement that $Qu_+ \in x^{-1/2}L^2(X)$. The statement $Qu_+ \in x^{-1/2}L^2(X)$ can be interpreted as the absence of channels at the maxima of V_0 and is closely related to the work of Herbst and Skibsted [6]; indeed this can be seen from the injectivity of $M_+(\lambda)$ in (I.35). Thus, on the one hand, our results strengthen theirs in this special case (i.e. when $\dim X = 2$) by showing that u_+ has additional decay away from the minima of V_0 and gives the precise asymptotic form of u_+. On the other hand, it also shows that the decay is not rapid. One interpretation of this phenomenon, supported by the short exact sequence (I.26) and isomorphism (I.35), is that the L^2-theory does not 'see' the maxima of the potential (because u_+ is too small there), while the smooth theory (working modulo $\dot{\mathcal{C}}^\infty(X)$) does.

As well as the identification (I.35) arising from the outgoing parameterization there is the corresponding incoming parameterization, corresponding to $\nu < 0$. The scattering matrix is then the unitary operator given by the composite

$$S(\lambda) = M_+ M_-^{-1} : L^2(\mathbb{R}_r) \to L^2(\mathbb{R}_r). \tag{I.36}$$

In a future publication it will be shown to be a Fourier integral operator, of an appropriate type, associated with the limiting scattering relation arising from the vector field W.

The classical dynamical system underlying the eigenvalue problem is analyzed in § 1, and § 2 contains a brief description of the blow-up procedure used later to discuss expansions. In § 3 extension of results of Herbst on the limiting absorption principle to the present more general setting are discussed; they are proved by positive commutator methods in § 4. The basic properties of the (incoming and outgoing) microlocal eigenfunctions associated to critical points of the classical system (i.e. radial points) are presented in § 5 and an outline of the methods used to analyse them is given in § 6. The next four sections contain the detailed analysis of incoming microlocal eigenfunctions at radial points associated to minima and both incoming and outgoing eigenfunctions associated to maxima. The decomposition of smooth eigenfunctions in terms of these microlocal components is described in § 11 and this is used, together with the natural pairing on eigenfunctions, to describe all tempered eigenfunctions, the scattering matrix and various forms of asymptotic completeness in the final section.

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1. Classical problem

The classical system formally associated to the Schrödinger operator $\Delta + V$ on $\mathbb{R}^n$ is generated by the Hamiltonian function

$$p = |\zeta|^2 + V(z) \in \mathcal{C}^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta). \tag{1.1}$$

In fact, only the 'large momentum' (the usual semiclassical limit) $\zeta \to \infty$ or the 'large distance' limit $z \to \infty$ are relevant to the behaviour of solutions. Since we are interested in eigenfunctions

$$(\Delta + V - \lambda)u = 0, \tag{1.2}$$

for finite λ, only the latter limit is important and hence our microlocal analysis takes place in the vicinity of the finite (fixed) energy, or characteristic, surface which can be written formally

$$\Sigma(\lambda) = \left\{ |\zeta|^2 + V(z) = \lambda, \ |z| = \infty \right\}. \tag{1.3}$$

Setting $|z| = \infty$ is restriction to the sphere at infinity for the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$ so $\Sigma(\lambda) \subset \mathbb{S}^{n-1}_\omega \times \mathbb{R}^n$, $z = |z|\omega$. This phase space at infinity is a contact manifold where the contact form is

$$\alpha = -d\nu + \mu \cdot d\omega, \ \zeta = \mu \oplus \nu\omega, \ \mu \in \omega^\perp. \tag{1.4}$$

In the general case of a scattering metric on a compact manifold with boundary (see [13]) the corresponding phase space consists of

$$C_\partial = {}^{\mathrm{sc}}T^*_Y X, \tag{1.5}$$

the restriction of the scattering cotangent bundle to the boundary. Let (x, y) be local coordinates near a boundary point, where x is a local defining function and

y restrict to coordinates on Y. Then the forms dx/x^2 and dy/x are a basis for the fibre ${}^{\mathrm{sc}}T^*_pX$ at each point p in the coordinate chart, and hence one may write an arbitrary point $q \in {}^{\mathrm{sc}}T^*X$ in the form

$$q = -\nu \frac{dx}{x^2} + \sum_i \mu_i \frac{dy_i}{x}, \tag{1.6}$$

giving local coordinates (x, y, ν, μ) on ${}^{\mathrm{sc}}T^*X$. The characteristic variety of P at energy λ then is

$$\Sigma(\lambda) = \{|(\nu, \mu)|_y^2 + V(y) = \lambda\} \tag{1.7}$$

where $|\cdot|_y^2$ is the metric function at the boundary.

Again C_∂ is naturally a contact manifold. The contact structure arises as the 'boundary value' of the singular symplectic structure on ${}^{\mathrm{sc}}T^*X$, corresponding to the fact that the associated semiclassical model is the leading part, at the boundary, of the Hamiltonian sytem defined by the energy. In terms of local coordinates as above, the symplectic form, arising from the identification ${}^{\mathrm{sc}}T^*X^\circ \simeq T^*X^\circ$, is

$$\widetilde{\omega} = d(\nu d\frac{1}{x} + \mu \cdot \frac{dy}{x}) = (-d\nu + \mu \cdot dy) \wedge \frac{dx}{x^2} + \frac{d\mu \wedge dy}{x}, \tag{1.8}$$

and is hence singular at the boundary. Since the vector field $x^2\partial_x$ is well-defined at the boundary, modulo multiples, the contact form

$$\alpha = \widetilde{\omega}(x^2\partial_x, \cdot) = -d\nu + \mu \cdot dy \tag{1.9}$$

defines a line subbundle of T^*C_∂ which fixes the contact structure on C_∂.

The Hamilton vector field H_p of P, regarded as a vector field on ${}^{\mathrm{sc}}T^*X$, is of the form $x\tilde{W}$, where $\tilde{W}$ is tangent to $\partial({}^{\mathrm{sc}}T^*X)$. Here W has the form

$$\tilde{W} = -2\nu x\partial_x + W, \quad W \text{ a vector field on } {}^{\mathrm{sc}}T^*_{\partial X}X.$$

On $\Sigma(\lambda) = \{p = \lambda\}$, we have $x^{-1}H_p = x^{-1}H_{p-\lambda} = H_{x^{-1}(p-\lambda)} - (p - \lambda)H_{x^{-1}} = H_{x^{-1}(p-\lambda)}$. Since $x^{-1}(p - \lambda)$ is order -1, its Hamilton vector field W_λ restricted to $\partial({}^{\mathrm{sc}}T^*X)$ is the contact vector generated by the Hamiltonian $p - \lambda$ on $\partial({}^{\mathrm{sc}}T^*X)$. It is determined directly in terms of the contact form by

$$d\alpha(\cdot, W_\lambda) + \gamma\alpha = dp, \ \alpha(W_\lambda) = p - \lambda, \tag{1.10}$$

for some function γ. The vector field W_λ is tangent to $\Sigma(\lambda)$ whenever the latter is smooth, as follows by pairing (1.10) with W_λ. By the semiclassical model at energy λ we mean the flow defined by $W = W_\lambda$ on $\Sigma(\lambda)$.

Lemma 1.1. *For any scattering metric and any real $V \in \mathcal{C}^\infty(X)$ the characteristic surface $\Sigma(\lambda)$ is smooth whenever λ is not a critical value of V_0. For regular λ the critical set of the vector field W on $\Sigma(\lambda)$ is the union of the two radial sets*

$$\mathrm{RP}_\pm(\lambda) = \{(y, \nu, \mu) \in \Sigma(\lambda); \nu = \pm\sqrt{\lambda - V(p)}, \ \mu = 0, \ y = p \text{ where } d_YV(p) = 0\} \tag{1.11}$$

and projection onto Y gives bijections

$$\pi : \mathrm{RP}_\pm(\lambda) \longrightarrow \mathrm{Crit}(V) \cap \{y \in Y; V(y) < \lambda\} \tag{1.12}$$

for each sign.

Proof. The vector field is fixed by (1.10) so in any local coordinates y in the boundary it follows from (1.9) that

$$d\mu \wedge dy(\cdot, W) = \gamma(d\nu - \mu dy) + dp. \tag{1.13}$$

Given a boundary point q we may choose Riemannian normal coordinates based at q in the boundary. Thus the boundary metric is Euclidean to second order, so $p = \nu^2 + |\mu|_y^2 + O(|y|^2|\mu|^2) + V(y) - \lambda$ and we may easily invert (1.13) to find that at the fibre above q

$$W = 2\mu \cdot \partial_y - (V'(y) + 2\nu\mu) \cdot \partial_\mu + 2|\mu|_y^2 \partial_\nu + O(|y||\mu|), \quad \gamma = -2\nu. \tag{1.14}$$

This can only vanish when $\mu = 0$ and $dV_0(z) = 0$ giving (1.11). □

We remark that in the Euclidean setting, (1.11) amounts to

$$\mathrm{RP}_\pm(\lambda) = \{(y, \zeta) \in \Sigma(\lambda);\ y = \pm\zeta/|\zeta|,\ y = z \text{ where } d_Y V(z) = 0\}. \tag{1.15}$$

If $V_i = V\big|_{Y_i}$ is perfect Morse then the component of $\Sigma(\lambda)$ above Y_i,

$$\Sigma_i(\lambda) \text{ is a } \begin{cases} \text{sphere for} & \kappa_i = \min(V\big|_{Y_i}) < \lambda < K_i = \max(V\big|_{Y_i}), \\ \text{torus for} & \lambda > K_i. \end{cases} \tag{1.16}$$

In the general Morse case, $\Sigma_i(\lambda)$ is a union of disjoint spheres for non-critical $\lambda < K_i$ and a torus for $\lambda > K_i$.

In all cases if λ is not a critical value of $V\big|_{Y_i}$ then

$$I_i(\lambda) = \Sigma_i(\lambda) \cap \{\nu = 0\} \tag{1.17}$$

is a smooth curve (empty if $\lambda < \kappa_i$, generally with several components) to which W is transversal. The flow is symmetric under $\nu \to -\nu$, $\mu \to -\mu$.

Proposition 1.2. *Provided* $\dim X = 2$ *and* V_0 *is Morse the critical point* $q_\pm(\lambda) = (z, \pm\sqrt{\lambda - V(z)}, 0)$ *in* (1.11) *is*

$$\begin{cases} \text{a centre if } z \text{ is a minimum and } \lambda < \lambda_{\mathrm{Hess}}(z) \\ \text{a sink/source if } z \text{ is a minimum and } \lambda \geq \lambda_{\mathrm{Hess}}(z) \\ \text{a saddle if } z \text{ is a maximum,} \end{cases} \tag{1.18}$$

where $\lambda_{\mathrm{Hess}}(z) = V(z) + 2V''(z)$.

Proof. As already noted above, (1.14) is valid locally in geodesic boundary coordinates. If z is a critical point for V then $\mu = 0$ at the critical points $z_\pm(\lambda)$ of W on $\Sigma(\lambda)$ and we may use local coordinates y, μ in the characteristic variety nearby, since $\nu = \pm\sqrt{\lambda - V(y) - \mu^2}$ is smooth. In these coordinates the linearization of W inside $\Sigma(\lambda)$ at the critical point is

$$L = 2[\mu\partial_y - (ay + \tilde{\nu}\mu)\partial_\mu], \text{ where } V''(z) = 2a,\ \tilde{\nu} = \pm\sqrt{\lambda - V(z)}. \tag{1.19}$$

The eigenvalues of L are therefore the roots of

$$(\frac{s}{2})^2 + \tilde{\nu}(\frac{s}{2}) + a = 0. \tag{1.20}$$

It is convenient for future reference to write the eigenvalues s_j as

$$s_j = -2\tilde{\nu} r_j; \tag{1.21}$$

the r_j thus satisfy

$$r^2 - r + \frac{a}{\tilde{\nu}^2} = 0. \tag{1.22}$$

If $a < 0$, so z is a local maximum then the eigenvalues of L are real, and are given by

$$r_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{a}{\tilde{\nu}^2}} < 0 < 1 < r_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{a}{\tilde{\nu}^2}}, \tag{1.23}$$

so the critical point is a saddle. If $a > 0$, so z is a local minimum, then the discriminant

$$1 - \frac{4a}{\tilde{\nu}^2} \text{ is } \begin{cases} \text{negative} & \text{if } \lambda < \lambda_{\text{Hess}} \\ \text{zero} & \text{if } \lambda = \lambda_{\text{Hess}} \\ \text{positive} & \text{if } \lambda > \lambda_{\text{Hess}} \end{cases}, \quad \lambda_{\text{Hess}} = V(z) + 2V''(z). \tag{1.24}$$

Correspondingly the eigenvalues are of the form $-2\tilde{\nu}r_j$, $j = 1, 2$, where, respectively,

$$\begin{gathered} r_1 = \frac{1}{2} + i\sqrt{\frac{a}{\tilde{\nu}^2} - \frac{1}{4}}, \ r_2 = \frac{1}{2} - i\sqrt{\frac{a}{\tilde{\nu}^2} - \frac{1}{4}}, \\ r_1 = r_2 = 1/2, \\ 0 < r_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{a}{\tilde{\nu}^2}} < \frac{1}{2} < r_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{a}{\tilde{\nu}^2}} < 1 \end{gathered} \tag{1.25}$$

and the critical point is a centre, a degenerate center, or a source/sink. □

Remark 1.3. If $\lambda \neq \lambda_{\text{Hess}}(z)$, a basis for the eigenvectors of L is

$$e_j = (s_j/2 + \tilde{\nu})\, dy + d\mu = \tilde{\nu}(1 - r_j)\, dy + d\mu. \tag{1.26}$$

If $\lambda = \lambda_{\text{Hess}}$, L has only one eigenvector, $\frac{1}{2}\tilde{\nu}\, dy + d\mu$, but the generalized eigenspace (of eigenvalue $-\tilde{\nu}$) is of course two-dimensional.

We also need a result about integral curves associated to the eigenvectors at a sink.

Proposition 1.4. *Suppose that $q \in \text{Min}_\pm(\lambda)$, and let e_j be the eigenvectors of L as in* (1.26). *If $r_2/r_1 > 1$ is not an integer, then there are two smooth one-dimensional submanifolds L_j through z such that W is tangent to L_j in a neighbourhood of q and $e_j \in N^*L_j$. If $r_2/r_1 = N$ is an integer, then there is one smooth Legendre curve L_1 as above and another smooth curve, L_2, such that $W = W_t + a\tilde{W}$, with W_t tangent to L_2, and where a vanishes at z to order N i.e. $a \in \mathcal{I}^N$, $\mathcal{I}$ being the ideal of functions vanishing at z.*

Proof. By a linear change of variables, $(y, \mu) \to (v_1, v_2)$, we may assume that W takes the form

$$W = r_1 v_1 \frac{\partial}{\partial v_1} + r_2 v_2 \frac{\partial}{\partial v_2} + O(|v|^2)$$

near $v = 0$. In these coordinates, the eigenvector e_i corresponds to dv_i. Suppose first that r_2/r_1 is not an integer. Then the eigenvalues are nonresonant, and are both positive, so by the Sternberg linearization theorem [16] there is a smooth change of variables so that W takes the form

$$W = r_1 v_1 \frac{\partial}{\partial v_1} + r_2 v_2 \frac{\partial}{\partial v_2}.$$

Then we may take $L_i = \{v_i = 0\}$. In the resonant case, $r_2/r_1 = N \geq 2$, we still obtain a normal form, but we are unable to remove resonant terms on the right hand side. In this case, there is just one term, so we find [16] that there is a smooth change of variables so that W takes the form

$$W = \frac{1}{N+1} v_1 \frac{\partial}{\partial v_1} + \frac{N}{N+1} v_2 \frac{\partial}{\partial v_2} + c v_1^N \frac{\partial}{\partial v_2}.$$

Again, if we take $L_i = \{v_i = 0\}$ then we satisfy the conditions of the theorem. $\square$

The corresponding result at saddle points is the stable/unstable manifold theorem.

Proposition 1.5. *Suppose that $q = z_\pm(\lambda)$ is a saddle, and let e_j be the eigenvectors of L as in* (1.26)*. Then there are two smooth Legendre curves L_j through z such that W is tangent to L_j and $e_j \in N^* L_j$. Moreover, if $\nu(q) < 0$, then $\nu \geq \nu(q)$ on L_1 and $\nu \leq \nu(q)$ on L_2, while if $\nu(q) > 0$, then $\nu \leq \nu(q)$ on L_1 and $\nu \geq \nu(q)$ on L_2.*

Proof. The first part follows from the stable/unstable manifold theorem, while the second part follows from (1.27). $\square$

Remark 1.6. In both cases, there are coordinates (v_1, v_2) on $\Sigma(\lambda)$ near q such that W restricts to $r_2 v_2 \partial_{v_2}$ on L_1 and $r_1 v_1 \partial_{v_1}$ on L_2 near q.

Below we make use of open neighbourhoods of the critical points which are well-behaved in terms of W.

Definition 1.7. By a W-balanced neighbourhood of a critical point $q \in \Sigma(\lambda)$, $\lambda \notin \mathrm{Cv}(V)$, we shall mean an open neighbourhood, O of q in C_∂ which contains no other radial point, which meets $\Sigma(\lambda)$ in a W-convex set (that is, each integral curve of W meets $\Sigma(\lambda)$ in a single interval, possibly empty) and is such that the closure of each integral curve of W in O meets $\nu = \nu(q)$.

Lemma 1.8. *Any critical point $q \in \mathrm{RP}(\lambda)$ for $\lambda > V(\pi(q))$ has a neighbourhood basis of W-balanced open sets.*

Proof. In the case of sinks or sources this is clear, since arbitrarily small balls around these that are convex with respect to W are W-balanced. In the case of saddle points we may use the stable manifold theorem. It suffices to suppose that $\nu(q) > 0$. Taking a neigbourhood of $L_2 \cap \{\nu = \nu(q) - \epsilon\}$, for $\epsilon > 0$ but small, a W-balanced neighbourhood is given by the W-flow-out of this set in the direction of increasing ν strictly between $\nu = \nu(q) - \epsilon$ and $\nu = \nu(q) + \epsilon$, together with the parts of L_1 and L_2 in $|\nu - \nu(q)| < \epsilon$. $\square$

Consider the global structure of the dynamics of W. From (1.14) it follows directly that

$$W(\nu) = 2|\mu|_y^2 \geq 0, \text{ i.e. } \nu \text{ is non-decreasing along integral curves of } W. \tag{1.27}$$

In fact, along maximally extended integral curves of W, ν is only constant if the curve reduces to a critical point. It is also easy to see that W has no non-trivial periodic orbits and every maximally extended bicharacteristic $\gamma : \mathbb{R}_t \to \Sigma(\lambda)$ tends to a point in $\mathrm{RP}_+(\lambda) \cup \mathrm{RP}_-(\lambda)$ as $t \to \pm\infty$. Indeed, $\lim_{t\to\pm\infty} \nu(\gamma(t)) = \nu_\pm$ exists by the monotonicity of ν, and any sequence $\gamma_k : [0,1] \to \Sigma(\lambda)$, $\gamma_k(t) = \gamma(t_k + t)$, $t_k \to +\infty$, has a uniformly convergent subsequence, which is then an integral curve $\tilde\gamma$ of W. Then ν is constant along this bicharacteristic, hence h is identically 0. In view of the ∂_μ component of W in (1.14), $\partial_y V$ is identically 0 along $\tilde{\gamma}$, hence y is a critical point of V and the limit is a point in $\mathrm{RP}_+(\lambda) \cup \mathrm{RP}_-(\lambda)$.

We can now define the forward, resp. backward, bicharacteristic relation as follows.

Definition 1.9. The forward bicharacteristic relation $\Phi_+ \subset \Sigma(\lambda) \times \Sigma(\lambda)$ is the set of (ξ', ξ) for which there exist bicharacteristics γ_j, $j = 1, \ldots, N$, $N \geq 1$, and $T_-, T_+ \in \mathbb{R}$, $T_- \leq T_+$, such that $\gamma_1(T_-) = \xi'$, $\gamma_N(T_+) = \xi$, and $\lim_{t \to +\infty} \gamma_j(t) = \lim_{t \to -\infty} \gamma_{j+1}(t)$ for $j = 1, 2, \ldots, N-1$.

The backward bicharacteristic relation Φ_- is defined similarly, with the role of ξ and ξ' reversed. The relations $\Phi_\pm$ depend on λ, but this is not indicated in notation.

In view of the previous observations, $\lim_{t \to \pm\infty} \gamma_j(t) \in \mathrm{RP}_+(\lambda) \cup \mathrm{RP}_-(\lambda)$, so the content of the definition is to allow ξ and ξ' to be connected by a chain of bicharacteristics in the forward direction. Note that Φ_+ is reflexive and transitive, but not symmetric.

Also, if $\xi, \xi' \in \mathrm{RP}_+(\lambda) \cup \mathrm{RP}_-(\lambda)$, and there exists a bicharacteristic γ such that $\lim_{t \to +\infty} \gamma(t) = \xi$, $\lim_{t \to -\infty} \gamma(t) = \xi'$, then $(\xi', \xi) \in \Phi_+$. Indeed, we can take γ_1 and γ_3 to be appropriately parameterized constant bicharacteristics with image ξ', resp. ξ, and $\gamma_2 = \gamma$. Of course, γ may be replaced by a chain of bicharacteristics.

We recall that the image of a set $K \subset {}^{\mathrm{sc}}T^*_Y X$ under Φ_+ is the set

$$\Phi_+(K) = \{\xi \in {}^{\mathrm{sc}}T^*_Y X; \exists\, \xi' \in K \text{ s.t. } (\xi', \xi) \in \Phi_+\}. \tag{1.28}$$

We will be interested in the forward flow-out $\Phi_+(\mathrm{RP}_+(\lambda))$ of the outgoing radial set. Since $\nu > 0$ on $\mathrm{RP}_+(\lambda)$, and is increasing along bicharacteristics, $\nu > 0$ on $\Phi_+(\mathrm{RP}_+(\lambda))$ as well.

Lemma 1.10. *If $q \in \mathrm{RP}_+(\lambda)$ is a center or sink, then $\Phi_+(\{q\}) = \{q\}$. On the other hand, if q is a saddle, then with the notation of Proposition 1.5, $\Phi_+(\{q\})$ is locally (in $\Sigma(\lambda)$) given by the local unstable curve L_2, i.e. by the unique smooth Legendre manifold L_2 such that $\nu \geq \nu(q)$ on L_2, $q \in L_2$ and W is tangent to L_2.*

Remark 1.11. Suppose that $\lambda \notin \mathrm{Cv}(V)$. Then on $\Phi_+(\mathrm{RP}_+(\lambda))$, $\nu \geq a(\lambda)$, and on $\Phi_-(\mathrm{RP}_-(\lambda))$, $\nu \leq -a(\lambda)$, where $a(\lambda) = \min\{\nu(q); q \in \mathrm{RP}_+(\lambda)\} > 0$.

2. Blow-ups and resolution of singularities of flows

As indicated in the Introduction, to aid in the description of the asymptotic behaviour of eigenfunctions it is very convenient to modify the underlying manifold with boundary by blowing it up at the minima of V_0 and with homogeneity r with respect to the boundary. We give a brief discussion of the blow-up of a stable critical point of a vector field with respect to the vector field.

Suppose that W is a real $\mathcal{C}^\infty$ vector field on a manifold M, and that $o \in M$, with $W(o) = 0$, is a linearly stable critical point. That is, the eigenvalues of the linearization of W at o all have negative real parts. We handle unstable critical points by changing the sign of the vector field.

If $\Phi : M \times \mathbb{R}_t \to M$ denotes the flow generated by W, then o has a neighbourhood O such that for $p \in O$, $\lim_{t \to +\infty} \Phi(p, t) = o$. There always exists a closed embedded submanifold, diffeomorphic to a sphere, $S \subset O$ which is transversal to W. It also necessarily satisfies

$$\lim_{t \to +\infty} \Phi(S, t) = \{o\}. \tag{2.1}$$

Then Φ restricts to a diffeomorphism of $S \times [0, +\infty)$ to $O' \setminus \{o\}$ where O' is the union of S and the neighbourhood of o consisting of one of the components of $M \setminus S$.

We may compactify $S \times [0, +\infty)_t$ by embedding it as a dense subset

$$S \times [0, +\infty)_t \hookrightarrow S \times [0, 1], \ (s, t) \longmapsto (s, e^{-t}).$$

This makes e^{-t} a defining function for a boundary hypsersurface. Using the diffeomorphism Φ this compactifies $O' \setminus \{o\}$ to a manifold with boundary. Finally then, replacing O' as a subset of M by $S \times [0, 1]$ using Φ to identify $O' \setminus \{o\}$ with $S \times (0, 1)$ we obtain a compactification of $M \setminus \{o\}$ which is a compact manifold with boundary.

Lemma 2.1. *The compact manifold with boundary obtained by blow-up, with respect to a vector field W, of a linearly stable critical point, $\{o\}$, is independent of the transversal S to W, satisfying* (2.1)*, used to define it.*

We may therefore denote the blown up manifold by $[M, \{o\}]_W$.

Proof. Since there are spheres transversal to W in any neighbourhood of $\{o\}$ it suffices to show that the two manifolds obtained using a transversal S and a second transversal $\tilde{S} \subset O'$ are naturally diffeomorphic where O' is the component of $M \setminus S$ containing o. Since it lies within S, S' is precisely the image of Φ on $\{(s, T(s)); s \in S\}$ for some smooth map $T : S \longrightarrow (0, \infty)$. In terms of the compactification this replaces e^{-t} by $e^{-t}e^{-T(s)}$ which induces a diffeomorphism of the neighbourhoods of the boundaries of the two compactifications. $\square$

Thus the abstract manifold, $[M, \{o\}]_W$, defined in this way is, as a set, the union of $M \setminus \{o\}$ with an abstract sphere as boundary. The boundary can be realized more concretely by using the flow of W to identify any two transversals to W which are sufficiently close to $\{o\}$. The Lemma above shows that $[M, \{o\}]_W$ has a natural $\mathcal{C}^\infty$ structure as a compact manifold with boundary with interior which is canonically diffeomorphic to $M \setminus \{o\}$. The boundary hypersurface is called the front face of the blow-up and is usually denoted below as ff, or more precisely as $\mathrm{ff}([M; \{o\}]_W)$. Note that it is *not* generally the case that the natural map

$$[M, \{o\}]_W \longrightarrow M, \tag{2.2}$$

under which $\mathrm{ff} \longmapsto \{o\}$, is smooth.

More generally, if X is a smooth manifold with boundary and W is a smooth vector field on X which is tangent to the boundary then we may use the construction above to define $[X, \{o\}]_W$ even for a stable critical point of W on the boundary. Namely, we may simply extend W across the boundary to the double of X and then observe that the closure of the preimage of $X \setminus \{o\}$ in $[2X, \{o\}]_{\tilde{W}}$ is independent of the extension $\tilde{W}$ of W.

One particular use of this construction is to define inhomogeneous blow-ups. Let X be a 2-dimensional manifold with boundary, with $o \in \partial X$. Choose local coordinates (x, y) near o such that $x \geq 0$ is a boundary defining function and o is given by $x = 0$, $y = 0$. Let $r \in (0, 1)$ be a given homogeneity. We wish to blow up o in X so that y is homogeneous of degree 1 and x is homogeneous of degree $1/r$. To do so consider the vector field $-W = r^{-1}x\partial_x + y\partial_y$, and apply the construction above, denoting the result $[X, \{o\}]_r$; clearly it does not depend on how W is extended outside a neighbourhood of o. Observe that y/x^r and $x/|y|^{1/r}$ are homogeneous of degree 0, i.e. are annihilated by W, where they are bounded, so they can be regarded as variables on the transversal S. Thus, local coordinates on the blown-up space, in the lift of the region $|y/x^r| < C$ are given by y/x^r and x^r, while local coordinates in the lift of the region $x/|y|^{1/r} < C'$ are given by $x/|y|^{1/r}$ and y.

In fact the manifold $[X;\{o\}]_r$ does depend on the choice of W, and so on the choice of coordinates (x, y). However the space of classical conormal functions, with respect to the boundary, is defined independently of choices. This is in essence because x has homogeneity greater than that of the boundary variable. More explicitly, any change of coordinates takes the form $y' = a(x, y)y + b(x, y)x$, where $x' = c(x, y)x$, $c(0, 0) > 0$ and $a(0, 0) \neq 0$. So, for example, $y'/(x')^r = (a/c^r)(y/x^r)+ (b/c^r)x^{1-r}$ which is bounded (in $x \leq x_0$, $x_0 > 0$, $|y| \leq y_0$, $y_0 > 0$) if and only if y/x^r is bounded (since $r \in (0, 1)$). Such calculations show that the blow-ups using (x, y) and using (x', y') coincide as topological manifolds. Their $\mathcal{C}^\infty$ structures are different as can be seen from the appearance of terms involving x^{1-r} in the transformation law. However, it is easy to see that the class of conormal functions, and also that of polyhomogeneous conormal functions, on $[X;\{o\}]_r$, is well-defined, independent of the choice of local coordinates used in the definition (although the orders may need to be appropriately adjusted). Thus, $[X;\{o\}]_r$ may be thought of as a 'conormal manifold' rather than as a $\mathcal{C}^\infty$ manifold, i.e. the algebra of smooth functions should be replaced by the algebra of polyhomogeneous conormal functions as the basic object of interest.

Such inhomogeneous blow-ups may be generalized to higher dimensions, although there one needs extra structure to make it independent of the choice of coordinates. Since we do not need this notion in the present paper, we do notpursue it further.

3. Limiting absorption principle

In later sections we prove, by positive commutator methods, various results of propagation of singularities type. Here we summarize, in four theorems, the main conclusions concerning the limit of the resolvent on the real axis and the behaviour of L^2 eigenfunctions. The scattering wavefront set is discussed in [13] as are the scattering Sobolev spaces $H^{m,r}_{\mathrm{sc}}(X)$; the characteristic variety $\Sigma(\lambda)$, scattering relation Φ_+ and radial sets $\mathrm{RP}_\pm(\lambda)$ are defined in Section 1 above.

First, we recall Hörmander's theorem on the propagation of singularities; in the present setting this is [13, Proposition 7].

Theorem 3.1 (Propagation of singularities). *For $u \in \mathcal{C}^{-\infty}(X)$, and any real m and r,*

$$\mathrm{WF}^{m,r-1}_{sc}(u) \setminus \mathrm{WF}^{m,r}_{sc}((P-\lambda)u) \ \textit{is a union of maximally extended bicharacteristics of } P-\lambda \textit{ in } \Sigma(\lambda) \setminus \mathrm{WF}^{m,r}_{sc}((P-\lambda)u).$$

Similarly, if u_t, for $t \in G \subset (0, 1]$ is a bounded family in $H^{M,R}_{sc}(X)$, then

$$\mathrm{WF}^{m,r-1}_{\mathrm{sc},L^\infty(G_t)}(u_t) \setminus \mathrm{WF}^{m,r}_{\mathrm{sc},L^\infty(G_t)}((P-(\lambda+it))u_t) \ \textit{is a union of maximally forward-extended bicharacteristics of } P-\lambda \textit{ in } \Sigma(\lambda) \setminus \mathrm{WF}^{m,r}_{\mathrm{sc},L^\infty(G_t)}((P-\lambda)u).$$

Theorem 3.2 (Decay of outgoing eigenfunctions)**.** *If* $\lambda \notin \mathrm{Cv}(V)$ *then the space* $E_{\mathrm{pp}}(\lambda)$ *is finite dimensional, is contained in* $\dot{\mathcal{C}}^\infty(X)$*, and may be characterized by*

$$(3.1)\quad E_{\mathrm{pp}}(\lambda) = \{u \in \mathcal{C}^{-\infty}(X); (P-\lambda)u = 0 \text{ and } \mathrm{WF}_{sc}^{m,l}(u) \cap \mathrm{RP}_+(\lambda) = \emptyset \text{ for some } l > -1/2\}.$$

The set of eigenvalues $\sigma_{\mathrm{pp}}(P) = \{\lambda; E_{\mathrm{pp}}(\lambda) \neq \{0\}\}$ *is discrete outside* $\mathrm{Cv}(V)$ *and is contained in* $(-\inf_X V, \sup \mathrm{Cv}(V)]$.

As an immediate consequence of Theorem 3.2,

$$(3.2)\qquad E_{\mathrm{ess}}^{-\infty}(\lambda) = \big\{u \in \mathcal{C}^{-\infty}(X); (P-\lambda)u = 0,\ \langle u, v\rangle = 0\ \forall\ v \in E_{\mathrm{pp}}(\lambda)\big\}$$

is well defined for $\lambda \notin \mathrm{Cv}(V)$ as are spaces such as

$$H_{\mathrm{sc}}^{m,r}(X) \ominus E_{\mathrm{pp}}(\lambda) = \{f \in H_{\mathrm{sc}}^{m,r}(X); \langle f, \phi\rangle = 0\ \forall\ \phi \in E_{\mathrm{pp}}(\lambda)\}.$$

Theorem 3.3 (Limiting absorption principle)**.** *The resolvent*

$$(3.3)\qquad R(\lambda + it) = (P - (\lambda + it))^{-1},\ 0 \neq t \in \mathbb{R},\ \lambda \notin \mathrm{Cv}(V)$$

extends continuously to the real axis, i.e. $R(\lambda \pm i0)$ *exist, as bounded operators*

$$(3.4)\qquad H_{sc}^{m,r}(X) \ominus E_{\mathrm{pp}}(\lambda) \longrightarrow H_{sc}^{m+2,l}(X) \ominus E_{\mathrm{pp}}(\lambda),\ \forall\ m \in \mathbb{R},\ r > 1/2,\ l < -1/2.$$

Theorem 3.4 (Forward propagation)**.** *For* $\lambda \notin \mathrm{Cv}(V)$ *and* $f \in \dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda)$,

$$(3.5)\qquad \mathrm{WF}_{sc}(R(\lambda + i0)f) \subset \Phi_+(\mathrm{RP}_+(\lambda)) \subset \{\nu \geq 0\}$$

and $R(\lambda + i0)$ *extends by continuity to*

$$(3.6)\quad R(\lambda + i0) : \{v \in \mathcal{C}^{-\infty}(X) \ominus E_{\mathrm{pp}}(\lambda); \mathrm{WF}_{sc}(v) \cap \Phi_-(\mathrm{RP}_-(\lambda)) = \emptyset\} \longrightarrow \mathcal{C}^{-\infty}(X),$$

with wavefront set bound given by

$$(3.7)\qquad \mathrm{WF}_{sc}(R(\lambda + i0)v) \subset \mathrm{WF}_{sc}(v) \cup \Phi_+(\mathrm{WF}_{sc}(v) \cap \Sigma(\lambda)) \cup \Phi_+(\mathrm{RP}_+(\lambda)).$$

We next show that various statements in the Introduction are direct consequences of these results.

Proposition 3.5. *For* $\lambda \notin \mathrm{Cv}(V)$ *the space* $(P-\lambda)\dot{\mathcal{C}}^\infty(X)$ *is closed in* $\dot{\mathcal{C}}^\infty(X)$ *and for* $\lambda \in \sigma_{\mathrm{pp}}(P) \setminus \mathrm{Cv}(V)$, $(P-\lambda)\dot{\mathcal{C}}^\infty(X)$ *is* L^2*-orthogonal to* $E_{\mathrm{pp}}(\lambda)$ *and*

$$(3.8)\qquad R(\lambda + i0)f = R(\lambda - i0)f \quad \forall\ f \in (P-\lambda)\dot{\mathcal{C}}^\infty(X).$$

Proof. By assumption $\lambda \notin \mathrm{Cv}(V)$ so $E_{\mathrm{pp}}(\lambda) \subset \dot{\mathcal{C}}^\infty(X)$. The self-adjointness of P gives

$$\langle (P-\lambda)v, \phi\rangle = \langle v, (P-\lambda)\phi\rangle = 0 \text{ if } \phi \in E_{\mathrm{pp}}(\lambda)$$

so $(P-\lambda)\dot{\mathcal{C}}^\infty(X)$ is L^2 orthogonal to $E_{\mathrm{pp}}(\lambda)$.

Suppose u is in the closure of $(P-\lambda)\dot{\mathcal{C}}^\infty(X)$ in $\dot{\mathcal{C}}^\infty(X)$. Thus $u_j = (P-\lambda)v_j \to u$ with $v_j \in \dot{\mathcal{C}}^\infty(X)$. By the finite dimensionality of $E_{\mathrm{pp}}(\lambda)$ we may assume that $v_j \perp E_{\mathrm{pp}}(\lambda)$. Now consider $w_j = R(\lambda + i0)u_j$ which exist by Theorem 3.3 and by Theorem 3.4 satisfy $\mathrm{WF}_{\mathrm{sc}}(w_j) \subset \Phi_+(\mathrm{RP}_+(\lambda))$. Thus,

$$(P-\lambda)(v_j - w_j) = 0 \text{ and } \mathrm{WF}_{\mathrm{sc}}(v_j - w_j) \subset \Phi_+(\mathrm{RP}_+(\lambda))$$

so, by Theorem 3.2, $v_j - w_j \in E_{\mathrm{pp}}(\lambda)$, to which both terms are orthogonal, so $v_j = w_j = R(\lambda + i0)u_j$. Since $u_j \to u$ in $\dot{\mathcal{C}}^\infty(X)$, $v_j \to R(\lambda + i0)u$ in $\mathcal{C}^{-\infty}(X)$. The same argument shows that $v_j \to R(\lambda - i0)u$ in $\mathcal{C}^{-\infty}(X)$, so $v = R(\lambda + i0)u = R(\lambda - i0)u$. By (3.5) applied with each sign, $\mathrm{WF}_{\mathrm{sc}}(v)$ is disjoint from both $\{\nu \leq 0\}$ and $\{\nu \geq 0\}$ and hence is empty. Thus $v \in \dot{\mathcal{C}}^\infty(X)$, and since $u = (P - \lambda)v$, $(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ is closed in $\dot{\mathcal{C}}^\infty(X)$; this also proves (3.8). $\square$

Corollary 3.6. *For $\lambda \notin \mathrm{Cv}(V)$, the space $\dot{\mathcal{C}}^\infty(X)/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ is a Fréchet space, as is $(\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda))/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$.*

An abstract parameterization of generalized eigenfunctions, which has little specific to do with our particular problem, may be obtained via the two terms of (I.12). Namely, let

$$\mathrm{RR}^\infty_{\mathrm{ess},\pm}(\lambda) = [R(\lambda \pm i0)(\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda))]/(\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda)) \tag{3.9}$$

be the range of the incoming $(-)$, resp. outgoing $(+)$, boundary value of the resolvent acting on Schwartz functions, modulo Schwartz functions. (The abbreviation RR stands for 'relative range'.) It is now immediate that the map

$$(P - \lambda) : \mathrm{RR}^\infty_{\mathrm{ess},\pm}(\lambda) \longrightarrow (\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda))/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$$

is an isomorphism (for each sign). This induces a Fréchet topology on $\mathrm{RR}^\infty_{\mathrm{ess},\pm}(\lambda)$.

Proposition 3.7. *For $\lambda \notin \mathrm{Cv}(V) \cup \sigma_{\mathrm{pp}}(P)$ the nullspace of*

$$\mathrm{Sp}(\lambda) = \frac{1}{2\pi i}\left(R(\lambda + i0) - R(\lambda - i0)\right) : \dot{\mathcal{C}}^\infty(X) \longrightarrow E^\infty_{\mathrm{ess}}(\lambda)$$

is $(P - \lambda)\dot{\mathcal{C}}^\infty(X)$; if $\lambda \in \sigma_{\mathrm{pp}}(P) \setminus \mathrm{Cv}(V)$, the same remains true for the map $(2\pi i)^{-1}\left(R(\lambda + i0) - R(\lambda - i0)\right) : \dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda) \to E^\infty_{\mathrm{ess}}(\lambda)$, for $\lambda \in \sigma_{\mathrm{pp}}(P) \setminus \mathrm{Cv}(V)$.

Proof. By (3.8), $(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ lies in the nullspace of $\mathrm{Sp}(\lambda)$. Conversely, suppose that $f \in \dot{\mathcal{C}}^\infty(X)$, $\mathrm{Sp}(\lambda)f = 0$, so $v = R(\lambda + i0)f = R(\lambda - i0)f$. Since $\mathrm{WF}_{\mathrm{sc}}(R(\lambda + i0)(f))$ and $\mathrm{WF}_{\mathrm{sc}}(R(\lambda - i0)(f))$ are disjoint ($\nu < 0$ on one, $\nu > 0$ on the other), both of these are empty, so $v \in \dot{\mathcal{C}}^\infty(X)$, and $(P - \lambda)v = f$ is in $(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ as claimed. $\square$

Corollary 3.8. *If $\lambda \notin \mathrm{Cv}(V) \cup \sigma_{\mathrm{pp}}(P)$ the spaces $E^\infty_{\mathrm{ess}}(\lambda)$, $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$, $\mathrm{RR}^\infty_{\mathrm{ess},-}(\lambda)$ and $\dot{\mathcal{C}}^\infty(X)/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ are all isomorphic; this remains true for $\lambda \in \sigma_{\mathrm{pp}}(P) \setminus \mathrm{Cv}(V)$ with $\dot{\mathcal{C}}^\infty(X)/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$ replaced by $(\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda))/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$.*

Explicitly, the isomorphisms

$$I_\pm(\lambda) : E^\infty_{\mathrm{ess}}(\lambda) \to \mathrm{RR}^\infty_{\mathrm{ess},\pm}(\lambda) \tag{3.10}$$

are the maps

$$\begin{aligned} &I_+(\lambda) : u = [R(\lambda + i0) - R(\lambda - i0)]f \mapsto R(\lambda + i0)f, \\ &I_-(\lambda) : u = [R(\lambda + i0) - R(\lambda - i0)]f \mapsto -R(\lambda - i0)f,\ f \in \dot{\mathcal{C}}^\infty(X). \end{aligned}$$

Although f is not well-defined in $\dot{\mathcal{C}}^\infty(X)$, it is in $\dot{\mathcal{C}}^\infty(X)/(P - \lambda)\dot{\mathcal{C}}^\infty(X)$, hence the image of $R(\lambda + i0)f$ in $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$ is also well defined. By Theorem 3.4,

$$\mathrm{WF}_{\mathrm{sc}}(I_\pm(\lambda)u) \subset \Phi_\pm(\mathrm{RP}_\pm(\lambda)).$$

Thus, $I_-(\lambda)$ maps $u \in E^\infty_{\mathrm{ess}}(\lambda)$ to its 'incoming part', while $I_+(\lambda)$ maps u to its outgoing part. We define the abstract scattering matrix as the map

$$S(\lambda) : \mathrm{RR}^\infty_{\mathrm{ess},-}(\lambda) \to \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda), \quad S(\lambda) = I_+(\lambda)I_-(\lambda)^{-1}. \tag{3.11}$$

In Section 11 we identify $\mathrm{RR}^\infty_{\mathrm{ess},\pm}(\lambda)$ geometrically. We use this identification in Section 12 to extend $S(\lambda)$ to a unitary operator on a Hilbert space, giving rise to the definition of the S-matrix in the introduction in (I.36).

We now turn to the relationship between $E^\infty_{\mathrm{ess}}(\lambda)$ and $E^{-\infty}_{\mathrm{ess}}(\lambda)$ which has two facets. On the one hand, as we show below, $E^\infty_{\mathrm{ess}}(\lambda)$ is dense in $E^{-\infty}_{\mathrm{ess}}(\lambda)$ in the topology of $\mathcal{C}^{-\infty}(X)$, indeed in a stronger sense. On the other hand they are dual to each other with respect to a form of 'Green's pairing' that we describe in Section 12.

By Theorem 3.4, $\mathrm{Sp}(\lambda)$ may be applied to any distribution which is L^2 orthogonal to $E_{\mathrm{pp}}(\lambda)$, provided its scattering wavefront set is contained in $\{|\nu| < \epsilon\}$ for $\epsilon > 0$ sufficiently small, namely $\epsilon < a(\lambda)$,

$$a(\lambda) = \min\{\nu(q); q \in \mathrm{RP}_+(\lambda)\} > 0.$$

Proposition 3.9. *If $\lambda \notin \mathrm{Cv}(V)$ and $\epsilon > 0$ then every $u \in E^{-\infty}_{\mathrm{ess}}(\lambda)$ is of the form $\mathrm{Sp}(\lambda)f$ for some $f \in \mathcal{C}^{-\infty}(X)$ with $\mathrm{WF}_{sc}(f) \subset \{|\nu| < \epsilon\}$ and $f \perp E_{pp}(\lambda)$.*

Of course if $\lambda \notin \sigma_{\mathrm{pp}}(P)$ the orthogonality condition is void.

Proof. Suppose first that $\lambda \notin \sigma_{\mathrm{pp}}(P)$.

Let $A \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ be a pseudodifferential operator with boundary symbol given by $\psi_1(\sigma_\partial(P-\lambda))\psi_2(\nu/a(\lambda))$, where $\psi_1(t)$ is supported in $|t| < \epsilon$ and $\psi_2(t)$ is supported in $t > 1/4$, and equal to 1 when $t > 1/2$, with $\nu \geq \frac{a(\lambda)}{4}$ on $\mathrm{WF}'(A)$. Then, by Theorem 3.4, we may apply $R(\lambda - i0)$ to $(P-\lambda)(\mathrm{Id} - A)u$ and $R(\lambda + i0)$ to $(P-\lambda)Au$. Since

$$(P-\lambda)\big((\mathrm{Id} - A)u - R(\lambda - i0)(P-\lambda)(\mathrm{Id} - A)u\big) = 0,$$

and this function has scattering wavefront contained in $\nu \leq a(\lambda)/2$, we conclude using Theorem 3.2 that $(\mathrm{Id} - A)u = R(\lambda - i0)(P-\lambda)(\mathrm{Id} - A)u$. Similarly, $Au = R(\lambda + i0)(P-\lambda)Au$. Since $(P-\lambda)Au = [P,A]u = -(P-\lambda)(\mathrm{Id} - A)u$, we may take $f = (2\pi i)[P,A]u$.

If $\lambda \in \sigma_{\mathrm{pp}}(P)$, only a minor modification is necessary. Namely, we consider ΠA in place of A where Π is orthogonal projection off $E_{\mathrm{pp}}(\lambda)$. Note that $\mathrm{Id} - \Pi \in \Psi^{-\infty,\infty}_{\mathrm{sc}}(X)$. Then the rest of the argument goes through. □

In fact, this proof shows more.

Proposition 3.10. *Suppose $\lambda \notin \mathrm{Cv}(V)$ and that $A \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ has*

$$\mathrm{WF}'_{sc}(A) \subset \{\nu > -a(\lambda)\} \text{ and } \mathrm{WF}'_{sc}(\mathrm{Id} - A) \subset \{\nu < a(\lambda)\}$$

where $0 < a(\lambda) = \min\{\nu(q); q \in \mathrm{RP}_+(\lambda)\}$; if Π is the orthogonal projection off $E_{\mathrm{pp}}(\lambda)$ then

$$2\pi i \,\mathrm{Sp}(\lambda)\Pi[P,A] : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \tag{3.12}$$

is continuous, with range in $E^{-\infty}_{\mathrm{ess}}(\lambda)$ and the restriction of (3.12) to $E^{-\infty}_{\mathrm{ess}}(\lambda)$ is the identity map.

Proof. By assumption $-a(\lambda) < \nu < a(\lambda)$ on $\mathrm{WF}'_{\mathrm{sc}}(A) \cap \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - A)$ so this is also holds on $\mathrm{WF}'_{\mathrm{sc}}([P, A])$. The continuity of (3.12) follows. The final statement is a consequence of the proof of the preceding proposition, namely that, given $u \in E^{-\infty}_{\mathrm{ess}}(\lambda)$ one can take $f = 2\pi i \Pi[P, A]u$ and conclude that $u = \mathrm{Sp}(\lambda) f$. □

This result allows us to characterize the topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ three different ways. Let $\mathcal{T}_1$ be the topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ as in Corollary 3.8, induced by $\mathrm{Sp}(\lambda) : (\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda)) \to E^{\infty}_{\mathrm{ess}}(\lambda)$. Fix any A as in Proposition 3.10 and let $\mathcal{T}_2$ be the topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ induced by the topology of $\dot{\mathcal{C}}^\infty(X)$ on $E^{\infty}_{\mathrm{ess}}(\lambda)$ via the map $[P, A] : E^{\infty}_{\mathrm{ess}}(\lambda) \to \dot{\mathcal{C}}^\infty(X)$. Finally let $r < -1/2$, and let $B \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ be such that

$$B \text{ is elliptic at } \{\nu = 0\} \cap \Sigma(\lambda), \text{ with } \mathrm{WF}'_{\mathrm{sc}}(B) \subset \{|\nu| < a(\lambda)\}, \tag{3.13}$$

and let $\mathcal{T}_3$ be the weakest topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ stronger than both the $x^r L^2_{\mathrm{sc}}(X)$ topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ and the topology induced by the map $B : E^{\infty}_{\mathrm{ess}}(\lambda) \to \dot{\mathcal{C}}^\infty(X)$.

Proposition 3.11. *The topologies $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are equivalent. In particular, $\mathcal{T}_2$ and $\mathcal{T}_3$ are independent of choices.*

Proof. The equivalence of $\mathcal{T}_1$ and $\mathcal{T}_2$ follows from $2\pi i \,\mathrm{Sp}(\lambda)\Pi[P, A] = \mathrm{Id}$ on $E^{\infty}_{\mathrm{ess}}(\lambda)$. By the propagation of singularities, $\mathcal{T}_3$ is stronger that $\mathcal{T}_2$. Finally, by the forward propagation, $\mathcal{T}_1$ is stronger than $\mathcal{T}_3$. □

We can also topologize

$$E^s_{\mathrm{ess}}(\lambda) = \{u \in E^{-\infty}_{\mathrm{ess}}(\lambda);\ \mathrm{WF}^{0,s-1/2}_{\mathrm{sc}}(u) \cap \{\nu = 0\} = \emptyset\},\ s \in \mathbb{R}, \tag{3.14}$$

similarly. Note first that

$$E^s_{\mathrm{ess}}(\lambda) \subset x^r L^2_{\mathrm{sc}}(X) \text{ for } r < \min(-\frac{1}{2}, s - \frac{1}{2}).$$

For $s \in \mathbb{R}$, we let $\mathcal{T}^s_2$ be the topology on $E^s_{\mathrm{ess}}(\lambda)$ induced by $[P, A] : E^s_{\mathrm{ess}}(\lambda) \to x^{s+1/2} L^2_{\mathrm{sc}}(X)$, so

$$\|u\|_{\mathcal{T}^s_2} = \|u\|_2 = \|[P, A]u\|_{x^{s+1/2} L^2_{\mathrm{sc}}(X)}.$$

In the same setting, let $\mathcal{T}^s_3$ be the topology on $E^s_{\mathrm{ess}}(\lambda)$ induced by the $x^r L^2_{\mathrm{sc}}(X)$ topology on $E^s_{\mathrm{ess}}(\lambda)$, for r with $r < -1/2$ and $r < s - 1/2$, and by the operator B from (3.13), that is by the norm

$$\|u\|_{\mathcal{T}^s_3} = \|u\|_3 = \left(\|u\|^2_{x^r L^2_{\mathrm{sc}}(X)} + \|Bu\|^2_{x^{s-1/2} L^2_{\mathrm{sc}}(X)}\right)^{1/2}.$$

For $s > 0$, we deduce the following as above.

Proposition 3.12. *For each $s > 0$, the topologies $\mathcal{T}^s_2$ and $\mathcal{T}^s_3$ are equivalent and are independent of all choices, making $E^s_{\mathrm{ess}}(\lambda)$ into a Hilbert space.*

The inclusion map $E^s_{\mathrm{ess}}(\lambda) \hookrightarrow E^{s'}_{\mathrm{ess}}(\lambda)$, with norm $\|.\|_3$, is bounded for $s' \leq s$.

We consider $s = 0$ when we study Green's pairing in Section 12.

¿From the density of $\dot{\mathcal{C}}^\infty(X)$ in the subspace of $\mathcal{C}^{-\infty}(X)$ with wave front set in $\{|\nu| < \epsilon\} \cap \Sigma(\lambda)$, with the topology of Hörmander, and the continuity of $\mathrm{Sp}(\lambda)$ from this space to $\mathcal{C}^{-\infty}(X)$ we conclude (using Proposition 3.9) that the smooth eigenfunctions, $E^{\infty}_{\mathrm{ess}}(\lambda)$, are dense in $E^{-\infty}_{\mathrm{ess}}(\lambda)$.

More explicitly, this can be seen by considering $u_r = \phi(x/r)u$, $r > 0$ where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ is identically 1 on $[1, +\infty)$ and 0 on $[0, 1/2)$. Then $\mathrm{Sp}(\lambda)\Pi[P, A]u_r \in E^{\infty}_{\mathrm{ess}}(\lambda)$ for $r > 0$, and $u_r \to u$ in $\mathcal{C}^{-\infty}(X)$, hence

$$E^{\infty}_{\mathrm{ess}}(\lambda) \ni 2\pi i \,\mathrm{Sp}(\lambda)\Pi[P, A]u_r \to u$$

in $\mathcal{C}^{-\infty}(X)$. If $u \in E^s_{\mathrm{ess}}(\lambda)$, $s > 0$, then $[P, A]u_r \to [P, A]u$ in $x^{s+1/2}L^2_{\mathrm{sc}}(X)$, hence

$$[P, A]2\pi i \operatorname{Sp}(\lambda)\Pi[P, A]u_r \to [P, A]2\pi i \operatorname{Sp}(\lambda)\Pi[P, A]u = [P, A]u$$

in $x^{s+1/2}L^2_{\mathrm{sc}}(X)$. More generally, suppose that $u \in E^s_{\mathrm{ess}}(\lambda)$, $s \in \mathbb{R}$, and let $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ be such that $\mathrm{WF}'_{\mathrm{sc}}(Q) \subset \{|\nu| < a(\lambda)\}$, $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q) \cap \mathrm{WF}'_{\mathrm{sc}}([P, A]) = \emptyset$. Let p be such that $p < s - 1/2$, $p < -1/2$. Then $[P, A]u_r \to [P, A]u$ in $x^{s+1/2}L^2_{\mathrm{sc}}(X)$ shows that $2\pi i \operatorname{Sp}(\lambda)\Pi Q[P, A]u_r \to 2\pi i \operatorname{Sp}(\lambda)\Pi Q[P, A]u$ in $x^p L^2_{\mathrm{sc}}(X)$ and $(\mathrm{Id} - Q)[P, A]u_r \to (\mathrm{Id} - Q)[P, A]u$ in $\dot{\mathcal{C}}^\infty(X)$ so $2\pi i \operatorname{Sp}(\lambda)\Pi[P, A]u_r \to u$ in $x^p L^2_{\mathrm{sc}}(X)$, and, with B from (3.13), $2\pi i B \operatorname{Sp}(\lambda)\Pi[P, A]u_r \to Bu$ in $x^{s-1/2}L^2_{\mathrm{sc}}(X)$. We have thus proved the following.

Corollary 3.13. *For each $\lambda \notin \mathrm{Cv}(V)$, $E^\infty_{\mathrm{ess}}(\lambda)$ is dense in $E^{-\infty}_{\mathrm{ess}}(\lambda)$ in the topology of $\mathcal{C}^{-\infty}(X)$. Moreover, for each $\lambda \notin \mathrm{Cv}(V)$, $s \in \mathbb{R}$, $E^\infty_{\mathrm{ess}}(\lambda)$ is dense in $E^s_{\mathrm{ess}}(\lambda)$ in the topology $\mathcal{T}^s_3$.*

4. Propagation of singularities

In this section, we derive wavefront set bounds on eigenfunctions, or more generally of solutions of $(P - \lambda)u = f$, and use these to prove the results of the previous section. Let us first consider $\lambda \in \mathbb{C} \setminus [\inf V_0, +\infty)$. Then the symbol $p - \lambda$ of $\sigma_\partial(P - \lambda)$ never vanishes, so $P - \lambda$ has a parametrix in the scattering calculus, i.e. there is $G(\lambda) \in \Psi^{-2,0}_{\mathrm{sc}}(X)$ such that $E(\lambda) = (P - \lambda)G(\lambda) - \mathrm{Id} \in \Psi^{-1,1}_{\mathrm{sc}}(X)$. Thus $E(\lambda)$ is compact on L^2. By analytic Fredholm theory we conclude that P has discrete spectrum in $(-\infty, \inf V_0)$. Also ellipticity of $p - \lambda$ at the boundary implies that $\mathrm{WF}_{\mathrm{sc}}(u)$ is empty if $(P - \lambda)u = 0$, so $u \in \dot{\mathcal{C}}^\infty(X)$.

In the rest of the section we consider $\lambda > \inf V_0$, and use positive commutator estimates to derive wavefront set bounds. First we recall some standard results in the scattering calculus. Since $P \in \Psi^{*,0}_{\mathrm{sc}}(X)$, the Hamilton vector field H_p of P vanishes to first order at $x = 0$, we define the scattering Hamilton vector field ${}^{\mathrm{sc}}H_p = x^{-1}H_p$, which is smooth and nonvanishing up to the boundary. In fact, ${}^{\mathrm{sc}}H_p = W - 2\nu x\partial_x$ in local coordinates near the boundary, where W is the contact vector field from Section 1. If $A \in \Psi^{*,-l-1}_{\mathrm{sc}}(X)$ is a scattering pseudodifferential operator, with symbol $x^{-l-1}a$, then the symbol of the commutator $i[P, A]$ is

$$\sigma_\partial(i[P, A]) = H_p(x^{-l-1}a) = x^{-l}\big(Wa + 2(l + 1)\nu a\big). \tag{4.1}$$

We also recall from [13, Definition 4] the notion of wavefront set of a family of functions, $\mathrm{WF}^{m,l}_{\mathrm{sc},L^\infty(G_t)}(u_t)$, where $t \in G$ is a parameter, which is *only defined* for families $u_t \in L^\infty(G_t; \mathcal{C}^{-\infty}(X))$. Hence, the use of the notation $\mathrm{WF}^{m,l}_{\mathrm{sc},L^\infty(G_t)}(u_t)$ in a lemma, etc., implies that $u_t \in L^\infty(G_t; \mathcal{C}^{-\infty}(X))$ is assumed. For a single function v, the statement $q \notin \mathrm{WF}^{m,l}_{\mathrm{sc}}(v)$ is equivalent to the existence of $A \in \Psi^{m,-l}_{\mathrm{sc}}(X)$ which is elliptic at q and such that $Av \in L^2(X)$, while the statement $q \notin \mathrm{WF}^{m,l}_{\mathrm{sc},L^\infty(G_t)}(u_t)$ is equivalent to the existence of A as above such that the L^2 norm, $\|Au_t\|$, is *uniformly* bounded.

We begin with a Lemma describing the basic structure of positive commutator estimates.

Lemma 4.1. *Suppose $a \geq 0 \in x^{-l-1}\mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$ satisfies*

$${}^{\mathrm{sc}}H_p a = -x^{-l-1}b^2 + x^{-l-1}e, \tag{4.2}$$

with $a^{1/2}b$, b *and* $e \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$. *Suppose*

$$A \in \Psi_{\mathrm{sc}}^{-\infty,-l-1}(X),\ B \in \Psi_{\mathrm{sc}}^{-\infty,-l-1/2}(X) \text{ and } E \in \Psi_{\mathrm{sc}}^{-\infty,-2l-1}(X)$$

have principal symbols a, $a^{1/2}b$ *and* ae *respectively, with* $\mathrm{WF}'_{sc}(E) \subset \operatorname{supp} e$, *and assume that*

$$a \leq Cb^2 \text{ on } \{e = 0\} \text{ for some } C > 0. \tag{4.3}$$

Then there exists $F \in \Psi_{\mathrm{sc}}^{-\infty,-2l}(X)$ *such that for all* $u \in H_{sc}^{*,l+1/2}(X)$,

$$\|Bu\|^2 + 2t\|Au\|^2 \leq |\langle u, Eu\rangle| + |\langle u, Fu\rangle| + 2|\langle u, A^*A(P - (\lambda + it))u\rangle|. \tag{4.4}$$

If $G \subset (0,1]_t$, *and* u_t, $t \in G$, *is a (not necessarily bounded) family in* $H_{sc}^{*,l+1/2}(X)$ *then*

$$\mathrm{WF}^{*,l}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \operatorname{supp} a = \emptyset,\ \mathrm{WF}^{*,l+3/2}_{\mathrm{sc},L^\infty(G_t)}((P - (\lambda + it))u_t) \cap \operatorname{supp} a = \emptyset$$
$$\text{and } \mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \operatorname{supp} e = \emptyset$$
$$\Longrightarrow \mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) \subset \{b = 0\}.$$

The assumptions $t > 0$, $G \subset (0,1]$ can be replaced by $t < 0$, $G \subset [-1,0)$ if we arrange that $0 \leq a \in x^{-l-1}\psi_0(p)\mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$ satisfies

$${}^{\mathrm{sc}}H_p a = +x^{-l-1}b^2 + x^{-l-1}e,$$

with $a^{1/2}b$, b and $ae \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$.

Proof. Let $A \in \Psi_{\mathrm{sc}}^{-\infty,-l-1}(X)$ be a quantization of a, i.e. be such that $\sigma_{\partial,-l-1}(A) = x^{l+1}a$. ¿From the symbol calculus, and (4.2),

$$ix^{l+1/2}[A^*A, P]x^{l+1/2} = 2x^{l+1/2}(B^*B + E + F)x^{l+1/2}, \tag{4.5}$$

where

$$\begin{aligned} &F \in \Psi_{\mathrm{sc}}^{-\infty,-2l}(X),\ \mathrm{WF}'_{\mathrm{sc}}(F) \subset \operatorname{supp}(x^{l+1}a), \\ &B \in \Psi_{\mathrm{sc}}^{-\infty,-l-1/2}(X) \text{ and } \sigma_{\partial,-l-1/2}(B) = ba^{1/2}. \end{aligned} \tag{4.6}$$

For $v \in H_{\mathrm{sc}}^{*,l+1}(X)$, $t \in \mathbb{R}$,

$$\langle v, i[A^*A, P]v\rangle = 2\operatorname{Im}\langle v, A^*A(P - (\lambda + it))v\rangle - 2t\|Av\|^2. \tag{4.7}$$

Now if $u \in H_{\mathrm{sc}}^{*,l+1/2}(X)$, consider this for $v = u_r = \phi(x/r)u$ where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ is identically 1 on $[2,+\infty)$ and identically 0 on $[0,1]$. Writing $(P - (\lambda + it))u_r = \phi(x/r)(P-(\lambda+it))u + [P, \phi(x/r)]u$, observe that $A^*A[P, \phi(x/r)] \in \Psi_{\mathrm{sc}}^{-\infty,\infty}(X)$ is, for $r \in (0,1]$, uniformly bounded in $\Psi_{\mathrm{sc}}^{-\infty,-2l-1}(X)$, and indeed converges to 0 strongly as an operator $H_{\mathrm{sc}}^{*,l+1/2}(X) \longrightarrow H_{\mathrm{sc}}^{*,-l-1/2}(X)$, while $u_r \to u$ in $H_{\mathrm{sc}}^{*,l+1/2}(X)$. Taking the limit $r \to 0$, we deduce that

$$\langle u, i[A^*A, P]u\rangle = 2\operatorname{Im}\langle u, A^*A(P - (\lambda + it))u\rangle - 2t\|Au\|^2. \tag{4.8}$$

Combining this with (4.5), we conclude that (4.4) holds.

Now consider the uniform statement for the family. Suppose first that

$$\mathrm{WF}^{*,l+2}_{\mathrm{sc},L^\infty(G_t)}((P - \lambda + it)u_t) \cap \operatorname{supp} a = \emptyset.$$

Since $t > 0$, the second term on the left in (4.4) can be dropped. Since u_t is microlocally uniformly bounded in $H_{\mathrm{sc}}^{0,l}(X)$ on $\operatorname{supp} a$ and $f_t = (P - (\lambda + it))u_t$ is microlocally uniformly bounded in $H_{\mathrm{sc}}^{0,l+2}(X)$ by our assumption, it follows that the right hand side of (4.4) stays bounded as $t \to 0$. Thus, Bu_t is uniformly bounded in $L^2_{\mathrm{sc}}(X)$. This proves that $\operatorname{supp} b$ is disjoint from $\mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t)$.

In the general case, when we only assume that $\mathrm{WF}^{*,l+3/2}_{\mathrm{sc},L^\infty(G_t)}((P-\lambda+it)u_t) \cap \operatorname{supp} a = \emptyset$, we need to remove u from the last term of (4.4). To do so, we apply Cauchy-Schwarz to the last term and estimate it by

$$2\|x^{1/2}Au_t\|_2 \;\; \|x^{1/2}A(P-(\lambda+it))u_t\|_2 \leq \\ \epsilon\|x^{1/2}Au_t\|_2^2 + \epsilon^{-1}\|x^{-1/2}A(P-(\lambda+it))u_t\|_2^2,$$

for any $\epsilon > 0$. By the assumption $\mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \operatorname{supp} e = \emptyset$, we can find $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ such that $\mathrm{WF}'_{\mathrm{sc}}(Q) \cap \mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset$ (hence Qu_t is uniformly bounded in $H^{*,l+1/2}_{\mathrm{sc}}(X)$), while $\operatorname{supp} e \cap \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q) = \emptyset$, so $e = 0$ on $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$. By assumption (4.3), we have $a \leq Cb^2$ on $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$. Then we may estimate

$$\epsilon\langle x^{1/2}Au_t, x^{1/2}Au_t\rangle = \epsilon\Big(\langle x^{1/2}AQ^2u_t, x^{1/2}Au_t\rangle + \langle x^{1/2}A(\mathrm{Id}-Q^2)u_t, x^{1/2}Au_t\rangle\Big) \\ \leq \epsilon C'\Big(\|x^{1/2}AQu_t\|_2^2 + \|Bu_t\|_2^2 + |\langle \tilde F u_t, u_t\rangle|\Big), \tag{4.9}$$

for some $C' > 0$ and $\tilde F \in \Psi^{*,-2l}_{\mathrm{sc}}(X)$. Choose $\epsilon = (2C')^{-1}$. Then we may remove the $\|Bu_t\|_2^2$ term from the right hand side of (4.4) by replacing $\|Bu_t\|^2$ by $1/2\|Bu_t\|^2$ on the left hand side, obtaining

$$\frac{\|Bu_t\|^2}{2} + t\|Au_t\|^2 \\ \leq C''\Big(\|x^{1/2}AQu_t\|_2^2 + |\langle u_t, Fu_t\rangle| + \epsilon^{-1}\|x^{-1/2}A(P-(\lambda+it))u_t\|^2\Big). \tag{4.10}$$

The argument of the preceding paragraph now applies and completes the proof. □

Remark 4.2. The conclusions of Lemma 4.1 hold if we weaken the assumption that $u \in H^{*,l+1/2}_{\mathrm{sc}}(X)$ to $u \in H^{*,k}_{\mathrm{sc}}(X)$, for some $k \in \mathbb{R}$, and $\mathrm{WF}^{*,l+1/2}_{\mathrm{sc}}(u) \cap \operatorname{supp} a = \emptyset$. We may then apply operators $C \in \Psi^{*,-l-1/2}_{\mathrm{sc}}(X)$ with essential support contained in $\operatorname{supp} a$ to u. This is done by the prescription

$$Cu = CQu + C(\mathrm{Id}-Q)u,$$

where $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ is such that $\mathrm{WF}'_{\mathrm{sc}}(Q) \cap \mathrm{WF}'_{\mathrm{sc}}(C) = \emptyset$, and $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q) \cap \mathrm{WF}^{*,l+1/2}_{\mathrm{sc}}(u) = \emptyset$. This is independent of the choice of pseudodifferential operator Q. This observation allows the iterative use of the Lemma to gain regularity, as is done below.

If ν has a fixed sign, then so does the coefficient $-2\nu x$ of ∂_x in the Hamilton vector field of p. This enables us to prove the second part of the conclusion of the lemma above without requiring the existence of a t-dependent family u_t satisfying various conditions, i.e. we can work directly with $t = 0$.

Lemma 4.3. *Suppose that a, b, e, etc., are as in Lemma 4.1, and suppose that $\nu > 0$ on $\operatorname{supp} a$ and that $u \in H^{*,l}_{sc}(X)$. Then*

$$\mathrm{WF}^{*,l+3/2}_{\mathrm{sc}}((P-\lambda)u) \cap \operatorname{supp} a = \emptyset \textit{ and } \mathrm{WF}^{*,l+1/2}_{\mathrm{sc}}(u) \cap \operatorname{supp} e = \emptyset \\ \Rightarrow \mathrm{WF}^{*,l+1/2}_{\mathrm{sc}}(u) \subset \{b = 0\}.$$

The same conclusion holds if $a \geq 0$, $\nu < 0$ *on* $\operatorname{supp} a$ *and*

$$^{\mathrm{sc}}H_p a = x^{-l-1}\psi_0(p)b^2 + x^{-l-1}\psi_0(p)e,$$

with $a^{1/2}b$, b *and* $e \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$.

Remark 4.4. This theorem will be used to analyze functions whose wavefront set is concentrated near a radial point of ${}^{\mathrm{sc}}H_p$. Away from the radial points, the sign of ν is irrelevant and Theorem 3.1 tells us how the wavefront set of u propagates.

Proof. For $r > 0$ set $\chi_r = (1 + r/x)^{-1/2} = (\frac{r+x}{x})^{-1/2}$. Let A, B, E and F be as before, and let $A_r = A\chi_r$, $B_r = B\chi_r$ and $E = E\chi_r$. We repeat the arguments of the previous proposition with A_r in place of A. Now, ${}^{\mathrm{sc}}H_p\chi_r^2 = -2\nu(r/x)(1+r/x)^{-2} = -c_r^2$ is negative in $\nu > 0$ and so on $\operatorname{supp} a$. Thus, for $r > 0$,

$$ix^{l+1/2}[A_r^*A_r, P]x^{l+1/2} = x^{l+1/2}(B_r^*B_r + A^*C_r^*C_rA + E_r + F_r)x^{l+1/2}, \tag{4.11}$$

where C_r is the operator multiplication by c_r, $F_r \in \Psi_{\mathrm{sc}}^{-\infty,-2l+1}(X)$ for $r > 0$, is uniformly bounded in $\Psi_{\mathrm{sc}}^{-\infty,-2l}(X)$ as $r \to 0$ and $\mathrm{WF}'_{\mathrm{sc}}(F_r) \subset \operatorname{supp}(x^{l+1}a)$ uniformly as $r \to 0$.

Note that $A_r \in \Psi_{\mathrm{sc}}^{-l-1/2}(X)$ for $r > 0$, and it is uniformly bounded in $\Psi_{\mathrm{sc}}^{-l-1}(X)$ as $r \to 0$. Since $u \in H_{\mathrm{sc}}^{*,l}(X)$ by our assumption, the proof of (4.8) (with l replaced by $l + 1/2$) is applicable. Combining this with (4.11), we deduce that for $r > 0$

$$\|B_ru\|^2 + \|C_rAu\|^2 \leq |\langle u, F_ru\rangle| + |\langle u, E_ru\rangle| + 2|\langle u, A_r^*A_r(P-\lambda)u\rangle|. \tag{4.12}$$

We now drop the term $\|C_rAu\|^2$, and apply Cauchy-Schwarz to the last term as in the previous proof, obtaining

$$\frac{\|B_ru\|^2}{2} \leq C''\Big(|\langle E_ru, u\rangle| + |\langle u, \tilde{F}_ru\rangle| + \epsilon^{-1}\|x^{-1/2}A_r(P-\lambda)u\|^2\Big). \tag{4.13}$$

The right hand side of (4.13) remains uniformly bounded as $r \to 0$, so we conclude that $Bu \in H_{\mathrm{sc}}^{0,0}(X)$, which implies the conclusion of the Lemma. □

For basic spectral and scattering theory, partially microlocal estimates suffice, i.e. one does not need full microlocal results. These partially microlocal results (namely, microlocal in the ν variable, global in y and μ) are closely related to the two-body resolvent estimates of Isozaki and Kitada [11], and the corresponding many-body estimates of Gérard, Isozaki and Skibsted [4].

Proposition 4.5. *Suppose that, for* $t \in G \subset (0,1]$, *some* $l > -1$, $\nu_0 < 0$ *and* $r > -1/2$, *the function* $(P - (\lambda + it))u_t = f_t$ *satisfies*

$$\mathrm{WF}^{*,l}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu < \nu_0\} = \emptyset \text{ and}$$
$$\mathrm{WF}^{*,r+1}_{\mathrm{sc},L^\infty(G_t)}(f_t) \cap \{\nu < \nu_0\} = \emptyset,$$

then $\mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu < \nu_0\} = \emptyset$.

Proof. The key point is that ${}^{\mathrm{sc}}H_p$ is radial at $\mathrm{RP}(\lambda) = \mathrm{RP}_+(\lambda) \cup \mathrm{RP}_-(\lambda)$. That is, the tangential component W vanishes there, so we need to exploit the normal component, which is $-2\nu x\partial_x$. We obtain a positive commutator by using a weight x^{-l-1}, since $-l-1 < 0$. Fix $\epsilon > 0$, $\delta > 0$, $M > 0$, and let

$$a = x^{-l-1}\chi(\nu)\tilde{\psi}(x)\psi_0(p) \geq 0 \tag{4.14}$$

where $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0 and is supported in a bigger neighbourhood of 0 (it is simply a cutoff near ∂X), $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ supported in $(\lambda - \delta, \lambda + \delta)$,

$\chi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ is identically one on $(-\infty, \nu_0 - 3\epsilon)$, vanishes on $(\nu_0 - \epsilon, \infty)$, $\chi' \le 0$, and $\chi(t) = e^{M/(t-(\nu_0-\epsilon))}$ on $(\nu_0 - 2\epsilon, \nu_0 - \epsilon)$. Then

$$
{}^{\mathrm{sc}}H_p a = 2((l+1)\nu\chi(\nu) + |\mu|_y^2 \chi'(\nu)) x^{-l-1} \psi_0(p) = -b^2 x^{-l-1}, \tag{4.15}
$$

and b is $\mathcal{C}^\infty$ by the construction of χ. Thus, if $r < l + 3/2$, then Lemma 4.1 gives us the result (with ν_0 replaced by $\nu_0 - \epsilon$). Otherwise, from the Lemma we gain a power of $x^{1/2}$ in the wavefront set estimate for u_t, i.e. , we have

$$
\mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu < \nu_0 - \epsilon\} = \emptyset.
$$

We may now repeat the argument with l replaced by $l+1/2$. Iterating the argument a finite number of times (cf. the remark after the proof of Lemma 4.1), then sending $\epsilon \to 0$, we obtain our estimate. □

A similar argument works in $\nu \ge 0$, but this time we only need to assume boundedness in $H^{*,l}_{\mathrm{sc}}(X)$ for an arbitrary $l \in \mathbb{R}$ in this region, i.e. we do not need $l \in (-1, -1/2)$. On the other hand, we are only able to deduce boundedness in $H^{*,r}_{\mathrm{sc}}(X)$ for $r < -1/2$. In addition, we need to assume that the desired boundedness, i.e. in $H^{*,r}_{\mathrm{sc}}(X)$ for $r < -1/2$, holds for ν near $\nu_0 > 0$.

Proposition 4.6. *Suppose $f_t = (P - (\lambda + it))u_t$, where $t \in G \subset (0,1]$, $\nu_0 > 0$, $l \in \mathbb{R}$ and $r < -1/2$. Then*

$$
\begin{aligned}
\mathrm{WF}^{*,l}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu \ge \nu_0\} = \emptyset,\ \mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu = \nu_0\} &= \emptyset \\
\text{and } \mathrm{WF}^{*,r+1}_{\mathrm{sc},L^\infty(G_t)}(f_t) \cap \{\nu \ge \nu_0\} &= \emptyset \\
\Rightarrow \mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\nu \ge \nu_0\} &= \emptyset.
\end{aligned}
$$

Proof. The proof is nearly identical to that of Proposition 4.5. We may assume $l < -1$ and $l < r$ without loss of generality. Also, the estimate is clear away from $\Sigma(\lambda)$ by elliptic regularity, so we may restrict our attention to a small neighbourhood O of $\Sigma(\lambda)$ whose closure $\overline{O}$ is compact. Take a as in (4.14), where now $\chi' \ge 0$, χ vanishes on $(-\infty, \nu_0 - \epsilon]$, is identically 1 on $[\nu_0 + \epsilon, +\infty)$, and given by $e^{M/(\nu-(\nu_0+\epsilon))}$ on $(\nu_0, \nu_0+\epsilon)$. We assume that $\epsilon > 0$ is so small that $\mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \{\chi' \ne 0\} \cap \overline{O} = \emptyset$ and $\mathrm{WF}^{*,r+1}_{\mathrm{sc},L^\infty(G_t)}(f_t) \cap \{\chi \ne 0\} \cap \overline{O} = \emptyset$. Such a choice is possible since the wave front set is closed and $\overline{O}$ is compact. Now $l+1 < 0$ and $\nu > 0$, so the sign of the first term in (4.15) is unchanged, while the sign of the second term is reversed. Hence, the second term becomes an error term, e, which satisfies (4.3). By construction we have $\mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) \cap \operatorname{supp} e = \emptyset$. Thus, we can apply Lemma 4.1, following the argument of the previous Proposition, to complete the proof. □

Versions of these results with no parameter t can also be obtained. In this case, there is no distinction between $\nu > 0$ and $\nu < 0$.

Proposition 4.7. *Let $\nu_0 < 0$ and $l > -1/2$. Suppose that $Pu = f$, where*

$$
\mathrm{WF}^{*,l}_{\mathrm{sc}}(u) \cap \{\nu \le \nu_0\} = \emptyset \text{ and } \mathrm{WF}^{*,r+1}_{\mathrm{sc}}(f) \cap \{\nu \le \nu_0\} = \emptyset.
$$

Then

$$
\mathrm{WF}^{*,r}_{\mathrm{sc}}(u) \cap \{\nu \le \nu_0\} = \emptyset.
$$

A similar result holds with $\nu_0 > 0$ with all inequalities $\nu \le \nu_0$ replaced by $\nu \ge \nu_0$.

Proof. The proof uses the symbol a from Proposition 4.5, and proceeds via a regularization argument similar to Lemma 4.3. Now, however, ${}^{\mathrm{sc}}H_p a$ and ${}^{\mathrm{sc}}H_p \chi_r^2$ have opposite signs, so the two terms cannot be treated separately. Instead, we use the argument of the lemma with a replaced by $x^{1/2}a$ and $\chi_r = (1+r/x)^{-1/2}$ replaced by $x^{-1/2}\chi_r = (x+r)^{-1/2}$. Now ${}^{\mathrm{sc}}H_p(x^{1/2}a)$ and ${}^{\mathrm{sc}}H_p(x\chi_r^2)$ have the same sign, so the proof of Lemma 4.3 is applicable, and completes the proof of the proposition. Note that the power of the factor of x in $x^{1/2}a$ is $x^{-l-1/2}$; this is where we use that $l > -1/2$. □

The combination of Theorem 3.1 and Proposition 4.7 leads to

Proposition 4.8. *Suppose that $l > -1/2$, $\lambda_0 \notin \mathrm{Cv}(V)$. Then there exist operators $F_j \in \Psi_{\mathrm{sc}}^{-1,1}(X)$, $j = 1, \dots, N$, and constants $C > 0$, $\delta > 0$, such that for $u \in H_{sc}^{m,l}(X)$, $(P-\lambda)u \in H_{sc}^{m,l+1}(X)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$,*

$$\|u\|_{H_{sc}^{m,l}(X)} \leq \sum_{j=1}^{N} \|F_j u\|_{H_{sc}^{m,l}(X)} + C\|(P-\lambda)u\|_{H_{sc}^{m,l+1}(X)}.$$

Proof. Choose a small $\nu_0 > 0$ so that the Hamiltonian flow never vanishes for $|\nu| \leq 2\nu_0$, on the chacteristic surfaces $\Sigma(\lambda)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. We then decompose the identity operator in the form $\mathrm{Id} = Q_0 + Q_1 + B_+ + B_-$, where all operators are in $\Psi_{\mathrm{sc}}^{0,0}(X)$, and are microlocalized as follows: Q_0 is microsupported in $|\nu| \leq 2\nu_0$, $B_\pm$ are of the form that arise in the proof of Proposition 4.7, thus microsupported in $\{\pm\nu > \nu_0\}$, and Q_1 is microsupported away from the characteristic sets $\Sigma(\lambda)$. Since the Hamilton flow does not vanish on the microsupport of Q_0, we may write Q_0 as a commutator with $P - \lambda$, modulo lower order terms, and hence obtain an estimate

$$\|Q_0 u\|^2_{H_{\mathrm{sc}}^{m,l}(X)} \leq C\Big(\|(P-\lambda)u\|^2_{H_{\mathrm{sc}}^{m,l+1}(X)} + |\langle Eu, u\rangle|\Big),$$

where $E \in \Psi_{\mathrm{sc}}^{2m,-2l}(X)$ satisfies $\mathrm{WF}'_{\mathrm{sc}}(E) \subset \{|\nu| \geq \nu_0\}$. We add to this a sufficiently large multiple of the estimates for $\|B_\pm u\|_{H_{\mathrm{sc}}^{m,l}(X)}$, obtained from the proof of Proposition 4.7, and the estimate for $\|Q_1 u\|_{H_{\mathrm{sc}}^{m,l}(X)}$ obtained from ellipticity. This allows one to absorb the error term $|\langle Eu, u\rangle|$ above. □

This immediately yields the usual results on the spectrum of P. For the last statement we also need a unique continuation result.

Proposition 4.9. *(See Theorem 3.2) For $\lambda \notin \mathrm{Cv}(V)$ the space W of L^2 eigenfunctions with eigenvalue λ is a finite dimensional subspace of $\dot{C}^\infty(X)$. The pure point spectrum, $\sigma_{pp}(P)$, is disjoint from* $(\sup \mathrm{Cv}(V), +\infty)$ *and can only accumulate at* $\mathrm{Cv}(V)$.

Proof. Let F_k be as in Proposition 4.8. Since these operators are compact, there exists a finite dimensional subspace W_0 of W such that for $u \in W$ orthogonal to W_0, $\|F_k u\|_{H_{\mathrm{sc}}^{0,0}(X)} \leq \|u\|_{H_{\mathrm{sc}}^{0,0}(X)}/2N$, N as in Proposition 4.8. Suppose that $u \in W$, $\|u\|_{H_{\mathrm{sc}}^{0,0}(X)} = 1$. Applying the proposition with $\lambda_0 = \lambda$ we deduce that

$$1 = \|u\|_{H_{\mathrm{sc}}^{0,0}(X)} \leq \sum_{k=1}^{N} \|F_k u\|_{H_{\mathrm{sc}}^{0,0}(X)} \leq 1/2,$$

which is a contradiction. Hence $W = W_0$, so W is finite dimensional.

More generally, let $\tilde{W}$ be the sum of the L^2 eigenspaces of P with eigenvalue $\tilde{\lambda} \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2]$. By compactness of the F_k, there is a finite dimensional subspace $\tilde{W}_0$ of $\tilde{W}$ such that for $u \in \tilde{W}$ orthogonal to $\tilde{W}_0$, $\|F_k u\|_{H^{0,0}_{sc}(X)} \leq \|u\|_{H^{0,0}_{sc}(X)}/2N$. Again applying Proposition 4.8 with $\tilde{\lambda}$ in place of λ in the last term, $\|(P - \tilde{\lambda})u\|_{H^{m,l+1}_{sc}(X)} = 0$, shows that $\tilde{W} = \tilde{W}_0$, so $\tilde{W}$ is also finite dimensional.

That an L^2 eigenfunction u is in $\dot{\mathcal{C}}^\infty(X)$ follows from the following Proposition. The statement that $\sigma_{\mathrm{pp}}(P)$ is disjoint from $(\sup \mathrm{Cv}(V), +\infty)$ follows from the unique continuation theorem of Froese and Herbst [3], as generalized to scattering metrics in [17]. □

Proposition 4.10. *(Theorem 3.2) Suppose that $u \in \mathcal{C}^{-\infty}(X)$, $(P - \lambda)u = 0$, $\mathrm{WF}^{m,l}_{sc}(u) \cap \mathrm{RP}_-(\lambda) = \emptyset$ for some $l > -1/2$. Then $u \in \dot{\mathcal{C}}^\infty(X)$.*

Proof. This is essentially Isozaki's uniqueness theorem from [10], Lemma 4.5, proved the same way in this setting. From Theorem 3.1 and Proposition 4.7, we have

$$\mathrm{WF}_{\mathrm{sc}}(u) \subset \{\nu \geq a(\lambda) > 0\} \text{ and } u \in H^{m',l'}_{\mathrm{sc}}(X)\ \forall\ m' \in \mathbb{R},\ l' < -1/2, \tag{4.16}$$

where $a(\lambda)$ is as in Remark 1.11. Let $l \in (-1/2, 0)$ and let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, 1]$ and 1 on $[2, \infty)$. For $r > 0$ let

$$\chi_r(x) = r^{-2l-1} \int_0^{x/r} \phi^2(s) s^{-2l-2}\, ds. \tag{4.17}$$

Thus, $\chi_r \in \mathcal{C}^\infty_c(\mathrm{int}(X))$ and

$$x^2 \partial_x \chi_r(x) = x^{-2l} \phi^2(x/r). \tag{4.18}$$

Note that χ_r is not uniformly bounded in $\Psi^{m',l'}_{\mathrm{sc}}(X)$ for any m' and l', but $x^2\partial_x \chi_r$ is. Since χ_r enters the commutator $[\chi_r(x), P]$ only via $x^2 \partial_x \chi_r$, that boundedness will suffice for us. Now, by [13, Equation 3.7]

$$\Delta = (x^2 D_x)^2 + i(n-1)x^3 D_x + x^2 \Delta_h + x^3\, \mathrm{Diff}^2_{\mathrm{b}}(X), \tag{4.19}$$

so

$$-i[\chi_r(x), P] = 2x^{-2l}\phi^2(x/r)(x^2 D_x) + F'_r \tag{4.20}$$

where F'_r is bounded in $\Psi^{1,-2l+1}_{\mathrm{scc}}(X)$. Let $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ be supported close to λ and be identically 1 near λ. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, a(\lambda)/4)$ and be 1 on $(a(\lambda)/2, \infty)$. Let $B \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ and $E \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ have symbols $\sigma_{\partial,0}(B) = \sqrt{\nu}\rho(\nu)\psi(p) \in \mathcal{C}^\infty$ and $\sigma_{\partial,0}(E) = \nu(1 - \rho^2(\nu))\psi^2(p)$. Then

$$-i[\chi_r(x)\psi^2(P), P] = 2x^{-l}\phi(x/r)(B^2 + E)\phi(x/r)x^{-l} + F_r \tag{4.21}$$

where $\mathrm{WF}'_{\mathrm{sc}}(E)$ is disjoint from $\{\nu \geq a(\lambda)/2\}$ and F_r is bounded in $\Psi^{-\infty,-2l+1}_{\mathrm{sc}}(X)$. Now, for $r > 0$

$$\langle u, [\chi_r(x), P]u\rangle = -2i\, \mathrm{Im}\langle u, \chi_r(x)(P - \lambda)u\rangle = 0. \tag{4.22}$$

Hence,

$$\|x^{-l}\phi(x/r)Bu\|^2 \leq |\langle x^{-l}\phi(x/r)u, Ex^{-l}\phi(x/r)u\rangle| + |\langle u, F_r u\rangle|. \tag{4.23}$$

In view of (4.16) the right hand side stays bounded as $r \to 0$, so we conclude that $x^{-l}Bu \in L^2_{\mathrm{sc}}(X)$. Then, by (4.16), it follows that $u \in H^{\infty,l}_{\mathrm{sc}}(X)$. Since $l > -1/2$, we conclude from Theorem 3.1 and Proposition 4.7 that $u \in \dot{\mathcal{C}}^\infty(X)$. □

As mentioned above, $\chi_r(x)$ is not bounded in $\Psi_{\mathrm{sc}}^{m',l'}(X)$ for any m' and l', so the place where we have used the assumption $(P-\lambda)u=0$ is really in the elimination of the term on the right hand side of (4.22) from the right hand side of (4.23).

Theorem 4.11. *(Theorem 3.3) The resolvent $R(\lambda+it)$, $t>0$, $\lambda\notin\Lambda=\mathrm{Cv}(V)\cup\sigma_{pp}(P)$ extends continuously to the real axis, i.e. $R(\lambda+i0)$ exists, as a bounded operator $H_{sc}^{m,r}(X)\longrightarrow H_{sc}^{m+2,l}(X)$ for any $r>1/2$ and $l<-1/2$.*

Proof. Let $f\in H_{\mathrm{sc}}^{m,r}(X)$, for $r>1/2$, let $-1<l<-1/2$, and suppose that $u_t=R(\lambda+it)f$ is not bounded in $H_{\mathrm{sc}}^{m+2,l}(X)$ as $t\to 0$. We can then take a sequence $t_j\to 0$ with $\|u_{t_j}\|_{H_{\mathrm{sc}}^{m+2,l}(X)}\to\infty$. Now consider $v_{t_j}=u_{t_j}/\|u_{t_j}\|_{H_{\mathrm{sc}}^{m+2,l}(X)}$, so v_{t_j} remains bounded in $H_{\mathrm{sc}}^{m+2,l}(X)$, and $(P-(\lambda+it))v_{t_j}\to 0$ in $H_{\mathrm{sc}}^{m,r}(X)$. We may then pass to a convergent subsequence in $H_{\mathrm{sc}}^{m+2-\delta,l-\delta}(X)$, $\delta>0$, with limit v. Thus $(P-\lambda)v=0$, and by Proposition 4.5, $\mathrm{RP}_-(\lambda)\cap\mathrm{WF}^{*,l'}_{\mathrm{sc},L^\infty(G_t)}(v_t)=\emptyset$ for some $l'>-1/2$. By the t-dependent part of Theorem 3.1 on the propagation of singularities $\mathrm{WF}^{*,l'}_{\mathrm{sc},L^\infty(G_t)}(v_t)\subset\{\nu>0\}$, hence the same holds for v. The uniqueness result, Theorem 3.2, then shows that $v=0$. However this contradicts the fact that $\|v_{t_j}\|_{H_{\mathrm{sc}}^{m+2,l'}(X)}=1$. Hence, there exists $C>0$ such that $\|u_t\|_{H_{\mathrm{sc}}^{m+2,l}(X)}\le C$ for all t. Then, by Proposition 4.5 and the t-dependent part of Theorem 3.1, $\mathrm{WF}^{*,l'}_{\mathrm{sc},L^\infty(G_t)}(u_t)\subset\{\nu>0\}$.

Now suppose that $t_j\to 0$. By decreasing $m+2$ and l, we can pass to a convergent subsequence with limit u, which then satisfies $(P-\lambda)u=f$, and $\mathrm{WF}^{*,l'}_{\mathrm{sc}}(u)\subset\{\nu>0\}$. Taking any other sequence t'_j and any convergent subsequence, we obtain another distribution u' with the same properties, hence by considering their difference and using Theorem 3.2, $u=u'$. If now t''_j is a sequence such that $t''_j\to 0$ and $\|u_{t''_j}-u\|_{H_{\mathrm{sc}}^{m+2,l}(X)}\ge\epsilon>0$, then taking a subsequence of t''_j with $u_{t''_j}$ converging in $H_{\mathrm{sc}}^{m+2-\delta,l-\delta}(X)$ to some u'', we deduce that $u=u''$, which is a contradiction. This shows convergence of $R(\lambda+it)f$ as $t\to 0$, and similar arguments easily give boundedness as an operator, and continuity in λ as well. □

We next refine this theorem by combining it with the propagation estimates. To do so we need a microlocal version of Proposition 4.5, which we state afterwards.

Theorem 4.12. *(Theorem 3.4) If $\lambda\notin\Lambda$ and $f\in\dot{\mathcal{C}}^\infty(X)$ then $\mathrm{WF}_{sc}(R(\lambda+i0)f)\subset\Phi_+(\mathrm{RP}_+(\lambda))$. Moreover $R(\lambda+i0)$ extends by continuity to $v\in\mathcal{C}^{-\infty}(X)$ with $\mathrm{WF}_{sc}(v)\cap\Phi_-(\mathrm{RP}_-(\lambda))=\emptyset$, and for such v, $R(\lambda+i0)v$ satisfies (3.7).*

Proof. Let $u=R(\lambda+i0)f$. It follows from Proposition 4.5 that $\mathrm{WF}_{\mathrm{sc}}(u)\subset\{\nu\ge 0\}$. Then Theorem 3.1 shows that $\mathrm{WF}_{\mathrm{sc}}(u)\subset\Phi_+(\mathrm{RP}_+(\lambda))$. Indeed, given $q\in\Sigma(\lambda)$, let γ be the bicharacteristic through q. Then $q_0=\lim_{t\to-\infty}\gamma(t)\in\mathrm{RP}_+(\lambda)\cup\mathrm{RP}_-(\lambda)$. If $q_0\in\mathrm{RP}_+(\lambda)$, there is nothing to prove, since then $q\in\Phi_+(\mathrm{RP}_+(\lambda))$. If $q_0\in\mathrm{RP}_-(\lambda)$, there exist points on γ where ν is negative, hence these are not in $\mathrm{WF}_{\mathrm{sc}}(u)$. But then by Theorem 3.1, $q\notin\mathrm{WF}_{\mathrm{sc}}(u)$.

A duality argument immediately gives the extension of $R(\lambda+i0)$ to the class of distributions described above. The wave front set bound is a consequence of Theorem 3.1 and the following fully microlocal family version of Proposition 4.5, stated below. □

Proposition 4.13. *Suppose that $q\in\mathrm{RP}_-(\lambda)$, $f_t=(P-(\lambda+it))u_t$, for $t\in G\subset(0,1]$, and O is a W-balanced neighbourhood (see Definition 1.7) of q. Then for*

$r > -1/2$, and $l > -1$,

$$O \cap \mathrm{WF}^{*,r+1}_{\mathrm{sc},L^\infty(G_t)}(f_t) = \emptyset,\ O \cap \mathrm{WF}^{*,l}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset \text{ and } (O \setminus \Phi_+(\{q\})) \cap \mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset \Rightarrow O \cap \mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset.$$

Proof. Let $q = (y_0, \nu_0, 0)$, $\nu_0 < 0$, in our local coordinates. Due to our assumptions, and Theorem 3.1, we only need to make a commutator positive on $\Phi_+(\{q\})$.

For $l > -1$ consider a symbol a of the form

$$a = x^{-l-1}\chi(\nu)\phi(y,\mu)\tilde{\psi}(x)\psi_0^2(p) \geq 0 \tag{4.24}$$

where $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0 and is supported in a bigger neighbourhood of 0 (it is simply a cutoff near ∂X), $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ supported in $(\lambda-\delta,\lambda+\delta)$, $\phi \geq 0$ is $\mathcal{C}^\infty$, supported near $(y_0, 0)$, identically 1 in a neighbourhood of this point, $\chi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ supported in $[\nu_0-\epsilon, \nu_0+\epsilon]$, identically 1 near ν_0, $\chi' \leq 0$ on $[\nu_0 - \epsilon/2, \nu_0+\epsilon]$, $\chi' \geq 0$ on $[\nu_0-\epsilon, \nu_0-\epsilon/2]$ and $\chi(\nu) = e^{M/(\nu-(\nu_0+\epsilon))}$ on $(\nu_0+\epsilon/2, \nu_0+\epsilon)$. In addition, we choose $\epsilon > 0$ so that $[\nu_0 - 2\epsilon, \nu_0+2\epsilon] \times \operatorname{supp} d\phi \subset O \setminus \Phi_+(\{q\})$. This can be achieved by fixing ϕ and making $\epsilon > 0$ small since $\{\nu_0\} \times \operatorname{supp} d\phi$ is disjoint from the compact set $\Phi_+(\{q\}) \cap \overline{O}$.

Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 0 on $(-\infty, \nu_0 - 3\epsilon/8)$, identically 1 on $(\nu_0 - \epsilon/4, +\infty)$. Notice that $\chi' \leq 0$ on the support of ρ. Then, near $x = 0$,

$$\begin{aligned}
{}^{\mathrm{sc}}H_p a &= 2((l+1)\nu\chi(\nu) + |\mu|_y^2\chi'(\nu))\phi(y,\mu)x^{-l-1}\psi_0^2(p) + x^{-l-1}\chi(\nu)\psi_0^2(p)\ {}^{\mathrm{sc}}H_p\phi \\
&= -b^2x^{-l-1} + ex^{-l-1}, \\
b^2 &= \Big(-2\nu(l+1)\chi(\nu) - 2\rho^2(\nu)|\mu|_y^2\chi'(\nu)\Big)\phi(y,\mu)\psi_0^2(p), \\
e &= \Big(2(1-\rho^2(\nu))|\mu|_y^2\chi'(\nu))\phi(y,\mu) + \chi(\nu)\ {}^{\mathrm{sc}}H_p\phi\Big)\psi_0^2(p),
\end{aligned} \tag{4.25}$$

and b is $\mathcal{C}^\infty$ by the construction of χ. Note that $\operatorname{supp} e$ is disjoint from $\mathrm{WF}^{*,r}_{\mathrm{sc}}(u)$, and a, b, e satisfy (4.3). Thus, by Lemma 4.1, we gain half a power of x in our regularity of u_t near q, i.e. , we get $\mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset$ near q. By Theorem 3.1 this gives $O \cap \mathrm{WF}^{*,l+1/2}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset$. Iterating the argument, we get $O \cap \mathrm{WF}^{*,r}_{\mathrm{sc},L^\infty(G_t)}(u_t) = \emptyset$ after a finite number of steps. □

5. Microlocal eigenfunctions

With each zero q of the vector field W, when $\lambda > V(\pi(q))$, we associate spaces of microlocally incoming, respectively outgoing eigenfunctions. These are microfunctions near q with respect to the scattering wavefront set, satisfy the eigenfunction equation microlocally near q, and have wave front set in $\nu < \nu(q)$, resp. $\nu > \nu(q)$.

For a W-balanced neighbourhood $O \ni q$, as in Definition 1.7, the spaces of incoming and outgoing microlocal eigenfunctions, $\tilde{E}_{\mathrm{mic},\pm}(q,\lambda)$, are defined in (I.15). The propagation of singularities in regions of real principal type (i.e. , where $W \neq 0$) shows that stronger restrictions on $\mathrm{WF}_{\mathrm{sc}}(u)$ follow directly.

Lemma 5.1. *If $O \ni q$ is, for $q \in \mathrm{RP}(\lambda)$, a W-balanced neighbourhood then for every $u \in \tilde{E}_{\mathrm{mic},\pm}(O,\lambda)$*

$$\mathrm{WF}_{sc}(u) \cap O \subset \Phi_\pm(\{q\}) \text{ and}$$
$$\mathrm{WF}_{sc}(u) \cap O = \emptyset \Longleftrightarrow q \notin \mathrm{WF}_{sc}(u).$$

Thus, we could have defined $\tilde{E}_{\mathrm{mic},\pm}(O,\lambda)$ by strengthening the restriction on the wavefront set to $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \Phi_\pm(\{q\})$. With such a definition there is no need for O to be W-balanced; the only relevant bicharacteristics would be those contained in $\Phi_\pm(\{q\})$.

Proof. We may assume that $q \in \mathrm{RP}_+(\lambda)$. For the sake of definiteness also assume that $u \in \tilde{E}_{\mathrm{mic},+}(O,\lambda)$; the other case follows similarly. Consider $\zeta \in O \setminus \{q\}$. If $\nu(\zeta) < \nu(q)$, then $\zeta \notin \mathrm{WF}_{\mathrm{sc}}(u)$ by the definition of $\tilde{E}_{\mathrm{mic},+}(O,\lambda)$, so we may suppose that $\nu(\zeta) \geq \nu(q)$. Since $q \in \Phi_+(\{q\})$ we may also suppose that $\zeta \neq q$.

Let $\gamma : \mathbb{R} \to \Sigma(\lambda)$ be the bicharacteristic through ζ with $\gamma(0) = \zeta$. As O is W-convex, and $\mathrm{WF}_{\mathrm{sc}}((P-\lambda)u) \cap O = \emptyset$, Theorem 3.1 shows that

$$\zeta \in \mathrm{WF}_{\mathrm{sc}}(u) \Rightarrow \gamma(\mathbb{R}) \cap O \subset \mathrm{WF}_{\mathrm{sc}}(u).$$

As O is W-balanced, there exists $\zeta' \in \overline{\gamma(\mathbb{R})} \cap O$ such that $\nu(\zeta') = \nu(q)$.

If $\zeta' = \gamma(t_0)$ for some $t_0 \in \mathbb{R}$, then for $t < t_0$, $\nu(\gamma(t)) < \nu(\gamma(t_0)) = \nu(q)$, and for sufficiently small $|t - t_0|$, $\gamma(t) \in O$ as O is open. Thus, $\gamma(t) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ by the definition of $\tilde{E}_{\mathrm{mic},+}(O,\lambda)$, and hence we deduce that $\zeta \notin \mathrm{WF}_{\mathrm{sc}}(u)$.

On the other hand, if $\zeta' = \lim_{t\to-\infty} \gamma(t)$, then $\zeta' \in \mathrm{RP}(\lambda)$, and $\zeta' \in O$, so the fact that O is W-balanced shows that $\zeta' = q$, hence $\zeta \in \Phi_+(\{q\})$.

The last statement of the lemma follows from Theorem 3.1, and the fact that $\mathrm{WF}_{\mathrm{sc}}(u)$ is closed. □

Corollary 5.2. *Let O be a W-balanced neighbourhood of $q \in \mathrm{RP}_+(\lambda)$. If $\pi(q)$ is a minimum of $V|_Y$ and $u \in \tilde{E}_{\mathrm{mic},+}(O,\lambda)$ then $\mathrm{WF}_{sc}(u) \cap O \subset \{q\}$. If $\pi(q)$ is a maximum of $V|_Y$ and $u \in \tilde{E}_{\mathrm{mic},\pm}(O,\lambda)$ then $\mathrm{WF}_{sc}(u) \cap O \subset L_\pm$. Here L_+ is that one of the Legendrians L_1, L_2 from Proposition 1.5 which satisfies $L_+ \subset \{\nu \geq \nu(q)\}$, and L_- is the other one of these.*

To avoid truly microlocal arguments in the analysis of these spaces, such as microlocal solvability along the lines of Hörmander [8], we introduce, as a technical device, an operator $\tilde{P}$ which arises from P by altering V appropriately, and use global analysis of $\tilde{P}$.

Lemma 5.3. *Fix $\lambda > \min V_0$, and $\tilde{\nu} > 0$, and set $K = {V_0}^{-1}((-\infty, \lambda - \tilde{\nu}^2])$. There exists $\tilde{V} \in \mathcal{C}^\infty(X)$ with $\tilde{V}_0$ Morse such that the minima of V_0 and $\tilde{V}_0$ coincide, V and $\tilde{V}$ themselves coincide in a neighbourhood of their minima, $\tilde{V}_0 \geq V_0$, $\tilde{V}|_K = V|_K$ and no maximum value of $\tilde{V}$ lies in the interval $(\lambda - \tilde{\nu}^2, \lambda)$.*

Let $\tilde{P} = \Delta + \tilde{V}$ be the corresponding Hamiltonian, $\tilde{\Sigma}(\lambda)$ its characteristic variety at eigenvalue λ and $\widetilde{\mathrm{RP}_+}(\lambda)$ the set of its radial points in $\nu > 0$. If $q \in \widetilde{\mathrm{RP}_+}(\lambda)$ with $\nu(q) < \tilde{\nu}$, then $\pi(q)$ is a minimum of $\tilde{V}$ and the full symbols of P and $\tilde{P}$ coincide in a neighbourhood of q; in particular,

$$\Sigma(\lambda) \cap \{\nu \geq \tilde{\nu}\} = \tilde{\Sigma}(\lambda) \cap \{\nu \geq \tilde{\nu}\}. \tag{5.1}$$

Proof. It suffices to modify V in small neighbourhood U_z of each maximum z of V_0 with value in the range $(\lambda - \tilde{\nu}^2, \lambda]$ (if there are any). We do this so that the new potential has $\tilde{V} > \lambda - \tilde{\nu}^2$ on these sets, has only one critical point in each and at that maximum takes a value greater than λ.

Now $q \in \widetilde{\mathrm{RP}_+}(\lambda)$ implies that $z = \pi(q)$ is a critical point of $\tilde{V_0}$ and $\nu(q)^2 + \tilde{V}(z) = \lambda$. Suppose that z is a maximum of $\tilde{V}$ and $0 < \nu(q) < \tilde{\nu}$. Then $\nu(q)^2 + \tilde{V}(z) = \lambda$ shows that $\lambda - \tilde{\nu}^2 < \tilde{V}(z) < \lambda$, which is contradicted by the construction of $\tilde{V}$. Thus, z is a minimum of $\tilde{V}$.

Next, on $\Sigma(\lambda) \cap \{\nu \geq \tilde{\nu}\}$, $\nu^2 + |\mu|_y^2 + V_0 = \lambda$, hence $V_0 \leq \lambda - \tilde{\nu}^2$, so $V_0 = \tilde{V_0}$, and therefore $\Sigma(\lambda) \cap \{\nu \geq \tilde{\nu}\} \subset \tilde{\Sigma}(\lambda)$. With the converse direction proved similarly, (5.1) follows. □

Remark 5.4. The point of this Lemma is that it allows one to assume, in any argument concerning $q \in \mathrm{RP}_+(\lambda)$, that there is no $q' \in \mathrm{Max}_+(\lambda)$ with $\nu(q') < \nu(q)$. The virtue of this is illustrated in the proof of the following continuation result.

Lemma 5.5. *Suppose $u \in \mathcal{C}^{-\infty}(X)$ satisfies*

$$\mathrm{WF}_{sc}(u) \subset \{\nu \geq \nu_1\} \text{ and } \mathrm{WF}_{sc}((P-\lambda)u) \subset \{\nu \geq \nu_2\},$$

for some $0 < \nu_1 < \nu_2$, then there exists $\tilde{u} \in \mathcal{C}^{-\infty}(X)$ with $\mathrm{WF}_{sc}(u - \tilde{u}) \subset \{\nu \geq \nu_2\}$ and $(P - \lambda)\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$.

Proof. One is tempted to try to solve this by adding $R(\lambda + i0)((P - \lambda)u)$ to u. This does not quite work, however, since the wavefront set of this is contained in $\mathrm{WF}_{\mathrm{sc}}((P - \lambda)u) \cup \Phi_+(\mathrm{WF}_{\mathrm{sc}}((P - \lambda)u \cap \Sigma(\lambda))) \cup \Phi_+(\mathrm{RP}_+(\lambda))$, and only the first two sets are contained in $\{\nu \geq \nu_2\}$. The third is contained in the set $\mathrm{Min}_+(\lambda)$, together with the flowouts along L_2 from $\mathrm{Max}_+(\lambda)$. The flowouts are a nuisance here because they 'bump into' the rest of the wavefront set at $\{\nu \geq \nu_2\}$. To avoid dealing with this problem, we use the operator $\tilde{P}$ where such points are eliminated.

Let $\tilde{\nu}$ be such that $\nu_1 < \tilde{\nu} < \nu_2$, and $\tilde{\nu}$ is sufficiently close to ν_1 so that there are no critical points in $\Sigma(\lambda)$ with $\nu \in [\tilde{\nu}, \nu_2)$. Choose an operator $A \in \Psi^0_{\mathrm{sc}}(X)$ with $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - A) \cap \Sigma(\lambda) \subset \{\nu \geq \tilde{\nu}\}$ and $\mathrm{WF}_{\mathrm{sc}}(A) \subset \{\nu \leq \nu_2\}$. Then $f = (P - \lambda)Au$ has wavefront set confined to $\Sigma(\lambda) \cap \{\tilde{\nu} < \nu < \nu_2\}$.

Let $\tilde{V}$ be as in Lemma 5.3, and let $\tilde{\Pi}$ be the orthogonal projection off the L^2-nullspace of $\tilde{P} - \lambda$; $\mathrm{Id} - \tilde{\Pi}$ is a finite rank projection onto a subspace of $\dot{\mathcal{C}}^\infty(X)$. By (5.1), $\mathrm{WF}_{\mathrm{sc}}(f) \subset \tilde{\Sigma}(\lambda) \cap \{\tilde{\nu} < \nu < \nu_2\}$. Now consider $v = \tilde{R}(\lambda + i0)\tilde{\Pi} f$ where $\tilde{R}(\lambda)$ is the resolvent for $\tilde{P}$. This has wavefront set in $\tilde{\Sigma}(\lambda) \cap \{\nu > 0\}$. Furthermore in $\nu < \tilde{\nu}$, where f has no wavefront set, its only wavefront set is associated to critical points. These are all minima, by construction, so in $\nu < \tilde{\nu}$, $\mathrm{WF}_{\mathrm{sc}}(v)$ only has these as isolated points. Using microlocal cut-offs near these points we may excise the wavefront set and so obtain a solution of $(\tilde{P} - \lambda)v' = f + f'$, where $f' \in \dot{\mathcal{C}}^\infty(X)$ and $\mathrm{WF}_{\mathrm{sc}}(v')$ is confined to $\nu \geq \tilde{\nu}$ so

$$\mathrm{WF}_{\mathrm{sc}}(v') \subset \tilde{\Sigma}(\lambda) \cap \{\nu \geq \tilde{\nu}\} = \Sigma(\lambda) \cap \{\nu \geq \tilde{\nu}\}.$$

Since

$$\mathrm{WF}_{\mathrm{sc}}((\tilde{P} - P)v') \subset \Sigma(\lambda) \cap \{\nu \geq \tilde{\nu}\} \cap \mathrm{WF}'_{\mathrm{sc}}(\tilde{P} - P) = \emptyset,$$

it follows that $(P - \lambda)v' = f + f''$ with $f'' \in \dot{\mathcal{C}}^\infty(X)$. Finally then set $\tilde{u} = Au - v'$. By construction this reduces to u microlocally in $\nu < \tilde{\nu}$ and satisfies $(P - \lambda)\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$. Since there are no critical points with $\nu \in [\tilde{\nu}, \nu_2)$ it follows from Theorem 3.1 that $\mathrm{WF}_{\mathrm{sc}}(u - \tilde{u})$ is contained in $\{\nu \geq \nu_2\}$. □

If O_1 and O_2 are two W-balanced neighbourhoods of q then

$$O_1 \subset O_2 \Longrightarrow \tilde{E}_{\mathrm{mic},\pm}(O_2,\lambda) \subset \tilde{E}_{\mathrm{mic},\pm}(O_1,\lambda). \tag{5.2}$$

Since $\{u \in \mathcal{C}^{-\infty}(X); \mathrm{WF}_{\mathrm{sc}}(u) \cap O = \emptyset\} \subset \tilde{E}_{\mathrm{mic},\pm}(O,\lambda)$ for all O and this linear space decreases with O, the inclusions (5.2) induce similar maps on the quotients

$$\begin{gathered} E_{\mathrm{mic},\pm}(q,\lambda) = \tilde{E}_{\mathrm{mic},\pm}(O,\lambda)/\{u \in \mathcal{C}^{-\infty}(X); \mathrm{WF}_{\mathrm{sc}}(u) \cap O = \emptyset\}, \\ O_1 \subset O_2 \Longrightarrow E_{\mathrm{mic},\pm}(O_2,\lambda) \longrightarrow E_{\mathrm{mic},\pm}(O_1,\lambda). \end{gathered} \tag{5.3}$$

Lemma 5.6. *Provided O_i, for $i = 1, 2$, are W-balanced neighbourhoods of q and the closure of O_2 contains no radial points other than q, the map in (5.3) is an isomorphism.*

Proof. We may assume $q \in \mathrm{RP}_+(\lambda)$. We prove the lemma first for $E_{\mathrm{mic},+}(O_i,\lambda)$; we need a different method for $E_{\mathrm{mic},-}(O_i,\lambda)$.

If $q \in \mathrm{Min}_\pm(\lambda)$, the lemma follows directly from Corollary 5.2, so we may assume that $q \in \mathrm{Max}_\pm(\lambda)$. Let L_+ be that one of the Legendrians L_1, L_2 from Proposition 1.5 with $L_+ \subset \{\nu \geq \nu(q)\}$.

The map in (5.3) is injective since any element u of its kernel has a representative $\tilde{u} \in \tilde{E}_{\mathrm{mic},+}(O_2,\lambda)$ which satisfies $q \notin \mathrm{WF}_{\mathrm{sc}}(\tilde{u})$, hence $\mathrm{WF}_{\mathrm{sc}}(\tilde{u}) \cap O_2 = \emptyset$ by the second half of Lemma 5.1, so $u = 0$ in $E_{\mathrm{mic},+}(O_2,\lambda)$.

The surjectivity follows from Lemma 5.5. The assumption that $\overline{O_2}$ contains no radial points other than q allows us to find an open set $O \supset \overline{O_2}$ which is W-balanced. Now let $A \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ be microlocally the identity on $L_+ \cap \overline{O_1}$ and supported in a small neighbourhood of L_+ inside O. Then there exists $\nu_2 > \nu(q)$ such that $\nu > \nu_2$ on $L_+ \setminus O_1$, and $\mathrm{WF}'_{\mathrm{sc}}(A) \setminus O_1 \subset \{\nu \geq \nu_2\}$. Since $\mathrm{WF}_{\mathrm{sc}}(u) \cap O_1 \subset L_+$, $\mathrm{WF}_{\mathrm{sc}}(Au - u) \cap O_1 = \emptyset$. In addition, $\mathrm{WF}_{\mathrm{sc}}(Au) \subset \mathrm{WF}'_{\mathrm{sc}}(A) \cap \mathrm{WF}_{\mathrm{sc}}(u)$, hence $\nu \geq \nu(q)$ on $\mathrm{WF}_{\mathrm{sc}}(Au)$. Moreover, $\mathrm{WF}_{\mathrm{sc}}(Au - u) \cap O_1 = \emptyset$ implies that

$$\mathrm{WF}_{\mathrm{sc}}((P-\lambda)Au) \cap O_1 = \mathrm{WF}_{\mathrm{sc}}((P-\lambda)Au - (P-\lambda)u) \cap O_1 = \emptyset,$$

so $\mathrm{WF}_{\mathrm{sc}}((P-\lambda)Au) \subset \mathrm{WF}'_{\mathrm{sc}}(A) \setminus O_1$, hence is contained in $\{\nu \geq \nu_2\}$. Hence by Lemma 5.5, there exists $\tilde{u} \in \mathcal{C}^{-\infty}(X)$ such that $\nu \geq \nu_2$ on $\mathrm{WF}_{\mathrm{sc}}(\tilde{u} - Au)$ and $(P-\lambda)\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$. In particular, $\nu \geq \nu(q)$ in $\mathrm{WF}_{\mathrm{sc}}(\tilde{u})$, so $\tilde{u} \in \tilde{E}_{\mathrm{mic},+}(O_2,\lambda)$. Moreover, $\nu \geq \nu_2$ on $\mathrm{WF}_{\mathrm{sc}}(\tilde{u} - u) \cap O_1$, hence by the second half of Lemma 5.1 $\mathrm{WF}_{\mathrm{sc}}(\tilde{u} - u) \cap O_1 = \emptyset$, so $\tilde{u}$ and u have the same image in $E_{\mathrm{mic},+}(O_1,\lambda)$. This shows surjectivity.

Rather applying Lemma 5.5, we could have used Hörmander's existence theorem in the real principal type region [8], to find a microlocal solution v of $(P-\lambda)v = (P-\lambda)Au$ on O, with $\mathrm{WF}_{\mathrm{sc}}(v) \subset \Phi_+(\mathrm{WF}_{\mathrm{sc}}((P-\lambda)Au))$, and then taken $\tilde{u} = Au - v$. This method also allows us to deal with $E_{\mathrm{mic},-}(O_i,\lambda)$. □

It follows from this Lemma that the quotient space $E_{\mathrm{mic},\pm}(q,\lambda)$ in (5.3) is well-defined, as the notation already indicates, and each element is determined by the behaviour microlocally 'at' q.

Corollary 5.7. *If $q \in \mathrm{Min}_+(\lambda)$ then*

$$E_{\mathrm{mic},+}(q,\lambda) \simeq \left\{u \in \mathcal{C}^{-\infty}(X); (P-\lambda)u \in \dot{\mathcal{C}}^\infty(X),\ \mathrm{WF}_{sc}(u) \subset \{q\}\right\} / \dot{\mathcal{C}}^\infty(X). \tag{5.4}$$

If $q \in \mathrm{Max}_+(\lambda)$ then every element of $E_{\mathrm{mic},+}(q,\lambda)$ has a representative u such that $(P-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$, and $\mathrm{WF}_{sc}(u) \subset \Phi_+(\{q\})$.

Proof. Suppose first that $q \in \mathrm{Min}_+(\lambda)$ then any microlocal eigenfunction, $u \in \tilde{E}_{\mathrm{mic},+}(O,\lambda)$, has q as (at most) the only point in its wavefront set within O. Using a microlocal cutoff around q with support in O gives a map which realizes the isomorphism in (5.4).

If $q \in \mathrm{Max}_+(\lambda)$ then $\tilde{u}$ constructed in the proof of Lemma 5.6 provides a representative such that $(P-\lambda)\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$, with $\mathrm{WF}_{\mathrm{sc}}(\tilde{u})$ contained in the union of $\Phi_+(\{q\})$ and $\Phi_+(\{q'\})$, for $q' \in \mathrm{RP}_+(\lambda)$ with $\nu(q') > \nu(q)$. If we choose q' from the set

$$\{q' \in \mathrm{RP}_+(\lambda) \cap \mathrm{WF}_{\mathrm{sc}}(\tilde{u}) \mid \nu(q') > \nu(q),\ q' \notin \Phi_+(\{q\})\}, \tag{5.5}$$

with $\nu(q')$ minimal, then by localizing $\tilde{u}$ near q' we have an element v of $E_{\mathrm{mic},+}(q')$. By subtracting from $\tilde{u}$ a representative for v given by Lemma 5.6, we remove the wavefront set near q'. Inductively choosing radial points from (5.5) and performing this procedure, we remove all wavefront set from $\tilde{u}$ except that contained in $\Phi_+(\{q\})$. □

At radial points corresponding to maxima these results can be strengthened by using a positive commutator estimate which is dual, in some sense, to Proposition 4.13.

Proposition 5.8. *Assume that $q \in \mathrm{RP}_+(\lambda)$, and let O be a W-balanced neighbourhood of q. If, for some $r < -1/2$, $O \cap \mathrm{WF}_{sc}^{*,r+1}((P-\lambda)u) = \emptyset$ and $O \cap \mathrm{WF}_{sc}^{*,r}(u) \subset \Phi_+(\{q\})$, then $O \cap \mathrm{WF}_{sc}^{*,r}(u) = \emptyset$. The same holds true if $q \in \mathrm{RP}_-(\lambda)$, $u \in E_{\mathrm{mic},-}(q,\lambda)$, and $O \cap \mathrm{WF}_{sc}^{*,r}(u) \subset \Phi_-(\{q\})$.*

Proof. Assume that $q \in \mathrm{Max}_+(\lambda)$ for the sake of definiteness, and let L_+ be that one of the Legendrians L_1, L_2 from Proposition 1.5 with $L_+ \subset \{\nu \geq \nu(q)\}$. The argument for $q \in \mathrm{Min}_+(\lambda)$ is similar, and slightly easier since in that case $\Phi_+(\{q\})$ consists of the single point q. The case $q \in \mathrm{RP}_-(\lambda)$ follows by taking the complex conjugate.

Let $l < -1$, $l < r$, be such that $u \in H_{\mathrm{sc}}^{*,l}(X)$, let $\nu_0 = \nu(q) > 0$ and let $y_0 = y(q)$ in some local coordinates. Let a, χ, ϕ, $\tilde{\psi}$ and ψ_0 be as in Proposition 4.13 (except that here $\nu_0 > 0$). Then we may write

$$^{\mathrm{sc}}H_p a = -b^2 x^{-l-1} + e x^{-l-1} \tag{5.6}$$

as in Proposition 4.13, since now $l+1$ has changed signs, but so has ν (from negative to positive) on the support of χ. If we use Lemma 4.3 instead of Lemma 4.1 (since we are dealing here with a single function rather than a family), the argument from Proposition 4.13 applies, except we can only reach values of r less than $-1/2$ since we must have $l < -1$. □

Corollary 5.9. *If $q \in \mathrm{RP}_+(\lambda)$, O is a W-balanced neighbourhood of q and $u \in \tilde{E}_{\mathrm{mic},+}(O,\lambda)$, then $\mathrm{WF}_{sc}^{*,r}(u) \cap O = \emptyset$ for all $r < -1/2$.*

6. Test modules and iterative regularity

To investigate the regularity of microlocal eigenfunctions we use 'test modules' $\mathcal{M}$ of scattering pseudodifferential operators. This approach is a generalization of the description of conormality by stability under iterated application of vector fields, [12]. In this section we first describe the basic properties of test modules. It may help the reader to refer to the following sections, namely to (7.3), (8.1), (9.2) and (10.2), for the particular test modules we use here. These are all described at the beginning of the corresponding sections, so a quick inspection of these should be possible.

In the second half of the section we turn to the study of *methods* of proving $\mathcal{M}$-regularity of certain distributions. Our main technique in this paper will be the use of positive commutator estimates. These are more delicate versions of the real principal type estimates that already played in important role in the preceding sections. Their abstract framework is desribed starting with Lemma 6.5. However, we emphasize that there are other methods for obtaining $\mathcal{M}$-regularity. Indeed, for incoming eigenfunctions at an outgoing saddle, we use a combination of constructive techniques and duality to obtain a complete description in Theorem 12.5.

Since we work microlocally we consider scattering pseudodifferential operators microlocally supported in an open set $O \subset C = \partial({}^{\mathrm{sc}}\overline{T}^* X)$. In particular we shall use the notation

$$\Psi_{\mathrm{sc}}^{m,l}(O) = \left\{A \in \Psi_{\mathrm{sc}}^{m,l}(X); \mathrm{WF}_{\mathrm{sc}}'(A) \subset O\right\}. \tag{6.1}$$

Since we are only interested in boundary regularity here we shall suppose that $O \subset C_\partial = {}^{\mathrm{sc}}T^*_{\partial X} X$ (see (1.5)) and so we may take $m = -\infty$.

Definition 6.1. A *test module* in an open set $O \subset C_\partial$ is a linear subspace $\mathcal{M} \subset \Psi_{\mathrm{sc}}^{-\infty,-1}(O)$ which contains and is a module over $\Psi_{\mathrm{sc}}^{-\infty,0}(O)$, which is closed under commutators and which is finitely generated in the sense that there exist finitely many $A_i \in \Psi_{\mathrm{sc}}^{*,-1}(X)$, $i = 0, 1, \ldots, N$, $A_0 = \mathrm{Id}$, such that each $A \in \mathcal{M}$ can be written

$$A = \sum_{j=0}^{N} Q_i A_i, \ Q_i \in \Psi_{\mathrm{sc}}^{-\infty,0}(O). \tag{6.2}$$

Since we have assumed that $\mathcal{M} \subset \Psi_{\mathrm{sc}}^{-\infty,-1}(X)$, $A \in \mathcal{M}$, $C \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ implies that $[A, C] \in \Psi^{-\infty,0}(O) \subset \mathcal{M}$. The structure of the module is thus determined by the commutators of the generators. It certainly suffices to have

$$[A_i, A_j] = \sum_{k=0}^{N} C_{ijk} A_k + E'_{jk}, \ C_{ijk} \in \Psi_{\mathrm{sc}}^{0,0}(X), \tag{6.3}$$
$$E'_{jk} \in \Psi_{\mathrm{sc}}^{0,-1}(X), \ \mathrm{WF}_{\mathrm{sc}}'(E'_{jk}) \cap O = \emptyset.$$

This in turn follows from the purely symbolic condition on principal symbols:

$$\{a_i, a_j\} = \sum_{k=0}^{N} c_{ijk} a_k + e_{ij} \text{ in a neighbourhood of } \overline{O} \tag{6.4}$$

where the e_{ij} are symbols of order 0.

In our arguments below, we frequently shrink O, so the condition (6.3) is in essence necessary. When $O' \subset O$ is another open set we consider the restricted module over O'

$$\mathcal{M}(O') = \sum_{i=0}^{N} \Psi^{-\infty,0}(O') \cdot A_i. \tag{6.5}$$

which is easily seen to be independent of the choice of generators.

Definition 6.2. Let $\mathcal{M}$ be a test module. For $u \in \mathcal{C}^{-\infty}(X)$ we say $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ if $\mathcal{M}^m u \subset H_{\mathrm{sc}}^{\infty,s}(X)$ for all m. That is, for all m and for all $B_i \in \mathcal{M}$, $i = 1, \ldots, m$, we have $\prod_{i=1}^m B_i u \in H^{\infty,s}_{\mathrm{sc}}(X)$. We may define the finite regularity spaces by saying $u \in I^{(s),M}_{\mathrm{sc}}(O,\mathcal{M})$ if $\mathcal{M}^m u \subset H^{\infty,s}_{\mathrm{sc}}(X)$ for all $m \le M$.

Note that $I^{(s)}_{\mathrm{sc}}(O,\mathcal{M})$ is a well-defined space of microfunctions over O since if $u \in \mathcal{C}^{-\infty}(X)$ and $\mathrm{WF}_{\mathrm{sc}}(u) \cap O = \emptyset$ then $Au \in \dot{\mathcal{C}}^\infty(X)$ for all $A \in \Psi^{*,*}_{\mathrm{sc}}(X)$ such that $\mathrm{WF}'_{\mathrm{sc}}(A) \subset O$; in particular this is the case for all $A \in \mathcal{M}$. Clearly

$$I^{(s)}_{\mathrm{sc}}(O,\mathcal{M}) = \bigcap_{M=0}^{\infty} I^{(s),M}_{\mathrm{sc}}(O,\mathcal{M}). \tag{6.6}$$

¿From the definition of the restriction in (6.5) it also follows that if $O' \subset O$ then

$$I^{(s),k}_{\mathrm{sc}}(O,\mathcal{M}) \subset I^{(s),k}_{\mathrm{sc}}(O',\mathcal{M}). \tag{6.7}$$

Since $\mathrm{Id} \in \mathcal{M}$,

$$\Psi^{-\infty,0}_{\mathrm{sc}}(O) = \mathcal{M}^0 \subset \mathcal{M} \subset \mathcal{M}^2 \subset \dots.$$

Definition 6.2 above then reduces to

$$u \in I^{(s),M}_{\mathrm{sc}}(O,\mathcal{M}) \Longleftrightarrow \mathcal{M}^M(O)u \subset H^{\infty,s}_{\mathrm{sc}}(X). \tag{6.8}$$

If two test modules $\mathcal{M}_1$ and $\mathcal{M}_2$ define equivalent filtrations of the same set of operators they define the same space of iteratively-regular distributions. That is,

$$\mathcal{M}_1(O) \subset (\mathcal{M}_2(O))^{k_1} \text{ and } \mathcal{M}_2(O) \subset (\mathcal{M}_1(O))^{k_2} \Longrightarrow \\ I^{(s)}_{\mathrm{sc}}(O,\mathcal{M}_1) = I^{(s)}_{\mathrm{sc}}(O,\mathcal{M}_2),\ \forall\ s \in \mathbb{R}. \tag{6.9}$$

Lemma 6.3. *If A_i, $0 \le i \le N$, are generators for $\mathcal{M}$ in the sense of Definition 6.2 with $A_0 = \mathrm{Id}$, then*

$$\mathcal{M}^k = \left\{ \sum_{|\alpha| \le k} Q_\alpha \prod_{i=1}^N A_i^{\alpha_i},\ Q_\alpha \in \Psi^{-\infty,0}(O) \right\} \tag{6.10}$$

where α runs over multiindices $\alpha : \{1,\dots,N\} \longrightarrow \mathbb{N}_0$ and $|\alpha| = \alpha_1 + \dots + \alpha_N$.

Proof. This is just a consequence of the fact that $\mathcal{M}$ is a Lie algebra and module. Thus, we proceed by induction over k since (6.10) certainly holds for $k=1$, and by definition for $k=0$. It therefore suffice to prove the equality (6.10) modulo $\mathcal{M}^{k-1}$. Then (6.10) just corresponds to $[\mathcal{M}^r,\mathcal{M}^s] \subset \mathcal{M}^{r+s-1}$, applied with $r+s \le k$, which allows the factors to be freely rearranged. □

In the modules we consider, one of the generating elements is $A_N = x^{-1}(P-\lambda)$ and we deal with approximate eigenfunctions, so that typically $(P-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$. We may reorder the basis so that this element comes last. In this case it is enough to consider the action on u of the remaining generators in an inductive proof of regularity since if an operator as on the right hand side of (6.10) with $\alpha_N \ne 0$ is applied to u then we obtain an element of $\dot{\mathcal{C}}^\infty(X)$. We therefore use the 'reduced' multiindex notation, with an additional index to denote the weight, and set

$$A_{\alpha,s} = x^{-s} \prod_{i=1}^{N-1} A_i^{\alpha_i},\ \alpha_i \in \mathbb{N}_0,\ 1 \le i \le N-1. \tag{6.11}$$

Corollary 6.4. *Suppose $\mathcal{M}$ is a test module, $A_N = x^{-1}(P-\lambda)$ is a generator, $u \in \mathcal{C}^{-\infty}$ satisfies $(P-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$ and $u \in I^{(s),m-1}_{\mathrm{sc}}(O,\mathcal{M})$. Then for $O' \subset O$, $u \in I^{(s),m}_{\mathrm{sc}}(O',\mathcal{M})$ if for each multiindex α, with $|\alpha| = m$, $\alpha_N = 0$, there exists $Q_\alpha \in \Psi^{-\infty,0}(X)$, elliptic on O' such that $Q_\alpha A_{\alpha,s} u \in L^2_{\mathrm{sc}}(X)$.*

In the case of interest here, O will be a W-balanced neighbourhood of $q \in \mathrm{RP}(\lambda)$ and the generators A_i will be certain operators, related to the geometry of the classical dynamics near q, with principal symbols vanishing at q. The last operator, as mentioned above, will be $A_N = x^{-1}(P-\lambda)$.

Our main tool in the demonstration of regularity with respect to such a test module is the use of positive commutator techniques. These depend on finer structure of the module. In particular the basic source of positivity is the commutation relation

$$\begin{gathered} i[x^{-s-\frac12}, P-\lambda] = x^{-s+\frac12} C_0, \\ \text{where } C_0 \in \Psi^{1,0}_{\mathrm{sc}}(X) \text{ has principal symbol } -(2s+1)\nu, \end{gathered} \tag{6.12}$$

so has a definite sign in a neighbourhood of a radial point $q \in \mathrm{RP}_\pm(\lambda)$ if $s \neq -1/2$ and $\lambda \notin \mathrm{Cv}(V)$. This strict sign of $\sigma_0(C_0)$ allows us to deal with error terms. To fix signs we shall assume that $q \in \mathrm{RP}_+(\lambda)$ and $\lambda \notin \mathrm{Cv}(V)$ in the remainder of this section.

So suppose that $f(\nu, y, \mu)$ vanishes at $q \in \mathrm{RP}_+(\lambda)$, and df is an eigenvector for the linearization of W at q with eigenvalue $-2\nu r$, where $\nu = \nu(q) > 0$. Then

$$ {}^{\mathrm{sc}}H_p(x^m f) = x^m[-2\nu(m+r)f + f_2 + \mathcal{O}(x)], \tag{6.13}$$

where f_2 is smooth and vanishes quadratically at the critical point q. Thus,

$$ {}^{\mathrm{sc}}H_p(x^{2m} f^2) = 2x^{2m}[-2\nu(m+r)f^2 + f_2 f + \mathcal{O}(x)].$$

If $m + r \neq 0$ and f_2 is of the form $f\tilde{f}_2$, then this has a fixed sign near q. In general, f_2 does not factor this way, so we need to consider a sum of such commutators corresponding to functions f, f' such that df and df' span $T^*_q\Sigma(\lambda)$; then $x^{-2m}\,{}^{\mathrm{sc}}H_p(x^{2m}f^2) + x^{-2m}\,{}^{\mathrm{sc}}H_p(x^{2m}(f')^2)$ has a fixed sign near q on $\Sigma(\lambda)$. We use this as a basis for the construction of positive commutators, though it is more convenient to describe our conditions on the generators A_i, and then spell out in detail in the following sections why these conditions hold.

The basic condition we require is that the remaining generators of the test module, A_i, $i = 1, \ldots, l = N-1$, satisfy

$$\begin{gathered} x^{-1} i[A_i, P-\lambda] = \sum_{j=0}^N C_{ij} A_j,\ C_{ij} \in \Psi^{*,0}_{\mathrm{sc}}(X), \\ \text{where } \sigma_\partial(C_{ij})(q) = 0,\ j \neq i, \\ \text{and } \operatorname{Re}\sigma_\partial(C_{jj})(q) \geq 0,\ 0 < j \leq l,\ q \in \mathrm{RP}_+(\lambda),\ s < -\frac12. \end{gathered} \tag{6.14}$$

However, we will occasionally need slightly weaker conditions than (6.14). These weaker conditions allow $\sigma_\partial(C_{ij})(q) \neq 0$ even for $i \neq j$, but they require that the matrix $\sigma_\partial(C_{ij})(q)$ is lower triangular. Hence, in this case, the ordering of the A_i

(by their index) is not arbitrary. Our weaker condition is then

$$
\begin{gathered}
x^{-1} i[A_i, P-\lambda] = \sum_{j=0}^{N} C_{ij} A_j,\ C_{ij} \in \Psi_{\mathrm{sc}}^{*,0}(X), \\
\text{where } \sigma_\partial(C_{ij})(q) = 0,\ j > i, \\
\text{and } \operatorname{Re} \sigma_\partial(C_{jj})(q) \geq 0,\ 0 < j \leq l,\ q \in \mathrm{RP}_+(\lambda),\ s < -\frac{1}{2}..
\end{gathered}
\tag{6.15}
$$

We need the following notation. Suppose that α, β are multiindices. We say that $\alpha \sim \beta$ if there exist $i \neq j$ such that for $k \notin \{i,j\}$, $\alpha_k = \beta_k$, while $\alpha_i = \beta_i + 1$, $\alpha_j = \beta_j - 1$. This means that A_α has an additional factor of A_i, and one fewer factor of A_j than A_β. If $\alpha \sim \beta$, we write $\alpha < \beta$ provided that $i < j$.

The following is a technical Lemma we need for Proposition 6.8 below, which gives a method for showing membership of $u \in I_{\mathrm{sc}}^{(s),m}(O, \mathcal{M})$.

Lemma 6.5. *Let $q \in \mathrm{RP}_+(\lambda)$ and $s < -\frac{1}{2}$, where $\lambda \notin \mathrm{Cv}(V)$. Suppose $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O)$, let $K_\alpha > 0$ be arbitrary positive constants, and let C_0 and C_{ij} be given by* (6.12)*, resp.* (6.14)*. Then, assuming* (6.14)*, and using the notation of* (6.11)*, we have*

$$
\begin{aligned}
\sum_{|\alpha|=m} & i K_\alpha [A^*_{\alpha,s+1/2} Q^* Q A_{\alpha,s+1/2}, P-\lambda] \\
&= \sum_{|\alpha|,|\beta|=m} A^*_{\alpha,s} Q^* C'_{\alpha\beta} Q A_{\beta,s} \\
&+ \sum_{|\alpha|=m} \left(A^*_{\alpha,s} Q^* E_{\alpha,s} + E^*_{\alpha,s} Q A_{\alpha,s} \right) + \sum_{|\alpha|=m} A^*_{\alpha,s} i [Q^* Q, P-\lambda] A_{\alpha,s},
\end{aligned}
\tag{6.16}
$$

where

$$
E_{\alpha,s} = x^{-s} E_\alpha,\ E_\alpha \in \mathcal{M}^{m-1} + \mathcal{M}^{m-1} A_N,\ \mathrm{WF}'_{sc}(E_\alpha) \subset \mathrm{WF}'_{sc}(Q),
\tag{6.17}
$$

$$
\begin{gathered}
C'_{\alpha\beta} \in \Psi_{\mathrm{sc}}^{1,0}(X),\ \sigma_\partial(C'_{\alpha\alpha})(q) = 2K_\alpha \left(-(2s+1)\nu + \sum_j \alpha_j \operatorname{Re} \sigma_\partial(C_{jj})(q) \right) > 0 \\
\textit{and } \sigma_\partial(C'_{\alpha\beta})(q) = 0 \textit{ for } \alpha \neq \beta.
\end{gathered}
\tag{6.18}
$$

In addition, $C'_{\alpha\beta} = 0$ unless either $\alpha = \beta$ or $\alpha \sim \beta$.

Assuming instead (6.15) *instead of* (6.14)*, the conclusions hold with the last equation of* (6.18) *replaced by*

$$
\begin{gathered}
\sigma_\partial(C'_{\alpha\beta})(q) = K_\beta \beta_j \overline{\sigma_\partial(C_{ji})(q)} \textit{ for } \alpha \sim \beta,\ \alpha < \beta, \\
\sigma_\partial(C'_{\alpha\beta})(q) = K_\alpha \alpha_i \sigma_\partial(C_{ij})(q) \textit{ for } \alpha \sim \beta,\ \alpha > \beta.
\end{gathered}
\tag{6.19}
$$

Remark 6.6. The first term on the right hand side of (6.16) is the principal term, in terms of $\mathcal{M}$-order, since both the $A^*_{\alpha,s}$ and $A_{\beta,s}$ terms have $\mathcal{M}$-order equal to m. Note that this term has nonnegative principal symbol. In the second term, the E terms have $\mathcal{M}$-order $m-1$ which allows these to be treated as error terms. To deal with the third term, involving the commutator $[Q^*Q, P-\lambda]$, we need additional information about Q, discussed below.

Proof. We obtain the identity (6.16) by computing

$$[QA_{\alpha,s+1/2}, P-\lambda] = [Qx^{-s-1/2}A_1^{\alpha_1}\dots A_{N-1}^{\alpha_{N-1}}, P-\lambda], \quad Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O),\ |\alpha| = m. \tag{6.20}$$

The commutator with $P-\lambda$ distributes over the product, in each term replacing a single factor A_i by $\sum xC_{ij}A_j$. After rearrangement of the order of the factors, giving an error included in E_α below, it becomes

$$\begin{aligned}(C_0 + \sum_j \alpha_j C_{jj})Qx^{-s+1/2}A_\alpha + \sum_{|\beta|=m,\ \beta\neq\alpha} C_{\alpha\beta}Qx^{-s+1/2}A_\beta \\ + x^{-s+1/2}E_\alpha + i[Q, P-\lambda]A_{\alpha,s+1/2},\end{aligned} \tag{6.21}$$

where $\sigma_\partial(C_{\alpha\beta})(q) = 0$, $C_{\alpha\beta} = 0$ unless $\alpha = \beta$ or $\alpha \sim \beta$, and the E_α are as in (6.17) – using in particular the inclusion $x\mathcal{M}^m \subset \mathcal{M}^{m-1}$. In addition, for $\alpha \sim \beta$,

$$\sigma_\partial(C_{\alpha\beta})(q) = \alpha_i\sigma_\partial(C_{ij})(q).$$

This implies that

$$\begin{aligned} i[A^*_{\alpha,s+1/2}&Q^*QA_{\alpha,s+1/2}, P-\lambda] \\ &= A^*_{\alpha,s}Q^*(C_0 + C_0^* + \sum_j \alpha_j(C_{jj} + C^*_{jj}))QA_{\alpha,s} \\ &+ \sum_\beta A^*_{\alpha,s}Q^*C_{\alpha\beta}QA_{\beta,s} + \sum_\beta A^*_{\beta,s}Q^*C^*_{\alpha\beta}QA_{\alpha,s} + A^*_{\alpha,s}Q^*E_{\alpha,s} \\ &+ E^*_{\alpha,s}QA_{\alpha,s} + A^*_{\alpha,s}i[Q^*Q, P-\lambda]A_{\alpha,s};\end{aligned} \tag{6.22}$$

here $E_{\alpha,s} = x^{-s}E_\alpha$ and all sums over β are understood as $|\beta| = m$, $\beta \neq \alpha$. Let

$$\begin{aligned} C'_{\alpha\alpha} &= K_\alpha\left(C_0 + C_0^* + \sum_j \alpha_j(C_{jj} + C^*_{jj})\right) \\ C'_{\alpha\beta} &= K_\alpha\alpha_i C_{ij} + K_\beta\beta_j C^*_{ji} \text{ if } \alpha \sim \beta.\end{aligned} \tag{6.23}$$

If we assume (6.14), we deduce that (6.18) holds, while if we assume (6.15), (6.19) follows. This finishes the proof. □

We now consider $C' = (C'_{\alpha\beta})$, $|\alpha| = m = |\beta|$ as a matrix of operators, or rather as an operator on a trivial vector bundle with fiber $\mathbb{R}^{|M_m|}$ over a neighbourhood of $z = \pi(q)$ in X, where $|M_m|$ denotes the number of elements of the set M_m of multiindices α with $|\alpha| = m$. Let $c' = \sigma_\partial(C')(q)$, $c'_{\alpha\beta} = \sigma_\partial(C'_{\alpha\beta})(q)$.

Suppose first that (6.14) holds. Then for any choice of $K_\alpha > 0$, e.g. $K_\alpha = 1$ for all α, $c' = \sigma_\partial(C')(q)$ is positive or negative definite with the sign of $\sigma_\partial(C_0)(q)$ (whose sign for $q \in \mathrm{RP}_+(\lambda)$ is that of $-s - 1/2$). The same is therefore true microlocally near q. For the sake of definiteness, suppose that $\sigma_\partial(C_0)(q) > 0$. Then there exist a neighbourhood O_m of q, depending on $|\alpha| = m$, and $B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, $G \in \Psi_{\mathrm{sc}}^{-\infty,1}(X)$, with $\sigma_\partial(B) > 0$ on O_m such that

$$Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \mathrm{WF}'_{\mathrm{sc}}(Q) \subset O_m \Rightarrow Q^*C'Q = Q^*(B^*B + G)Q. \tag{6.24}$$

Assume now that (6.15) holds. There is a natural partial order on M_m given as follows: $\alpha < \beta$ if there exist $\alpha^{(j)}$, $j = 0, 1, \dots, n$, $n \geq 1$, $\alpha^{(0)} = \alpha$, $\alpha^{(n)} = \beta$, $\alpha^{(j)} \sim \alpha^{(j+1)}$ and $\alpha^{(j)} < \alpha^{(j+1)}$ (which has already been defined if $\alpha^{(j)} \sim \alpha^{(j+1)}$) for

$j = 0, 1, \dots, n-1$. This is clearly transitive, and if $\alpha < \beta$ then $\sum_j j\alpha_j < \sum_j j\beta_j$, so $\alpha < \beta$ and $\beta < \alpha$ cannot hold at the same time. We place a total order $<'$ on M_m which is compatible with $<$, so $\alpha < \beta$ implies $\alpha <' \beta$. For the sake of definiteness, suppose that $\sigma_\partial(C_0)(q) > 0$. We then choose K_α inductively starting at the maximal element of $<'$, making $\sigma_\partial(C')(q)$ positive definite on $\oplus_{\alpha \leq' \beta}\mathbb{R}_\beta$. We can for example take $K_\alpha = 1$ for the maximal element α. Once K_β has been chosen for all β with $\alpha <' \beta$, so that c' is positive definite on $\oplus_{\alpha <' \beta}\mathbb{R}_\beta$, we remark that of all entries $c'_{\beta\gamma}$ with $\alpha \leq \beta, \gamma$, only $c'_{\alpha\alpha}$ depends on K_α. Expanding $(v, c'v)$ for $v = (v', v'') \in \mathbb{R}_\alpha \oplus (\oplus_{\alpha <' \beta}\mathbb{R}_\beta)$, and using Cauchy-Schwarz for the cross terms involving both v' and v'', we deduce that for any $\epsilon > 0$,

$$(v, c'v) \geq c'_{\alpha\alpha}\|v'\|^2 + (v'', c'v'') - \epsilon\|c'v''\|^2 - \epsilon^{-1}\|v'\|^2.$$

Since c' is positive definite on $\oplus_{\alpha <' \beta}\mathbb{R}_\beta$, we can choose first $\epsilon > 0$ sufficiently small, then $K_\alpha > 0$ sufficiently large (depending on ϵ) such that c' is positive definite on $\oplus_{\alpha \leq' \beta}\mathbb{R}_\beta$. Thus, for suitable constants K_α, $c' = \sigma_\partial(C')(q)$ is positive definite, and we again conclude that there exist a neighbourhood O_m of q, depending on $|\alpha| = m$, and $B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, $G \in \Psi_{\mathrm{sc}}^{-\infty,1}(X)$, with $\sigma_\partial(B) > 0$ on O_m such that

$$Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \mathrm{WF}'_{\mathrm{sc}}(Q) \subset O_m \Rightarrow Q^*C'Q = Q^*(B^*B + G)Q. \tag{6.25}$$

Equations (6.12) and (6.14) (or (6.12) and (6.15)) are all that is needed in the two cases of centers and sinks, where we know *a priori* that $\mathrm{WF}_{\mathrm{sc}}(u) \subset \{q\}$. In these cases we take Q such that $q \notin \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q)$, which implies that

$$i[Q^*Q, P - \lambda] = x^{1/2}\tilde{F}x^{1/2},\ \text{where } \tilde{F} \in \Psi_{\mathrm{sc}}^{0,0}(O),\ q \notin \mathrm{WF}'_{\mathrm{sc}}(\tilde{F}). \tag{6.26}$$

In the case of a saddle point, i.e. $q \in \mathrm{Max}_+(\lambda)$, we need a more general setup and we shall arrange that $q \notin \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q)$,

$$\begin{gathered} i[Q^*Q, P - \lambda] = x^{1/2}(\tilde{B}^*\tilde{B} + \tilde{G})x^{1/2} + x^{1/2}\tilde{F}x^{1/2},\ \text{where} \\ \tilde{B},\ \tilde{F} \in \Psi_{\mathrm{sc}}^{0,0}(O),\ \tilde{G} \in \Psi_{\mathrm{sc}}^{0,1}(X),\ q \notin \mathrm{WF}'_{\mathrm{sc}}(\tilde{F}), \end{gathered} \tag{6.27}$$

and in addition, $\tilde{F}$ satisfies $\mathrm{WF}'_{\mathrm{sc}}(\tilde{F}) \subset \{\nu < \nu(q)\}$. This condition on $\tilde{F}$ will ensure that $\mathrm{WF}'_{\mathrm{sc}}(\tilde{F}) \cap \mathrm{WF}_{\mathrm{sc}}(u) = \emptyset$ for the application in section 9.

Proposition 6.7. *Suppose that $m > 0$, $s < -1/2$, $q \in \mathrm{RP}_+(\lambda)$, $\lambda \notin \mathrm{Cv}(V)$, either (6.14) or (6.15) hold, and let O_m be as in (6.24) (or (6.25)). Suppose that $u \in I_{\mathrm{sc}}^{(s),m-1}(O_m, \mathcal{M})$, $\mathrm{WF}_{sc}((P - \lambda)u) \cap O_m = \emptyset$ and that there exists $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O_m)$ elliptic at q that satisfies (6.27) with $\mathrm{WF}'_{sc}(\tilde{F}) \cap \mathrm{WF}_{sc}(u) = \emptyset$. Then $u \in I_{\mathrm{sc}}^{(s),m}(O', \mathcal{M})$ where O' is the elliptic set of Q.*

Proof. First consider $u' \in I_{\mathrm{sc}}^{(s),m}(O_m, \mathcal{M})$. Let $Au' = (QA_{\alpha,s}u')_{|\alpha|=m}$, regarded as a column vector of length $|M_m|$. Now consider

$$\begin{aligned} &\sum_{|\alpha|=m} K_\alpha \langle u', i[A^*_{\alpha,s+1/2}Q^*QA_{\alpha,s+1/2}, P - \lambda]u'\rangle \\ &= \|BAu'\|^2 + \langle Au', GAu'\rangle + \sum_{|\alpha|=m}\Big(\langle QA_{\alpha,s}u', E_{\alpha,s}u'\rangle + \langle E_{\alpha,s}u', QA_{\alpha,s}u'\rangle\Big) \\ &+ \sum_{|\alpha|=m}\Big(\|\tilde{B}A_{\alpha,s}u'\|^2 + \langle A_{\alpha,s}u', \tilde{F}A_{\alpha,s}u'\rangle + \langle A_{\alpha,s}u', \tilde{G}A_{\alpha,s}u'\rangle\Big). \end{aligned} \tag{6.28}$$

Hence, dropping the term involving $\tilde{B}$ and applying the Cauchy-Schwarz inequality to the terms with $E_{\alpha,s}$, G and $\tilde{G}$, we have for any $\epsilon > 0$,

$$
\begin{aligned}
(6.29)\quad \|BAu'\|^2 \leq \sum_\alpha \Big|\langle u', i[A^*_{\alpha,s+1/2}Q^*QA_{\alpha,s+1/2}, P-\lambda]u'\rangle\Big| \\
+ \epsilon\Big(\|Au'\|^2 + \sum_\alpha(\|QA_{\alpha,s}u'\|^2 + \|A_{\alpha,s}u'\|^2)\Big) \\
+ \epsilon^{-1}\Big(\|GAu'\|^2 + \sum_\alpha(\|E_{\alpha,s}u'\|^2 + \|K_\alpha\tilde{G}A_{\alpha,s}u'\|^2)\Big) + |\langle A_{\alpha,s}u', \tilde{F}A_{\alpha,s}u'\rangle|.
\end{aligned}
$$

Choosing $\epsilon > 0$ small enough, the second term on the right can be absorbed in the left hand side (since B is strictly positive), and we get

$$
\begin{aligned}
\frac{1}{2}\|BAu'\|^2 \leq \sum_\alpha |\langle u', i[A^*_{\alpha,s+1/2}Q^*QA_{\alpha,s+1/2}, P-\lambda]u'\rangle| \\
+ \epsilon^{-1}\Big(\|GAu'\|^2 + \sum_\alpha(\|E_{\alpha,s}u'\|^2 + \|K_\alpha\tilde{G}A_{\alpha,s}u'\|^2)\Big) \\
+ |\langle A_{\alpha,s}u', \tilde{F}A_{\alpha,s}u'\rangle|.
\end{aligned}
\tag{6.30}
$$

We apply this with u' replaced by $u_r = (1+r/x)^{-1}u = \frac{x}{x+r}u$ $r > 0$, where now $u \in I^{(s),m-1}_{\mathrm{sc}}(O_m, \mathcal{M})$. Then letting $r \to 0$, using the strong convergence of $(1+r/x)^{-1}$ to the identity as in the argument of Lemma 4.3, and the assumption that $\mathrm{WF}'_{\mathrm{sc}}(\tilde{F}) \cap \mathrm{WF}_{\mathrm{sc}}(u) = \emptyset$, shows that $BAu \in L^2_{\mathrm{sc}}(X)$, finishing the proof. □

We may arrange that for all m,

$$\overline{O_{m+1}} \subset O_m \text{ and } Q = Q_m \text{ is elliptic on } O_{m+1}. \tag{6.31}$$

By induction, the proposition implies that if $\mathrm{WF}^{*,s}_{\mathrm{sc}}(u) \cap O_1 = \emptyset$ for some $s < -1/2$, and $\mathrm{WF}_{\mathrm{sc}}((P-\lambda)u) \cap O_1 = \emptyset$ then for all m, $u \in I^{(s),m}_{\mathrm{sc}}(O_{m+1})$. Since we may have $\cap_{m=1}^\infty O_m = \{q\}$, we need an additional assumption, amounting to 'propagation of regularity', to extend the result to $I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ for some neighbourhood O of q. Of course, there is no need for such an assumption if $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \{q\}$ for then $u \in I^{(s),m}_{\mathrm{sc}}(O_{m+1}, \mathcal{M})$ implies that $u \in I^{(s),m}_{\mathrm{sc}}(O, \mathcal{M})$. We state these two cases separately, but prove them together.

Proposition 6.8. *Suppose that $s < -1/2$, $q \in \mathrm{RP}_+(\lambda)$, $\lambda \notin \mathrm{Cv}(V)$, either (6.14) or (6.15) hold, and let O be a neighbourhood of q. Then for any $u \in \mathcal{C}^{-\infty}(X)$,*

$$
\begin{aligned}
\mathrm{WF}^{*,s}_{sc}(u) \cap O = \emptyset,\ \mathrm{WF}_{sc}(u) \cap O \subset \{q\} \text{ and } \mathrm{WF}_{sc}((P-\lambda)u) \cap O = \emptyset \\
\Rightarrow u \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M}).
\end{aligned}
$$

In the next proposition we assume 'propagation of regularity' to show that $u \in I^{(s),m}_{\mathrm{sc}}(O, \mathcal{M})$.

Proposition 6.9. *Suppose that $s < -1/2$, $q \in \mathrm{RP}_+(\lambda)$, $\lambda \notin \mathrm{Cv}(V)$, either (6.14) or (6.15) hold, and let Q and O_m be as in (6.24) (or (6.25)), (6.27) and (6.31). Suppose also that there exists a neighbourhood O of q such that for all m,*

$$u \in I^{(s),m}_{\mathrm{sc}}(O_{m+1}, \mathcal{M}) \text{ and } \mathrm{WF}_{sc}((P-\lambda)u) \cap O = \emptyset \Rightarrow u \in I^{(s),m}_{\mathrm{sc}}(O, \mathcal{M}). \tag{6.32}$$

Then for any $u \in \mathcal{C}^{-\infty}(X)$,

$$\mathrm{WF}^{*,s}_{sc}(u) \cap O = \emptyset \text{ and } \mathrm{WF}_{sc}((P-\lambda)u) \cap O = \emptyset \Rightarrow u \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M}).$$

Proof. By hypothesis, $u \in I^{(s),0}_{\mathrm{sc}}(O,\mathcal{M})$. We now show that for $m \geq 1$, $u \in I^{(s),m-1}_{\mathrm{sc}}(O,\mathcal{M})$ implies that $u \in I^{(s),m}_{\mathrm{sc}}(O,\mathcal{M})$. But indeed, $u \in I^{(s),m-1}_{\mathrm{sc}}(O,\mathcal{M})$ implies that $u \in I^{(s),m-1}_{\mathrm{sc}}(O_m,\mathcal{M})$, hence $u \in I^{(s),m}_{\mathrm{sc}}(O_{m+1},\mathcal{M})$ by Proposition 6.7 and (6.31). In Proposition 6.8, $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \{q\}$, so $u \in I^{(s),m}_{\mathrm{sc}}(O_{m+1},\mathcal{M})$ implies that $u \in I^{(s),m}_{\mathrm{sc}}(O,\mathcal{M})$. In Proposition 6.9, by (6.32), $u \in I^{(s),m}_{\mathrm{sc}}(O,\mathcal{M})$. This completes the inductive step. $\square$

Remark 6.10. If q is a sink or center, we use Proposition 6.8. If q is a saddle, we use Proposition 6.9. In that case, we take O to be a W-balanced neighbourhood of q. Then $u \in I^{(s),m}_{\mathrm{sc}}(O,\mathcal{M})$ is a finite Legendre regularity condition, and (6.32) holds by propagation of Legendre regularity outside critical points of the bicharacteristic flow, which is an immediate consequence of the parametrix construction of Duistermaat and Hörmander. The propagation of Legendre regularity away from critical points can also be proved by the use of test modules as in this section. In fact, the use of (6.32) can be avoided altogether if we construct Q more carefully and use the term with $\tilde{B}$ as well in (6.28). Namely, fix a small neighbourhood O' of q. Next, given m, we first choose $K_\alpha > 0$ so that C' is positive definite at q. Then we construct Q such that $q \notin \mathrm{WF}_{\mathrm{sc}}(\mathrm{Id}-Q)$, Q elliptic on O' and (6.27) holds, and in addition $(\tilde{B}^*\tilde{B})_{|\alpha|=m} + Q^*C'Q$ (the first term denoting a diagonal matrix) can be written as $B^*B + F + G$ with $\mathrm{WF}'_{\mathrm{sc}}(F) \cap \mathrm{WF}_{\mathrm{sc}}(u) = \emptyset$, $G \in \Psi^{-\infty,1}_{\mathrm{sc}}(X)$. We can do this since for any constant $N = N_m$, we can arrange that $Nq_0 \leq -\,{}^{\mathrm{sc}}H_p q_0$ on a neighborhood of $\mathrm{WF}_{\mathrm{sc}}(u)$ outside a small m-dependent neighborhood of q where $q_0 = \sigma_\partial(Q)$. Then the proof of Proposition 6.7 applies directly and shows that $u \in I^{(s),m}_{\mathrm{sc}}(O',\mathcal{M})$ for all m.

7. Outgoing eigenfunctions at a center

We first analyze the structure of elements of $E_{\mathrm{mic},+}(q,\lambda)$ when $q \in \mathrm{Min}_+(\lambda)$ is a center for the vector field W, i.e. when

$$2V''(z) > \lambda - V(z),\ z = \pi(q). \tag{7.1}$$

As shown in Corollary 5.7, by microlocalizing the solution near q we may simply assume that

$$(P-\lambda)u = f \in \dot{\mathcal{C}}^\infty(X),\ u \in \mathcal{C}^{-\infty}(X),\ \mathrm{WF}_{\mathrm{sc}}(u) \subset \{q\}. \tag{7.2}$$

We proceed in three steps, first using the commutator methods outlined above to obtain iterative regularity, with respect to the $\Psi^{-\infty,0}_{\mathrm{sc}}(O)$-module

(7.3)
$$\mathcal{M} \text{ generated by } \mathrm{Id},\ \{x^{-\frac{1}{2}}A,\ A \in \Psi^{-\infty,0}_{\mathrm{sc}}(O),\ \sigma_\partial(A)(q) = 0\} \text{ and } x^{-1}(P-\lambda).$$

Then we pass to a blown up space where this regularity becomes conormal regularity, once a phase is factored out. Finally, using the equation again, we find the expansion (I.20).

Proposition 7.1. *The module $\mathcal{M}$ in* (7.3) *is a test module in the sense of Definition 6.2, which satisfies* (6.14). *If u satisfies* (7.2) *at q, then $u \in I^{(-1/2-\epsilon)}_{\mathrm{sc}}(O,\mathcal{M})$ for all $\epsilon > 0$.*

Proof. Choose a complex symbol $a \in \mathcal{C}^\infty_c({}^{\mathrm{sc}}T^*X)$ such that $da_j(q)$, restricted to $\Sigma(\lambda)$ at q, is a non-trivial eigenvector for the linearization of W with eigenvalue $-2\tilde{\nu}r_1 \in \mathbb{C}$; here $\tilde{\nu} = \nu(q)$. Then $\bar{a}$ has the same property with eigenvalue $-2\tilde{\nu}r_2 =$

$-2\tilde{\nu}\overline{r_1}$ and together they generate, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions vanishing at q. Recall from (1.25) that $\operatorname{Re} r_1 = \operatorname{Re} r_2 = 1/2$. Certainly $A \in x^{-\frac{1}{2}}\Psi_{\mathrm{sc}}^{-\infty,0}(X)$ with symbol $x^{-\frac{1}{2}}a$ is in the set in (7.3) and indeed $\mathcal{M}$ is generated by $A_0 = \mathrm{Id}$, $A_1 = A$, $A_2 = A^*$ and $A_3 = x^{-1}(P-\lambda)$.

By (6.13), the commutator $i[A_j, P-\lambda] \in \Psi_{\mathrm{sc}}^{-\infty,-1/2}(X)$ has principal symbol

$$-\,{}^{\mathrm{sc}}H_p(x^{-1/2}a_j) = 2\tilde{\nu}x^{-1/2}(i\operatorname{Im}(r_j)a_j + e_j),\ j = 1,2, \tag{7.4}$$

where the e_j vanish to second order at q. Thus

$$ix^{-1}[A_j, P-\lambda] = C_{j0}x^{1/2} + C_{j1}A_1 + C_{j2}A_2 + x^{1/2}C_{j3}A_3,\ j = 1,2, \tag{7.5}$$

with $C_{jk} \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, $\sigma_\partial(C_{jk})(q) = 0$ if $j \neq k$, and where $\sigma_\partial(C_{jj})(q) = i2\tilde{\nu}\operatorname{Im}(r_j)$ is pure imaginary. In particular $\mathcal{M}$ is closed under commutators and hence is a test module satisfying (6.14).

To prove the statement about u, note that $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\epsilon}(u) \cap O = \emptyset$ for any W-balanced neighbourhood O of q, by Corollary 5.9. If we take any $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O)$ with $q \notin \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$, then (6.26) is satisfied, so Proposition 6.8 yields the result. □

Remark 7.2. The factor $x^{-1/2}$ in (7.3) is chosen precisely that the real part of r_j in (7.4) is cancelled, allowing the application of Proposition 6.8. A similar cancellation is crucial in (8.2) in the next section.

Next we reinterpret this iterative regularity more geometrically by introducing the parabolically blown up space

$$X_z = [X;\{z\}]_{1/2},\ \beta : X_z \longrightarrow X. \tag{7.6}$$

Such parabolic blow-ups are defined in considerable generality, including this case, in [2]. We recall a form of the construction, sufficient for our purposes here, in Section 2. Recall that X_z is a compact manifold with corners, of dimension two. The two boundary curves are the front face $\mathrm{ff} = \beta^{-1}(z)$, created by the blowup, and the closure of the pullback of $\partial X \setminus \{z\}$ which we shall refer to as the old boundary. We shall denote boundary defining functions for these boundary curves by ρ_{ff} and ρ_{o}, respectively. We also have boundary defining functions for the other boundary curves of X, which lift unchanged to boundary curves of X_z; we shall denote the product of these as ρ' since they play no significant rôle here.

One class of natural Sobolev spaces on such a manifold with corners is the class of weighted b-Sobolev spaces, which we proceed to define. Let $\mathcal{V}_{\mathrm{b}}(X_z)$ be the space of smooth vector fields on X_z which are tangent to all boundary faces and let μ_{b} be a b-density on X_z, i.e. , a density of the form $(\rho'\rho_{\mathrm{o}}\rho_{\mathrm{ff}})^{-1}\mu$, where μ is a smooth, nonvanishing density. Then we set

$$\begin{gathered}(\rho')^a\rho_{\mathrm{o}}^b\rho_{\mathrm{ff}}^c H_{\mathrm{b}}^M(X_z) = \left\{u; u = (\rho')^a\rho_{\mathrm{o}}^b\rho_{\mathrm{ff}}^c v,\ v \in H_{\mathrm{b}}^M(X_z)\right\}\ ,\\ H_{\mathrm{b}}^M(X_z) = \left\{u \in L^2(X_z;\mu_{\mathrm{b}});\ \mathcal{V}_{\mathrm{b}}^k u \subset L^2(X_z;\mu_{\mathrm{b}})\ \forall\, k \leq M\right\}.\end{gathered} \tag{7.7}$$

These are the conormal functions (conormal with respect to the boundary) and we also use the short-hand notation

$$(\rho')^\infty\rho_{\mathrm{o}}^\infty\rho_{\mathrm{ff}}^c H_{\mathrm{b}}^\infty(X_z) = \bigcap_{a,b}(\rho')^a\rho_{\mathrm{o}}^b\rho_{\mathrm{ff}}^c H_{\mathrm{b}}^\infty(X_z) \tag{7.8}$$

for the subspaces which are rapidly decreasing up to the boundaries other than ff .

Proposition 7.3. *If q is a center for W in $\Sigma(\lambda)$, with $\mathcal{M}$ the test module given by (7.3) in a W-balanced neighbourhood of q then, for any s, multiplication gives an isomorphism*

$$\{u \in \mathcal{C}^{-\infty}(X); \mathrm{WF}_{sc}(u) \subset \{q\},\ u \in I^{(s)}(O,\mathcal{M})\} \ni u \longmapsto e^{-i\nu(q)/x}u \in (\rho')^{\infty}\rho_{\mathrm{o}}^{\infty}\rho_{\mathrm{ff}}^{2s+3/2}H_{\mathrm{b}}^{\infty}(X_z). \tag{7.9}$$

Proof. We shall work in Riemannian normal coordinates in the boundary based at the minimum under discussion. In terms of these coordinates the boundary symbol of $P-\lambda$ is

$$p-\lambda = \nu^2+\mu^2+V_0(0)+y^2a(y)-\lambda+O(x),\ y^2a(y)=V_0(y)-V_0(0). \tag{7.10}$$

Here $\mu=0$, $y=0$ and $\nu=\tilde{\nu}=\sqrt{\lambda-V_0(0)}>0$ and $a(0)=V_0''(0)/2>0$ at the radial point. The module $\mathcal{M}$ is therefore generated by Id, $x^{\frac12}D_y$, $x^{-\frac12}y$ and

$$x^{-1}\left((x^2D_x)^2+x^2D_y^2-\tilde{\nu}^2+y^2a(y)\right). \tag{7.11}$$

Consider the effect of conjugation by $e^{i\tilde{\nu}/x}$. This maps each $\Psi_{\mathrm{sc}}^{m,l}(X)$ isomorphically to itself and hence

$$\widetilde{\mathcal{M}} = e^{-i\tilde{\nu}/x}\mathcal{M}e^{i\tilde{\nu}/x} \tag{7.12}$$

is another test module, but in the open set

$$\tilde{O} = \{(\nu-\tilde{\nu},y,\mu);(\nu,y,\mu)\in O\} \ni \tilde{q}=(0,0,0), \tag{7.13}$$

$\tilde{q}$ being the image of q. It is generated by the conjugates of the generators, namely Id, $x^{\frac12}D_y$, $x^{-\frac12}y$ and

$$\frac{1}{x}\left((x^2D_x-\tilde{\nu})^2-\tilde{\nu}^2\right)+xD_y^2+x^{-1}y^2a(y). \tag{7.14}$$

The last two terms in (7.14) are in $\widetilde{\mathcal{M}}^2$. So, following the observation (6.9), we may drop them without changing the spaces $I^{(s)}(\tilde{O},\widetilde{\mathcal{M}})$. The new generator can then be written

$$\frac{1}{x}\left((x^2D_x-\tilde{\nu})^2-\tilde{\nu}^2\right) = (xD_xx-2\tilde{\nu})xD_x.$$

The factor $xD_xx-2\tilde{\nu}\in\Psi_{\mathrm{sc}}^{1,0}(X)$ is elliptic at the base point $(0,0,0)$ so can be dropped without changing the test module. We can also add the product of the other two generators, again using (6.9). Thus we see that

$$I^{(s)}(O,\mathcal{M}) = e^{i\tilde{\nu}/x}\cdot I^{(s)}(\tilde{O},\mathcal{M}') \text{ with } \mathcal{M}' \text{ generated by Id},\ x^{\frac12}D_y,\ x^{-\frac12}y \text{ and } 2xD_x+yD_y. \tag{7.15}$$

Now, consider the effect of the parabolic blow up of z, passing from X to X_z on

$$v\in x^sL^2_{\mathrm{sc}}(X) \text{ with } \mathrm{WF}_{\mathrm{sc}}(v)\subset\{\tilde{q}\},\ v\in I^{(s)}(\tilde{O},\mathcal{M}'). \tag{7.16}$$

As a neighbourhood of the front face we may take

$$[0,\delta)_\rho\times\overline{\mathbb{R}_Y} \tag{7.17}$$

where $\rho=(x+|y|^2)^{\frac12}=x^{\frac12}(1+|Y|^2)^{-\frac12}$, $Y=x^{-\frac12}y$ and $\overline{\mathbb{R}_Y}$ is the radial compactification of the line (so that on $\overline{\mathbb{R}_Y}$, $1/|Y|$ is a defining function for 'infinity').

Thus we may take $\rho_{\mathrm{ff}} = \rho$ and $\rho_{\mathrm{o}} = (1+Y^2)^{-1/2}$ for explicit boundary defining functions. In terms of coordinates (ρ, Y), the generators of $\mathcal{M}'$ in (7.15) become

$$\text{(7.18)}\quad \mathrm{Id},\ x^{\frac{1}{2}}D_y = D_Y + Y(1+|Y|^2)^{-1}\rho D_\rho,\ x^{-\frac{1}{2}}y = Y,\ 2xD_x + yD_y = \rho D_\rho.$$

We may replace the second generator by just D_Y. Computing the pull-back of the Riemannian measure we see that in region (7.17) we have

$$\text{(7.19)}\qquad Y^k D_Y^l(\rho D_\rho)^m \beta^* v \in \rho_{\mathrm{o}}^{2s+2}\rho_{\mathrm{ff}}^{2s+3/2} L^2_{\mathrm{b}}([0,\delta)_\rho \times \overline{\mathbb{R}_Y})\ \forall\ k,l,m \in \mathbb{N}_0 .$$

Full iterative regularity with respect to Y and D_Y in a polynomially weighted L^2 space is equivalent to regularity in the Schwartz space, and hence to smoothness up to, and rapid vanish at, the old boundary $Y = \infty$. Thus (7.19) reduces to the statement that $\beta^* v$ is Schwartz in Y with values in the conormal space in $\rho = \rho_{\mathrm{ff}}$ which is precisely the content of (7.9). □

Theorem 7.4. *Let $q \in \mathrm{Min}_+(\lambda)$ be a center for W, i.e. (7.1) holds, and let (x,y) be local coordinates centered at z, with y given by arclength along ∂X. Let $X = x^{\frac{1}{2}}$ and $Y = x^{-\frac{1}{2}}y$, let $h = V''(z)$, and let Q be the operator*

$$\text{(7.20)}\qquad Q = D_Y^2 - \frac{\tilde{\nu}}{2}(YD_Y + D_Y Y) + \frac{1}{2}hY^2,$$

on $L^2(\mathrm{ff}, dY)$, with eigenvalues β_j and normalized eigenfunctions v_j, $j = 1,2,\ldots$.

Suppose that u satisfies (7.2). Then for all N there exist sequences γ_j, $\gamma_{j,k}$, $1 \le k \le N$ which are rapidly decreasing in j (i.e. for all s, there exists $C_s > 0$ such that $|\gamma_j|, |\gamma_{j,k}| \le C_s j^{-s}$, $j \ge 1$) and such that

(7.21)
$$u = e^{i\tilde{\nu}/x}v,\quad v = \sum_j X^{1/2}X^{i(\beta_j + V_1(0))/\tilde{\nu}}\left(\gamma_j + \sum_{k=1}^N \gamma_{j,k}X^k + \gamma'_{j,N}(X)\right)v_j(Y),$$
$$\text{where}\qquad \tilde{\nu} = \nu(q),\ V_1(y) = \frac{\partial V}{\partial x}(0,y),\ \gamma'_{j,N} \in \mathcal{S}(\mathbb{R}; X^N H^\infty_{\mathrm{b}}([0,\delta))).$$

The map from microfunctions u satisfying (7.2) to rapidly decreasing sequences $\{\gamma_j\}$ is a bijection.

Proof. Combining Propositions 7.1 and 7.3 we have now established the conormal regularity of the outgoing eigenfunctions at a center on the resolved space X_z. To deduce the existence of expansions as in (I.20) we consider the lift of the operator P to X_z. Using the coordinates $X = x^{\frac{1}{2}}$ and $Y = x^{-\frac{1}{2}}y$, arising from Riemannian normal coordinates in the boundary, we find that $\tilde{P} - \lambda = e^{-i\tilde{\nu}/x}(P-\lambda)e^{i\tilde{\nu}/x}$ takes the form

(7.22)
$$\begin{aligned}\tilde{P} - \lambda = e^{-i\tilde{\nu}/x}\Big((x^2D_x)^2 + ix^3D_x + x^2D_y^2 + x^2A(x,y,x^2D_x) + xB(x,y,xD_y)(xD_y)\\ + xC(x,y)(x^2D_x)(xD_y) + V(x,y) - \lambda\Big)e^{i\tilde{\nu}/x}\\ = X^2\left(-\tilde{\nu}(XD_X + i/2) + Q + V_1(0)\right) + X^3B'(x,y,XD_X,D_Y),\end{aligned}$$

where Q is as in (7.20), B' is a differential operator in XD_X and D_Y of order at most two with coefficients smooth in (x,y), and $V_1(y) = \partial_x V\big|_{x=0}$.

The operator Q is a harmonic oscillator; in fact, conjugation by $e^{i\tilde{\nu}Y^2/4}$ gives

$$e^{i\tilde{\nu}Y^2/4}Qe^{-i\tilde{\nu}Y^2/4} = D_Y^2 + \alpha^2Y^2, \quad \alpha = \sqrt{\frac{V_0''(0)}{2} - \frac{\tilde{\nu}^2}{4}} > 0, \tag{7.23}$$

where α is real by (7.1). Thus, Q has discrete spectrum with eigenvalues $\beta_j = \alpha(2j+1)$, and eigenvalues $v_j \in \mathcal{S}(\mathbb{R})$ which we take to be normalized in L^2.

The conclusion of (7.9) is that if u is a microlocal eigenfunction then $v = e^{-i\tilde{\nu}/x}u$ has conormal regularity on X_z and vanishes rapidly at the old boundary. Even though $X = x^{\frac{1}{2}}$, and $Y = x^{-\frac{1}{2}}y$ are singular coordinates at the old boundary, the rapid vanishing there means that $v(Y, X)$ is Schwartz in Y with values in a conormal space $X^rH_{\mathrm{b}}^{\infty}([0,\delta))$ in X. The remainder term in (7.22) maps this space into $\mathcal{S}(\mathbb{R}_Y; X^{r+3}H_{\mathrm{b}}^{\infty}([0,\delta)))$. Thus the condition $(P-\lambda)u \in \dot{\mathcal{C}}^{\infty}(X)$ becomes the iterative equation

$$\begin{aligned}(-\tilde{\nu}(XD_X + i/2) + Q + V_1(z))v = XB'(X^2, XY, XD_X, D_Y)v \\ \in \mathcal{S}(\mathbb{R}_Y; X^{r+1}H_{\mathrm{b}}^{\infty}([0,\delta)).\end{aligned}$$

Since v is Schwartz in Y with values in a conormal space $X^rH_{\mathrm{b}}^{\infty}([0,\delta))$ in X, the eigenfunction expansion

$$v = \sum_j \gamma_j(X)v_j(Y)$$

converges in the space $\mathcal{S}(\mathbb{R}_Y; X^rH_{\mathrm{b}}^{\infty}([0,\delta)))$, and the coefficients are Schwartz in j,

$$\{\gamma_j(.)\}_{j\in\mathbb{N}} \in \mathcal{S}(\mathbb{N}_j; X^rH_{\mathrm{b}}^{\infty}([0,\delta))), \tag{7.24}$$

and satisfy

$$(\tilde{\nu}XD_X + i\tilde{\nu}/2 - \beta_j - V_1(z))\gamma_j(X) = f_j(X) \in X^{r+1}H_{\mathrm{b}}^{\infty}([0,\delta). \tag{7.25}$$

Here (7.24) means that for every seminorm of $\gamma_j(.)$ in $X^rH_{\mathrm{b}}^{\infty}([0,\delta))$ and for every k there exists a constant $C > 0$ such that the seminorm of $\gamma_j(.)$ is bounded by Cj^{-k}. It follows from (7.25) that

$$\gamma_j(X) = X^{1/2}X^{i(\beta_j+V_1(z))/\tilde{\nu}}\gamma_j + \gamma_j'(X)$$

with γ_j constant and $\gamma_j' \in X^{r+1}H_{\mathrm{b}}^{\infty}([0,\delta))$. In fact the sequence f_j is rapidly decreasing in j with values in the space in (7.25) so the same is true of the terms γ_j' which come from integration. The γ_j, being the difference of two rapidly decreasing sequences, therefore also form a rapidly decreasing sequence. These arguments can be iterated, giving the asymptotic expansion, meaning that for any N

$$\begin{aligned}v = \sum_j X^{1/2}X^{i(\beta_j+V_1(z))/\tilde{\nu}}\left(\gamma_j + \sum_{k=1}^{N}\gamma_{j,k}X^k + \gamma_{j,N}'(X)\right)v_j(Y), \\ \gamma_{j,N}' \in \mathcal{S}(\mathbb{R}; X^NH_{\mathrm{b}}^{\infty}([0,\delta)))\end{aligned} \tag{7.26}$$

where the series in j converges rapidly. If all the leading constants γ_j vanish then repeated integration shows that $v \in \dot{\mathcal{C}}^{\infty}(X_z) = \dot{\mathcal{C}}^{\infty}(X)$ is rapidly decreasing.

Conversely the γ_j in (7.26) form an arbitrary rapidly decreasing sequence since v may be constructed iteratively as in (7.26) and then asymptotic summation, which can be made uniform in the Schwartz parameter j gives a corresponding microlocal eigenfunction.

This completes the proof of the theorem. ∎

8. Outgoing eigenfunctions at a sink

In this section we consider the structure of microlocally outgoing eigenfunctions at a radial point $q \in \mathrm{Min}_+(\lambda)$ which is a sink for W, which occurs when

$$\lambda > \lambda_{\mathrm{Hess}}(z) = \lambda_{\mathrm{Hess}} = V(z) + 2V''(z), \quad z = \pi(q).$$

In section 1 it was shown that in this case the linearization of W at q has two negative eigenvalues, $-2\tilde{\nu}r_1$ and $-2\tilde{\nu}r_2$, where $0 < r_1 < 1/2 < r_2 < 1$, $r_1 + r_2 = 1$ and $\tilde{\nu} = \nu(q)$. The discussion here closely parallels that in the previous section, so emphasis is placed on the differences. At the end of the section we also discuss the degenerate case, where $\lambda = \lambda_{\mathrm{Hess}}$ (and $r_1 = r_2 = 1/2$).

As in the previous section, we may assume the microlocal eigenfunctions satisfy (7.2). We again use commutator methods to deduce iterative regularity, now with respect to the $\Psi_{\mathrm{sc}}^{-\infty,0}(X)$-module

$$\begin{aligned}\mathcal{M} \text{ generated by } \mathrm{Id},\ &\{x^{-r_1}B,\ B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \sigma_\partial(B)(q) = 0\},\\ &\{x^{-r_2}B,\ B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \sigma_\partial(B)|_{L_2} = 0\} \text{ and } x^{-1}(P-\lambda).\end{aligned} \tag{8.1}$$

Recall here that L_2 is a smooth curve given by Proposition 1.4.

We separate the resonant ($r_2/r_1 = 2, 3, 4, \dots$) and non-resonant cases, although the proof for the resonant case would go through in the non-resonant case as well, since the proof in the latter case is more transparent.

Proposition 8.1. *Suppose q is a sink for W, with eigenvalues $-2\tilde{\nu}r_i$, as above, and suppose that r_2/r_1 is not an integer. Then the module $\mathcal{M}$ in* (8.1) *is a test module in the sense of Definition 6.2, which satisfies* (6.14). *If u satisfies* (7.2) *at q, then $u \in I_{\mathrm{sc}}^{(-1/2-\epsilon)}(O, \mathcal{M})$ for all $\epsilon > 0$.*

Proof. Choose real symbols $a_j \in \mathcal{C}_c^\infty({}^{\mathrm{sc}}T^*X)$ such that $da_j(q)$ is, restricted to $\Sigma(\lambda)$ at q, a non-trivial eigenvector for the linearization of W with eigenvalue $-2\tilde{\nu}r_j \in \mathbb{R}$ and with $a_2 = 0$ on L_2 near q. Then a_1, a_2 generate, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions vanishing at q. Similarly, a_2 generates, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions which vanish on L_2. Then if $A_j \in \Psi_{\mathrm{sc}}^{-\infty,-r_j}(X)$ have symbols $x^{-r_j}a_j$, $j = 1, 2$, $\mathcal{M}$ is generated by $A_0 = \mathrm{Id}$, A_1, A_2 and $A_3 = x^{-1}(P-\lambda)$ (notice that the operator with symbol $x^{-r_1}a_2$ is in $\mathcal{M}$, since $r_1 < r_2$).

By (6.13), the commutator $ix^{-1}[A_j, P-\lambda] \in \Psi_{\mathrm{sc}}^{-\infty,-r_j}(X)$ has principal symbol

$$-\,{}^{\mathrm{sc}}H_p(x^{-r_j}a_j) = x^{-r_j}e_j,\ j = 1, 2, \tag{8.2}$$

where the e_j vanish to second order at q (here we exploit a cancellation, similar to that in Remark 7.2). Since ${}^{\mathrm{sc}}H_p$ is tangent to L_2, e_2 in fact vanishes on L_2. Thus

$$\begin{aligned} ix^{-1}[A_1, P-\lambda] &= C_{10}x^{1-r_1} + C_{11}A_1 + x^{r_2-r_1}C_{12}A_2 + x^{1-r_1}C_{13}A_3,\\ ix^{-1}[A_2, P-\lambda] &= C_{20}x^{1-r_2} + C_{22}A_2 + x^{1-r_2}C_{23}A_3, \end{aligned} \tag{8.3}$$

with $C_{jk} \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, $\sigma_\partial(C_{jk})(q) = 0$ for all j and all $k \geq 1$. Note that $[A_1, A_2] \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ as $r_1 + r_2 = 1$. So $\mathcal{M}$ is closed under commutators, and hence is a test module satisfying (6.14).

The statement about u follows exactly as in Proposition 7.1. □

Next we deal with the case when $r_2/r_1 = N$ is an integer, so $r_1^{-1} = 1/(N+1)$. Let $\mathcal{I}$ be the ideal of $\mathcal{C}^\infty$ functions on C_∂ vanishing at q. Now consider the $\Psi_{\mathrm{sc}}^{-\infty,0}(X)$-module

$$
\begin{aligned}
\text{(8.4)}\quad & \mathcal{M} \text{ generated by Id}, \ \{x^{-k/(N+1)}B,\ B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \sigma_\partial(B) \in \mathcal{I}^k\},\ k = 1, \dots, N \\
& \{x^{-N/(N+1)}B,\ B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \sigma_\partial(B)|_{L_2} \in \mathcal{I}^N\} \text{ and } x^{-1}(P-\lambda).
\end{aligned}
$$

Proposition 8.2. *Suppose q is a sink for W, with eigenvalues $-2\tilde{\nu} r_i$, and suppose that $r_2/r_1 = N$ is an integer. Then the module $\mathcal{M}$ in* (8.4) *is a test module in the sense of Definition 6.2, which satisfies* (6.15). *If u satisfies* (7.2) *at q, then $u \in I_{\mathrm{sc}}^{(-1/2-\epsilon)}(O, \mathcal{M})$ for all $\epsilon > 0$.*

Proof. Assume first that $r_1 < 1/2 < r_2$, i.e. that $\lambda > \lambda_{\mathrm{Hess}}$. Choose real symbols $a_j \in \mathcal{C}_c^\infty({}^{\mathrm{sc}}T^*X)$ such that $da_j(q)$ is, restricted to $\Sigma(\lambda)$ at q, a non-trivial eigenvector for the linearization of W with eigenvalue $-2\tilde{\nu} r_j \in \mathbb{R}$, and a_2 vanishes on L_2. Then a_1, a_2 generate, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions vanishing at q. Similarly, a_2 generates, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions vanishing on L_2. Certainly $A_k \in \Psi_{\mathrm{sc}}^{-\infty,-kr_1}(X)$ with symbol $x^{-kr_1}a_1^k$, $k = 1, 2, \dots, N$, are in the set in (8.4), as is A_{N+1} with symbol $x^{-r_2}a_2$, and indeed $\mathcal{M}$ is generated by $A_0 = \mathrm{Id}$, $A_1, \dots, A_{N+1}$ and $A_{N+2} = x^{-1}(P-\lambda)$.

By (6.13), the commutator $ix^{-1}[A_1, P-\lambda] \in \Psi_{\mathrm{sc}}^{-\infty,-r_1}(X)$ has principal symbol

$$
-\,{}^{\mathrm{sc}}H_p(x^{-r_1}a_1) = x^{-r_1}e_1,
$$

and $ix^{-1}[A_{N+1}, P-\lambda] \in \Psi_{\mathrm{sc}}^{-\infty,-r_2}(X)$ has principal symbol

$$
-\,{}^{\mathrm{sc}}H_p(x^{-r_2}a_2) = x^{-r_2}e_2,
$$

where the e_j vanish to second order at q. Since W is tangent to L_2 to order N, $e_2 = e_2' + e_2''$, e_2' vanishing on L_2 and $e_2'' \in \mathcal{I}^N$. Thus

$$
\begin{aligned}
ix^{-1}[A_1, P-\lambda] =& C_{10}x^{1-r_1} + C_{11}A_1 + x^{r_2-r_1}C_{1,N+1}A_{N+1} \\
& + x^{1-r_1}C_{1,N+2}A_{N+2}, \\
ix^{-1}[A_k, P-\lambda] =& C_{k0}x^{1-kr_1} + C_{kk}A_k + x^{r_2-kr_1}C_{k,N+1}A_{N+1} \\
& + x^{1-kr_1}C_{k,N+2}A_{N+2},\ k = 2, 3, \dots, N \\
ix^{-1}[A_{N+1}, P-\lambda] =& C_{20}x^{1-r_2} + C_{N+1,N}A_N + C_{N+1,N+1}A_{N+1} \\
& + x^{1-r_2}C_{N+1,N+2}A_{N+2},
\end{aligned}
\tag{8.5}
$$

with $C_{jk} \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$,

$$
\sigma_\partial(C_{jk})(q) = 0 \text{ for all } j \le k \text{ and } k \ge 1,
$$

but $\sigma_\partial(C_{N+1,N})(q)$ usually does not vanish. It is not hard to check that $\mathcal{M}$ is closed under commutators, and hence is a test module satisfying (6.15).

The statement about u follows exactly as in Proposition 7.1.

If $r_1 = r_2 = 1/2$, there is only one eigenvector, which we may arrange to be $da_1(q)$. We choose any a_2 with $da_2(q)$ linearly independent. The argument as above still works, with $N = 1$. □

Remark 8.3. In the case that r_2/r_1 is an integer, W is not tangent to the curve L_2, and it is necessary to add extra generators $A_2, \dots, A_N$ to make the module closed under commutators, as the proof above shows. Nevertheless, the enveloping algebra of $\mathcal{M}$ in (8.4) is the same as the enveloping algebra of the module in (8.1), so for regularity considerations, we work with (8.1) below, instead of (8.4), even when r_2/r_1 is integral.

We can again interpret this interative regularity more geometrically by introducing a blown up space, although with blow-up of different homogeneities in different variables, as discussed in Section 2. Namely, we consider the blow-up of X at p along the vector field $\tilde{V} = r_1^{-1} x\partial_x + y\partial_y$:

$$X_{z,r_1} = [X;\{z\}]_{\tilde{V}},\ \tilde{V} = r_1^{-1} x\partial_x + y\partial_y, \tag{8.6}$$

with blow-down map $\beta = \beta_{r_1}$. This space is different to that considered in the previous section (unless $r_1 = 1/2$), but it 'looks similar', and we use similar notation to describe it. It is again a compact manifold with corners, with two boundary curves, the front face $\mathrm{ff} = \beta^{-1}(z)$ and the old boundary . The two boundary curves are the front face $\mathrm{ff} = \beta^{-1}(z)$, created by the blowup, and the closure of the pullback of $\partial X \setminus \{z\}$ which we shall refer to as the old boundary. We shall again denote boundary defining functions for these boundary curves by ρ_{ff} and ρ_{o}, and denote the product of boundary defining functions for the other boundary curves of X by ρ'.

We next describe the appropriate conjugation. Since L_2 is Legendre and has full rank projection to ∂X near q, we can arrange that nearby L_2 is given by $\nu = \Phi_2(y)$, $\mu = \Phi_2'(y)$, so inside $\Sigma(\lambda)$, $\mu - \Phi_2'(y)$ is a defining function for L_2.

Proposition 8.4. *If $q \in \mathrm{Min}_+(\lambda)$ is a sink for W in $\Sigma(\lambda)$, with $\mathcal{M}$ the test module given by* (8.1) *or* (8.4) *in a W-balanced neighbourhood of q then for any s, multiplication gives an isomorphism*

$$\begin{multline}\{u \in \mathcal{C}^{-\infty}(X); \mathrm{WF}_{sc}(u) \subset \{q\},\ u \in I^{(s)}(O,\mathcal{M})\} \ni u \longmapsto \\ e^{-i\Phi_2(y)/x} u \in (\rho')^\infty \rho_{\mathrm{o}}^\infty \rho_{\mathrm{ff}}^{(s+1)/r_1 - 1/2} H_{\mathrm{b}}^\infty(X_{z,r_1}).\end{multline} \tag{8.7}$$

Remark 8.5. Φ_2 can be replaced by any other smooth function $\tilde{\Phi}_2$ such that $\Phi_2 - \tilde{\Phi}_2 = \mathcal{O}(y^{1/r_1})$.

Proof. As in the beginning of the proof of Proposition 7.3 we have coordinates in which (7.10) holds. In this case the module $\mathcal{M}$ is therefore generated by $A_0 = \mathrm{Id}$, $A_1 = x^{-r_1} y$, $A_2 = x^{-r_2}(xD_y - \Phi_2'(y))$ and $A_3 = x^{-1}(P - \lambda)$. Note that $\Phi_2(y) - \nu$ also vanishes on L_2, hence $x^{-r_2}(x^2 D_x + \Phi_2(y)) \in \mathcal{M}$ as well.

Consider the effect of conjugation by $e^{i\Phi_2(y)/x}$. This maps each $\Psi_{\mathrm{sc}}^{m,l}(X)$ isomorphically to itself and hence

$$\widetilde{\mathcal{M}} = e^{-i\Phi_2(y)/x} \mathcal{M} e^{i\Phi_2(y)/x} \tag{8.8}$$

is another test module, but in the open set

$$\tilde{O} = \{(\nu - \Phi_2(y), y, \mu - \Phi_2'(y)); (\nu, y, \mu) \in O\} \ni \tilde{q} = (0,0,0), \tag{8.9}$$

$\tilde{q}$ being the image of q. It is generated by the conjugates of the generators, namely Id, $x^{-r_1} y$, $x^{-r_2}(xD_y)$ and

$$x^{-1}\tilde{P} = x^{-1} e^{-i\Phi_2(y)/x} (P - \lambda) e^{i\Phi_2(y)/x} \tag{8.10}$$

To compute this generator, we write $P-\lambda$ as in (7.22) and express (8.10) as
(8.11)
$$\begin{aligned}x^{-1}\Big((x^2D_x-\Phi_2)^2+ix(x^2D_x-\Phi_2)+(xD_y+\partial_y\Phi_2)^2+x^2A(x,y,x^2D_x-\Phi_2)\\+xB(x,y,xD_y+\partial_y\Phi_2)(xD_y+\partial_y\Phi_2)\\+xC(x,y)(x^2D_x-\Phi_2)(xD_y+\partial_y\Phi_2)+V(x,y)-\lambda\Big)\\=x^{-1}\Big(\Phi_2^2+(\partial_y\Phi_2)^2+V_0(y)-\lambda\Big)\\-2\Phi_2(xD_x)-i\Phi_2+2(\partial_y\Phi_2)D_y-i\partial_y^2\Phi_2+B(x,y,\partial_y\Phi_2)\partial_y\Phi_2\\-C(x,y)\Phi_2\partial_y\Phi_2+V_1(y)+B_1y^2D_y+B_2xD_y^2+B_3x^3D_x^2\\+B_4x^2D_x+B_5xD_y+B_6y+B_7x,\ B_j\in\mathrm{Diff}^2_{\mathrm{sc}}(X).\end{aligned}$$

In the nonresonant case, the term proportional to x^{-1} vanishes identically, since Φ_2 satisfies the eikonal equation $\Phi^2+(\partial_y\Phi)^2+V-\lambda=0$. Eliminating all terms which are in the span of the generators Id, $x^{-r_1}y$, $x^{-r_2}(xD_y)$ over $\Psi^{*,0}_{\mathrm{sc}}(X)$, as well as (following the observation (6.9)) elements of $\mathcal{M}^2$, we are left with the generator
$$-2\Phi_2(xD_x)+2(\partial_y\Phi_2)D_y.$$
A Taylor series analysis of Φ_2 gives
$$\Phi_2(y)=\tilde\nu(1-r_1y^2/2+O(y^3)). \tag{8.12}$$
Thus, $2(\partial_y\Phi_2)D_y$ is of the form $a(y)yD_y$. This may be written $a(y)(x^{-r_1}y)(x^{r_1}D_y)$, the product of two generators and a smooth function, so we may eliminate this term, again following the observation (6.9). Thus, in the nonresonant case (8.10) may be replaced by the generator xD_x. This is true in the resonant case as well, since, although the term proportional to x^{-1} in (8.11) does not vanish, it is $O(y^{1/r_1})$, and $x^{-1}y^{1/r_1}\in\mathcal{M}^{1/r_1}$. In summary, we have shown
$$\begin{aligned}I^{(s)}(O,\mathcal{M})=e^{i\Phi_2(y)/x}\cdot I^{(s)}(\tilde O,\mathcal{M}')\text{ with}\\\mathcal{M}'\text{ generated by Id, }x^{-r_1}y,\ x^{-r_2}(xD_y)\text{ and }xD_x.\end{aligned}\tag{8.13}$$

Now, consider the effect of the inhomogeneous blow up of z, passing from X to X_z on
$$v\in x^sL^2_{\mathrm{sc}}(X)\text{ with }\mathrm{WF}_{\mathrm{sc}}(v)\subset\{\tilde q\},\ v\in I^{(s)}(\tilde O,\mathcal{M}'). \tag{8.14}$$
As a neighbourhood of the front face we may take
$$[0,\delta)_\rho\times\overline{\mathbb{R}_Y} \tag{8.15}$$
where $\rho=(x^{2r_1}+y^2)^{1/2}$, $Y=x^{-r_1}y$ and $\overline{\mathbb{R}_Y}$ is the radial compactification of the line, where now $1/|Y|^{1/r_1}$ is a defining function for 'infinity' on $\overline{\mathbb{R}_Y}$. In terms of these coordinates, the generators of $\mathcal{M}'$ in (8.13) become
(8.16)
$$\mathrm{Id},\ x^{-r_1}y=Y,\ x^{r_1}D_y=D_Y+(1+|Y|^2)^{-1}Y\rho D_\rho,\ r_1(1+Y^2)^{-1}\rho D_\rho-r_1YD_Y.$$
A suitable linear combination of the last two generators, namely $yD_y+r_1^{-1}xD_x$ from (8.13), is ρD_ρ. Using (6.9) as before to simplify the third generator and computing the pull-back measure we see that in (8.15)
$$Y^kD_Y^l(\rho D_\rho)^m\beta^*v\in\rho_{\mathrm{o}}^{(s+1)/r_1}\rho_{\mathrm{ff}}^{(s+1)/r_1-1/2}L^2_{\mathrm{b}}([0,\delta)_\rho\times\overline{\mathbb{R}_Y})\ \forall\ k,l,m\in\mathbb{N}_0. \tag{8.17}$$

Full iterative regularity with respect to Y and D_Y in a polynomially weighted L^2 space is equivalent to regularity in the Schwartz space, and hence to smoothness up to, and rapid vanish at, the old boundary $Y = \infty$. Thus (8.17) reduces to the statement that $\beta^* v$ is Schwartz in Y with values in the conormal space in ρ_o which is precisely the content of (8.7). $\square$

We now show that outgoing microlocal eigenfunctions have full asymptotic expansions on X_{z,r_1}. For this we recall from [12] that if $I \subset \mathbb{R}$ is discrete, $I \cap (-\infty, a]$ finite for $a \in \mathbb{R}$, and $I + \mathbb{N}_0 \subset I$ then $\mathcal{A}^I_{\mathrm{phg},\mathrm{ff}}(X_{z,r_1})$ is the space of polyhomogeneous conormal distributions with index set I, that vanish to infinite order everywhere but on the front face ff. That is, $u \in \mathcal{A}^I_{\mathrm{phg},\mathrm{ff}}(X_{z,r_1})$ means there exist $\phi_t \in \mathcal{C}^\infty(X_{z,r_1})$, $t \in I$, vanishing to infinite order everywhere but on the front face, such that

$$u \sim \sum_{t \in I} \phi_t \rho_{\mathrm{ff}}^t.$$

Here the summation is understood as asymptotic summation, i.e. with the notation of (8.17), the difference $v = u - \sum_{t\in I,\ t\leq a} \phi_t \rho_{\mathrm{ff}}^t$ satisfies estimates

$$Y^k D_Y^l (\rho D_\rho)^m v \in \rho_{\mathrm{ff}}^a L^2_{\mathrm{b}}([0,\delta)_\rho \times \overline{\mathbb{R}_Y}) \ \forall\ k, l, m \in \mathbb{N}_0. \tag{8.18}$$

Theorem 8.6. *Suppose that $q \in \mathrm{Min}_+(\lambda)$ is a sink of W, and u satisfies* (7.2)*. Let $\beta = r_2/2 + iV_1(z)/2\tilde\nu$. If r_2/r_1 is not an integer, then*

$$u_0 = x^{-\beta} e^{-i\Phi_2/x} u \in \mathcal{A}^I_{phg,\mathrm{ff}}(X_{z,r_1}),\ \ I = \mathbb{N}_0 + \mathbb{N}_0 \frac{1}{r_1} + \mathbb{N}_0 \frac{2r_2 - 1}{r_1} + \mathbb{N}_0 \frac{r_2}{r_1}. \tag{8.19}$$

In particular, u_0 vanishes to infinite order off the front face, and is continuous up to the front face. The map sending microfunctions u satisfying (7.2) *to $u_0|_{\mathrm{ff}}$ is a bijection.*

If $1/r_1$ is an integer, $r_1 < 1/2$, then the same conclusions hold with (8.19) *replaced by*

$$u_0 = x^{-\beta} x^{icY^{1/r_1}} e^{-i\Phi_2/x} u, \quad Y = y/x^{r_1}, \tag{8.20}$$

where c is a constant determined by the Taylor series, up to order $1/r_1$, of V and g at z.

Proof. Suppose first that r_2/r_1 is not an integer. Let $\delta = \min(2r_2 - 1, r_1) > 0$. We see from (8.11) and (8.12) that

$$x^{-1}\tilde P = -2\tilde\nu\big(xD_x + r_1 y D_y + i\beta\big) + \rho_{\mathrm{ff}}^{\delta/r_1} P_2, \tag{8.21}$$

where P_2 is a differential operator of degree at most two generated by vector fields tangent to the boundary of X_{z,r_1}. By Proposition 8.4, $v = e^{-i\Phi_2/x}u$ is invariant under such vector fields. Thus, we deduce that

$$x^{-1}\tilde P v = -2\tilde\nu(xD_x + r_1 y D_y + i\beta)v + \rho_{\mathrm{ff}}^{\delta/r_1} P_2 v \in \dot{\mathcal{C}}^\infty(X),$$

so with $xD_{x|Y} = xD_x + r_1 y D_y$ denoting the derivative keeping Y, rather than y, fixed, we have by Proposition 8.4

$$(xD_{x|Y} + i\beta)v \in x^{-1/2+\delta'} L^2_{\mathrm{sc}}(X) \quad \forall\, 0 < \delta' < \delta. \tag{8.22}$$

Writing $v = x^\beta \tilde v$, and noting that $\mathrm{Re}\,\beta = r_2/2$, this yields

$$xD_{x|Y}\tilde v \in x^{-1/2+\delta'-r_2/2} L^2_{\mathrm{sc}}(X) = x^{-1/2+\delta'-r_2/2} L^2\big(\frac{dx\,dy}{x^3}\big).$$

Taking into account the smoothness in Y, and changing the measure, this means for each fixed Y,

$$\partial_x \tilde{v} \in x^{-1/2+\delta'} L^2([0,1)_x; dx) \subset x^{\delta''} L^1([0,1)_x; dx), \ \forall \delta'' < \delta'.$$

But that implies that $\tilde{v}$ is continuous to $X = 0$, and after subtracting $\tilde{v}(0, Y)$, the result is bounded by $Cx^{\delta''}$. This gives that, modulo $x^{-1/2+\epsilon'} L^2_{\mathrm{sc}}(X)$, $\epsilon' > 0$ small, microlocally near the critical point, u has the form $x^\beta e^{i\Phi_2(y)/x} u_0$, where u_0 is smooth on the blown-up space and rapidly vanishing off the front face. A simple asymptotic series construction then yields the asymptotic series described before, and then the uniqueness result Proposition 4.10 shows that u is actually given by such a series.

Now suppose that r_2/r_1 is an integer. We only need minor modifications of the argument given above; we indicate these below. Equation (8.21) must be replaced by

$$x^{-1}\tilde{P} = -2\tilde{\nu}\big(xD_x + r_1 y D_y + cx^{-1} y^{1/r_1} + i\beta\big) + x^\delta P_2,$$

where here $\delta = \min(2r_2 - 1, r_1) > 0$. Equation (8.22) must be replaced by

$$(xD_{x|Y} + cY^{1/r_1} + i\beta)v \in x^{-1/2+\delta'} L^2_{\mathrm{sc}}(X) \quad \forall 0 < \delta' < \delta. \tag{8.23}$$

Thus, we reach the same conclusion as before if we redefine

$$\tilde{v} = x^{-\beta} x^{icY^{1/r_1}} v.$$

□

Remark 8.7. First, we remark that if $1/r_1 \notin \mathbb{N}$, the form of the parameterization (8.19) is unchanged if we replace Φ_2, parameterizing the smooth Legendrian L_2, by $\tilde{\Phi}_2 \in \mathcal{C}^\infty(X)$ with $\Phi_2 - \tilde{\Phi}_2 \in \mathcal{I}^N$, $N > 1/r_1$. Indeed, then

$$(\Phi_2 - \tilde{\Phi}_2)/x = (y/x^{r_1})^N x^{r_1 N - 1} a, \quad a \in \mathcal{C}^\infty(X),$$

so both the form of (8.19) and $u_0|_{\mathrm{ff}}$ are unaffected by this change of phase functions. In particular, if $r_1 > 1/N$, but we let $r_1 \to 1/N$, the factor $x^{r_1 N - 1}$ decays less and less, and indeed in the limit it takes a different form (8.20).

If now we allow $\Phi_2 - \tilde{\Phi}_2 \in \mathcal{I}^N$, $|r_1^{-1} - N| < 1$ for some $N \in \mathbb{N}$, then (8.21) is replaced by

$$x^{-1} e^{-i\tilde{\Phi}_2/x}(P - \lambda) e^{i\tilde{\Phi}_2/x} = -2\tilde{\nu}(xD_x + r_1 y D_y) + cy^N/x + i\beta + \rho_{\mathrm{ff}}^\delta P_2, \tag{8.24}$$

where P_2 is in the enveloping algebra of $\widetilde{\mathcal{M}}$ and $\delta > 0$. (Note that the y^N/x term can be included in P_2 if $N > 1/r_1$, but not otherwise, and we are interested in taking $r_1 \to 1/N$!) Proceeding with the argument as in the resonant case, (8.23) is replaced by

$$(xD_{x|Y} + cx^{r_1 N - 1} Y^N + i\beta)v \in x^{-1/2+\delta} L^2_{\mathrm{sc}}(X). \tag{8.25}$$

If $r_1 N - 1 \neq 1$, we can remove the $(cx^{r_1 N - 1} Y^N + i\beta)$ terms by introducing an integrating factor, namely by writing $v = x^{-\beta} e^{-ic(r_1 N - 1)^{-1} x^{r_1 N - 1} Y^N} \tilde{v}$, to obtain that $\tilde{v}$ is continuous to $x = 0$, and the asymptotics take the form

$$u = x^\beta e^{-ic(r_1 N - 1)^{-1} x^{r_1 N - 1} Y^N} e^{i\Phi_2/x} u_0, \quad Y = y/x^{r_1}, \tag{8.26}$$

with u_0 continuous up to the front face of X_{z,r_1}, and (8.20) is the limiting case as $r_1 N \to 1$. In particular, the special role played by $1/r_1 \in \mathbb{N}$ in the statement of the theorem is partly due to its formulation. Here if $N > 1/r_1$, the factor

$e^{-ic(r_1N-1)^{-1}x^{r_1N-1}Y^N}$ tends to 1 in the interior of ff, i.e. does not affect the form of the asymptotics, but if $N < 1/r_1$, it introduces an oscillatory factor with phase $-c(r_1N-1)^{-1}y^N/x$.

Finally, we analyze what happens at the Hessian threshold, i.e. when $r_1 = r_2 = 1/2$. The correct space to describe the asymptotics is not quite the parabolic blow-up $[X;\{z\}]_{\frac{1}{2}}$, on which $Y = yx^{-1/2}$ is smooth, rather the space on which

$$\frac{y}{x^{1/2}\log x} = \frac{Y}{\log x}$$

is smooth. To simplify the statement we only consider the top order of the asymptotics.

Theorem 8.8. *Suppose that $q \in \mathrm{Min}_+(\lambda)$, and λ is at the Hessian threshold for q, so that $r_1(q) = r_2(q) = \frac{1}{2}$. Let β as in Theorem 8.6, let $\Phi_2(y) = \tilde\nu(1-y^2/4)$, where $\tilde\nu = \nu(q)$, and let $Y = x^{-1/2}y$. Suppose that u satisfies (7.2). Then*

$$u_0 = x^{-\beta}(\log x)^{1/2}e^{-i\Phi_2/x}e^{i\frac{\tilde\nu}{4}\frac{Y^2}{\log x}}u \tag{8.27}$$

is such that

$$u_0 - g(Y/\log x) \in (\log x)^{-1}L^\infty(X) \text{ for some } g \in \mathcal{S}(\mathbb{R}). \tag{8.28}$$

The map from microfunctions u satisfying (7.2) to g is a bijection.

The full asymptotic expansion, which is not stated above, is in terms of powers of $\log x$ and it arises from a stationary phase argument as can be seen from the proof given below. One could instead give a description of u as an oscillatory integral, analogous to how u is described as an oscillatory sum in Theorem 7.4, with an error term in $x^{\frac{1}{2}-\delta}L^\infty$ for all $\delta > 0$. In particular, not only are u, hence all terms of the (logarithmic) asymptotic expansion, determined by g, but the terms only depend on the principal symbol of $P-\lambda$ near q, modulo $\mathcal{I}^2$, and the subprincipal symbol at q.

Proof. We use the computations of Theorem 7.4. Thus, with $\tilde P = e^{-i\tilde\nu/x}Pe^{i\tilde\nu/x}$, and $X = x^{1/2}$,

$$\tilde P = X^2\left(-\tilde\nu(XD_X+i/2) + Q + V_1(z)\right) + X^3B'(X, XD_X, Y, D_Y), \quad \text{with } Q = e^{i\tilde\nu Y^2/4}D_Y^2e^{-i\tilde\nu Y^2/4}, \tag{8.29}$$

since α in (7.23) is zero at the Hessian threshold. Thus

$$\begin{aligned}\tilde P' &= e^{-i\Phi_2/x}(P-\lambda)e^{i\Phi_2/x}\\ &= X^2\left(-\tilde\nu(XD_X+i/2) + D_Y^2 + V_1(z)\right) + X^3B''(X, XD_X, Y, D_Y).\end{aligned} \tag{8.30}$$

We thus deduce that

$$X^{-3}\left(\tilde P' - X^2(-\tilde\nu XD_X + D_Y^2 - 2i\tilde\nu\beta)\right)$$

is in the enveloping algebra of $\widetilde{\mathcal{M}}$. Thus, with $u = e^{i\Phi_2/x}X^\beta v$,

$$(-\tilde\nu XD_X + D_Y^2)v \in x^{1-\delta}L^2_{\mathrm{sc}}(X),\ \forall\delta > 0.$$

The operator on the left hand side is the Schrödinger operator $-D_t + D_Y^2$ after a logarithmic change of variables $t = \frac{1}{\tilde\nu}\log X$. This can now be solved explicitly by taking the Fourier transform in Y, using the fact that v is Schwartz in Y for

X bounded away from 0. The solution thus has the form of the inverse Fourier transform of $X^{i\eta^2/\tilde{\nu}}\tilde{g}(\eta)$, plus faster vanishing terms, where $\tilde{g}$ is Schwartz, and η is the dual variable of Y. This gives an expression of the form (8.27). □

9. Outgoing eigenfunctions at a saddle

We next analyze microlocal eigenfunctions $u \in E_{\mathrm{mic},+}(q,\lambda)$ when $q \in \mathrm{Max}_+(\lambda)$, hence q is an outgoing saddle point for W. Recall from section 1 that in this case there are always two smooth Legendre curves through q which are tangent to W, L_1 and L_2, and there is a local coordinate v_i on L_i with $v_i = 0$ at q such that W takes the form $-2\tilde{\nu}r_1 v_2 \partial_{v_2}$ on L_2 and $-2\tilde{\nu}r_2 v_1 \partial_{v_1}$ on L_1. Here $\tilde{\nu} = \nu(q)$ is positive, and r_i satisfy $r_1 < 0$ and $r_2 > 1$. Hence $L_2 \subset \Phi_+(\{q\})$ is the outgoing Legendrian in this case, and by microlocalizing $u \in E_{\mathrm{mic},+}(q,\lambda)$ to a W-balanced neighbourhood O of q, we may assume that

$$u \in \mathcal{C}^{-\infty}(X),\ O \cap \mathrm{WF}_{\mathrm{sc}}((P-\lambda)u) = \emptyset,\ \mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset L_2. \tag{9.1}$$

As before we first use commutator methods to deduce iterative regularity, now with respect to the $\Psi_{\mathrm{sc}}^{-\infty,0}(X)$-module

$$\mathcal{M} \text{ generated by } \mathrm{Id},\ \{x^{-1}B,\ B \in \Psi_{\mathrm{sc}}^{-\infty,0}(X),\ \sigma_\partial(B)|_{L_2} = 0\}. \tag{9.2}$$

In particular, $x^{-1}(P-\lambda) \in \mathcal{M}$, since $\sigma_\partial(P-\lambda)$ vanishes on L_2.

Proposition 9.1. *The module $\mathcal{M}$ in* (9.2) *is a test module in the sense of Definition 6.2, which satisfies* (6.14)*. If u satisfies* (9.1) *at q, then $u \in I_{\mathrm{sc}}^{(-1/2-\epsilon)}(O,\mathcal{M})$ for all $\epsilon > 0$. That is, u is a Legendre distribution microlocally near q.*

Proof. Choose real symbols $a_j \in \mathcal{C}_c^\infty({}^{\mathrm{sc}}T^*X)$ such that a_j vanishes on L_j in a neighbourhood of q, and such that $da_j(q)$ is, restricted to $\Sigma(\lambda)$ at q, a non-trivial eigenvector for the linearization of W with eigenvalue $-2\tilde{\nu}r_j \in \mathbb{R}$. Then a_1, a_2 generate, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions vanishing at q. Similarly, a_i generates, over $\mathcal{C}_c^\infty(\Sigma(\lambda))$ and locally near q, the ideal of functions also with differentials vanishing on L_i. Certainly $A_1 \in x^{-1}\Psi_{\mathrm{sc}}^{-\infty,0}(X)$ with symbol $x^{-1}a_2$, is in the set in (9.2) and indeed $\mathcal{M}$ is generated by $A_0 = \mathrm{Id}$, A_1 and $A_2 = x^{-1}(P-\lambda)$.

By (6.13), the commutator $i[A_1, P-\lambda] \in \Psi_{\mathrm{sc}}^{-\infty,-1}(X)$ has principal symbol

$$-\,{}^{\mathrm{sc}}H_p(x^{-1}a_2) = x^{-1}(2\tilde{\nu}(r_2-1)a_2 + e_2).$$

where e_2 vanishes to second order at q. Since ${}^{\mathrm{sc}}H_p$ is tangent to L_2, e_2 in fact vanishes on L_2. Thus

$$ix^{-1}[A_1, P-\lambda] = C_{10} + C_{11}A_1 + C_{12}A_2, \tag{9.3}$$

with $C_{jk} \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, $\sigma_\partial(C_{12})(q) = 0$, and $\sigma_\partial(C_{11})(q) = 2\tilde{\nu}(r_2-1) > 0$. In particular $\mathcal{M}$ is closed under commutators, and hence is a test module satisfying (6.14).

To prove the statement about u, note that $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2-\epsilon}(u) \cap O = \emptyset$ for any W-balanced neighbourhood O of q, by Corollary 5.9. To apply Proposition 6.7, we need to construct a Q satisfying (6.27). We take $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ such that

$$\sigma_\partial(Q) = q = \chi_1(a_1^2)\chi_2(a_2)\psi(p), \tag{9.4}$$

where $\chi_1, \chi_2, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi_1, \chi_2 \geq 0$ are supported near 0, ψ supported near σ, $\chi_1, \chi_2 \equiv 1$ near 0 and $\chi_1' \leq 0$ in $[0, \infty)$. Note that $a_2 = 0$ on L_2, so $\operatorname{supp} d(\chi_2 \circ a_2) \cap L_2 = \emptyset$. On the other hand,

$$ {}^{\mathrm{sc}}H_p \chi_1(a_1^2) = 2a_1({}^{\mathrm{sc}}H_p a_1)\chi_1'(a_1^2) = -4\tilde{\nu} a_1(r_1 a_1 + e_1)\chi_1'(a_1^2), \tag{9.5}$$

with e_1 vanishing quadratically at q. Moreover, on $\operatorname{supp} \chi_1'$, a_2 is bounded away from 0, and $r_1 < 0$, so $r_1 a_1 + e_1 \neq 0$, provided $\operatorname{supp} \chi_1'$ is sufficiently small. Since $r_1 < 0$ and $\chi_1' \leq 0$, $4\tilde{\nu} r_1 a_1^2 \chi_1'(a_1^2)$ is positive. Hence, choosing χ_1 such that $(-\chi_1\chi_1')^{1/2}$ is $\mathcal{C}^\infty$, and $\operatorname{supp} \chi_1$ is sufficiently small, we can write

$$\begin{aligned} &\sigma_\partial(i[Q^*Q, P - \lambda]) = -\,{}^{\mathrm{sc}}H_p q^2 = 4\tilde{\nu}\tilde{b}^2 + \tilde{f}, \\ &\tilde{b} = (a_1(r_1 a_1 + e_1)\chi_1'(a_1^2)\chi_1(a_1^2))^{1/2}\chi_2(a_2)\psi(p), \ \operatorname{supp} \tilde{f} \cap L_2 = \emptyset. \end{aligned} \tag{9.6}$$

Now we can apply Proposition 6.9, using the remark that follows it, to finish the proof. □

Let Φ_2 parameterize L_2 near q, as in the previous section. Multiplication by $e^{-i\Phi_2(y)/x}$ maps L_2 to the zero section $\nu = 0$, $\mu = 0$. The set of Legendre distributions associated to the zero section is exactly that of distibutions conormal to the boundary. The test module corresponding to this class (which is the conjugate of $\mathcal{M}$ under $e^{-i\Phi_2/x}$), is $\widetilde{\mathcal{M}} = \mathcal{V}_{\mathrm{b}}(X)$, generated by xD_x and D_y, microlocalized in $\tilde{O}$, the image of O under the multiplication. We thus deduce the following corollary.

Corollary 9.2. *Suppose that u satisfies (9.1) for $q \in \mathrm{Max}_+(\lambda)$. Then $e^{-i\Phi_2(y)/x}u \in I_{\mathrm{sc}}^{(-1/2-\epsilon)}(\tilde{O}, \widetilde{\mathcal{M}})$ for all $\epsilon > 0$, i.e. its microlocalization to $\tilde{O}$ is in $x^{1/2-\epsilon}H_{\mathrm{b}}^\infty(X)$ for all $\epsilon > 0$.*

Remark 9.3. The L^2 conormal space $x^s H_{\mathrm{b}}^\infty(X)$ is contained in the weighted L^∞ space with the weight shifted by ϵ; that is, $v \in x^s H_{\mathrm{b}}^\infty(X)$ gives $v \in x^{s-\epsilon}L^\infty(X)$ for all $\epsilon > 0$ (by Sobolev embedding). Conversely, if $v \in x^m L^\infty(X) \cap x^s H_{\mathrm{b}}^\infty(X)$ then by interpolation, $v \in x^{m-\epsilon}H_{\mathrm{b}}^\infty(X)$ for all $\epsilon > 0$.

The equation $(P - \lambda)u \in \dot{\mathcal{C}}^\infty(X)$ for Legendre functions u reduces to a transport equation. We can obtain this transport equation by a very explicit conjugation of $P - \lambda$ as the following proposition shows.

Proposition 9.4. *There exists an operator A of the form*

$$A = e^{i\Phi_2/x} \circ F^* \circ x^\beta e^{if},$$

where $\beta = r_2/2 + iV_1(z)/2\tilde{\nu}$, $f \in \mathcal{C}^\infty(X)$, and F^* is pullback by a diffeomorphism F on X, such that

$$A^{-1}(P - \lambda)A - \tilde{P}_0 \in x^2\widetilde{\mathcal{M}}^2, \quad \tilde{P}_0 = -2\tilde{\nu}((x^2 D_x) + r_1 y(xD_y)). \tag{9.7}$$

Proof. By (8.11) and (8.12), we have

$$\begin{aligned} &e^{-i\Phi_2/x}(P - \lambda)e^{i\Phi_2/x} = \\ &-2\tilde{\nu}\big((x^2 D_x) + r_1 y(xD_y) + ix\beta(y)\big) + yF(y)(x^2 D_x) + y^2 G(y)(xD_y) \mod x^2\widetilde{\mathcal{M}}^2, \end{aligned}$$

where $\beta(0) = \beta$. Since $r_1 < 0$, the vector field $x\partial_x + r_1 y\partial_y$ is nonresonant. Hence by a change of coordinates $x' = a(y)x$, $y' = b(y)y$, we may arrange that the vector field becomes

$$-2\tilde{\nu}(((x')^2 D_{x'}) + r_1 y(x' D_{y'})),$$

modulo terms in $x^2\widetilde{\mathcal{M}}^2$ and subprincipal terms. We therefore deduce that there is a local diffeomorphism F such that

$$\begin{aligned}(F^{-1})^* e^{-i\Phi_2/x}(P-\lambda)e^{i\Phi_2/x}F^* - \tilde{P}_1 \in x^2\widetilde{\mathcal{M}}^2,\\ \tilde{P}_1 = -2\tilde{\nu}((x^2D_x) + r_1 y(xD_y) + ix\beta(y)).\end{aligned} \tag{9.8}$$

Now we let $\tilde{\beta}(y) = (\beta(y)-\beta(0))/y$ and set $f(y) = \int_0^y \tilde{\beta}(z)\,dz$. This yields

$$x^{-\beta}e^{-if}\tilde{P}_1 e^{if}x^{\beta} - \tilde{P}_0 \in x^2\widetilde{\mathcal{M}}^2, \tag{9.9}$$

which completes the proof of the proposition. □

Remark 9.5. We have not conjugated the operator $P-\lambda$ microlocally to $\tilde{P}_0$; we have only conjugated it to the correct form 'along L_2' – this suffices because of the a priori conormal estimates. Note also that $e^{-i\Phi_2/x}u \in x^s H_{\mathrm{b}}^\infty(X)$ is equivalent to $A^{-1}u \in x^{s+r_2/2}H_{\mathrm{b}}^\infty(X)$.

Note that $(yx^{-r_1})^n$, for $n \geq 0$ integral, satisfies $\tilde{P}_0(yx^{-r_1})^n = 0$, and $(yx^{-r_1})^n \in x^s H_{\mathrm{b}}^\infty(X)$ for all $s < -r_1 n$ (where s is increasing with n, since $r_1 < 0$). These can be modified to obtain microlocal solutions of $A^{-1}(P-\lambda)Av \in \dot{\mathcal{C}}^\infty(X)$.

Proposition 9.6. *For each integer $n \geq 0$, there exists $v_n \in \cap_{s<-r_1 n} x^s H_{\mathrm{b}}^\infty(X)$ such that $v_n - (yx^{-r_1})^n \in x^{s'}H_{\mathrm{b}}^\infty(X)$ for some $s' > -r_1 n$, and such that $q \notin \mathrm{WF}_{sc}(A^{-1}(P-\lambda)Av_n)$.*

Proof. Applying $A^{-1}(P-\lambda)A$ to $x^\alpha y^n$ yields a function of the form $x^{\alpha+2}f'$, $f' \in \mathcal{C}^\infty(X)$. The equation $\tilde{P}_0 x^{\alpha+1}g = x^{\alpha+2}f'$ has a smooth solution g modulo $x^{\alpha+3}f''$, $f'' \in \mathcal{C}^\infty(X)$, provided $\alpha+1$ is not an integer multiple of r_1 (and then only a logarithmic factor in x is needed). Applying this argument with $\alpha = -r_1 n$, etc., we deduce that the error terms arising from $(yx^{-r_1})^n$ can be solved away iteratively. □

This result shows that $A^{-1}(P-\lambda)A$, hence $P-\lambda$, have a series of approximate generalized eigenfunctions v_n, resp. $u_n = Av_n$, where u_n has the form $u_n = e^{i\Phi_2/x}x^{\beta - r_i n}\tilde{v}_n$, with $\tilde{v}_n$ polyhomogeneous conormal and continuous up to the boundary.

We proceed to show that for any u satisfying (9.1) there exist constants a_n such that $u \sim \sum_{n=0}^\infty a_n u_n$. The main technical result is the following lemma.

If $U \subset X$ is open, we denote by $x^s H_{\mathrm{b}}^\infty(U)$ the space of functions consisting of the restrictions of $x^s H_{\mathrm{b}}^\infty(X)$ to U.

Lemma 9.7. *Suppose that U_0 is a neighbourhood of z in X. There exists a neighbourhood $U \subset U_0$ of z such that the following hold.*

Suppose that $v|_U \in x^r H_{\mathrm{b}}^\infty(U)$ and $x^{-1}\tilde{P}_0 v|_U \in x^s H_{\mathrm{b}}^\infty(U)$, $s > r$. Then

1. *If $s \leq 0$ then $v|_U \in x^{s-\epsilon}H_{\mathrm{b}}^\infty(U)$ for all $\epsilon > 0$.*
2. *If $s > 0$, then there exists a constant a_0 such that $v|_U - a_0 \in x^{r'}H_{\mathrm{b}}^\infty(U)$ for any $r' < \min(-r_1, s)$.*
3. *If $r > 0$, then $v|_U \in x^{r'}H_{\mathrm{b}}^\infty(U)$ for all $r' < \min(-r_1, s)$.*
4. *More generally, if $n \geq 0$ is an integer, $r > -(n-1)r_1$, $s > -nr_1$, then there exists a constant a_n such that $v|_U - a_n(x^{-r_1}y)^n \in x^{r'}H_{\mathrm{b}}^\infty(U)$ for any $r' < \min(-(n+1)r_1, s)$.*

(5) *If* $s > r$, $r > (n-1)r_1$, $n \geq 0$ *an integer, then* $v|_U \in x^{r'}H_{\mathrm{b}}^\infty(U)$ *for all* $r' < \min(-nr_1, s)$.

Proof. Let I be a small open interval, $0 \in I$, and let $x_0 > 0$ be small, so that $U = [0, x_0]_x \times I_y \subset U_0$ is a coordinate neighbourhood of z. Thus, $v|_U \in x^r H_{\mathrm{b}}^\infty(U)$ and $x^{-1}\tilde{P}_0 v|_U \in x^s H_{\mathrm{b}}^\infty(U)$, $s > r$. The integral curves of the vector field $\tilde{P}_0$ are given by

$$x^{-r_1}y = \text{constant}.$$

Since $-r_1 > 0$, if x is increasing along an integral curve then $|y|$ is decreasing, and vice versa. Hence, the integral curve γ through $(x, y) \in U$ satisfies $\gamma(T) \in U$ provided $x_0 \leq x(\gamma(T)) \leq x$.

Since $\tilde{P}_0$ is a vector field, $v - v|_{x=x_0}$ is given by the integral of $f = x^{-1}\tilde{P}_0 v$ along integral curves of $\tilde{P}_0$. Namely, the solution is given by

$$v(x,y) = -(2\tilde{\nu})^{-1}\int_{x_0}^{x} f(t, \left(\frac{x}{t}\right)^{-r_1} y)\,\frac{dt}{t} + v(x_0, \left(\frac{x}{x_0}\right)^{-r_1} y).$$

Note that the second term, which solves the homogeneous equation, is certainly a polyhomogeneous function on (x, y) down to $x = 0$, since it only evaluates v at $x = x_0$. Thus, it has a full asymptotic expansion, corresponding to the Taylor series of v at $x = x_0$ around $y = 0$ of the form $\sum_{j=0}^\infty c_j(x^{-r_1}y)^j$.

So suppose that $s \leq 0$ first. Then the first term (when restricted to U) is in $x^{s-\delta}L^\infty$ for all $\delta > 0$ since $f \in x^s L^\infty$ along the restriction of γ to $x \leq x \circ \gamma \leq x_0$. Hence, $v \in x^{s-\delta}L^\infty$ for all $\delta > 0$, so $v \in x^{s-\epsilon}H_{\mathrm{b}}^\infty(X)$ for all $\epsilon > 0$.

On the other hand, suppose that $f \in x^s L^\infty$ for some $s > 0$. The first term gives a convergent integral, hence is L^∞; indeed, it is continuous to $x = 0$ with limit $\int_{x_0}^0 f(t, 0)\,\frac{dt}{t}$. Let $a_0 = v(x_0, 0) + \int_{x_0}^0 f(t, 0)\,\frac{dt}{t}$. Then $v - a_0 \in x^{\min(-r_1, s-\epsilon)}L^\infty(X)$ for all $\epsilon > 0$, so $v - a_0 \in x^{\min(-r_1, s)-\epsilon}H_{\mathrm{b}}^\infty(X)$ by Remark 9.3. To see this, we write

$$v - a_0 = \int_{x_0}^{x} [f(t,0) - f(t, \left(\frac{x}{t}\right)^{-r_1} y)]\,\frac{dt}{t} = \int_{x_0}^{x}\int_0^1 \left(\frac{x}{t}\right)^{-r_1} y\frac{\partial f}{\partial y}(t, \rho\left(\frac{x}{t}\right)^{-r_1} y)\,d\rho\,\frac{dt}{t},$$

and estimate the integrand directly. A similar argument works for arbitrary n. $\square$

Corollary 9.8. *Let* v_n, $n \geq 0$ *integer, be given by Proposition 9.6. Suppose that* U_0 *is a neighbourhood of* z. *There exists a neighbourhood* $U \subset U_0$ *of* z *with the following properties.*

Suppose that $v \in x^r H_{\mathrm{b}}^\infty(U)$ *and* $x^{-1}A^{-1}(P-\lambda)Av \in \dot{\mathcal{C}}^\infty(U)$. *Then there exists a constant* a_0 *such that* $v - a_0 v_0 \in x^{r'}H_{\mathrm{b}}^\infty(X)$ *for any* $r' < -r_1$. *More generally, if* $n \geq 0$ *is an integer,* $r > -(n-1)r_1$, *then there exists a constant* a_n *such that* $v - a_n v_n \in x^{r'}H_{\mathrm{b}}^\infty(X)$ *for any* $r' < -(n+1)r_1$.

Proof. First, by Proposition 9.4, $A^{-1}(P-\lambda)A = \tilde{P}_0 + x^2 E$, $E \in \widetilde{\mathcal{M}}^2$.

Let $r_0 = \sup\{r'; v \in x^{r'}H_{\mathrm{b}}^\infty(U)\}$. Thus, $v \in x^{r'}H_{\mathrm{b}}^\infty(U)$ for all $r' < r$, hence $Ev \in x^{r'}H_{\mathrm{b}}^\infty(U)$. Thus, $x^{-1}\tilde{P}_0 v \in x^{r'+1}H_{\mathrm{b}}^\infty(U)$ for all $r' < r_0$.

We can now apply the previous lemma. Namely, suppose first that $r_0 < 0$, so $x^{-1}\tilde{P}_0 v \in x^{r_0+1-\epsilon}H_{\mathrm{b}}^\infty(U)$ for all $\epsilon > 0$. Part (i) of the lemma, applied with $s = \min(0, r_0 + 1 - \epsilon) \leq 0$, shows that $v \in x^{s-\epsilon}H_{\mathrm{b}}^\infty(U)$ for all $\epsilon > 0$. Thus, either $v \in x^{-\epsilon}H_{\mathrm{b}}^\infty(U)$ for all $\epsilon > 0$, but this contradicts $r_0 < 0$, or $v \in x^{r_0+1-\epsilon}H_{\mathrm{b}}^\infty(U)$ for all $\epsilon > 0$, which in turn contradicts the definition of r_0. Thus, $r_0 \geq 0$.

By part (ii) of the lemma, there exists a constant a_0 such that $v - a_0 \in x^{r'} H_{\mathrm{b}}^{\infty}(U)$ for any $r' < \min(-r_1, 1)$.

Let $v' = v - a_0 v_0$, with v_0 given by Proposition 9.6. Then $v' \in x^{r'} H_{\mathrm{b}}^{\infty}(U)$ for some $r' > 0$ still and it satisfies $x^{-1} A^{-1}(P - \lambda) A v' \in \dot{\mathcal{C}}^{\infty}(U)$. Now let $r_0' = \sup\{r'; v' \in x^{r'} H_{\mathrm{b}}^{\infty}(U)\} > 0$. As above, this yields $x^{-1} \tilde{P}_0 v' \in x^{r'+1} H_{\mathrm{b}}^{\infty}(U)$ for all $r' < r_0'$. Suppose that $r_0' < -r_1$. Then, by the part (iii) of the lemma, applied with $s = r_0' + 1 - \epsilon$, $\epsilon > 0$, $v' \in x^{r'} H_{\mathrm{b}}^{\infty}(U)$ for all $r' < \min(r_1, r_0' + 1 - \epsilon)$, hence for some $r' > r_0'$, since we assumed $r_0' < -r_1$. Therefore $v' = v - a_0 \in x^{r'} H_{\mathrm{b}}^{\infty}(U)$ for any $r' < -r_1$, finishing the proof of the first part.

The general case, with n arbitrary, is analogous. □

Theorem 9.9. *Suppose $q \in \mathrm{Max}_+(\lambda)$, O is a W-balanced neighbourhood of q and u satisfies* (9.1). *Let $u_n = A v_n$ be the local approximate solutions constructed in Proposition 9.6. Then there exist unique constants a_n such that for any u' with $u' \sim \sum_{n=0}^{\infty} a_n u_n$, $O \cap \mathrm{WF}_{sc}(u - u') = \emptyset$. That is,*

$$u \sim \sum_{n=0}^{\infty} a_n u_n \tag{9.10}$$

microlocally near q. The map $u \mapsto \{a_n\}_{n \in \mathbb{N}}$ is a bijection from microfunctions satisfying (9.1) *to complex-valued sequences. Thus, $E_{\mathrm{mic},+}(q, \lambda)$ is isomorphic to the space of arbitrary complex-valued sequences, i.e. to $\mathbb{C}[[x]]$.*

Remark 9.10. The theorem implies that microlocally near q, $u \in x^{r_2/2} L^{\infty}$, hence $u \in x^s L^2_{\mathrm{sc}}(X)$ for all $s < r_2/2 - 1$. Since $r_2 > 1$, this is an improvement over $x^{-1/2-\epsilon} L^2_{\mathrm{sc}}(X)$, $\epsilon > 0$.

Proof. Let $O_1 \ni q$ be any open set with $\overline{O_1} \subset O$, and choose some $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ such that $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q) \cap \overline{O_1} = \emptyset$, $\mathrm{WF}'_{\mathrm{sc}}(Q) \subset O$. Let $\tilde{u} = Qu$. Then

$$\begin{gathered} \mathrm{WF}_{\mathrm{sc}}(\tilde{u}) \subset O \cap \mathrm{WF}_{\mathrm{sc}}(u) \subset O \cap L_2, \\ \mathrm{WF}_{\mathrm{sc}}((P - \lambda)\tilde{u}) \subset \mathrm{WF}'_{\mathrm{sc}}([P, Q]) \cap \mathrm{WF}_{\mathrm{sc}}(u) \subset (O \setminus \overline{O_1}) \cap L_2, \end{gathered} \tag{9.11}$$

hence $q \notin \mathrm{WF}_{\mathrm{sc}}((P - \lambda)\tilde{u})$.

By Corollary 9.2 and Proposition 9.4, $v = A^{-1}\tilde{u} \in x^r H_{\mathrm{b}}^{\infty}(U')$, for all $r < (1 - r_2)/2 = r_1/2 < 0$, and by (9.11), $A^{-1}(P - \lambda)\tilde{u}$ is in $\dot{\mathcal{C}}^{\infty}(U')$ for a sufficiently small neighbourhood U' of the critical point $z \in \partial X$. Hence we can apply Corollary 9.8 to $v = A^{-1}\tilde{u}$. We deduce that there exists a constant a_0 such that $A^{-1}\tilde{u} - a_0 v_0 \in x^{r'} H_{\mathrm{b}}^{\infty}(X)$ for all $r' < -r_1$. Proceeding iteratively now proves (9.10). Surjectivity follows by asymptotically summing the u_n given by Proposition 9.6, and injectivity follows from Corollary 9.8. □

10. Incoming eigenfunctions at an outgoing saddle

Let $q \in \mathrm{Max}_+(\lambda)$ as before, but now consider microlocally incoming functions u at q, i.e. , $u \in E_{\mathrm{mic},-}(q, \lambda)$. We have $\Phi_-(\{q\}) \subset L_1$ locally near q, so by microlocalizing we may assume that

$$u \in \mathcal{C}^{-\infty}(X),\ O \cap \mathrm{WF}_{\mathrm{sc}}((P - \lambda)u) = \emptyset,\ \mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset L_1 \tag{10.1}$$

in some W-balanced neighbourhood O of q. Thus, in this section L_1 plays the role that L_2 played in the previous section. Essentially all the results go through as before. In particular, let Φ_1/x parameterize L_1, and consider $\Psi_{sc}^{-\infty,0}(X)$-module

$$\mathcal{M} \text{ generated by Id}, \ \{x^{-1}B, \ B \in \Psi_{sc}^{-\infty,0}(X), \ \sigma_\partial(B)|_{L_1} = 0\}. \tag{10.2}$$

This is a test module, which does *not* satisfy the positivity estimates of (6.14)-(6.15) – unless we take $s > -1/2$, which is of no interest since we will *not* have $u \in x^s L^2_{sc}(X)$ for such s. However, this defect is of no consequence, since below we only use $\mathcal{M}$ to describe an iterative construction, and its result, rather than to provide a framework for positive commutator estimates.

The test module conjugate to $\mathcal{M}$ under $e^{-i\Phi_1/x}$ is $\widetilde{\mathcal{M}} = \mathcal{V}_b(X)$, generated by xD_x and D_y. We then deduce the following analog of Proposition 9.4.

Proposition 10.1. *There exists an operator $\tilde{A}$ having the same structure as in Proposition 9.4 such that*

$$\tilde{A}^{-1}(P-\lambda)\tilde{A} - \tilde{P}_0 \in x^2\widetilde{\mathcal{M}}^2, \quad \tilde{P}_0 = -2\tilde{\nu}((x^2D_x) + r_2 y(xD_y)). \tag{10.3}$$

Now $r_2 > 1$, so the structure of the Legendre solutions of $\tilde{P}_0 w = 0$ is different from that considered in the previous section. In particular, $(yx^{-r_2})^n$, $n \geq 0$, integer satisfies $\tilde{P}_0(yx^{-r_2})^n = 0$, and $(yx^{-r_2})^n \in x^s H_b^\infty(X)$ for all $s < -r_2 n$. Since $r_2 > 0$, these become larger at ∂X as $n \to \infty$. These can still be modified to obtain microlocal solutions of $\tilde{A}^{-1}(P-\lambda)\tilde{A}v \in \dot{\mathcal{C}}^\infty(X)$, with the proof essentially identical to that presented in the previous section.

Proposition 10.2. *For each integer $n \geq 0$, there exists $w_n \in \cap_{s<-r_2 n} x^s H_b^\infty(X)$ such that $w_n - (yx^{-r_2})^n \in x^{s'} H_b^\infty(X)$ for some $s' > -r_2 n$, and such that $q \notin \mathrm{WF}_{sc}(\tilde{A}^{-1}(P-\lambda)\tilde{A}w_n)$.*

Note that only *finite* linear combinations of these are tempered distributions, since every $u \in \mathcal{C}^{-\infty}(X)$ lies in $x^s H_b^m(X)$ for some s, m. In fact, the space of these finite linear combinations is isomorphic to $E_{\mathrm{mic},-}(q,\lambda)$ as we prove in Section 12.

11. Microlocal Morse decomposition

Equipped with the microlocal eigenspaces $E_{\mathrm{mic},\pm}(q,\lambda)$ defined below (I.15), we can decompose $R(\lambda + i0)(\dot{\mathcal{C}}^\infty(X))$ or more precisely $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$ defined in (3.9) into microlocal components. Using the identification (3.10) of the space $E^\infty_{\mathrm{ess}}(\lambda)$ of 'smooth eigenfunctions' (see (I.12)) with $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$, this decomposes elements of $E^\infty_{\mathrm{ess}}(\lambda)$ into microlocal building blocks, and in particular proves Theorem I.2.

Let us start with the perfect Morse case, with $z_{\min}$ and $z_{\max}$ denoting the minimum and maximum of V_0. Denote by ι the identification map

$$\iota : E_{\mathrm{mic},+}(q^+_{\min},\lambda) \longrightarrow \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda). \tag{11.1}$$

Namely, by Corollary 5.7, $u \in E_{\mathrm{mic},+}(q^+_{\min},\lambda)$, may be regarded as a distribution with $\mathrm{WF}_{sc}(u) \subset \{q^+_{\min}\}$ and $f = (P-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$, determined up to addition of an element of $\dot{\mathcal{C}}^\infty(X)$. Then $u = R(\lambda+i0)f$ since both sides are outgoing, and their difference is a generalized eigenfunction of P, so u can indeed be identified with an element of $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$. The map

$$r : \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda) \longrightarrow E_{\mathrm{mic},+}(q^+_{\max},\lambda)$$

arises by microlocal restriction. That is, we choose $Q \in \Psi^{0,0}_{sc}(X)$ with $\sigma(Q) = 1$ in a neighbourhood of $q^+_{\max}$, and map $u \in \mathrm{RR}_{\mathrm{ess},+}(\lambda)$ to $Qu \in E_{\mathrm{mic},+}(q^+_{\max})$.

Theorem 11.1 (Theorem I.2)**.** *Suppose that* $\dim X = 2$, ∂X *is connected and* V_0 *is perfect Morse. For* $\kappa < \lambda < K$, ι *is an isomorphism. If* $K < \lambda$ *then* r *is surjective and has null space which restricts isomorphically onto* $E_{\mathrm{mic},+}(q^+_{\min}, \lambda)$ *leading to the short exact sequence*

$$0 \longrightarrow E_{\mathrm{mic},+}(q^+_{\min}, \lambda) \longrightarrow R^{\infty}_{\mathrm{ess},+}(\lambda) \longrightarrow E_{\mathrm{mic},+}(q^+_{\max}, \lambda) \longrightarrow 0.$$

Proof. The injectivity of ι is clear.

If $\lambda < K$, then any $u \in \mathrm{RR}^{\infty}_{\mathrm{ess},+}(\lambda)$ has scattering wavefront set in $\{\nu \geq 0\} \cap \Sigma(\lambda)$, and is invariant under bicharacteristic flow. Thus, we must have $\mathrm{WF}_{\mathrm{sc}}(u) \subset \{q^+_{\min}\}$. If Q is in $\Psi^{0,0}_{\mathrm{sc}}(X)$ with $\sigma(Q) = 1$ in a neighbourhood of $q^+_{\min}$, then $Qu \in E_{\mathrm{mic},+}(q^+_{\min}, \lambda)$ and $\iota(Qu) = u$, which shows that ι is an isomorphism in this range.

Next suppose that $\lambda > K$. Then $r \circ \iota = 0$ since $q^+_{\max} \notin \mathrm{WF}_{\mathrm{sc}}(\iota(u))$ for $u \in E_{\mathrm{mic},+}(q^+_{\min}, \lambda)$. Conversely, if $v \in \mathrm{RR}^{\infty}_{\mathrm{ess},+}(\lambda)$ maps to 0 under r, then by definition $\nu \geq \nu(q^+_{\max})$ on $\mathrm{WF}_{\mathrm{sc}}(v)$, and $q^+_{\max} \notin \mathrm{WF}_{\mathrm{sc}}(v)$, hence by Hörmander's theorem on the propagation of singularities, $\mathrm{WF}_{\mathrm{sc}}(v) \subset \{q^+_{\min}\}$. Thus, v is a representative of an element of $E_{\mathrm{mic},+}(q^+_{\min}, \lambda)$, and is indeed in the range of ι. This shows exactness at $\mathrm{RR}^{\infty}_{\mathrm{ess},+}(\lambda)$. Finally, surjectivity of r can be seen as follows. Any element of $E_{\mathrm{mic},+}(q^+_{\max}, \lambda)$ has a representative u as in Corollary 5.7. In particular, $f = (P - \lambda)u \in \dot{\mathcal{C}}^{\infty}(X)$, $\mathrm{WF}_{\mathrm{sc}}(u) \subset \{\nu \geq \nu(q^+_{\max})\}$, so $u = R(\lambda + i0)f$, showing surjectivity of r. $\square$

The general situation, i.e. when $\dim X = 2$ and V_0 is Morse, is not substantially different if none of the bicharacteristic curves $L_+ = L_+(q)$, emanating from $q \in \mathrm{Max}_+(\lambda)$, hits another $q' \in \mathrm{Max}_+(\lambda)$. For each $q \in \mathrm{Min}_+(\lambda)$ there is an identification map

$$\iota_q : E_{\mathrm{mic},+}(q, \lambda) \to \mathrm{RR}^{\infty}_{\mathrm{ess},+}(\lambda) \tag{11.2}$$

defined as described after (11.1). Similarly, for each $q \in \mathrm{Max}_+(\lambda)$, there is a restriction map

$$R_q : \mathrm{RR}^{\infty}_{\mathrm{ess},+}(\lambda) \to E_{\mathrm{mic},+}(q)$$

as described above. Then similar arguments to above yield

Proposition 11.2. *Suppose that* $\dim X = 2$, V_0 *is Morse and none of the curves* $L_+(q)$, $q \in \mathrm{Max}_+(\lambda)$, *hit another* $q' \in \mathrm{Max}_+(\lambda)$. *Then*

$$0 \longrightarrow \bigoplus_{q \in \mathrm{Min}_+(\lambda)} E_{\mathrm{mic},+}(q, \lambda) \longrightarrow R^{\infty}_{\mathrm{ess},+}(\lambda) \longrightarrow \bigoplus_{q' \in \mathrm{Max}_+(\lambda)} E_{\mathrm{mic},+}(q', \lambda) \longrightarrow 0$$

is a short exact sequence.

Note that in case V_0 is perfect Morse, we simply recover Theorem I.2.

In complete generality, i.e. for general Morse V_0, the relationship between critical points is more complicated. We introduce a partial order on $\mathrm{RP}_+(\lambda)$ corresponding to the flow-out under W.

Definition 11.3. If q, $q' \in \mathrm{RP}_+(\lambda)$ we say that $q \leq q'$ if $q' \in \Phi_+(\{q\})$ and $q < q'$ if $q \leq q'$ but $q' \neq q$. A subset $\Gamma \subset \mathrm{RP}_+(\lambda)$ is closed under $\leq$ if for all $q \in \Gamma$, we have $\{q' \in \mathrm{RP}_+(\lambda) \mid q \leq q'\} \subset \Gamma$. We call the set $\{q' \in \mathrm{RP}_+(\lambda) \mid q \leq q'\}$ the string generated by q.

This partial order relation between two radial points in $\mathrm{RP}_+(\lambda)$ corresponds to the existence of a sequence $q_j \in \mathrm{RP}_+(\lambda)$, $j = 0, \dots, k$, $k \geq 1$, with $q_0 = q$, $q_k = q'$ and such that for every $j = 0, \dots, k-1$, there is a bicharacteristic γ_j with $\lim_{t\to-\infty} \gamma_j = q_j$ and $\lim_{t\to+\infty} \gamma_j = q_{j+1}$.

We can now define a subspace of $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda)$ associated to a $\leq$-closed subset $\Gamma \subset \mathrm{RP}_+(\lambda)$ by setting

$$\begin{aligned}(11.3)\quad & \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \Gamma) \\ & = \{u \in \mathcal{C}^{-\infty}(X); (P-\lambda)u \in \dot{\mathcal{C}}^\infty(X),\ \mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{RP}_+(\lambda) \subset \Gamma\}/\dot{\mathcal{C}}^\infty(X).\end{aligned}$$

Notice that the assumption that $\Gamma \subset \mathrm{RP}_+(\lambda)$ implies that $\mathrm{WF}_{\mathrm{sc}}(u) \subset \{\nu > 0\}$. The 'trivial' case where $\Gamma = \mathrm{RP}_+(\lambda)$ is the space we are ultimately trying to describe:

$$\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \mathrm{RP}_+(\lambda)) \equiv \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda),\ \lambda \notin \mathrm{Cv}(V). \tag{11.4}$$

Proposition 11.4. *Suppose that $\Gamma \subset \mathrm{RP}_+(\lambda)$ is $\leq$-closed and q is a $\leq$-minimal element of Γ. Then with $\Gamma' = \Gamma \setminus \{q\}$*

$$0 \longrightarrow \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \Gamma') \xrightarrow{\iota} \mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \Gamma) \xrightarrow{r_q} E_{\mathrm{mic},+}(\lambda, q) \longrightarrow 0$$

is a short exact sequence.

Proof. The injectivity of ι follows from the definitions. The null space of the microlocal restriction map r_q, which can be viewed as restriction to a W-balanced neighbourhood of q, is precisely the subset of $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \Gamma)$ with wave front set disjoint from $\{q\}$, and this subset is $\mathrm{RR}^\infty_{\mathrm{ess},+}(\lambda, \Gamma')$. Thus it only remains to check the surjectivity of r_q.

We do so first for the strings generated by $q \in \mathrm{RP}_+(\lambda)$. For $q \in \mathrm{Min}_+(\lambda)$, the string just consists of q itself and the result follows trivially. So consider the string $S(q)$ generated by $q \in \mathrm{Max}_+(\lambda)$. By Corollary 5.7 any element of $E_{\mathrm{mic},+}(q, \lambda)$ has a representative $\tilde{u}$ satisfying $(P-\lambda)\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$ with $\mathrm{WF}_{\mathrm{sc}}(\tilde{u}) \subset \Phi_+(\{q\})$, which immediately gives surjectivity in this case.

For any $\leq$-closed set Γ and $\leq$-minimal element q, the string $S(q)$ is contained in Γ, so the surjectivity of r_q follows in general. □

Notice that we can always find a sequence $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \mathrm{RP}_+(\lambda)$, of $\leq$-closed sets with $\Gamma_j \setminus \Gamma_{j-1}$ consisting of a single point q_j which is $\leq$-minimal in Γ_j: we simply order the $q_i \in \mathrm{RP}_+(\lambda)$ so that $\nu(q_1) \geq \nu(q_2) \geq \ldots$, and set $\Gamma_i = \{q_1, \dots, q_i\}$. Then Proposition 11.4 implies the following

Theorem 11.5. *Suppose that $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \mathrm{RP}_+(\lambda)$, is as described in the previous paragraph. Then*

$$\{0\} \longrightarrow R^\infty_{\mathrm{ess},+}(\lambda, \Gamma_1) \hookrightarrow \ldots \hookrightarrow R^\infty_{\mathrm{ess},+}(\lambda, \Gamma_{n-1}) \hookrightarrow R^\infty_{\mathrm{ess},+}(\lambda), \tag{11.5}$$

with

$$R^\infty_{\mathrm{ess},+}(\lambda, \Gamma_j)/R^\infty_{\mathrm{ess},+}(\lambda, \Gamma_{j-1}) = E_{\mathrm{ess},+}(q_j, \lambda),\ j = 1, 2, \dots, n. \tag{11.6}$$

This theorem shows that there are elements of $E^\infty_{\mathrm{ess},+}(\lambda)$ associated to each radial point q. The parameterization of the microlocal eigenspaces shows that in some sense, there are 'more' eigenfunctions associated to the minima than to the maxima. One way to make this precise is to show that the former class is dense in $E^\infty_{\mathrm{ess}}(\lambda)$, in the topology of $\mathcal{C}^{-\infty}(X)$, or indeed in the topology of $E^s_{\mathrm{ess}}(\lambda)$ for $s>0$ small. We define

$$E^\infty_{\mathrm{Min},+}(\lambda)=\{u\in E^\infty_{\mathrm{ess}}(\lambda);\mathrm{WF}_{\mathrm{sc}}(u)\cap\mathrm{RP}_+(\lambda)\subset\mathrm{Min}_+(\lambda)\}\,. \tag{11.7}$$

Proposition 11.6. *Let $s_0=-1/2+\min\{r_2(q)/2;\ q\in\mathrm{Max}_+(\lambda)\}>0$. Then for any $\lambda\notin\mathrm{Cv}(C)$ and $0<s<s_0$, $E^\infty_{\mathrm{Min},+}(\lambda)$ is dense in $E^\infty_{\mathrm{ess}}(\lambda)$ in the topology of $E^s_{\mathrm{ess}}(\lambda)$ given by Proposition 3.12.*

Proof. We first suppose that

$$\lambda \text{ is not in the point spectrum of } P. \tag{11.8}$$

Let $u\in E^\infty_{\mathrm{ess}}(\lambda)$. We know from Theorem 9.9 (see the remark following it) that

$$\mathrm{WF}^{*,s-1/2}_{\mathrm{sc}}(u)\subset\mathrm{Min}_-(\lambda)\cup\mathrm{Min}_+(\lambda),\ s<s_0,$$

i.e. the wave front set of u relative to $x^{s-1/2}H^\infty_{\mathrm{sc}}(X)$ is localized at the radial points over the minima, provided that $s<s_0$; in particular we can take $s>0$.

Now choose $Q\in\Psi^{-\infty,0}_{\mathrm{sc}}(X)$ which has $\mathrm{WF}'_{\mathrm{sc}}(Q)$ localized near $\mathrm{Min}_+(\lambda)$ and $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$ disjoint from $\mathrm{Min}_+(\lambda)$. Then

$$\mathrm{WF}^{*,s-1/2}_{\mathrm{sc}}(u-Qu)\subset\mathrm{Min}_-(\lambda),\ s<s_0.$$

Let $\tilde P=\Delta+\tilde V$ be the operator constructed in Lemma 5.3 with $\tilde\nu$ large, so that $\lambda-\tilde\nu^2<\inf V$. Then every radial point $q\in\widetilde{\mathrm{RP}_+}(\lambda)$ of $\tilde P$ lies above a minimum of $\tilde V$ (i.e. $\widetilde{\mathrm{RP}_+}(\lambda)=\widetilde{\mathrm{Min}_+}(\lambda)$). Let $\tilde\Pi$ be the orthogonal projection off the L^2-nullspace of $\tilde P-ev$, so $\mathrm{Id}-\tilde\Pi$ is a finite rank projection to a subspace of $\dot{\mathcal{C}}^\infty(X)$. Since $\mathrm{WF}'_{\mathrm{sc}}([P,Q])\subset\mathrm{WF}'_{\mathrm{sc}}(Q)\cap\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$, we have

$$\mathrm{WF}^{*,s-1/2}_{\mathrm{sc}}(u)\cap\mathrm{WF}'_{\mathrm{sc}}([P,Q])=\emptyset,\ s<s_0.$$

Moreover,

$$\mathrm{WF}'_{\mathrm{sc}}(P-\tilde P)\cap\mathrm{WF}'_{\mathrm{sc}}(Q)=\emptyset,$$

and $(P-\lambda)Qu=[P,Q]u+Q(P-\lambda)u$, so

$$(P-\lambda)Qu,(\tilde P-\lambda)Qu\in x^{s+1/2}H^\infty_{\mathrm{sc}}(X),\ s<s_0.$$

Here we can take $s>0$, hence we can apply $R(\lambda-i0)$ to these functions, and

$$\mathrm{WF}^{*,s-1/2}_{\mathrm{sc}}(R(\lambda-i0)(P-\lambda)Qu)\subset\mathrm{Min}_-(\lambda).$$

We claim that

$$u=v-R(\lambda-i0)(P-\lambda)v,\ v=Qu. \tag{11.9}$$

To see this, let $\tilde u=u-(v-R(\lambda-i0)(P-\lambda)v)$. For $s<s_0$, $\mathrm{WF}^{*,s-1/2}_{\mathrm{sc}}(\tilde u)$ lies in $\mathrm{Min}_-(\lambda)$, since this is true both for $u-v$ and for $R(\lambda-i0)(P-\lambda)v$. Moreover, $(P-\lambda)\tilde u=0$, so the uniqueness theorem, Theorem 3.2, shows that $\tilde u=0$.

Since $v=Qu$ has wavefront set confined to $\{\nu>0\}$ and hence is outgoing,

$$\tilde\Pi v=\tilde R(\lambda+i0)(\tilde P-\lambda)v.$$

Thus,

$$u=(\mathrm{Id}-\tilde\Pi)Qu+\tilde R(\lambda+i0)w-R(\lambda-i0)(P-\lambda)\tilde R(\lambda+i0)w,$$
$$\text{where } w=(\tilde P-\lambda)Qu.$$

Now consider $u_j' = \phi(x/r_j)u$, where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ is identically 1 on $[2, +\infty)$, 0 on $[0, 1]$, and $r_j > 0$ satisfy $\lim_{j\to\infty} r_j = 0$. Thus, $u_j' \in \dot{\mathcal{C}}^\infty(X)$, and

$$
\begin{aligned}
u_j &= (\mathrm{Id} - \tilde{\Pi})Qu_j' + \tilde{R}(\lambda + i0)w_j - R(\lambda - i0)(P - \lambda)\tilde{R}(\lambda + i0)w_j,\\
w_j &= (\tilde{P} - \lambda)Qu_j' = (\tilde{P} - P)Qu_j' + [P, Q\phi(x/r_j)]u
\end{aligned}
\tag{11.10}
$$

satisfy $(P - \lambda)u_j = 0$ and $\mathrm{WF}_{\mathrm{sc}}(u_j) \cap \{\nu \geq 0\} \subset \mathrm{Min}_+(\lambda)$, so $u_j \in E^\infty_{\mathrm{Min},+}(\lambda) \subset E^\infty_{\mathrm{ess}}(\lambda)$. Since $w_j \to w$ and $(P - \lambda)Qu_j' \to (P - \lambda)Qu$ in $x^{s+1/2}L^2_{\mathrm{sc}}(X)$, we have $u_j \to u$ in $x^r L^2_{\mathrm{sc}}(X)$, for any $r < -1/2$ by Theorem 3.3. Moreover, if B is as in (3.13) then since $\mathrm{WF}'_{\mathrm{sc}}(B) \cap \mathrm{RP}(\lambda) = \emptyset$ and $\mathrm{WF}_{\mathrm{sc}}(w) \cap \mathrm{RP}(\lambda) = \emptyset$, we have $Bu_j \to Bu$ in $x^{s-1/2}L^2_{\mathrm{sc}}(X)$ by Theorem 3.1, proving convergence in $E^s_{\mathrm{ess}}(\lambda)$.

We now indicate the modifications necessary if (11.8) is not satisfied. Let Π denote the orthogonal projection off the L^2 λ-eigenspace. Equation (11.9) must be replaced by $u = \Pi v - R(\lambda - i0)(P - \lambda)v$. Then if we define u_j by

$$u_j = \Pi\tilde{R}(\lambda + i0)w_j - R(\lambda - i0)(P - \lambda)\tilde{R}(\lambda + i0)w_j,$$

instead of (11.10), the argument goes through. □

12. Pairing and duality

We define a basic version of 'Green's pairing' in this context on

$$\tilde{E}^{-\infty}(\lambda) = \left\{u \in \mathcal{C}^{-\infty}(X); (P - \lambda)u \in \dot{\mathcal{C}}^\infty(X)\right\}. \tag{12.1}$$

Namely

$$\tilde{B}(u_1, u_2) = i\left(\langle (P - \lambda)u_1, u_2\rangle - \langle u_1, (P - \lambda)u_2\rangle\right),\ u_i \in \tilde{E}^{-\infty}(\lambda). \tag{12.2}$$

The imaginary factor is inserted only to make the pairing sesquilinear,

$$\overline{\tilde{B}(u_1, u_2)} = \tilde{B}(u_2, u_1).$$

Observe that

$$\tilde{B}(u, v) = 0 \text{ if } u \in \dot{\mathcal{C}}^\infty(X)$$

since then integration by parts is permitted.

More significantly we define a sesquilinear form on $E^\infty_{\mathrm{ess}}(\lambda)$ using $\tilde{B}$. Assuming that $\lambda \notin \mathrm{Cv}(V)$, choose

$$
\begin{aligned}
&A \in \Psi^{0,0}_{\mathrm{sc}}(X) \text{ such that } \mathrm{WF}'_{\mathrm{sc}}(A) \cap \Sigma(\lambda) \subset \{\nu > -a(\lambda)\} \text{ and}\\
&\qquad\qquad \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - A) \cap \Sigma(\lambda) \subset \{\nu < a(\lambda)\} = \emptyset
\end{aligned}
\tag{12.3}
$$

where $a(\lambda)$ is as in Remark 1.11. If $u \in E^\infty_{\mathrm{ess}}(\lambda)$ then $\mathrm{WF}_{\mathrm{sc}}((P - \lambda)Au) = \emptyset$ so $Au \in \tilde{E}^{-\infty}(\lambda)$. This allows us to define

$$B : E^\infty_{\mathrm{ess}}(\lambda) \times E^\infty_{\mathrm{ess}}(\lambda) \longrightarrow \mathbb{C},\ B(u_1, u_2) = \tilde{B}(Au_1, Au_2). \tag{12.4}$$

Lemma 12.1. *If $\lambda \notin \mathrm{Cv}(V)$ and A satisfies (12.3) then B in (12.4) is independent of the choice of A and defines a non-degenerate sesquilinear form.*

Proof. The space of cut-off operators satisfying (12.3) is convex. If we consider the linear homotopy between two of them and denote the bilinear form B_t for $t \in [0, 1]$ then

$$\frac{d}{dt}B_t(u_1, u_2) = \tilde{B}(\frac{d}{dt}A_t u_1, A_t u_2) + \tilde{B}(A_t u_1, \frac{d}{dt}A_t u_2) = 0$$

since $\frac{d}{dt}A_t u_i \in \dot{\mathcal{C}}^\infty(X)$ for $i = 1, 2$.

The non-degeneracy of B follows by rewriting it

$$B(u_1,u_2) = i\langle (P-\lambda)Au_1, u_2\rangle,\ u_1,\ u_2 \in E^{\infty}_{\mathrm{ess}}(\lambda). \tag{12.5}$$

Indeed the difference between the right side of (12.5) and (12.4) is

$$i\left(\langle (P-\lambda)Au_1, (\mathrm{Id}-A)u_2\rangle + \langle Au_1, (P-\lambda)Au_2\rangle\right).$$

In the first term the scattering wavefront sets of Au_1 and $(\mathrm{Id}-A)u_2$ do not meet, so integration by parts is permitted and, u_2 being an eigenfunction, this difference vanishes. If (12.5) vanishes for all u_1 it follows that $\langle f, u_2\rangle = 0$ for all $f \in \dot{\mathcal{C}}^\infty(X)$ since it vanishes for $f \in (P-\lambda)\dot{\mathcal{C}}^\infty(X) \oplus E_{\mathrm{pp}}(\lambda)$ which, by Propositions 3.7 and 3.10 is a complement to $(P-\lambda)AE^{\infty}_{\mathrm{ess}}(\lambda)$ in $\dot{\mathcal{C}}^\infty(X)$. This proves the non-degeneracy. □

Lemma 12.2. *The pairing B extends to a non-degenerate pairing and topological duality*

$$B : E^{\infty}_{\mathrm{ess}}(\lambda) \times E^{-\infty}_{\mathrm{ess}}(\lambda) \longrightarrow \mathbb{C},\ B(u,v) = \tilde{B}(Au, Av). \tag{12.6}$$

Moreover, for all $s \in \mathbb{R}$, B extends to a nondegenerate pairing

$$B : E^{s}_{\mathrm{ess}}(\lambda) \times E^{-s}_{\mathrm{ess}}(\lambda) \longrightarrow \mathbb{C},\ B(u,v) = \tilde{B}(Au, Av). \tag{12.7}$$

Proof. The extension of the pairing to (12.6) follows directly from (12.5), since $Au_1 \in \dot{\mathcal{C}}^\infty(X)$ when $u_1 \in E^{\infty}_{\mathrm{ess}}(\lambda)$. The same argument shows that it is non-degenerate. To see that it is a topological pairing observe that the Fréchet topology on $E^{\infty}_{\mathrm{ess}}(\lambda)$ arises from its identification with the quotient of $\dot{\mathcal{C}}^\infty(X)$ by the closed subspace $(\Delta+V-\lambda)\dot{\mathcal{C}}^\infty(X)$. Thus an element of the dual space may be identified with a distribution $u \in \mathcal{C}^{-\infty}(X)$ which vanishes on $(\Delta+V-\lambda)\dot{\mathcal{C}}^\infty(X)$, i.e. is an element of $E^{-\infty}_{\mathrm{ess}}(\lambda)$. The pairing between $E^{-\infty}_{\mathrm{ess}}(\lambda)$ and $E^{\infty}_{\mathrm{ess}}(\lambda)$ defined in this way is precisely B.

A similar argument applies to the spaces of eigenfunctions of finite regularity. The square of the Hilbert norm on $E^{s}_{\mathrm{ess}}(\lambda)$ may be taken to be

$$\|Au\|^2_{x^{s-1/2}L^2} + \|u\|^2_{x^{-N}L^2} \tag{12.8}$$

where A is as in (12.3) and N is sufficiently large, in particular $N > \max\{-s+\frac{1}{2}, \frac{1}{2}\}$. Writing the pairing in the form

$$B(u_1,u_2) = i\langle [P,A]u_1, u_2\rangle \tag{12.9}$$

shows its continuity with respect to (12.8) (for a different A which is microlocally the identity on the essential support of the operator in (12.9).) The non-degeneracy follows much as before. □

As well as these global forms of the pairing, we can define versions of it on the components of the microlocal Morse decomposition. Namely, for $q \in \mathrm{RP}_+(\lambda)$, choose $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ with $q \notin \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q)$ and $\mathrm{WF}'_{\mathrm{sc}}(Q) \cap \Sigma(\lambda)$ supported in a W-balanced neighbourhood O of q. Then consider

$$\tilde{E}_{\mathrm{mic},+}(q,\lambda) \times \tilde{E}_{\mathrm{mic},-}(q,\lambda) \ni (u,v) \longmapsto \tilde{B}(Qu, Qv). \tag{12.10}$$

The pairings on the right are well defined. Indeed, for such (u,v),

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{WF}_{\mathrm{sc}}(v) \cap O \subset \{q\}, \tag{12.11}$$

since $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \Phi_+(\{q\})$ and $\mathrm{WF}_{\mathrm{sc}}(v) \cap O \subset \Phi_-(\{q\})$. Moreover, $q \notin \mathrm{WF}_{\mathrm{sc}}\left((P-\lambda)Qu\right)$ and $q \notin \mathrm{WF}_{\mathrm{sc}}\left((P-\lambda)Qv\right)$, so

$$\mathrm{WF}_{\mathrm{sc}}\left((P-\lambda)Qu\right) \cap \mathrm{WF}_{\mathrm{sc}}(Qv) = \mathrm{WF}_{\mathrm{sc}}\left((P-\lambda)Qv\right) \cap \mathrm{WF}_{\mathrm{sc}}(Qu) = \emptyset, \tag{12.12}$$

so B in (12.6) is well-defined.

If Q' is any other microlocal cutoff with the same properties then $q \notin \mathrm{WF}'_{\mathrm{sc}}(Q - Q')$, hence $\mathrm{WF}_{\mathrm{sc}}((Q-Q')u) \cap \mathrm{WF}_{\mathrm{sc}}(Qv) = \emptyset$, so $\tilde{B}(Qu - Q'u, Qv) = 0$. Similarly, $\tilde{B}(Qu, Qv - Q'v) = 0$, so $\tilde{B}(Qu, Qv) = \tilde{B}(Q'u, Q'v)$.

If we define a new pairing B' as in (12.6), but using the operator $\tilde{P}$ from Lemma 5.3 in place of P, then $B'(Qu, Qv) = \tilde{B}(Qu, Qv)$ provided that $O \cap \mathrm{WF}'_{\mathrm{sc}}(P - \tilde{P}) = \emptyset$.

Proposition 12.3. *For each $q \in \mathrm{RP}_+(\lambda)$ the sesquilinear form* (12.10) *descends to a non-degenerate pairing*

$$\tilde{B}_q : E_{\mathrm{mic},+}(q, \lambda) \times E_{\mathrm{mic},-}(q, \lambda) \longrightarrow \mathbb{C}. \tag{12.13}$$

Proof. The terms involved in the quotients all have wavefront set disjoint from $\mathrm{WF}'_{\mathrm{sc}}(Q)$, so integration by parts shows that (12.13) is independent of choices.

To show non-degeneracy, suppose that q is a minimal element of $\mathrm{RP}_+(\lambda)$ with respect to the partial order $\le$ of Definition 11.3. Then every element v of $\tilde{E}_{\mathrm{mic},-}(q, \lambda)$ has a representative in $E^{-\infty}_{\mathrm{ess}}(\lambda)$ with $\mathrm{WF}_{\mathrm{sc}}(v) \subset \{\nu \le \nu(q)\}$ and $\mathrm{WF}_{\mathrm{sc}}(v) \cap \{\nu = \nu(q)\} = \{q\}$. By Lemma 12.2, there exists $u \in E^{\infty}_{\mathrm{ess}}(\lambda)$ such that $\tilde{B}(u, v) \neq 0$. Since q is $\le$-minimal, we have $\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{WF}_{\mathrm{sc}}(v) \subset \{q\}$. Hence, $\tilde{B}(u, v) = \tilde{B}(Qu, Qv)$. Thus, Qu is an element of $E_{\mathrm{mic},+}(q, \lambda)$ with $\tilde{B}(Qu, v) \neq 0$.

If q is not $\le$-minimal, then we use Lemma 5.3 with $\tilde{\nu}$ smaller than $\nu(q)$, but larger than $\nu(q')$ for any $q' \in \mathrm{RP}_+(\lambda)$ with $\nu(q')$ smaller than $\nu(q)$. Then $q \in \widetilde{\mathrm{RP}_+}(\lambda)$, and $\tilde{V} = V$ in a neighbourhood of $\pi(q)$, so $E_{\mathrm{mic},\pm}(q, \lambda)$ is the same space irrespective of whether we consider P or $\tilde{P}$. However, q is $\le$-minimal for $\tilde{P}$, so the result follows from the previous paragraph. □

Proposition 12.4. *Suppose that $q \in \mathrm{Max}_+(\lambda)$ and O is a W-balanced neighbourhood of q in ${}^{\mathrm{sc}}T^*_Y X$. Let $u_n = Av_n \in E_{\mathrm{mic},+}(q, \lambda)$ and $\tilde{u}_n = \tilde{A}w_n \in E_{\mathrm{mic},-}(q, \lambda)$ be the local approximate outgoing, resp. incoming, solutions constructed in Proposition 9.6, resp. Proposition 10.2. Then*

$$\tilde{B}_q(u_n, \tilde{u}_m) = 0 \quad \text{if } n > m,$$
$$\tilde{B}_q(u_n, \tilde{u}_n) \neq 0 \ \forall\ n.$$

Proof. This follows from a straightforward calculation, using the asymptotic expansions, similarly to the proof of Proposition 12.6 below. □

We can thus renormalize $\tilde{u}_m$ inductively by letting

$$\tilde{u}'_m = \frac{1}{\tilde{B}_q(u_m, \tilde{u}_m)} \left(\tilde{u}_m - \sum_{n<m} \tilde{B}_q(u_n, \tilde{u}_m)\, \tilde{u}'_n \right).$$

Then $\tilde{B}_q(u_n, \tilde{u}'_m) = 0$ if $n \neq m$ and $\tilde{B}_q(u_m, \tilde{u}'_m) = 1$.

An immediate consequence of non-degeneracy is the following theorem.

Theorem 12.5. *Suppose that $q \in \mathrm{Max}_+(\lambda)$ and O is a W-balanced neighbourhood of q in ${}^{\mathrm{sc}}T^*_Y X$. If $u \in E_{\mathrm{mic},-}(q, \lambda)$ is a microlocally incoming eigenfunction at q, then there exist unique constants a_n such that $a_n \neq 0$ for only finitely many n, and*

$$O \cap \mathrm{WF}_{sc}(u - \sum a_n \tilde{u}_n) = \emptyset. \tag{12.14}$$

Thus, $E_{\mathrm{mic},-}(q,\lambda)$ is isomorphic to the space of finite complex-valued sequences, $\mathbb{C}[x]$.

Proof. Choose $s>0$ such that $u\in W'_s=E_{\mathrm{mic},-}(q,\lambda)\cap x^{-s-1/2}L^2(X)$. Since W'_s automatically annihilates $E_{\mathrm{mic},+}(q,\lambda)\cap x^{s-1/2}L^2(X)$, we deduce that $\tilde{B}_q(u_n,u)=0$ for n sufficiently large. Now $u'=u-\sum_n \tilde{B}_q(u_n,u)\tilde{u}'_n$ satisfies $\tilde{B}_q(u_n,u')=0$ for all n. The non-degeneracy of $\tilde{B}_q$ and the structure theorem for $E_{\mathrm{mic},+}(q,\lambda)$, Theorem 9.9, thus imply that $u'=0$. Since the $\tilde{u}'_n$ are finite linear combinations of $\tilde{u}_n$, the proof is complete. □

We can compute the form of B explicitly on the subspace $E^\infty_{\mathrm{Min},+}(\lambda)$. Recall that for each $q\in\mathrm{Min}_+(\lambda)$ there is a map

$$M_+(q,\lambda):E^\infty_{\mathrm{Min},+}(\lambda)\longrightarrow\mathcal{S}(\mathbb{R}_{r(q)})$$

given by (I.21) or (I.23), which is an isomorphism on $E_{\mathrm{mic},+}(q)$.

Proposition 12.6. *If $\lambda\notin\mathrm{Cv}(V)$ then for $u_1,u_2\in E^\infty_{\mathrm{Min},+}(\lambda)$,*

$$B(u_1,u_2)=2\sum_{q\in\mathrm{Min}_+(\lambda)}\sqrt{\lambda-V(\pi(q))}\int_{\mathbb{R}_{r(q)}}M_+(q,\lambda)(u_1)\overline{M_+(q,\lambda)(u_2)}\,\omega_q, \tag{12.15}$$

where ω_q is the density on the front face of the blow-up induced by the Riemannian density of h.

Proof. Let $\chi\in\mathcal{C}^\infty(\mathbb{R})$ be identically 0 near 0 and identically 1 on $[1,+\infty)$, and let $\chi_r(x)=\chi(x/r)$. Then, with A as in (12.3),

$$\begin{aligned}B(u_1,u_2)&=i\lim_{r\to0}(\langle\chi_r(P-\lambda)Au_1,Au_2\rangle-\langle\chi_rAu_1,(P-\lambda)Au_2\rangle)\\&=-i\lim_{r\to0}\langle[P,\chi_r]Au_1,Au_2\rangle=-i\lim_{r\to0}\int_X([P,\chi_r]Au_1)\overline{Au_2}\,dg,\end{aligned}\tag{12.16}$$

since for $r>0$,

$$\langle(P-\lambda)\chi_rAu_1,Au_2\rangle=\langle\chi_rAu_1,(P-\lambda)Au_2\rangle.$$

Since $[P,\chi_r]\to0$ strongly as a map $H^{1,s-1/2}_{\mathrm{sc}}(X)\longrightarrow H^{0,s+1/2}_{\mathrm{sc}}(X)$, for $s\in\mathbb{R}$, it follows that only microlocal regions where at least one of Au_1 and Au_2 is not in $H^{0,-1/2}_{\mathrm{sc}}(X)$ contribute. That is, we can insert microlocal cut-offs near the minimal radial points and thereby localize the computation to a single radial point.

So, let $q\in\mathrm{Min}_+(\lambda)$, let $z=\pi(q)$, and assume that λ is above the Hessian threshold for q. Using the asymptotic form of Au_1 and Au_2 given by Theorem 8.6, we need to compute

$$-i\lim_{r\to0}\int[\Delta,\chi_r](e^{i\Phi_2/x}x^\beta v_1)e^{-i\Phi_2/x}x^{\overline{\beta}}\overline{v_2}\,\frac{dx}{x^2}\,\frac{\omega}{x^{r_2}},$$

where ω is the density on the front face of the blow-up induced by the Riemannian density of h. Thus, $\omega=|dh(z)|/x^{r_1}$. Here, $|dx|$ is fixed at the boundary by the requirement that the scattering metric takes the form (I.1), so this is a well-defined density on the front face of X_{z,r_1}. In the computation the commutator $[\Delta,\chi_r]$ may be replaced by $2(x^2D_x\chi_r)(x^2D_x)$ modulo terms that vanish in the limit, and v_1, v_2

may be restricted to the front face. In addition, $x^2 D_x$ must fall on $e^{i\Phi_2/x}$, or the limit vanishes. Finally, $\operatorname{Re}\beta = r_2/2$. Hence, the limit is

$$2i\Phi_2(z)(\int v_1|_{\mathrm{ff}}\overline{v_2|_{\mathrm{ff}}}\,\omega)\,(\lim_{r\to 0}\int_0^\infty (D_x\chi_r)\,dx) = 2\sqrt{\lambda - V(z)}(\int v_1|_{\mathrm{ff}}\overline{v_2|_{\mathrm{ff}}}\,\omega).$$

Since $v_j|_{\mathrm{ff}} = M_+(q,\lambda)(u_j)$, this gives the stated result if λ is above the Hessian threshold of z.

Now suppose that the λ is below the Hessian threshold of q. Then the asymptotics of Au_i are given by Theorem 7.4 and take the form

$$e^{i\tilde\nu/x}\sum_j X^{1/2}X^{i((2j+1)\alpha+\gamma)}\gamma_j v_j(Y),\ \tilde\nu = \nu(z) = \sqrt{\lambda - V(z)},$$

modulo terms whose contribution vanishes in the limit $r \to 0$, and where the v_j, $j = 1, 2, \ldots$ are orthonormal eigenfunctions of a harmonic oscillator. Here $\gamma_j = \gamma_{1,j}$ (for Au_1) or $\gamma_{2,j}$ (for Au_2) is a Schwartz sequence in j, hence interchanging the order of various integrals and sums is permitted. Again, the commutator $[\Delta, \chi_r]$ may be replaced by $2(x^2D_x\chi_r)(x^2D_x)$ and x^2D_x must fall on $e^{i\tilde\nu/x}$, to yield terms that do not vanish as $r \to 0$. Now we interchange the integral in Y and the summations, and use the fact that the v_k are orthonormal, to conclude that the limit is

$$2\sqrt{\lambda - V(z)}\sum_{k=1}^\infty \gamma_{1,k}\,\overline{\gamma_{2,k}}.$$

Since this is also equal to

$$2\sqrt{\lambda - V(z)}\int M_+(q,\lambda)(u_1)\,\overline{M_+(q,\lambda)(u_2)},\quad M_+(q,\lambda)(u_i) = \sum_{j=1}^\infty \gamma_{i,j}v_j,$$

the proof is complete. □

Corollary 12.7. *For $\lambda \notin \mathrm{Cv}(V)$, the maps $M_+(q,\lambda)$ extend continously to*

$$M_+(q,\lambda) : E^0_{\mathrm{ess}}(\lambda) \longrightarrow L^2(\mathbb{R}_{r(q)}). \tag{12.17}$$

The pairing B is positive definite, hence it defines a norm on $E^0_{\mathrm{ess}}(\lambda)$ satisfying

$$0 \le \|u\|_0^2 = 2\sum_{q\in\mathrm{Min}_+(\lambda)}\sqrt{\lambda - V(\pi(q))}\|M_+(q,\lambda)u\|^2 = B(u,u) = i\langle[P-\lambda, A]u, u\rangle \tag{12.18}$$

for any $A \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ as in (12.3).

Proof. Equation (12.18) holds for $u \in E^\infty_{\mathrm{Min},+}(\lambda)$, with the positivity coming from Proposition 12.6. Since B extends to a pairing on $E^0_{\mathrm{ess}}(\lambda)$, which is continuous in the topology $\mathcal{T}^0_3$, (12.17) follows from the density statements of Proposition 11.6 and Corollary 3.13 (in $\mathcal{T}^0_3$ in the latter case). As all other expressions extend to continuous bilinear maps on $E^0_{\mathrm{ess}}(\lambda)$, the density statements of Proposition 11.6 and Corollary 3.13 show that (12.18) remains true. Since B is non-degenerate and it is a semi-norm, it follows that B is in fact positive definite, hence a norm. □

Lemma 12.8. *Suppose that $\lambda \notin \mathrm{Cv}(V)$. For $r < -1/2$ there exists $C_r > 0$ such that for $u \in E^0_{\mathrm{ess}}(\lambda)$, $\|u\|^2_{x^rL^2_{\mathrm{sc}}(X)} \le C_rB(u,u)$.*

Proof. For simplicity of notation, suppose first that $\lambda \notin \sigma_{\mathrm{pp}}(P)$.

For $f \in x^s L^2_{\mathrm{sc}}(X)$, $s > 1/2$, let

$$\tilde{u} = R(\lambda + i0)f - R(\lambda - i0)f.$$

For $u \in E^0_{\mathrm{ess}}(\lambda)$ then

$$i\langle u, f\rangle = \tilde{B}(u, R(\lambda + i0)f) = \tilde{B}(Au, R(\lambda + i0)f) = \tilde{B}(Au, \tilde{u}) = B(u, \tilde{u}).$$

By Cauchy-Schwarz and Corollary 12.7,

$$B(u, \tilde{u}) \leq B(u, u)^{1/2} B(\tilde{u}, \tilde{u})^{1/2}.$$

But

$$B(\tilde{u}, \tilde{u}) = \tilde{B}(R(\lambda + i0)f, R(\lambda + i0)f) = i(\langle f, R(\lambda + i0)f\rangle - \langle R(\lambda + i0)f, f\rangle),$$

so by the limiting absorption principle,

$$B(\tilde{u}, \tilde{u}) \leq \tilde{C}_s \|f\|^2_{x^s L^2}.$$

Combining these results gives that

$$|\langle u, f\rangle| \leq \tilde{C}_s^{1/2} B(u, u)^{1/2} \|f\|_{x^s L^2}. \tag{12.19}$$

Now for $r < -1/2$, $u \in E^0_{\mathrm{ess}}(\lambda)$, let $f = x^{-2r}u \in x^s L^2_{\mathrm{sc}}(X)$ for all $s < -2r - 1/2$, in particular for $s = -r > 1/2$. Applying (12.19) yields

$$\|u\|^2_{x^r L^2} \leq \tilde{C}_{-r}^{1/2} B(u, u)^{1/2} \|x^{-2r}u\|_{x^{-r}L^2} = \tilde{C}_{-r}^{1/2} B(u, u)^{1/2} \|u\|_{x^r L^2}.$$

Cancelling a factor of $\|u\|_{x^r L^2}$ from both sides proves the lemma if $\lambda \notin \sigma_{\mathrm{pp}}(P)$.

If $\lambda \in \sigma_{\mathrm{pp}}(P)$, the calculations up to and including (12.19) work with f replaced by Πf, Π the orthogonal projection off $E_{\mathrm{pp}}(\lambda)$. Take $f = x^{-2r}u$ again and use the identity $\Pi u = u$ to finish the proof in this case. □

An immediate consequence is the following proposition.

Proposition 12.9. *Suppose that $\lambda \notin \mathrm{Cv}(V)$. The norm (12.18) on $E^0_{\mathrm{ess}}(\lambda)$ is equivalent to the the norm*

$$\|u\|_{\mathcal{T}^0_3} = \|u\|_3 = \left(\|u\|^2_{x^r L^2_{\mathrm{sc}}(X)} + \|B'u\|^2_{x^{-1/2}L^2_{\mathrm{sc}(X)}}\right)^{1/2}, \ r < -1/2,$$

where B' is as in (3.13).

Proof. We may choose A as above such that, in addition, $i[P, A] \in \Psi^{-\infty,1}_{\mathrm{sc}}(X)$ is of the form $G^2 + E$, $G \in \Psi^{-\infty,1/2}_{\mathrm{sc}}(X)$, $E \in \Psi^{-\infty,2}_{\mathrm{sc}}(X)$. We can take, for example, A with $\sigma(A) = \chi(\nu)\phi^2(p)$, $p = \sigma(P)$, with $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ identically 1 near λ, $\chi \in \mathcal{C}^\infty(\mathbb{R})$ identically 0 on $(-\infty, -a(\lambda)/2)$, 1 on $(a(\lambda)/2, +\infty)$, and $\chi' \geq 0$ with $(\chi')^{1/2} \in \mathcal{C}^\infty(\mathbb{R})$. Then the principal symbol of $i[P, A]$ is $W\sigma(A)$, and $(W\sigma(A))^{1/2}$ is real and $\mathcal{C}^\infty$ by (1.27), so $i[P, A] = G^2 + E$ as above. Now

$$B(u, u) = |\langle [P - \lambda, A]u, u\rangle| \geq \|Gu\|^2 - |\langle u, Eu\rangle| \geq \|Gu\|^2 - C\|u\|^2_{x^{-1}L^2},$$

so by Lemma 12.8, $\|Gu\|^2 + \|u\|^2_{x^{-1}L^2} \leq C' B(u, u)$.

Conversely,

$$B(u, u) = |\langle [P - \lambda, A]u, u\rangle| \leq C''(\|Gu\|_{x^{-1/2}L^2} + \|u\|_{x^{-1}L^2})^2,$$

and G may be chosen so as to satisfy the conditions of (3.13). □

Corollary 12.10. *For $\lambda \notin \mathrm{Cv}(V)$, $q \in \mathrm{Min}_+(\lambda)$, the Poisson operators*

$$P_+(q,\lambda) : \mathcal{S}(\mathbb{R}) \to E^\infty_{\mathrm{ess}}(\lambda)$$
$$P_+(q,\lambda)(g) = u \Longleftrightarrow u \in E^\infty_{\mathrm{ess}}(\lambda),\ M_+(q,\lambda)u = g,\ M_+(q',\lambda)u = 0 \text{ for } q' \neq q$$

extend to continuous linear maps

$$P_+(q,\lambda) : L^2(\mathbb{R}) \to E^0_{\mathrm{ess}}(\lambda).$$

Proof. By Proposition 12.9, for $g \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\|P(q,\lambda)g\|_3^2 &\leq C|B(P(q,\lambda)g, P(q,\lambda)g)|\\ &= 2\sqrt{\lambda - V(\pi(q))}\int_{\mathbb{R}} |M_+(q,\lambda)(P(q,\lambda)g)|^2\,\omega_q = 2\sqrt{\lambda - V(\pi(q))}\int_{\mathbb{R}} |g|^2\,\omega_q,\end{aligned}$$

so the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ finishes the proof. □

The combination of these results now yields the following theorem.

Theorem 12.11 (Theorem I.3)**.** *For $\lambda \notin \mathrm{Cv}(V)$, the joint map*

(12.20)
$$M_+(\lambda) = E^0_{\mathrm{ess}}(\lambda) \to \oplus_{q\in\mathrm{Min}_+(\lambda)} L^2(\mathbb{R}),\ \ M_+(\lambda)u = (M_+(q,\lambda)(u))_{q\in\mathrm{Min}_+(\lambda)}$$

is an isomorphism with inverse

$$P_+(\lambda) : \oplus_{q\in\mathrm{Min}_+(\lambda)} L^2(\mathbb{R}) \to E^0_{\mathrm{ess}}(\lambda),\ \ P_+(\lambda)(g_q)_{q\in\mathrm{Min}_+(\lambda)} = \sum_q P_+(q,\lambda)g_q$$

We can also express the pairing (12.15) in terms of the expansions of microlocal eigenfunctions $u \in E_{\mathrm{mic},-}(q,\lambda)$, $q \in \mathrm{Min}_-(\lambda)$, i.e. in the incoming region. That leads to maps $M_-(q,\lambda)$ and $P_-(q,\lambda)$, for $q \in \mathrm{Min}_-(\lambda)$ and Theorem 12.11 holds with all plus signs changed to minus signs. It is also convenient to change the notation slightly and identify the spaces $L^2(\mathbb{R})$ for the incoming and outgoing radial points over $z \in \mathrm{Min}$.

Corollary 12.12. *For $\lambda \notin \mathrm{Cv}(V)$, the S-matrix may be identified as the unitary operator $S(\lambda) = M_+(\lambda)P_-(\lambda)$ on $\oplus_{z\in\mathrm{Min}} L^2(\mathbb{R})$.*

This theorem is essentially a pointwise version of asymptotic completeness in λ. Integrating in λ gives a version of the usual statement. Namely, let $I \subset \mathbb{R}\backslash\mathrm{Cv}(V)$ be a compact interval. For $f \in \dot{\mathcal{C}}^\infty(X)$ orthogonal to $E_{\mathrm{pp}}(I)$, $u = u(\lambda) = R(\lambda + i0)f$, as discussed above,

(12.21)
$$\begin{aligned}B(u,u) = B_\lambda(u(\lambda),u(\lambda)) &= i\left(\langle f, R(\lambda+i0)f\rangle - \langle R(\lambda+i0)f, f\rangle\right)\\ &= i\langle f, [R(\lambda+i0) - R(\lambda-i0)]f\rangle = 2\pi\langle f, \mathrm{Sp}(\lambda)f\rangle.\end{aligned}$$

Integrating over λ in I, denoting the spectral projection of P to I by Π_I, and writing $M_+(q,\lambda) = 0$ if $V(z) > \lambda$, $z = \pi(q)$, we deduce that

$$\|\Pi_I f\|^2 = \sum_{q\in\mathrm{Min}_+(\lambda)} 4\pi\sqrt{\lambda - V(z)}\int \|M_+(q,\lambda)R(\lambda+i0)f\|^2_{L^2(\mathbb{R}),\omega_q}\,d\lambda, \tag{12.22}$$

so $M_+ \circ R(.+i0)$ is an isometry on the orthocomplement of the finite dimensional space $E_{\mathrm{pp}}(I)$ in the range of Π_I.

Theorem 12.13 (Asymptotic completeness). *If $I \subset \mathbb{R} \setminus \mathrm{Cv}(V)$ is compact then*

$$M_+ \circ R(. + i0) : \mathrm{Ran}(\Pi_I) \ominus E_{\mathrm{pp}}(I) \to L^2(I \times \mathbb{R})$$

is unitary.

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