THE WAVE EQUATION ON ASYMPTOTICALLY ANTI-DE SITTER SPACES

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Abstract. In this paper we describe the behavior of solutions of the Klein-Gordon equation, $(\Box_g + \lambda)u = f$, on Lorentzian manifolds (X°, g) which are anti-de Sitter-like (AdS-like) at infinity. Such manifolds are Lorentzian analogues of the so-called Riemannian conformally compact (or asymptotically hyperbolic) spaces, in the sense that the metric is conformal to a smooth Lorentzian metric $\hat{g}$ on X, where X has a non-trivial boundary, in the sense that $g = x^{-2}\hat{g}$, with x a boundary defining function. The boundary is conformally time-like for these spaces, unlike asymptotically de Sitter spaces studied in [33, 6], which are similar but with the boundary being conformally space-like.

Here we show local well-posedness for the Klein-Gordon equation, and also global well-posedness under global assumptions on the (null)bicharacteristic flow, for λ below the Breitenlohner-Freedman bound, $(n-1)^2/4$. These have been known under additional assumptions, [8, 9, 15]. Further, we describe the propagation of singularities of solutions and obtain the asymptotic behavior (at ∂X) of regular solutions. We also define the scattering operator, which in this case is an analogue of the hyperbolic Dirichlet-to-Neumann map. Thus, it is shown that below the Breitenlohner-Freedman bound, the Klein-Gordon equation behaves much like it would for the conformally related metric, $\hat{g}$, with Dirichlet boundary conditions, for which propagation of singularities was shown by Melrose, Sjöstrand and Taylor [21, 22, 27, 24], though the precise form of the asymptotics is different.

1. Introduction

In this paper we consider asymptotically anti de Sitter (AdS) type metrics on n-dimensional manifolds with boundary X, $n \geq 2$. We recall the actual definition of AdS space below, but for our purposes the most important feature is the asymptotic of the metric on these spaces, so we start by making a bold general definition. Thus, an asymptotically AdS type space is a manifold with boundary X such that X° is equipped with a pseudo-Riemannian metric g of signature $(1, n-1)$ which near the boundary Y of X is of the form

$$g = \frac{-dx^2 + h}{x^2}, \tag{1.1}$$

h a smooth symmetric 2-cotensor on X such that with respect to some product decomposition of X near Y, $X = Y \times [0, \epsilon)_x$, $h|_Y$ is a section of $T^*Y \otimes T^*Y$

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(rather than merely[1] $T^*_Y X \otimes T^*_Y X$) and is a Lorentzian metric on Y (with signature $(1, n-2)$). Note that Y is time-like with respect to the conformal metric

$$\hat{g} = x^2 g, \text{ so } \hat{g} = -dx^2 + h \text{ near } Y,$$

i.e. the dual metric $\hat{G}$ of $\hat{g}$ is negative definite on N^*Y, i.e. on span$\{dx\}$, in contrast with the asymptotically de Sitter-like setting studied in [33] when the boundary is space-like. Let the wave operator $\Box = \Box_g$ be the Laplace-Beltrami operator associated to this metric, and let

$$P = P(\lambda) = \Box_g + \lambda$$

be the Klein-Gordon operator, $\lambda \in \mathbb{C}$. The convention with the positive sign for the 'spectral parameter' λ preserves the sign of λ relative to the dx^2 component of the metric in both the Riemannian conformally compact and the Lorentzian de Sitter-like cases, and hence is convenient when describing the asymptotics. We remark that if $n = 2$ then up to a change of the (overall) sign of the metric, these spaces are asymptotically de Sitter, hence the results of [33] apply. However, some of the results are different even then, since in the two settings the role of the time variable is reversed, so the formulation of the results differs as the role of 'initial' and 'boundary' conditions changes.

These asymptotically AdS-metrics are also analogues of the Riemannian 'conformally compact', or asymptotically hyperbolic, metrics, introduced by Mazzeo and Melrose [18] in this form, which are of the form $x^{-2}(dx^2 + h)$ with $dx^2 + h$ smooth Riemannian on X, and $h|_Y$ is a section of $T^*Y \otimes T^*Y$. These have been studied extensively, in part due to the connection to AdS metrics (so some phenomena might be expected to be similar for AdS and asymptotically hyperbolic metrics) and their Riemannian signature, which makes the analysis of related PDE easier. We point out that hyperbolic space actually solves the Riemannian version of Einstein's equations, while de Sitter and anti-de Sitter space satisfy the actual hyperbolic Einstein equations. We refer to the works of Fefferman and Graham [13], Graham and Lee [14] and Anderson [3] among others for analysis on conformally compact spaces. There is also a large body of literature on asymptotically de Sitter spaces. Among others, Anderson and Chruściel studied the geometry of asymptotically de Sitter spaces [1, 2, 4], while in [33] the asymptotics of solutions of the Klein-Gordon equation were obtained, and in [6] the forward fundamental solution was constructed as a Fourier integral operator. It should be pointed out that the de Sitter-Schwarzschild metric in fact has many similar features with asymptotically de Sitter spaces (in an appropriate sense, it simply has two de Sitter-like ends). A weaker version of the asymptotics in this case is contained in the part of works of Dafermos and Rodnianski [10, 12, 11] (they also study a non-linear problem), and local energy decay was studied by Bony and Häfner [7], in part based on the stationary resonance analysis of Sá Barreto and Zworski [25]; stronger asymptotics (exponential decay to constants) was shown in a series of papers with Antônio Sá Barreto and Richard Melrose [20, 19].

For the universal cover of AdS space itself, the Klein-Gordon equation was studied by Breitenlohner and Freedman [8, 9], who showed its solvability for $\lambda < (n-1)^2/4$, $n = 4$, and uniqueness for $\lambda < 5/4$, in our normalization. Analogoues

[1]In fact, even this most general setting would necessitate only minor changes, except that the 'smooth asymptotics' of Proposition 8.10 would have variable order, and the restrictions on λ that arise here, $\lambda < (n-1)^2/4$, would have to be modified.

of these results were extended to the Dirac equation by Bachelot [5]. Finally, for a class of perturbations of the universal cover of AdS, which still possess a suitable Killing vector field, Holzegel [15] recently showed well-posedness for $\lambda < (n-1)^2/4$ by imposing a boundary condition, see [15, Definiton 3.1]. He also obtained certain estimates on the derivatives of the solution, as well as pointwise bounds.

Below we consider solutions of $Pu = 0$, or indeed $Pu = f$ with f given. Before describing our results, first we recall a formulation of the conformal problem, namely $\hat{g} = x^2 g$, so $\hat{g}$ is Lorentzian smooth on X, and Y is time-like – at the end of the introduction we give a full summary of basic results in the 'compact' and 'conformally compact' Riemannian and Lorentzian settings, with space-like as well as time-like boundaries in the latter case. Let

$$\hat{P} = \Box_{\hat{g}};$$

adding λ to the operator makes no difference in this case (unlike for P). Suppose that $\mathcal{S}$ is a space-like hypersurface in X intersecting Y (automatically transversally). Then the Cauchy problem for the Dirichlet boundary condition,

$$\hat{P}u = f,\ u|_Y = 0,\ u|_{\mathcal{S}} = \psi_0,\ Vu|_{\mathcal{S}} = \psi_1,$$

f, ψ_0, ψ_1 given, V a vector field transversal to $\mathcal{S}$, is locally well-posed (in appropriate function spaces) near $\mathcal{S}$. Moreover, under a global condition on the generalized broken bicharacteristic (or GBB) flow and $\mathcal{S}$, which we recall below in Definition 1.1, the equation is globally well-posed.

Namely, the global geometric assumption is that

$$\text{(TF)}\qquad \begin{aligned}&\text{there exists } t \in \mathcal{C}^\infty(X) \text{ such that for every } \mathsf{GBB}\ \gamma,\ t \circ \rho \circ \gamma : \mathbb{R} \to \mathbb{R}\\ &\text{is either strictly increasing or strictly decreasing and has range } \mathbb{R},\end{aligned}$$

where $\rho : T^*X \to X$ is the bundle projection. In the above formulation of the problem, we would assume that $\mathcal{S}$ is a level set, $t = t_0$ – note that locally this is always true in view of the Lorentzian nature of the metric and the conditions on Y and $\mathcal{S}$. As is often the case in the presence of boundaries, see e.g. [16, Theorem 24.1.1] and the subsequent remark, it is convenient to consider the special case of the Cauchy problem with vanishing initial data and f supported to one side of $\mathcal{S}$, say in $t \geq t_0$; one can phrase this as solving

$$\hat{P}u = f,\ u|_Y = 0,\ \operatorname{supp} u \subset \{t \geq t_0\}.$$

This forward Cauchy problem is globally well-posed for $f \in L^2_{\mathrm{loc}}(X)$, $u \in \dot{H}^1_{\mathrm{loc}}(X)$, and the analogous statement also holds for the backward Cauchy problem. Here we use Hörmander's notation $\dot{H}^1(X)$, see [16, Appendix B], to avoid confusion with the 'zero Sobolev spaces' $H^s_0(X)$, which we recall momentarily. In addition, (without any global assumptions) singularities of solutions, as measured by the b-wave front set, WF_{b}, relative to either $L^2_{\mathrm{loc}}(X)$ or $\dot{H}^1_{\mathrm{loc}}(X)$, propagate along GBB as was shown by Melrose, Sjöstrand and Taylor [21, 22, 27, 24], see also [26] in the analytic setting. Here recall that in X°, bicharacteristics are integral curves of the Hamilton vector field H_p (on $T^*X^\circ \setminus o$) of the principal symbol $\hat{p} = \sigma_2(\hat{P})$ inside the characteristic set,

$$\Sigma = \hat{p}^{-1}(\{0\}).$$

We also recall that the notion of a $\mathcal{C}^\infty$ and an analytic GBB is somewhat different due to the behavior at diffractive points, with the analytic definition being more permissive (i.e. weaker). Throughout this paper we use the analytic definition, which we now recall.

First, we need the notion of the compressed characteristic set, $\dot{\Sigma}$ of $\hat{P}$. This can be obtained by replacing, in T^*X, T^*_YX by its quotient T^*_YX/N^*Y, where N^*Y is the conormal bundle of Y in X. One denotes then by $\dot{\Sigma}$ the image $\hat{\pi}(\Sigma)$ of Σ in this quotient. One can give a topology to $\dot{\Sigma}$, making a set O open if and only if $\hat{\pi}^{-1}(O)$ is open in Σ. This notion of the compressed characteristic set is rather intuitive, since working with the quotient encodes the law of reflection: points with the same tangential but different normal momentum at Y are identified, which, when combined with the conservation of kinetic energy (i.e. working on the characteristic set) gives the standard law of reflection. However, it is very useful to introduce another (equivalent) definition already at this point since it arises from structures which we also need.

The alternative point of view (which is what one needs in the proofs) is that the analysis of solutions of the wave equation takes place on the b-cotangent bundle, ${}^{\mathrm{b}}T^*X$ ('b' stands for boundary), introduced by Melrose. We refer to [23] for a very detailed description, [32] for a concise discussion. Invariantly one can define ${}^{\mathrm{b}}T^*X$ as follows. First, let $\mathcal{V}_{\mathrm{b}}(X)$ be the set of all $\mathcal{C}^\infty$ vector fields on X tangent to the boundary. If $(x, y_1, \ldots, y_{n-1})$ are local coordinates on X, with x defining Y, elements of $\mathcal{V}_{\mathrm{b}}(X)$ have the form

$$a\, x\partial_x + \sum_{j=1}^{n-1} b_j\, \partial_{y_j}, \tag{1.2}$$

with a and b_j smooth. It follows immediately that $\mathcal{V}_{\mathrm{b}}(X)$ is the set of all smooth sections of a vector bundle, ${}^{\mathrm{b}}TX$: x, y_j, a, b_j, $j = 1, \ldots, n-1$, give local coordinates in terms of (1.2). Then ${}^{\mathrm{b}}T^*X$ is defined as the dual bundle of ${}^{\mathrm{b}}TX$. Thus, points in the b-cotangent bundle, ${}^{\mathrm{b}}T^*X$, of X are of the form

$$\underline{\xi}\,\frac{dx}{x} + \sum_{j=1}^{n-1} \underline{\zeta}_j\, dy_j,$$

so $(x, y, \underline{\xi}, \underline{\zeta})$ give coordinates on ${}^{\mathrm{b}}T^*X$. There is a natural map $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ induced by the corresponding map between sections

$$\xi\, dx + \sum_{j=1}^{n-1} \zeta_j\, dy_j = (x\xi)\,\frac{dx}{x} + \sum_{j=1}^{n-1} \zeta_j\, dy_j,$$

thus

$$\pi(x, y, \xi, \zeta) = (x, y, x\xi, \zeta), \tag{1.3}$$

i.e. $\underline{\xi} = x\xi$, $\underline{\zeta} = \zeta$. Over the interior of X we can identify $T^*_{X^\circ}X$ with ${}^{\mathrm{b}}T^*_{X^\circ}X$, but this identification π becomes singular (no longer a diffeomorphism) at Y. We denote the image of Σ under π by

$$\dot{\Sigma} = \pi(\Sigma),$$

called the compressed characteristic set. Thus, $\dot{\Sigma}$ is a subset of the vector bundle ${}^{\mathrm{b}}T^*X$, hence is equipped with a topology which is equivalent to the one define by the quotient, see [32, Section 5]. The definition of *analytic* GBB then becomes:

Definition 1.1. *Generalized broken bicharacteristics,* or GBB, are continuous maps $\gamma : I \to \dot{\Sigma}$, where I is an interval, satisfying that for all $f \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X)$ real valued,

$$\liminf_{s\to s_0} \frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0} \geq \inf\{\mathsf{H}_p(\pi^* f)(q) :\ q\in\pi^{-1}(\gamma(s_0))\cap\Sigma\}.$$

Since the map $p \mapsto \mathsf{H}_p$ is a derivation, $\mathsf{H}_{ap} = a\mathsf{H}_p$ at Σ, so bicharacteristics are merely reparameterized if p is replaced by a conformal multiple. In particular, if P is the Klein-Gordon operator, $\Box_g + \lambda$, for an asymptotically AdS-metric g, the bicharacteristics over X° are, up to reparameterization, those of $\hat{g}$. We make this into our definition of GBB.

Definition 1.2. The compressed characteristic set $\dot{\Sigma}$ of P is that of $\Box_{\hat{g}}$.
Generalized broken bicharacteristics, or GBB, of P are GBB *in the analytic sense* of the smooth Lorentzian metric $\hat{g}$.

We now give a formulation for the global problem. For this purpose we need to recall one more class of differential operators in addition to $\mathcal{V}_{\mathrm{b}}(X)$ (which is the set of $\mathcal{C}^\infty$ vector fields *tangent to the boundary*). Namely, we denote the set of $\mathcal{C}^\infty$ vector fields *vanishing at the boundary* by $\mathcal{V}_0(X)$. In local coordinates (x,y), these have the form

$$a\,x\partial_x + \sum_{j=1}^n b_j(x\partial_{y_j}), \tag{1.4}$$

with $a, b_j \in \mathcal{C}^\infty(X)$; cf. (1.2). Again, $\mathcal{V}_0(X)$ is the set of all $\mathcal{C}^\infty$ sections of a vector bundle, ${}^0TX$, which over X° can be naturally identified with $T_{X^\circ}X$; we refer to [18] for a detailed discussion of 0-geometry and analysis, and to [33] for a summary. We then let $\mathrm{Diff}_{\mathrm{b}}(X)$, resp. $\mathrm{Diff}_0(X)$, be the set of differential operators generated by $\mathcal{V}_{\mathrm{b}}(X)$, resp. $\mathcal{V}_0(X)$, i.e they are locally finite sums of products of these vector fields with $\mathcal{C}^\infty(X)$-coefficients. In particular,

$$P = \Box_g + \lambda \in \mathrm{Diff}_0^2(X),$$

which explains the relevance of $\mathrm{Diff}_0(X)$. This can be seen easily from g being in fact a non-degenerate smooth symmetric bilinear form on ${}^0TX$; the conformal factor x^{-2} compensates for the vanishing factors of x in (1.4), so in fact this is *exactly* the same statement as $\hat{g}$ being Lorentzian on TX.

Let $H_0^k(X)$ denote the zero-Sobolev space relative to

$$L^2(X) = L^2_0(X) = L^2(X, dg) = L^2(X, x^{-n}\,d\hat{g}),$$

so if $k \geq 0$ is an integer then

$$u \in H_0^k(X) \text{ iff for all } L \in \mathrm{Diff}_0^k(X),\ Lu \in L^2(X);$$

negative values of k give Sobolev spaces by dualization. For our problem, we need a space of 'very nice' functions corresponding to $\mathrm{Diff}_{\mathrm{b}}(X)$. We obtain this by replacing $\mathcal{C}^\infty(X)$ with the space of conormal functions to the boundary relative to a fixed space of functions, in this case $H_0^k(X)$, i.e. functions $v \in H^k_{0,\mathrm{loc}}(X)$ such that $Qv \in H^k_{0,\mathrm{loc}}(X)$ for every $Q \in \mathrm{Diff}_{\mathrm{b}}(X)$ (of any order). The finite order regularity version of this is $H^{k,m}_{0,\mathrm{b}}(X)$, which is given for $m \geq 0$ integer by

$$u \in H^{k,m}_{0,\mathrm{b}}(X) \iff u \in H_0^k(X) \text{ and } \forall Q \in \mathrm{Diff}^m_{\mathrm{b}}(X),\ Qu \in H^k_0(X),$$

while for $m < 0$ integer, $u \in H^{k,m}_{0,\mathrm{b}}(X)$ if $u = \sum Q_j u_j$, $u_j \in H^{k,0}_{0,\mathrm{b}}(X)$, $Q_j \in \mathrm{Diff}^m_{\mathrm{b}}(X)$. Thus, $H^{-k,-m}_{0,\mathrm{b}}(X)$ is the dual space of $H^{k,m}_{0,\mathrm{b}}(X)$, relative to $L^2_0(X)$.

Although the finite speed of propagation means that the wave equation has a local character in X, and thus compactness of the slices $t = t_0$ is immaterial, it is convenient to assume

$$\text{the map } t : X \to \mathbb{R} \text{ is proper.} \tag{PT}$$

Even as stated, the propagation of singularities results (which form the heart of the paper) do not assume this, and the assumption is made elsewhere merely to make the formulation and proof of the energy estimates and existence slightly simpler, in that one does not have to localize in spatial slices this way.

Suppose $\lambda < (n-1)^2/4$. Suppose

$$f \in H^{-1,1}_{0,\mathrm{b,loc}}(X),\ \operatorname{supp} f \subset \{t \geq t_0\}. \tag{1.5}$$

We want to find $u \in H^1_{0,\mathrm{loc}}(X)$ such that

$$Pu = f,\ \operatorname{supp} u \subset \{t \geq t_0\}. \tag{1.6}$$

We show that this is locally well-posed near $\mathcal{S}$. Moreover, under the previous global assumption on GBB, this problem is globally well-posed:

Theorem 1.3. *(See Theorem 4.16.) Assume that (TF) and (PT) hold. Suppose $\lambda < (n-1)^2/4$. The forward Dirichlet problem,* (1.6)*, has a unique global solution $u \in H^1_{0,\mathrm{loc}}(X)$, and for all compact $K \subset X$ there exists a compact $K' \subset X$ and a constant $C > 0$ such that for all f as in* (1.5)*, the solution u satisfies*

$$\|u\|_{H^1_0(K)} \leq C\|f\|_{H^{-1,1}_{0,\mathrm{b}}(K')}.$$

Remark 1.4. In fact, one can be quite explicit about K' in view of (PT), since $u|_{t\in[t_0,t_1]}$ can be estimated by $f|_{t\in I}$, I open containing $[t_0, t_1]$.

We also prove microlocal elliptic regularity and describe the propagation of singularities of solutions, as measured by WF_{b} relative to $H^1_{0,\mathrm{loc}}(X)$. We define this notion in Definition 5.9 and discuss it there in more detail. However, we recall the definition of the standard wave front set WF on manifolds without boundary X that immediately generalizes to the b-wave front set WF_{b}. Thus, one says that $q \in T^*X \setminus o$ is *not* in the wave front set of a distribution u if there exists $A \in \Psi^0(X)$ such $\sigma_0(A)(q)$ is invertible and $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}(X)$ – this is equivalent to $Au \in \mathcal{C}^\infty(X)$ by the Sobolev embedding theorem. Here $L^2(X)$ can be replaced by $H^m(X)$ instead, with m arbitrary. Moreover, WF^m can also be defined analogously: we require $Au \in L^2(X)$ for $A \in \Psi^m(X)$ elliptic at q. Thus, $q \notin \mathrm{WF}(u)$ means that u is 'microlocally $\mathcal{C}^\infty$ at q', while $q \notin \mathrm{WF}^m(u)$ means that u is 'microlocally H^m at q'.

In order to microlocalize $H^{k,m}_{0,\mathrm{b}}(X)$, we need pseudodifferential operators, here extending $\mathrm{Diff}_{\mathrm{b}}(X)$ (as that is how we measure regularity). These are the b-pseudodifferential operators $A \in \Psi^m_{\mathrm{b}}(X)$ introduced by Melrose, their principal symbol $\sigma_{\mathrm{b},m}(A)$ is a homogeneous degree m function on ${}^{\mathrm{b}}T^*X \setminus o$; we again refer to [23, 32]. Then we say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}^{k,\infty}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^0_{\mathrm{b}}(X)$ with $\sigma_{\mathrm{b},0}(A)(q)$ invertible and such that Au is H^k_0-conormal to the boundary. One also defines $\mathrm{WF}^{k,m}_{\mathrm{b}}(u)$: $q \notin \mathrm{WF}^m_{\mathrm{b}}(u)$ if there exists $A \in \Psi^m_{\mathrm{b}}(X)$

with $\sigma_{\mathrm{b},0}(A)(q)$ invertible and such that $Au \in H^k_{0,\mathrm{loc}}(X)$. One can also extend these definitions to $m < 0$.

With this definition we have the following theorem:

Theorem 1.5. *(See Proposition 7.7 and Theorem 8.8.) Suppose that $P = \Box_g + \lambda$, $\lambda < (n-1)^2/4$, $m \in \mathbb{R}$ or $m = \infty$. Suppose $u \in H^{1,k}_{0,\mathrm{b,loc}}(X)$ for some $k \in \mathbb{R}$. Then*

$$\mathrm{WF}_{\mathrm{b}}^{1,m}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu).$$

Moreover,

$$(\mathrm{WF}_{\mathrm{b}}^{1,m}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,m+1}(Pu)$$

is a union of maximally extended generalized broken bicharacteristics of the conformal metric $\hat{g}$ in

$$\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,m+1}(Pu).$$

In particular, if $Pu = 0$ then $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \subset \dot{\Sigma}$ is a union of maximally extended generalized broken bicharacteristics of $\hat{g}$.

As a consequence, we obtain the following more general, and precise, well-posedness result.

Theorem 1.6. *(See Theorem 8.12.) Assume that (TF) and (PT) hold. Suppose that $P = \Box_g + \lambda$, $\lambda < (n-1)^2/4$, $m \in \mathbb{R}$, $m' \leq m$. Suppose $f \in H^{-1,m+1}_{0,\mathrm{b,loc}}(X)$. Then* (1.6) *has a unique solution in $H^{1,m'}_{0,\mathrm{b,loc}}(X)$, which in fact lies in $H^{1,m}_{0,\mathrm{b,loc}}(X)$, and for all compact $K \subset X$ there exists a compact $K' \subset X$ and a constant $C > 0$ such that*

$$\|u\|_{H_0^{1,m}(K)} \leq C \|f\|_{H_{0,\mathrm{b}}^{-1,m+1}(K')}.$$

While we prove this result using the propagation of singularities, thus a relatively sophisticated theorem, it could also be derived without full microlocalization, i.e. without localizing the propagation of energy in phase space.

We also generalize propagation of singularities to the case $\operatorname{Im}\lambda \neq 0$ ($\operatorname{Re}\lambda$ arbitrary), in which case we prove one sided propagation depending on the sign of $\operatorname{Im}\lambda$. Namely, if $\operatorname{Im}\lambda > 0$, resp. $\operatorname{Im}\lambda < 0$,

$$(\mathrm{WF}_{\mathrm{b}}^{1,m}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,m+1}(Pu)$$

is a union of *maximally forward, resp. backward*, extended generalized broken bicharacteristics of the conformal metric $\hat{g}$. There is no difference between the case $\operatorname{Im}\lambda = 0$ and $\operatorname{Re}\lambda < (n-1)^2/4$, resp. $\operatorname{Im}\lambda \neq 0$, at the elliptic set, i.e. the statement

$$\mathrm{WF}_{\mathrm{b}}^{1,m}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu).$$

holds even if $\operatorname{Im}\lambda \neq 0$. We refer to Proposition 7.7 and Theorem 8.9 for details.

These results indicate already that for $\operatorname{Im}\lambda \neq 0$ there are many interesting questions to answer, and in particular that one cannot think of λ as 'small'; this will be the focus of future work.

In particular, if f is conormal relative to $H^1_0(X)$ then $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) = \emptyset$. Let $\sqrt{}$ denote the branch square root function on $\mathbb{C} \setminus (-\infty, 0]$ chosen so that takes positive values on $(0, \infty)$. If we assume e.g. $f \in \dot{\mathcal{C}}^\infty(X)$, then

$$u = x^{s_+(\lambda)} v,\ v \in \mathcal{C}^\infty(X),\ s_+(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda},$$

as we show in Proposition 8.10. Since the indical roots of $\Box_g + \lambda$ are

$$s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}, \tag{1.7}$$

this explains the interpretation of this problem as a 'Dirichlet problem', much like it was done in the Riemannian conformally compact case by Mazzeo and Melrose [18]: asymptotics corresponding to the growing indicial root, $x^{s_-(\lambda)} v_-$, $v_- \in \mathcal{C}^\infty(X)$, is ruled out.

For $\lambda < (n-1)^2/4$, one can then easily solve the problem with inhomogeneous 'Dirichlet' boundary condition, i.e. given $v_0 \in \mathcal{C}^\infty(Y)$ and $f \in \dot{\mathcal{C}}^\infty(X)$, both supported in $\{t \geq t_0\}$,

$$Pu = f,\ u|_{t<t_0} = 0,\ u = x^{s_-(\lambda)} v_- + x^{s_+(\lambda)} v_+,\ v_\pm \in \mathcal{C}^\infty(X),\ v_-|_Y = v_0,$$

if $s_+(\lambda) - s_-(\lambda) = 2\sqrt{\frac{(n-1)^2}{4} - \lambda}$ is not an integer. If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in \mathcal{C}^\infty(X)$ by $v_- = \mathcal{C}^\infty(X) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$; see Theorem 8.11.

The operator $v_-|_Y \to v_+|_Y$ is the analogue of the Dirichlet-to-Neumann map, or the scattering operator. In the De Sitter setting the setup is somewhat different as both pieces of scattering data are specified either at past or future infinity, see [33]. Nonetheless, one expects that the result of [33, Section 7], that the scattering operator is a Fourier integral operator associated to the GBB relation can be extended to the present setting, at least if the boundary is totally geodesic with respect to the conformal metric $\hat{g}$. Indeed, in an ongoing project, Baskin and the author are extending Baskin's construction of the forward fundamental solution on asymptotically De Sitter spaces, [6], to the totally geodesic asymptotically AdS setting. In addition, it is an interesting question what is the 'best' problem to pose when $\operatorname{Im}\lambda \neq 0$; the results of this paper suggest that the global problem (rather than local, Cauchy data versions) is the best behaved. One virtue of the parametrix construction is that we expect to be able answer Lorentzian analogues of questions related to the work of Mazzeo and Melrose [18], which would bring the Lorentzian world of AdS spaces significantly closer (in terms of results) to the Riemannian world of conformally compact spaces. We singled out the totally geodesic condition since it holds on actual AdS space, which we now discuss.

We now recall the structure of the actual AdS space to justify our terminology. Consider $\mathbb{R}^{n+1}$ with the pseudo-Riemannian metric of signature $(2, n-1)$ given by

$$-dz_1^2 - \ldots - dz_{n-1}^2 + dz_n^2 + dz_{n+1}^2,$$

with $(z_1, \ldots, z_{n+1})$ denoting coordinates on $\mathbb{R}^{n+1}$, and the hyperboloid

$$z_1^2 + \ldots + z_{n-1}^2 - z_n^2 - z_{n+1}^2 = -1$$

inside it. Note that $z_n^2 + z_{n+1}^2 \geq 1$ on the hyperboloid, so we can (diffeomorphically) introduce polar coordinates in these two variables, i.e. we let $(z_n, z_{n+1}) = R\theta$, $R \geq 1$, $\theta \in \mathbb{S}^1$. Then the hyperboloid is of the form

$$z_1^2 + \ldots + z_{n-1}^2 - R^2 = -1$$

inside $\mathbb{R}^{n-1} \times (0,\infty)_R \times \mathbb{S}^1_\theta$. As dz_j, $j = 1, \ldots, n-1$, $d\theta$ and $d(z_1^2 + \ldots + z_{n-1}^2 - R^2)$ are linearly independent at the hyperboloid,

$$z_1, \ldots, z_{n-1}, \theta$$

give local coordinates on it, and indeed these are global in the sense that the hyperboloid X° is identified with $\mathbb{R}^{n-1} \times \mathbb{S}^1$ via these. A straightforward calculation shows that the metric on $\mathbb{R}^{n+1}$ restricts to give a Lorentzian metric g on the hyperboloid. Indeed, away from $\{0\} \times \mathbb{S}^1$, we obtain a convenient form of the metric by using polar coordinates (r, ω) in $\mathbb{R}^{n-1}$, so $R^2 = r^2 + 1$:

$$g = -(dr)^2 - r^2\, d\omega^2 + (dR)^2 + R^2\, d\theta^2 = -(1+r^2)^{-1}\, dr^2 - r^2\, d\omega^2 + (1+r^2)\, d\theta^2,$$

where $d\omega^2$ is the standard round metric; a similar description is easily obtained near $\{0\} \times \mathbb{S}^1$ by using the standard Euclidean variables.

We can compactify the hyperboloid by compactifying $\mathbb{R}^{n-1}$ to a ball $\overline{\mathbb{B}^{n-1}}$ via inverse polar coordinates (x, ω), $x = r^{-1}$,

$$(z_1, \dots, z_{n-1}) = x^{-1}\omega,\ 0 < x < \infty,\ \omega \in \mathbb{S}^{n-2}.$$

Thus, the interior of $\overline{\mathbb{B}^{n-1}}$ is identified with $\mathbb{R}^{n-1}$, and the boundary $\mathbb{S}^{n-2}$ of $\overline{\mathbb{B}^{n-1}}$ is added at $x = 0$ to compactify $\mathbb{R}^{n-1}$. We let

$$X = \overline{\mathbb{B}^{n-1}} \times \mathbb{S}^1$$

be this compactification of X°; a collar neighborhood of ∂X is identified with

$$[0,1)_x \times \mathbb{S}^{n-2}_\omega \times \mathbb{S}^1_\theta.$$

In this collar neighborhood the Lorentzian metric takes the form

$$g = \frac{1}{x^2}\Big(-(1+x^2)^{-1}\, dx^2 - d\omega^2 + (1+x^2)\, d\theta^2 \Big),$$

which is of the desired form, and the conformal metric is

$$\hat{g} = -(1+x^2)^{-1}\, dx^2 - d\omega^2 + (1+x^2)\, d\theta^2$$

with respect to which the boundary, $\{x = 0\}$, is indeed time-like. Note that the induced metric on the boundary is $-d\omega^2 + d\theta^2$, up to a conformal multiple.

As already remarked, $\hat{g}$ has the special feature that Y is totally geodesic, unlike e.g. the case of $\mathbb{B}^{n-1} \times \mathbb{S}^1$ equipped with a product Lorentzian metric, with $\mathbb{B}^{n-1}$ carrying the standard Euclidean metric.

For global results, it is useful to work on the universal cover $\tilde{X} = \overline{\mathbb{B}^{n-1}} \times \mathbb{R}_t$ of X, where $\mathbb{R}_t$ is the universal cover of $\mathbb{S}^1_\theta$; we use t to denote the time-like nature of this coordinate. The local geometry is unchanged, but now t provides a global parameter along generalized broken bicharacteristics, hence satisfies the assumptions for our main theorem.

We use this opportunity to summarize the results, already referred to earlier, for analysis on conformally compact Riemannian or Lorentzian spaces, including a comparison with the conformally related problem, i.e. for $\Delta_{\hat{g}}$ or $\Box_{\hat{g}}$. We assume Dirichlet boundary condition (DBC) when relevant for the sake of definiteness, and global hyperbolicity for the hyperbolic equations, and do not state the function spaces or optimal forms of regularity results.

(i) Riemannian: $(\Delta_{\hat{g}} - \lambda)u = f$, with DBC is well-posed for $\lambda \in \mathbb{C} \setminus [0, \infty)$; moreover, if $f \in \dot{\mathcal{C}}^\infty(X)$, then $u \in \mathcal{C}^\infty(X)$. (Also works outside a discrete set of poles λ in $[0, \infty)$.)

(ii) Lorentzian, $\partial X = Y_+ \cup Y_-$ is spacelike, f supported in $t \geq t_0$, $\lambda \in \mathbb{C}$: $(\Box_{\hat{g}} - \lambda)u = f$, u supported in $t \geq t_0$, is well-posed. If $f \in \dot{\mathcal{C}}^\infty(X)$, the solution is $\mathcal{C}^\infty$ up to $Y_\pm$.

(iii) Lorentzian, ∂X is timelike, f supported in $t \geq t_0$, $\lambda \in \mathbb{C}$: $(\Box_{\hat{g}} - \lambda)u = f$, with DBC at Y, u supported in $t \geq t_0$, is well-posed. If $f \in \dot{\mathcal{C}}^\infty(X)$, the solution is $\mathcal{C}^\infty$ up to $Y_\pm$.

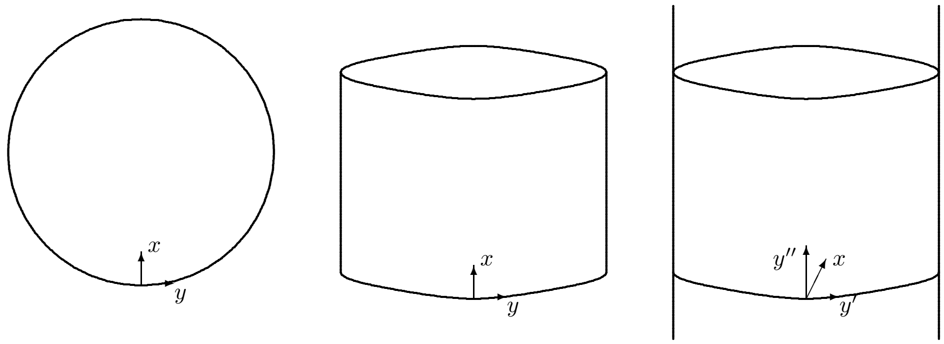

FIGURE 1. On the left, a Riemannian example, $\overline{\mathbb{B}^2}$, in the middle, an example of spacelike boundary, $[0,1]_x \times \mathbb{S}^1_y$ with x timelike, on the right, the case of timelike boundary, $\overline{\mathbb{B}^2_{x,y'}} \times \mathbb{R}_{y''}$, with y'' timelike.

We now go through the original problems. Let $s_\pm(\lambda)$ be as in (1.7).

(i) Asymptotically hyperbolic, $\lambda \in \mathbb{C} \setminus [0, +\infty)$: There is a unique solution of $(\Delta_g - \lambda)u = f$, $f \in \dot{\mathcal{C}}^\infty(X)$, such that $u = x^{s_+(\lambda)}v$, $v \in \mathcal{C}^\infty(X)$. (Analogue of DBC; Mazzeo and Melrose [18].) (Indeed, $u = (\Delta_g - \lambda)^{-1} f$, and this can be extended to $\lambda \in [0, +\infty)$, apart from finitely many poles in $[0, (n-1)^2/4]$, and analytically continued further.)

(ii) Asymptotically de Sitter, $\lambda \in \mathbb{C}$: For f supported in $t \geq t_0$, there is a unique solution of $(\Box_g - \lambda)u = f$ supported in $t \geq t_0$. Moreover, for $f \in \dot{\mathcal{C}}^\infty(X)$, $u = x^{s_+(\lambda)}v_+ + x^{s_-(\lambda)}v_-$, $v_\pm \in \mathcal{C}^\infty(X)$ and $v_\pm|_{Y_-}$ is specified, provided that $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$. (See [33].)

(iii) Asymptotically Anti de Sitter, $\lambda \in \mathbb{R} \setminus [(n-1)^2/4, +\infty)$: For $f \in \dot{\mathcal{C}}^\infty(X)$ supported in $t \geq t_0$, there is a unique solution of $(\Box_g - \lambda)u = f$ such that $u = x^{s_+(\lambda)}v$, $v \in \mathcal{C}^\infty(X)$ and $\operatorname{supp} u \subset \{t \geq t_0\}$.

The structure of this paper is the following. In Section 2 we prove a Poincaré inequality that we use to allow the sharp range $\lambda < (n-1)^2/4$ for λ real. Then in Section 3 we recall the structure of energy estimates on manifolds without boundary as these are then adapted to our 'zero geometry' in Section 4. In Section 5 we introduce microlocal tools to study operators such as P, namely the zero-differential-b-pseudodifferential calculus, $\mathrm{Diff}_0\Psi_\mathrm{b}(X)$. In Section 6 the structure of GBB is recalled. In Section 7 we study the Dirichlet form and prove microlocal elliptic regularity. Finally, in Section 8, we prove the propagation of singularities for P.

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2. Poincaré inequality

Let h be a conformally compact Riemannian metric, i.e. a positive definite inner product on ${}^0TX$, hence by duality on ${}^0T^*X$; we denote the latter by H. We denote the by corresponding space of L^2 sections of ${}^0T^*X$ by $L^2(X; {}^0T^*X) =$

$L^2_0(X;{}^0T^*X)$. While the inner product on $L^2(X;{}^0T^*X)$ depends on the choice of h, the corresponding norms are independent of h, at least over compact subsets K of X. We first prove a Hardy-type inequality:

Lemma 2.1. *Suppose $V_0 \in \mathcal{V}(X)$ is real with $V_0 x|_{x=0} = 1$, and let $V \in \mathcal{V}_{\mathrm{b}}(X)$ be given by $V = xV_0$. Given any compact subset K of X and $\tilde{C} < \frac{n-1}{2}$, there exists $x_0 > 0$ such that if $u \in \dot{\mathcal{C}}^\infty(X)$ is supported in K then for $\psi \in \mathcal{C}^\infty(X)$ supported in $x < x_0$,*

$$\tilde{C}\|\psi u\|_{L^2_0(X)} \le \|\psi V u\|_{L^2_0(X)}. \tag{2.1}$$

Proof. For any $V \in \mathcal{V}_{\mathrm{b}}(X)$ real, and $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(X)$, $u \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)$, we have, using $V^* = -V - \operatorname{div} V$,

$$\begin{aligned}\langle (V\chi)u, u\rangle &= \langle [V,\chi]u, u\rangle = \langle \chi u, V^* u\rangle - \langle Vu, \chi u\rangle \\ &= -\langle \chi u, Vu\rangle - \langle Vu, \chi u\rangle - \langle \chi u, (\operatorname{div} V)u\rangle.\end{aligned}$$

Now, if $V = xV_0$, $V_0 \in \mathcal{V}(X)$ transversal to ∂X, and if we write $dg = x^{-n}\,d\hat{g}$, $d\hat{g}$ a smooth non-degenerate density then in local coordinates z_j such that $d\hat{g} = J|dz|$, $V_0 = \sum V_0^j \partial_j$,

$$\begin{aligned}\operatorname{div} V &= x^n J^{-1} \sum \partial_j (x^{-n} J x V_0^j) \\ &= -(n-1)\sum_j V_0^j(\partial_j x) + xJ^{-1}\sum \partial_j(JV_0^j) = -(n-1)(V_0 x) + x \operatorname{div} V_0.\end{aligned}$$

Thus, assuming $V_0 \in \mathcal{V}(X)$ with $V_0 x|_{x=0} = 1$,

$$\operatorname{div} V = -(n-1) + xa,\ \ a \in \mathcal{C}^\infty(X).$$

Let $x_0' > 0$ be such that $V_0 x > \frac{1}{2}$ in $x \le x_0'$. Thus, if $0 \le \chi_0 \le 1$, $\chi_0 \equiv 1$ near 0, $\chi_0' \le 0$, χ_0 is supported in $x \le x_0'$, $\chi = \chi_0 \circ x$, then

$$V\chi = x(V_0 x)(\chi_0' \circ x) \le 0,$$

hence $\langle (V\chi)u, u\rangle \le 0$

$$\langle \chi((n-1) + xa)u, u\rangle \le 2\|\chi^{1/2}u\|\|\chi^{1/2}Vu\|,$$

and thus given any $\tilde{C} < (n-1)/2$ there is $x_0 > 0$ such that for u supported in K,

$$\tilde{C}\|\chi^{1/2}u\| \le \|\chi^{1/2}Vu\|,$$

namely we take $x_0 < x_0'/2$ such that $(n-1)/2 - \tilde{C} > (\sup_K |a|)x_0$, choose $\chi_0 \equiv 1$ on $[0, x_0]$, supported in $[0, 2x_0)$. This completes the proof of the lemma. □

Tha basic Poincaré estimate is:

Proposition 2.2. *Suppose $K \subset X$ compact, $K \cap \partial X \neq \emptyset$, O open with $K \subset O$, O arcwise connected to ∂X, $K' = \overline{O}$ compact. There exists $C > 0$ such that for $u \in H^1_{0,\mathrm{loc}}(X)$ one has*

$$\|u\|_{L^2_0(K)} \le C\|du\|_{L^2_0(O;{}^0T^*X)}, \tag{2.2}$$

where the norms are relative to the metric h.

Proof. It suffices to prove the estimate for $u \in \dot{\mathcal{C}}^\infty(X)$, for then the proposition follows by the density of $\dot{\mathcal{C}}^\infty(X)$ in $H^1_{0,\mathrm{loc}}(X)$ and the continuity of both sides in the $H^1_{0,\mathrm{loc}}(X)$ topology.

Let V_0, V be as in Lemma 2.1, and let $\phi_0 \in \mathcal{C}^\infty_{\mathrm{comp}}(Y)$ identically 1 on a neighborhood of $K \cap Y$, supported in O, and let $x_0 > 0$ be as in the Lemma with K replaced by K'. We pull back ϕ_0 to a function ϕ defined on a neighborhood of Y by the V_0-flow; thus, $V_0\phi = 0$. By decreasing x_0 if needed, we may assume that ϕ is defined and is $\mathcal{C}^\infty$ in $x < x_0$, and $\operatorname{supp}\phi \cap \{x < x_0\} \subset O$. Now, let $\psi \in \mathcal{C}^\infty(X)$ identically 1 where $x < x_0/2$, supported where $x < 3x_0/4$, and let $\tilde{\psi} \in \mathcal{C}^\infty(X)$ be identically 1 where $x < 3x_0/4$, supported in $x < x_0$; thus $\tilde{\psi}\phi \in \mathcal{C}^\infty_{\mathrm{comp}}(X)$. Then, by Lemma 2.1 applied to $\psi_0\phi u$,

$$\tilde{C}\|\psi\phi u\|_{L^2_0(X)} = \tilde{C}\|\psi\psi_0\phi u\|_{L^2_0(X)} \leq \|\psi V(\psi_0\phi u)\|_{L^2_0(X)} = \|\psi\phi V u\|_{L^2_0(X)}. \tag{2.3}$$

The full proposition follows by the standard Poincaré estimate and arcwise connectedness of K to Y (hence to $x < x_0/2$), since one can estimate $u|_{x>x_0/2}$ in L^2 in terms of $du|_{x>x_0/2}$ in L^2 and $u|_{x_0/4<x<x_0/2}$. □

We can get a more precise estimate of the constants if we restrict to a neighborhood of a space-like hypersurface $\mathcal{S}$; it is convenient to state the result under our global assumptions. *Thus, (TF) and (PT) are assumed to hold from here on in this section.*

Proposition 2.3. *Suppose $V_0 \in \mathcal{V}(X)$ is real with $V_0x|_{x=0} = 1$, $V_0t \equiv 0$ near Y and let $V \in \mathcal{V}_{\mathrm{b}}(X)$ be given by $V = xV_0$. Let I be a compact interval. Let $C < (n-1)/2$, $\gamma > 0$. Then there exist $\epsilon > 0$, $x_0 > 0$ and $C' > 0$ such that the following holds.*

For $t_0 \in I$, $0 < \delta < \epsilon$ and for $u \in H^1_{0,\mathrm{loc}}(X)$ one has

$$\begin{aligned} &\|u\|_{L^2_0(\{p:\ t(p)\in[t_0,t_0+\epsilon]\})} \\ &\leq C^{-1}\|Vu\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon],\ x(p)\leq x_0\})} + \gamma\|du\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})} \\ &\qquad + C'\|u\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0]\})}, \end{aligned} \tag{2.4}$$

where the norms are relative to the metric h.

Proof. We proceed as in the proof of Proposition 2.2, using that the t-preimage of the enlargement of the interval by distance ≤ 1 points is still compact by (PT); we always use $\epsilon < 1$ correspondingly. We simply let $\phi = \tilde{\phi} \circ t$, where $\tilde{\phi}$ is the characteristic function of $[t_0, t_0 + \epsilon]$. Thus $V_0\phi$ vanishes near Y; at the cost of possibly decreasing x_0 we may assume that it vanishes in $x < x_0$. By (2.3), with $C = \tilde{C} < (n-1)/2$, $\psi \equiv 1$ on $[0, x_0/4)$, supported in $[0, x_0/2)$,

$$\|\psi\phi u\|_{L^2_0(X)} \leq C^{-1}\|\psi V\phi u\| = C^{-1}\|\psi\phi V u\|. \tag{2.5}$$

Thus, it remains to give a bound for $\|(1-\psi)u\|_{L^2_0(\{p:\ t(p)\in[t_0,t_0+\epsilon]\})}$.

Let $\mathcal{S}$ be the space-like hypersurface in X given by $t = t_0$, $t_0 \in I$. Now let $W \in \mathcal{V}_{\mathrm{b}}(X)$ be transversal to $\mathcal{S}$. The standard Poincaré estimate (whose weighted version we prove below in Lemma 2.4) obtained by integrating from $t = t_0 - \delta$ yields that for $u \in \dot{\mathcal{C}}^\infty(X)$ with $u|_{t=t_0-\delta} = 0$,

$$\|u\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})} \leq C'(\epsilon+\delta)^{1/2}\|Wu\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})}, \tag{2.6}$$

with $C'(\epsilon+\delta)\to 0$ as $\epsilon+\delta\to 0$. Applying this with u supported where $x\in(x_0/8,\infty)$

$$\|u\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})}\leq C''(\epsilon+\delta)^{1/2}\|xWu\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})},\tag{2.7}$$

with $C''(\epsilon+\delta)\to 0$ as $\epsilon+\delta\to 0$. As we want $0<\delta<\epsilon$, we choose $\epsilon>0$ such that

$$C''(2\epsilon)^{1/2}<\gamma.$$

Let $\chi\in\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R};[0,1])$ be identically 1 on $[t_0,\infty)$, and be supported in $(t_0-\delta,\infty)$. Applying (2.6) to $\chi(t)u$,

$$\begin{aligned}&\|u\|_{L^2_0(\{p:\ t(p)\in[t_0,t_0+\epsilon]\})}\\&\quad\leq C''(\epsilon+\delta)^{1/2}\|xWu\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})}\\&\qquad+C''(\epsilon+\delta)^{1/2}\|x\chi'(t)(Wt)u\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0]\})}.\end{aligned}$$

In particular, this can be applied with u replaced by $(1-\psi)u$. This completes the proof. □

We also need a weighted version of this result. We first recall a Poincaré inequality with weights.

Lemma 2.4. *Let $C_0>0$. Suppose that $W\in\mathcal{V}_{\mathrm{b}}(X)$ real, $|\operatorname{div}W|\leq C_0$, $0\leq\chi\in\mathcal{C}^\infty_{\mathrm{comp}}(X)$, and $\chi\leq-\gamma(W\chi)$ for $t\geq t_0$, $0<\gamma<1/(2C_0)$. Then there exists $C>0$ such that for $u\in H^1_{0,\mathrm{loc}}(X)$ with $t\geq t_0$ on $\operatorname{supp}u$,*

$$\int|W\chi|\,|u|^2\,dg\leq C\gamma\int\chi|Wu|^2\,dg.$$

Proof. We compute, using $W^*=-W-\operatorname{div}W$,

$$\begin{aligned}\langle(W\chi)u,u\rangle&=\langle[W,\chi]u,u\rangle=\langle\chi u,W^*u\rangle-\langle Wu,\chi u\rangle\\&=-\langle\chi u,Wu\rangle-\langle Wu,\chi u\rangle-\langle\chi u,(\operatorname{div}W)u\rangle,\end{aligned}$$

so

$$\begin{aligned}&\int|W\chi|\,|u|^2\,dg=-\langle(W\chi)u,u\rangle\leq 2\|\chi^{1/2}u\|_{L^2}\|\chi^{1/2}Wu\|_{L^2}+C_0\|\chi^{1/2}u\|^2_{L^2}\\&\leq 2\left(\int\gamma|W\chi|\,|u|^2\,dg\right)^{1/2}\|\chi^{1/2}Wu\|_{L^2}+C_0\int\gamma|W\chi|\,|u|^2\,dg.\end{aligned}$$

Dividing through by $(\int|W\chi|\,|u|^2\,dg)^{1/2}$ and rearranging yields

$$(1-C_0\gamma)\left(\int|W\chi|\,|u|^2\,dg\right)^{1/2}\leq 2\gamma^{1/2}\|\chi^{1/2}Wu\|_{L^2},$$

hence the claim follows. □

Our Poincaré inequality (which could also be named Hardy, in view of the relationship of (2.1) to the Hardy inequality) is then:

Proposition 2.5. *Suppose $V_0\in\mathcal{V}(X)$ is real with $V_0x|_{x=0}=1$, $V_0t\equiv 0$ near Y, and let $V\in\mathcal{V}_{\mathrm{b}}(X)$ be given by $V=xV_0$. Let I be a compact interval. Let $C<(n-1)/2$. Then there exist $\epsilon>0$, $x_0>0$, $C'>0$, $\gamma_0>0$ such that the following holds.*

Suppose $t_0 \in I$, $0 < \gamma < \gamma_0$. Let $\chi_0 \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R})$, $\chi = \chi_0 \circ t$ and $0 \leq \chi_0 \leq -\gamma\chi_0'$ on $[t_0, t_0+\epsilon]$, χ_0 supported in $(-\infty, t_0+\epsilon]$, $\delta < \epsilon$. For $u \in H^1_{0,\mathrm{loc}}(X)$ one has

$$\begin{aligned} &\||\chi'|^{1/2}u\|_{L^2_0(\{p:\ t(p)\in[t_0,t_0+\epsilon]\})} \\ &\leq C^{-1}\||\chi'|^{1/2}Vu\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon],\ x(p)\leq x_0\})} \\ &\qquad + C'\gamma\|\chi^{1/2}du\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0+\epsilon]\})} \\ &\qquad + C'\|u\|_{L^2_0(\{p:\ t(p)\in[t_0-\delta,t_0]\})}, \end{aligned} \tag{2.8}$$

where the norms are relative to the metric h.

Proof. Let $\mathcal{S}$ be the space-like hypersurface in X given by $t = t_0$, $t_0 \in I$. We apply Lemma 2.4 with $W \in \mathcal{V}_{\mathrm{b}}(X)$ transversal to $\mathcal{S}$ as follows.

One has from (2.5) applied with ϕ replaced by $|\chi'|^{1/2}$ that

$$\|\psi|\chi'|^{1/2}u\|_{L^2_0(X)} \leq \tilde{C}^{-1}\|\psi|\chi'|^{1/2}Vu\|.$$

We now use Lemma 2.4 with χ replaced by $\chi\rho^2$, $\rho \equiv 1$ on $\mathrm{supp}(1-\psi)$, $\rho \in \mathcal{C}^\infty_{\mathrm{comp}}(X^\circ)$, to estimate $\|(1-\psi)|W\chi|^{1/2}u\|_{L^2_0(X)}$. We choose ρ so that in addition $W\rho = 0$; this can be done by pulling back a function ρ_0 from $\mathcal{S}$ under the W-flow. We may also assume that ρ is supported where $x \geq x_0/8$ in view of $x \geq x_0/4$ on $\mathrm{supp}(1-\psi)$ (we might need to shorten the time interval we consider, i.e. $\epsilon > 0$, to accomplish this). Thus, $W(\rho^2\chi) = \rho^2 W\chi$, and hence

$$\int \rho^2|W\chi|\,|u|^2\,dg \leq C\gamma\int \rho^2\chi|Wu|^2\,dg.$$

As $x \geq x_0/8$ on $\mathrm{supp}\,\rho$, one can estimate $\int \chi\rho^2|Wu|^2\,dg$ in terms of $\int \chi|du|^2_H\,dg$ (even though h is a Riemannian 0-metric!), giving the desired result. $\square$

3. Energy estimates

We recall energy estimates on manifolds without boundary in a form that will be particularly convenient in the next sections. Thus, we work on X°, equipped with a Lorentz metric g, and dual metric G; let $\Box = \Box_g$ be the d'Alembertian, so $\sigma_2(\Box) = G$. We consider a 'twisted commutator' with a vector field $V = -\imath Z$, where Z is a real vector field, typically of the form $Z = \chi W$, χ a cutoff function. Thus, we compute $\langle -\imath(V^*\Box - \Box V)u, u\rangle$ – the point being that the use of V^* eliminates zeroth order terms and hence is useful when we work not merely modulo lower order terms.

Note that $-\imath(V^*\Box - \Box V)$ is a second order, real, self-adjoint operator, so if its principal symbol agrees with that of d^*Cd for some real self-adjoint bundle endomorphism C, then in fact both operators are the same as the difference is 0th order and vanishes on constants. Correspondingly, there are no 0th order terms to estimate, which is useful as the latter tend to involve higher derivatives of χ, which in turn tend to be large relative to $d\chi$. The principal symbol in turn is easy to calculate, for the operator is

$$-\imath(V^*\Box - \Box V) = -\imath(V^* - V)\Box + \imath[\Box, V], \tag{3.1}$$

whose principal symbol is

$$-\imath\sigma_0(V^* - V)G + H_G\sigma_1(V).$$

In fact, it is easy to perform this calculation explicitly in local coordinates z_j and dual coordinates ζ_j. Let $dg = J\,|dz|$, so $J = |\det g|^{1/2}$. We write the components of the metric tensors as g_{ij} and G^{ij}, and $\partial_j = \partial_{z_j}$ when this does not cause confusion. We also write $Z = \chi W = \sum_j Z^j \partial_j$. *In the remainder of this section only, we adopt the standard summation convention.* Then

$$(-\imath Z)^* = \imath Z^* = -\imath J^{-1}\partial_j J Z^j,$$
$$-\Box = J^{-1}\partial_i J G^{ij}\partial_j,$$

so

$$\begin{aligned} &-\imath(V^* - V)u = -\imath((-\imath Z)^* + \imath Z)u = (Z^* + Z)u = (-J^{-1}\partial_j J Z^j + Z^j\partial_j)u \\ &\qquad = -J^{-1}(\partial_j J Z^j)u = -(\operatorname{div} Z)u, \\ &H_G = G^{ij}\zeta_i\partial_{z_j} + G^{ij}\zeta_j\partial_{z_i} - (\partial_{z_k}G^{ij})\zeta_i\zeta_j\partial_{\zeta_k}, \end{aligned}$$

(the first two terms of H_G are the same after summation, but it is convenient to keep them separate) hence

$$H_G\sigma_1(V) = G^{ij}(\partial_{z_j}Z^k)\zeta_i\zeta_k + G^{ij}(\partial_{z_i}Z^k)\zeta_j\zeta_k - Z^k(\partial_{z_k}G^{ij})\zeta_i\zeta_j.$$

Relabelling the indices, we deduce that

$$\begin{aligned} &-\imath\sigma_0(V^* - V)G + H_G\sigma_1(V) \\ &= (-J^{-1}(\partial_k J Z^k)G^{ij} + G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i) - Z^k\partial_k G^{ij})\zeta_i\zeta_j, \end{aligned}$$

with the first and fourth terms combining into $-J^{-1}\partial_k(JZ^kG^{ij})\zeta_i\zeta_j$, so

$$\begin{aligned} &-\imath(V^*\Box - \Box V) = d^*Cd,\ C_{ij} = g_{i\ell}B_{\ell j} \\ &B_{ij} = -J^{-1}\partial_k(JZ^kG^{ij}) + G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i), \end{aligned} \tag{3.2}$$

where C_{ij} are the matrix entries of C relative to the basis $\{dz_s\}$ of the fibers of the cotangent bundle.

We now want to expand B using $Z = \chi W$, and separate the terms with χ derivatives, with the idea being that we choose the derivative of χ large enough relative to χ to dominate the other terms. Thus,

$$\begin{aligned} B_{ij} &= G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i) - J^{-1}\partial_k(JZ^kG^{ij}) \\ &= (\partial_k\chi)(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k) \\ &\qquad + \chi(G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i) - J^{-1}\partial_k(JZ^kG^{ij})) \end{aligned} \tag{3.3}$$

and multiplying the first term on the right hand side by $\partial_i u\,\overline{\partial_j u}$ (and summing over i, j) gives

$$\begin{aligned} E_{W,d\chi}(du) &= (\partial_k\chi)(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k)\partial_i u\,\overline{\partial_j u} \\ &= (du, d\chi)_G\,\overline{du(W)} + du(W)\,(d\chi, du)_G - d\chi(W)(du, du)_G, \end{aligned} \tag{3.4}$$

which is twice the sesquilinear stress-energy tensor associated to the wave u. This is well-known to be positive definite in du, i.e. for covectors α, $E_{W,d\chi}(\alpha) \geq 0$ vanishing if and only if $\alpha = 0$, when W and $d\chi$ are both forward time-like for smooth Lorentz metrics, see e.g. [28, Section 2.7] or [16, Lemma 24.1.2]. In the present setting, the metric is degenerate at the boundary, but the analogous result still holds, as we show below.

If we replace the wave operator by the Klein-Gordon operator $P = \Box + \lambda$, $\lambda \in \mathbb{C}$, we obtain an additional term

$$\begin{aligned} -\imath\lambda(V^* - V) + 2\operatorname{Im}\lambda V &= -\imath\operatorname{Re}\lambda(V^* - V) + \operatorname{Im}\lambda(V + V^*) \\ &= -\imath\operatorname{Re}\lambda\operatorname{div}V + \operatorname{Im}\lambda(V + V^*) \end{aligned}$$

in

$$-\imath(V^*P - P^*V)$$

as compared to (3.1). With $V = -\imath Z$, $Z = \chi W$, as above, this contributes $-\operatorname{Re}\lambda(W\chi)$ in terms containing derivatives of χ to $-\imath(V^*P - P^*V)$. In particular,

$$\begin{aligned} &\langle -\imath(V^*P - P^*V)u, u\rangle \\ &\quad = \int E_{W,d\chi}(du)\,dg - \operatorname{Re}\lambda\langle (W\chi)u, u\rangle \\ &\qquad + \operatorname{Im}\lambda(\langle \chi Wu, u\rangle + \langle u, \chi Wu\rangle) + \langle \chi R\,du, du\rangle + \langle \chi R'u, u\rangle, \end{aligned} \tag{3.5}$$

$R \in \mathcal{C}^\infty(X^\circ; \operatorname{End}(T^*X^\circ))$, $R' \in \mathcal{C}^\infty(X^\circ)$.

Now suppose that W and $d\chi$ are either both time like (either forward or backward; this merely changes an overall sign). The point of (3.5) is that one controls the left hand side if one controls Pu (in the extreme case, when $Pu = 0$, it simply vanishes), and one can regard all terms on the right hand side after $E_{W,d\chi}(du)$ as terms one can control by a small multiple of the positive definite quantity $\int E_{W,d\chi}(du)\,dg$ due to the Poincaré inequality if one arranges that χ' is large relative to χ, and thus one can control $\int E_{W,d\chi}(du)\,dg$ in terms of Pu.

In fact, one does not expect that $d\chi$ will be non-degenerate time-like everywhere: then one decomposes the energy terms into a region Ω_+ where one has the desired definiteness, and a region Ω_- where this need not hold, and then one can estimate $\int E_{W,d\tilde{\chi}}(du)\,dg$ in Ω_+ in terms of its behavior in Ω_- and Pu: thus one propagates energy estimates (from Ω_- to Ω_+), provided one controls Pu. Of course, if u is supported in Ω_+, then one automatically controls u in Ω_-, so we are back to the setting that u is controlled by Pu. This easily gives uniqueness of solutions, and a standard functional analytic argument by duality gives solvability.

It turns out that in the asymptotically AdS case one can proceed similarly, except that the term $\operatorname{Re}\lambda\langle (W\chi)u, u\rangle$ is not negligible any more at ∂X, and neither is $\operatorname{Im}\lambda(\langle \chi Wu, u\rangle + \langle u, \chi Wu\rangle)$. In fact, the $\operatorname{Re}\lambda$ term is the 'same size' as the stress energy tensor at ∂X, hence the need for an upper bound for it, while the $\operatorname{Im}\lambda$ term is even larger, hence the need for the assumption $\operatorname{Im}\lambda = 0$ because although χ is not differentiated (hence in some sense 'small'), W is a vector field that is too large compared to the vector fields the stress energy tensor can estimate at ∂X: it is a b-vector field, rather than a 0-vector field: we explain these concepts now.

4. Zero-differential operators and b-differential operators

We start by recalling that $\mathcal{V}_{\mathrm{b}}(X)$ is the Lie algebra of $\mathcal{C}^\infty$ vector fields on X tangent to ∂X, while $\mathcal{V}_0(X)$ is the Lie algebra of $\mathcal{C}^\infty$ vector fields vanishing at ∂X. Thus, $\mathcal{V}_0(X)$ is a Lie subalgebra of $\mathcal{V}_{\mathrm{b}}(X)$. Note also that both $\mathcal{V}_0(X)$ and $\mathcal{V}_{\mathrm{b}}(X)$ are $\mathcal{C}^\infty(X)$-modules under multiplication from the left, and they act on $x^k\mathcal{C}^\infty(X)$, in the case of $\mathcal{V}_0(X)$ in addition mapping $\mathcal{C}^\infty(X)$ into $x\mathcal{C}^\infty(X)$. The Lie subalgebra property can be strengthened as follows.

Lemma 4.1. *$\mathcal{V}_0(X)$ is an ideal in $\mathcal{V}_{\mathrm{b}}(X)$.*

Proof. Suppose $V \in \mathcal{V}_0(X)$, $W \in \mathcal{V}_{\mathrm{b}}(X)$. Then, as V vanishes at ∂X, there exists $V' \in \mathcal{V}(X)$ such that $V = xV'$. Thus,

$$[V, W] = [xV', W] = [x, W]V' + x[V', W].$$

Now, as W is tangent to Y, $[x, W] = -Wx \in x\mathcal{C}^\infty(X)$, and as $V', W \in \mathcal{V}(X)$, $[V', W] \in \mathcal{V}(X)$, so $[V, W] \in x\mathcal{V}(X) = \mathcal{V}_0(X)$. □

As usual, $\mathrm{Diff}_0(X)$ is the algebra generated by $\mathcal{V}_0(X)$, while $\mathrm{Diff}_{\mathrm{b}}(X)$ is the algebra generated by $\mathcal{V}_{\mathrm{b}}(X)$. We combine these in the following definition, originally introduced in [33] (indeed, even weights x^r were allowed there).

Definition 4.2. Let $\mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^m(X)$ be the (complex) vector space of operators on $\dot{\mathcal{C}}^\infty(X)$ of the form

$$\sum P_j Q_j,\ P_j \in \mathrm{Diff}_0^k(X),\ Q_j \in \mathrm{Diff}_{\mathrm{b}}^k(X),$$

where the sum is locally finite, and let

$$\mathrm{Diff}_0\mathrm{Diff}_{\mathrm{b}}(X) = \cup_{k=0}^\infty \cup_{m=0}^\infty \mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^m(X).$$

We recall that this space is closed under composition, and that commutators have one lower order in the 0-sense than products, see [33, Lemma 4.5]:

Lemma 4.3. $\mathrm{Diff}_0\mathrm{Diff}_{\mathrm{b}}(X)$ *is a filtered ring under composition with*

$$A \in \mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^m(X),\ B \in \mathrm{Diff}_0^{k'}\mathrm{Diff}_{\mathrm{b}}^{m'}(X) \Rightarrow AB \in \mathrm{Diff}_0^{k+k'}\mathrm{Diff}_{\mathrm{b}}^{m+m'}(X).$$

Moreover, composition is commutative to leading order in Diff_0*, i.e. for* A, B *as above, with* $k + k' \geq 1$*,*

$$[A, B] \in \mathrm{Diff}_0^{k+k'-1}\mathrm{Diff}_{\mathrm{b}}^{m+m'}(X).$$

Here we need an improved property regarding commutators with $\mathrm{Diff}_{\mathrm{b}}(X)$ (which would a priori only gain in the 0-sense by the preceeding lemma). It is this lemma that necessitates the lack of weights on the $\mathrm{Diff}_{\mathrm{b}}(X)$-commutant.

Lemma 4.4. *For* $A \in \mathrm{Diff}_{\mathrm{b}}^s(X)$*,* $B \in \mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^m(X)$*,* $s \geq 1$*,*

$$[A, B] \in \mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^{s+m-1}(X).$$

Proof. We first note that only the leading terms in terms of $\mathrm{Diff}_{\mathrm{b}}$ order in both commutants matter for the conclusion, for otherwise the composition result, Lemma 4.3, gives the desired conclusion. We again write elements of $\mathrm{Diff}_0\mathrm{Diff}_{\mathrm{b}}(X)$ as locally finite sums of products of vector fields and functions, and then, using Lemma 4.3 and expanding the commutators, we are reduced to checking that

(i) $V \in \mathcal{V}_0(X)$, $W \in \mathcal{V}_{\mathrm{b}}(X)$, $[W, V] = -[V, W] \in \mathrm{Diff}_0^1(X)$, which follows from Lemma 4.1,
(ii) and for $W \in \mathcal{V}_{\mathrm{b}}(X)$, $f \in \mathcal{C}^\infty(X)$, $[W, f] = Wf \in \mathcal{C}^\infty(X) = \mathrm{Diff}_{\mathrm{b}}^0(X)$.

In both cases thus, the commutator drops b-order by 1 as compared to the product, completing the proof of the lemma. □

We also remark the following:

Lemma 4.5. *For each non-negative integer* l *with* $l \leq m$*,*

$$x^l\mathrm{Diff}_0^k\mathrm{Diff}_{\mathrm{b}}^m(X) \subset \mathrm{Diff}_0^{k+l}\mathrm{Diff}_{\mathrm{b}}^{m-l}(X).$$

Proof. This result is an immediate consequence of $x\mathcal{V}_{\mathrm{b}}(X) \subset x\mathcal{V}(X) = \mathcal{V}_0(X)$. □

Integer ordered Sobolev spaces, $H^{k,m}_{0,\mathrm{b}}(X)$ were defined in the introduction. It is immediate from our definitions that for $P \in \mathrm{Diff}^r_0 \mathrm{Diff}^s_{\mathrm{b}}(X)$,

$$P : H^{k,m}_{0,\mathrm{b}}(X) \to H^{k-r,s-m}_{0,\mathrm{b}}(X)$$

is continuous.

A particular consequence of Lemma 4.4 is that if $V \in \mathcal{V}_{\mathrm{b}}(X)$, $P \in \mathrm{Diff}^m_0(X)$, the $[P, V] \in \mathrm{Diff}^m_0(X)$.

We also note that for $Q \in \mathcal{V}_{\mathrm{b}}(X)$, $Q = -\imath Z$, Z real, we have $Q^* - Q \in \mathcal{C}^\infty(X)$, where the adjoint is taken with respect to the $L^2 = L^2_0(X)$ inner product. Namely:

Lemma 4.6. *Suppose $Q \in \mathcal{V}_{\mathrm{b}}(X)$, $Q = -\imath Z$, Z real. Then $Q^* - Q \in \mathcal{C}^\infty(X)$, and with*

$$Q = a_0(xD_x) + \sum a_j D_{y_j},$$

$$Q^* - Q = \operatorname{div} Q = J^{-1}(D_x(xa_0 J) + \sum D_{y_j}(a_j J)).$$

with the metric density given by $J\,|dx\,dy|$, $J \in x^{-n}\mathcal{C}^\infty(X)$.

Combining these results we deduce:

Proposition 4.7. *Suppose $Q \in \mathcal{V}_{\mathrm{b}}(X)$, $Q = -\imath Z$, Z real. Then*

$$-\imath(Q^*\Box - \Box Q) = d^* C d, \tag{4.1}$$

where $C \in \mathcal{C}^\infty(X; \mathrm{End}({}^0T^*X))$ and in the basis $\{\frac{dx}{x}, \frac{dy_1}{x}, \ldots, \frac{dy_{n-1}}{x}\}$,

$$C_{ij} = \sum_\ell g_{i\ell} \sum_k \Big(- J^{-1}\partial_k(J a_k \hat G^{\ell j}) + \hat G^{\ell k}(\partial_k a_j) + \hat G^{jk}(\partial_k a_\ell)\Big).$$

Proof. We write

$$-\imath(Q^*\Box - \Box Q) = -\imath(Q^* - Q)\Box - \imath[Q, \Box] \in \mathrm{Diff}^2_0(X),$$

and compute the principal symbol, which we check agrees with that of d^*Cd. One way of achieving this is to do the computation over X°; by continuity if the symbols agree here, they agree on ${}^0T^*X$. But over the interior this is the standard computation leading to (3.2); in coordinates z_j, with dual coordinates ζ_j, writing $Z = \sum Z^j \partial_{z_j}$, $G = \sum G^{ij}\partial_{z_i}\partial_{z_j}$, both sides have principal symbol

$$\sum_{ij} B_{ij}\zeta_i\zeta_j,\ B_{ij} = \sum_k \Big(- J^{-1}\partial_k(J Z^k G^{ij}) + G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i)\Big).$$

Now both sides of (4.1) are elements of $\mathrm{Diff}^2_0(X)$, are formally self-adjoint, real, and have the same principal symbol. Thus, their difference is a first order, self-adjoint and real operator; it follows that its principal symbol vanishes, so in fact this difference is zeroth order. Since it annihilates constants (as both sides do), it actually vanishes. □

We particularly care about the terms in which the coefficients a_j are differentiated, with the idea being that we write $Z = \chi W$, and choose the derivative of χ large enough relative to χ to dominate the other terms. Thus, as in (3.4),

$$\begin{aligned} B_{ij} &= \sum_k (\partial_k\chi)(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k) \\ &\qquad + \chi\sum_k (G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i) - J^{-1}\partial_k(J Z^k G^{ij})) \end{aligned} \tag{4.2}$$

and multiplying the first term on the right hand side by $\partial_i u\,\overline{\partial_j u}$ (and summing over i,j) gives

$$\sum_{i,j,k}(\partial_k\chi)(G^{ik}W^j+G^{jk}W^i-G^{ij}W^k)\partial_i u\,\overline{\partial_j u},$$

which is twice the sesquilinear stress-energy tensor $\frac{1}{2}E_{W,d\chi}(du)$ associated to the wave u. As we mentioned before, this is positive definite when W and $d\chi$ are both forward time-like for smooth Lorentz metrics. In the present setting, the metric is degenerate at the boundary, but the analogous result still holds since

(4.3)
$$\begin{aligned}E_{W,d\chi}(du)&=\sum_{i,j,k}(\partial_k\chi)(\hat G^{ik}W^j+\hat G^{jk}W^i-\hat G^{ij}W^k)(x\partial_i u)\,\overline{x\partial_j u}\\&=(x\,du,d\chi)_{\hat G}\,\overline{x\,du(W)}+x\,du(W)\,(d\chi,x\,du)_{\hat G}-d\chi(W)(x\,du,x\,du)_{\hat G},\end{aligned}$$

so the Lorentzian non-degenerate nature of $\hat G$ proves the (uniform) positive definiteness in $x\,du$, considered as an element of T^*_qX, hence in du, regarded as an element of ${}^0T^*_qX$. Indeed, we recall the quick proof here since we need to improve on this statement to get an optimal result below.

Thus, we wish to show that for $\alpha\in T^*_qX$, $W\in T_qX$, α and W forward time-like,

$$\hat E_{W,\alpha}(\beta)=(\beta,\alpha)_{\hat G}\,\overline{\beta(W)}+\beta(W)\,(\alpha,\beta)_{\hat G}-\alpha(W)(\beta,\beta)_{\hat G}$$

is positive definite as a quadratic form in β. Since replacing W by a positive multiple does not change the positive definiteness, we may assume, as we do below, that $(W,W)_{\hat G}=1$. Then we may choose local coordinates $(z_1,\ldots,z_n)$ such that $W=\partial_{z_n}$ and $\hat g|_q=dz_n^2-(dz_1^2+\ldots+dz_{n-1}^2)$, thus $\hat G|_q=\partial_{z_n}^2-(\partial_{z_1}^2+\ldots+\partial_{z_{n-1}}^2)$. Then $\alpha=\sum\alpha_j\,dz_j$ being forward time-like means that $\alpha_n>0$ and $\alpha_n^2>\alpha_1^2+\ldots+\alpha_{n-1}^2$. Thus,

(4.4)
$$\begin{aligned}\hat E_{W,\alpha}(\beta)&=(\beta_n\alpha_n-\sum_{j=1}^{n-1}\beta_j\alpha_j)\overline{\beta_n}+\beta_n(\alpha_n\overline{\beta_n}-\sum_{j=1}^{n-1}\alpha_j\overline{\beta_j})-\alpha_n(|\beta_n|^2-\sum_{j=1}^{n-1}|\beta_j|^2)\\&=\alpha_n\sum_{j=1}^{n}|\beta_j|^2-\beta_n\sum_{j=1}^{n-1}\alpha_j\overline{\beta_j}-\sum_{j=1}^{n-1}\beta_j\alpha_j\overline{\beta_n}\\&\geq\alpha_n\sum_{j=1}^{n}|\beta_j|^2-2|\beta_n|(\sum_{j=1}^{n-1}\alpha_j^2)^{1/2}(\sum_{j=1}^{n-1}|\beta_j|^2)^{1/2}\\&\geq\alpha_n\sum_{j=1}^{n}|\beta_j|^2-2|\beta_n|\alpha_n(\sum_{j=1}^{n-1}|\beta_j|^2)^{1/2}=\alpha_n\Big(|\beta_n|-(\sum_{j=1}^{n-1}|\beta_j|^2)^{1/2}\Big)^2\geq 0,\end{aligned}$$

with the last inequality strict if $|\beta_n|\neq(\sum_{j=1}^{n-1}|\beta_j|^2)^{1/2}$, and the preceding one (by the strict forward time-like character of α) strict if $\beta_n\neq 0$ and $\sum_{j=1}^{n-1}|\beta_j|^2\neq 0$. It is then immediate that at least one of these inequalities is strict unless $\beta=0$, which is the claimed positive definiteness.

We claim that we can make a stronger statement if $U \in T_qX$ and $\alpha(U) = 0$ and $(U,W)_{\hat{g}} = 0$ (thus U is necessarily space-like, i.e. $(U,U)_{\hat{g}} < 0$):

$$\hat{E}_{W,\alpha}(\beta) + c\,\frac{\alpha(W)}{(U,U)_{\hat{g}}}|\beta(U)|^2,\ c < 1,$$

is positive definite in β. Indeed, in this case (again assuming $(W,W)_{\hat{g}} = 1$) we can choose coordinates as above such that $W = \partial_{z_n}$, U is a multiple of ∂_{z_1}, namely $U = (-(U,U)_{\hat{g}})^{1/2}\partial_{z_1}$, $\hat{g}|_q = dz_n^2 - (dz_1^2 + \ldots + dz_{n-1}^2)$. To achieve this, we complete $e_n = W$ and $e_1 = (-(U,U)_{\hat{g}})^{-1/2}U$ (which are orthogonal by assumption) to a $\hat{g}$ normalized orthogonal basis $(e_1, e_2, \ldots, e_n)$ of T_qX, and then choose coordinates such that the coordinate vector fields are given by the e_j at q. Then α forward time-like means that $\alpha_n > 0$ and $\alpha_n^2 > \alpha_1^2 + \ldots + \alpha_{n-1}^2$, and $\alpha(U) = 0$ means that $\alpha_1 = 0$. Thus, with $c < 1$,

$$\begin{aligned}
&\hat{E}_{W,\alpha}(\beta) + c\,\frac{\alpha(W)}{(U,U)_{\hat{g}}}|\beta(U)|^2\\
&= (\beta_n\alpha_n - \sum_{j=2}^{n-1}\beta_j\alpha_j)\overline{\beta_n} + \beta_n(\alpha_n\overline{\beta_n} - \sum_{j=2}^{n-1}\alpha_j\overline{\beta_j})\\
&\qquad - \alpha_n(|\beta_n|^2 - \sum_{j=1}^{n-1}|\beta_j|^2) - c\alpha_n|\beta_1|^2\\
&\geq (1-c)\alpha_n|\beta_1|^2\\
&\qquad + \Big((\beta_n\alpha_n - \sum_{j=2}^{n-1}\beta_j\alpha_j)\overline{\beta_n} + \beta_n(\alpha_n\overline{\beta_n} - \sum_{j=2}^{n-1}\alpha_j\overline{\beta_j})\\
&\qquad\qquad - \alpha_n(|\beta_n|^2 - \sum_{j=2}^{n-1}|\beta_j|^2)\Big).
\end{aligned}$$

On the right hand side the term in the large paranthesis is the same kind of expression as in (4.4), with the terms with $j = 1$ dropped, thus is positive definite in $(\beta_2, \ldots, \beta_n)$, and for $c < 1$, the first term is positive definite in β_1, so the left hand side is indeed positive definite as claimed. Rewriting this in terms of G in our setting, we obtain that for $c < 1$

$$E_{W,d\chi}(du) - c(W\chi)|xUu|^2$$

is positive definite in du, considered an element of ${}^0T_q^*X$, when $q \in \partial X$, and hence is positive definite sufficiently close to ∂X.

Stating the result as a lemma:

Lemma 4.8. *Suppose $q \in \partial X$, $U, W \in T_qX$, $\alpha \in T_q^*X$ and $\alpha(U) = 0$ and $(U,W)_{\hat{g}} = 0$. Then*

$$E_{W,\alpha}(\beta) + c\,\frac{\alpha(W)}{(U,U)_{\hat{g}}}|\beta(xU)|^2,\ c < 1,$$

is positive definite in $\beta \in {}^0T_q^*X$.

At this point we modify the choice of our time function t so that we can construct U and W satisfying the requirements of the lemma.

Lemma 4.9. *Assume (TF) and (PT). Given $\delta_0 > 0$ and a compact interval I there exists a function $\tau \in \mathcal{C}^\infty(X)$ such that $|t-\tau| < \delta_0$ for $t \in I$, $d\tau$ is time-like in the same component of the time-like cone as dt, and $\hat{G}(d\tau, dx) = 0$ at $x = 0$.*

Proof. Let $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}([0,\infty))$, identically 1 near 0, $0 \le \chi \le 1$, $\chi' \le 0$, supported in $[0,1]$, and for $\epsilon, \delta > 0$ to be specified let

$$\tau = t - x\chi\left(\frac{x^\delta}{\epsilon}\right)\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}.$$

Note that on the support of $\chi\left(\frac{x^\delta}{\epsilon}\right)$, $x \le \epsilon^{1/\delta}$, so if $\epsilon^{1/\delta}$ is sufficiently small, $\hat{G}(dx,dx) < 0$, and bounded away from 0, there in view of (PT) and as $\hat{G}(dx,dx) < 0$ at Y.

At $x = 0$

$$d\tau = dt - \frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}\,dx,$$

so $\hat{G}(d\tau, dx) = 0$. As already noted, on the support of $\chi\left(\frac{x^\delta}{\epsilon}\right)$, $x \le \epsilon^{1/\delta}$, so for $t \in I$, I compact, in view of (PT),

$$|\tau - t| \le C\epsilon^{1/\delta}, \tag{4.5}$$

with C independent of ϵ, δ. Next,

$$d\tau = dt - \alpha\gamma\,dx - \tilde{\alpha}\gamma\,dx - \beta\mu,$$

where

$$\alpha = \chi\left(\frac{x^\delta}{\epsilon}\right),\ \gamma = \frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)},\ \tilde{\alpha} = \delta\frac{x^\delta}{\epsilon}\chi'\left(\frac{x^\delta}{\epsilon}\right),$$
$$\beta = x\chi\left(\frac{x^\delta}{\epsilon}\right),\ \mu = d\left(\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}\right).$$

Now,

$$\begin{aligned}\hat{G}(dt - \alpha\gamma dx, dt - \alpha\gamma dx) &= \hat{G}(dt,dt) - 2\alpha\gamma\hat{G}(dt,dx) + \alpha^2\gamma^2\hat{G}(dx,dx)\\ &= \hat{G}(dt,dt) - (2\alpha - \alpha^2)\frac{\hat{G}(dt,dx)^2}{\hat{G}(dx,dx)},\end{aligned}$$

which is $\ge \hat{G}(dt,dt)$ if $2\alpha - \alpha^2 \ge 0$, i.e. $\alpha \in [0,2]$. But $0 \le \alpha \le 1$, so

$$\hat{G}(dt - \alpha\gamma dx, dt - \alpha\gamma dx) \ge \hat{G}(dt,dt) > 0$$

indeed, i.e. $dt - \alpha\gamma dx$ is timelike. Since $dt - \rho\alpha\gamma\,dx$ is still time-like for $0 \le \rho \le 1$, $dt - \alpha\gamma dx$ is in the same component of time-like covectors as dt, i.e. is forward oriented. Next, observe that with $C' = \sup s|\chi'(s)|$,

$$|\tilde{\alpha}| \le C'\delta,\ |\beta| \le \epsilon^{1/\delta},$$

so over compact sets $\tilde{\alpha}\gamma\,dx + \beta\mu$ can be made arbitrarily small by first choosing $\delta > 0$ sufficiently small and then $\epsilon > 0$ sufficiently small. Thus, $\hat{G}(d\tau, d\tau)$ is forward time-like as well. Reducing $\epsilon > 0$ further if needed, (4.5) completes the proof. □

This lemma can easily be made global.

Lemma 4.10. *Assume (TF) and (PT). Given $\delta_0 > 0$ there exists a function $\tau \in \mathcal{C}^\infty(X)$ such that $|t-\tau| < \delta_0$ for $t \in \mathbb{R}$, $d\tau$ is time-like in the same component of the time-like cone as dt, and $\hat{G}(d\tau, dx) = 0$ at $x = 0$.*

In particular, τ also satisfies (TF) and (PT).

Proof. We proceed as above, but let
$$\tau = t - x\chi\left(\frac{x^{\delta(t)}}{\epsilon(t)}\right)\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}.$$
We then have two additional terms,
$$-x^{1-\delta(t)}\delta'(t)\log x\frac{x^{\delta(t)}}{\epsilon(t)}\chi'\left(\frac{x^{\delta}(t)}{\epsilon(t)}\right)\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}\,dt,$$
and
$$x\frac{\epsilon'(t)}{\epsilon(t)}\frac{x^{\delta(t)}}{\epsilon(t)}\chi'\left(\frac{x^{\delta(t)}}{\epsilon(t)}\right)\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}\,dt,$$
in $d\tau$. Note that on the support of both terms $x \le \epsilon(t)^{1/\delta(t)}$, while $\frac{x^{\delta(t)}}{\epsilon(t)}\chi'\left(\frac{x^{\delta(t)}}{\epsilon(t)}\right)$ is uniformly bounded. Thus, if $\delta(t) < 1/3$, $|\delta'(t)| \le 1$, $|\epsilon'(t)| \le 1$, the factors in front of dt in both terms is bounded in absolute value by $C\epsilon(t)\frac{\hat{G}(dt,dx)}{\hat{G}(dx,dx)}$. Now for any k there are $\delta_k, \epsilon_k > 0$, which we may assume are in $(0,1/3)$ and are decreasing with k, such that on $I = [-k,k]$, τ so defined, satisfies all the requirements if $0 < \epsilon(t) < \epsilon_k$, $0 < \delta(t) < \delta_k$ on I and $|\epsilon'(t)| \le 1$, $|\delta'(t)| \le 1$. But now in view of the bounds on ϵ_k and δ_k it is straightforward to write down $\epsilon(t)$ and $\delta(t)$ with the desired properties, e.g. by approximating the piecewise linear function which takes the value ϵ_k at $\pm(k-1)$, $k \ge 2$, to get $\epsilon(t)$, and similarly with δ, finishing the proof. □

From this point on, within this section, we assume that (TF) and (PT) hold. From now on we simply replace t by τ. We let $W = \hat{G}(dt,.)$, $U_0 = \hat{G}(dx,.)$. Thus, *at $x = 0$,*
$$dt(U_0) = \hat{G}(dx,dt) = 0,\ (U_0,W)_{\hat{g}} = \hat{G}(dx,dt) = 0.$$
We extend $U_0|_Y$ to a vector field U such that $Ut = 0$, i.e. U is tangent to the level surfaces of t. Then we have on all of X,
$$W(dt) = \hat{G}(dt,dt) > 0, \tag{4.6}$$
and
$$U(dx) = \hat{G}(dx,dx) < 0 \tag{4.7}$$
on a neighborhood of Y, with uniform upper and lower bounds (bounding away from 0) for both (4.6) and (4.7) on compact subsets of X.

We thus deduce that for $\chi = \tilde{\chi}\circ t$, $c < 1$, $\rho \in \mathcal{C}^\infty(X)$, identically 1 near Y, supported sufficiently close to Y, $Q = -\imath Z$, $Z = \chi W$,
$$\begin{aligned}\langle -\imath(Q^*P - P^*Q)u,u\rangle &\\ &= \int E_{W,d\chi}(du)\,dg - \operatorname{Re}\lambda\langle (W\chi)u,u\rangle\\ &\qquad + \operatorname{Im}\lambda(\langle \chi Wu,u\rangle + \langle u,\chi Wu\rangle) + \langle \chi R du,du\rangle + \langle \chi R'u,u\rangle\\ &= \langle (\chi' A + \chi R)du,du\rangle + \langle c\rho(W\chi)xUu,xUu\rangle - \operatorname{Re}\lambda\langle (W\chi)u,u\rangle\\ &\qquad + \operatorname{Im}\lambda(\langle \chi Wu,u\rangle + \langle u,\chi Wu\rangle) + \langle \chi R'u,u\rangle\end{aligned}\tag{4.8}$$

with $A, R \in \mathcal{C}^\infty(X; \mathrm{End}({}^0T^*X))$, $R' \in \mathcal{C}^\infty(X)$ and A is positive definite, all independent of χ. Here ρ is used since $E_{W,d\chi}(du) - c(W\chi)|xUu|^2$ is only positive definite near Y.

Fix $t_0 < t_0 + \epsilon < t_1$. Let $\chi_0(s) = e^{-1/s}$ for $s > 0$, $\chi_0(s) = 0$ for $s < 0$, $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ identically 1 on $[1, \infty)$, vanishing on $(-\infty, 0]$, Thus, $s^2\chi_0'(s) = \chi_0(s)$ for $s \in \mathbb{R}$. Now consider

$$\tilde{\chi}(s) = \chi_0(-F^{-1}(s - t_1))\chi_1((s - t_0)/\epsilon),$$

so

$$\operatorname{supp} \tilde{\chi} \subset [t_0, t_1]$$

and

$$s \in [t_0 + \epsilon, t_1] \Rightarrow \tilde{\chi}' = -F^{-1}\chi_0'(-F^{-1}(s - t_1)),$$

so

$$s \in [t_0 + \epsilon, t_1] \Rightarrow \tilde{\chi} = -F^{-1}(s - t_1)^2\tilde{\chi}',$$

so for $F > 0$ sufficiently large, this is bounded by a small multiple of $\tilde{\chi}'$, namely

$$s \in [t_0 + \epsilon, t_1] \Rightarrow \tilde{\chi} = -\gamma\tilde{\chi}', \ \gamma = (t_1 - t_0)^2 F^{-1}. \tag{4.9}$$

In particular, for sufficiently large F,

$$-(\chi' A + \chi R) \geq -\chi' A/2$$

on $[t_0 + \epsilon, t_1]$. In addition, by (2.8) and (4.9), for $\operatorname{Re} \lambda < (n - 1)^2/4$, and $c' > 0$ sufficiently close to 1

$$-\langle \operatorname{Re} \lambda (W\chi)u, u\rangle \leq c' \langle \rho(-W\chi)xUu, xUu\rangle + CF^{-1}\|\chi^{1/2}du\|^2$$

while

$$|\langle \chi R' u, u\rangle| \leq C\|\chi^{1/2}u\|^2$$

and

$$\begin{aligned} \|\chi^{1/2}u\|^2 &\leq CF^{-1}\langle(-W\chi)u, u\rangle \\ &\leq C''F^{-1}\langle(-W\chi)xUu, xUu\rangle + C''F^{-2}\|\chi^{1/2}du\|^2. \end{aligned} \tag{4.10}$$

However, $\operatorname{Im} \lambda(\langle \chi W u, u\rangle + \langle u, \chi W u\rangle)$ is too large to be controlled by the stress energy tensor since W is a b-vector field, but not a 0-vector field. Thus, in order to control the $\operatorname{Im} \lambda$ term for $t \in [t_0 + \epsilon, t_1]$, we need to assume that $\operatorname{Im} \lambda = 0$. Then, writing $Qu = Q^*u + (Q - Q^*)u$, and choosing $F > 0$ sufficiently large to absorb the first term on the right hand side of (4.10),

(4.11)

$$\begin{aligned} &\langle -\chi' A du, du\rangle/2 \leq -\langle -\imath Pu, Qu\rangle + \langle \imath Pu, Qu\rangle + \gamma\langle(-\chi')du, du\rangle \\ &\leq 2C\|\chi^{1/2}WPu\|_{H_0^{-1}(X)}\|\chi^{1/2}u\|_{H_0^1(X)} + 2C\|(-\chi')^{1/2}Pu\|_{L_0^2(X)}\|(-\chi')^{1/2}u\|_{L_0^2(X)} \\ &\qquad + C\gamma\|(-\chi')^{1/2}du\|^2 \\ &\leq 2C\delta^{-1}(\|WPu\|^2_{H_0^{-1}(X)} + \|Pu\|^2_{L_0^2(X)}) + 2C\delta(\|\chi^{1/2}u\|^2_{H_0^1(X)} + \|(-\chi')^{1/2}u\|^2_{L^2(X)}) \\ &\qquad + CF^{-1}\|(-\chi')^{1/2}du\|^2. \end{aligned}$$

For sufficiently small $\delta > 0$ and sufficiently large $F > 0$ we absorb all but the first paranthesized term on the right hand side into the left hand side by the positive definiteness of A and the Poincaré inequality, Proposition 2.5, to conclude that for u supported in $[t_0+\epsilon,t_1]$,

$$\|(-\chi')^{1/2}du\|_{L^2_0(X;{}^0T^*X)} \leq C\|Pu\|_{H^{-1,1}_{0,\mathrm{b}}(X)}. \tag{4.12}$$

In view of the Poincaré inequality we conclude:

Lemma 4.11. *Suppose $\lambda<(n-1)^2/4$, $t_0<t_0+\epsilon<t_1$, χ as above. For $u\in\dot{\mathcal{C}}^\infty(X)$ supported in $[t_0+\epsilon,t_1]$ one has*

$$\|(-\chi')^{1/2}u\|_{H^1_0(X)} \leq C\|Pu\|_{H^{-1,1}_{0,\mathrm{b}}(X)}. \tag{4.13}$$

Remark 4.12. Note that if I is compact then there is $T>0$ such that for $t_0\in I$ we can take any $t_1\in(t_0,t_0+T]$, i.e. the time interval over which we can make the estimate is uniform over such compact intervals I.

This lemma gives local in time uniqueness immediately, hence iterative application of the lemma, together with Remark 4.12, yields:

Corollary 4.13. *Suppose $\lambda<(n-1)^2/4$. For $f\in H^{-1,1}_{0,\mathrm{b},\mathrm{loc}}(X)$ supported in $t>t_0$, there is at most one $u\in H^1_{0,\mathrm{loc}}(X)$ such that $\operatorname{supp}u\subset\{p:t(p)\geq t_0\}$ and $Pu=f$.*

Via the standard functional analytic argument, we deduce from (4.12):

Lemma 4.14. *Suppose $\lambda<(n-1)^2/4$, I a compact interval. There is $\sigma>0$ such that for $t_0\in I$, and for $f\in H^{-1}_{0,\mathrm{loc}}(X)$ supported in $t>t_0$, there exists $u\in H^{1,-1}_{0,\mathrm{b},\mathrm{loc}}(X)$, $\operatorname{supp}u\subset\{p:t(p)\geq t_0\}$ and $Pu=f$ in $t<t_0+\sigma$.*

Proof. For any subspace $\mathfrak{X}$ of $\mathcal{C}^{-\infty}(X)$ let $\mathfrak{X}|_{[\tau_0,\tau_1]}$ consist of elements of $\mathfrak{X}$ restricted to $t\in[\tau_0,\tau_1]$, $\mathfrak{X}^\bullet_{[\tau_0,\tau_1]}$ consist of elements of $\mathfrak{X}$ supported in $t\in[\tau_0,\tau_1]$. In particular, an element of $\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[\tau_0,\tau_1]}$ vanishes to infinite order at $t=\tau_0,\tau_1$. Thus, the dot over $\mathcal{C}^\infty$ denotes the infinite order vanishing at ∂X, while the $\bullet$ denotes the infinite order vanishing at the time boundaries we artificially imposed.

We assume that f is supported in $t>t_0+\delta_0$. We use Lemma 4.11, with the role of t_0 and t_1 reversed (backward in time propagation), and our requirement on σ is that it is sufficiently small so that the backward version of the lemma is valid with $t_1=t_0+2\sigma$. (This can be done uniformly over I by Remark 4.12.) Let $T_1=t_1-\epsilon$ and t_1 be such that $t_0+\sigma=T_1'<T_1<t_1<t_0+2\sigma$. Applying the estimate (4.12), using $P=P^*$, with u replaced by $\phi\in\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$ with t_1 in the role of t_0 there (backward estimate), $\tau_0\in[t_0,T_1)$ in the role of t_0, we obtain:

$$\|(\chi')^{1/2}\phi\|_{H^1_0(X)|_{[\tau_0,T_1]}} \leq C\|P^*\phi\|_{H^{-1,1}_{0,\mathrm{b}}(X)|_{[\tau_0,T_1]}},\ \phi\in\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[\tau_0,T_1]}. \tag{4.14}$$

It is also useful to rephrase this as

$$\|\phi\|_{H^1_0(X)|_{[\tau_0',T_1]}} \leq C\|P^*\phi\|_{H^{-1,1}_{0,\mathrm{b}}(X)|_{[\tau_0,T_1]}},\ \phi\in\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[\tau_0,T_1]}, \tag{4.15}$$

when $\tau_0'>\tau_0$. By (4.14), $P^*:\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}\to\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$ is injective. Define

$$(P^*)^{-1}:\operatorname{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}}P^*\to\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$$

by $(P^*)^{-1}\psi$ being the unique $\phi\in\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$ such that $P^*\phi=\psi$. Now consider the conjugate linear functional on $\operatorname{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}}P^*$ given by

$$\ell:\psi\mapsto\langle f,(P^*)^{-1}\psi\rangle. \tag{4.16}$$

In view of (4.14), and the support condition on f (namely the support is in $t > t_0+\delta_0$) and ψ (the support is in $t \leq T_1$)[2],

$$\begin{aligned}|\langle f,(P^*)^{-1}\psi\rangle| &\leq \|f\|_{H_0^{-1}(X)|_{[t_0+\delta_0,T_1]}}\|(P^*)^{-1}\psi\|_{H_0^1(X)|_{[t_0+\delta_0,T_1]}}\\ &\leq C\|f\|_{H_0^{-1}(X)|_{[t_0+\delta_0,T_1]}}\|\psi\|_{H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0,T_1]}},\end{aligned}$$

so ℓ is a continuous conjugate linear functional if we equip $\mathrm{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}} P^*$ with the $H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0,T_1]}$ norm.

If we did not care about the solution vanishing in $t < t_0+\delta_0$, we could simply use Hahn-Banach to extend this to a continuous conjugate linear functional u on $H_{0,\mathrm{b}}^{-1,1}(X)^\bullet_{[t_0,T_1]}$, which can thus by identified with an element of $H_{0,\mathrm{b}}^{1,-1}(X)|_{[t_0,T_1]}$. This would give

$$Pu(\phi) = \langle Pu,\phi\rangle = \langle u, P^*\phi\rangle = \ell(P^*\phi) = \langle f,(P^*)^{-1}P^*\phi\rangle = \langle f,\phi\rangle,$$

$\phi \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$, so $Pu = f$.

We do want the vanishing of u in $(t_0, t_0+\delta_0)$, i.e. when applied to ϕ supported in this region. As a first step in this direction, let $\delta_0' \in (0,\delta_0)$, and note that if

$$\phi \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,t_0+\delta_0')} \cap \mathrm{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}} P^*$$

then $\ell(\phi) = 0$ directly by (4.16), namely the right hand side vanishes by the support condition on f. Correspondingly, the conjugate linear map L is well-defined on the algebraic sum

$$\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,t_0+\delta_0')} + \mathrm{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}} P^* \tag{4.17}$$

by

$$L(\phi+\psi) = \ell(\psi),\ \phi \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,t_0+\delta_0')},\ \psi \in \mathrm{Ran}_{\dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}} P^*.$$

We claim that the functional L is actually continuous when (4.17) is equipped with the $H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0,T_1]}$ norm. But this follows from

$$|\langle f,(P^*)^{-1}\psi\rangle| \leq C\|f\|_{H_0^{-1}(X)|_{[t_0+\delta_0,T_1]}}\|\psi\|_{H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0+\delta_0',T_1]}}$$

together with

$$\|\psi\|_{H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0+\delta_0',T_1]}} \leq \|\phi+\psi\|_{H_{0,\mathrm{b}}^{-1,1}(X)|_{[t_0,T_1]}}$$

since ϕ vanishes on $[t_0+\delta_0', T_1]$. Correspondingly, by the Hahn-Banach theorem, we can extend L to a continuous conjugate linear map

$$u : H_{0,\mathrm{b}}^{-1,1}(X)^\bullet_{[t_0,T_1]} \to \mathbb{C},$$

which can thus by identified with an element of $H_{0,\mathrm{b}}^{1,-1}(X)|_{[t_0,T_1]}$ This gives

$$Pu(\phi) = \langle Pu,\phi\rangle = \langle u, P^*\phi\rangle = \ell(P^*\phi) = \langle f,(P^*)^{-1}P^*\phi\rangle = \langle f,\phi\rangle,$$

$\phi \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,T_1]}$ supported in (t_0,T_1), so $Pu = f$, and in addition

$$u(\phi) = 0,\ \phi \in \dot{\mathcal{C}}^\infty_{\mathrm{comp}}(X)^\bullet_{[t_0,t_0+\delta_0']},$$

[2]We use below that we can thus regard f as an element of $H_0^{-1}(X)^\bullet_{[t_0+\delta_0,\infty)}$, while $(P^*)^{-1}\psi$ as an element of $H_0^1(X)^\bullet_{(-\infty,T_1]}$, so these can be naturally paired, with the pairing bounded in the appropriate norms. We then write these norms as $H_0^{-1}(X)|_{[t_0+\delta_0,T_1]}$ and $H_0^1(X)|_{[t_0+\delta_0,T_1]}$.

so

$$t \geq t_0 + \delta_0' \text{ on } \operatorname{supp} u. \tag{4.18}$$

In particular, extending u to vanish on $(-\infty, t_0 + \delta_0')$, which is compatible with the existing definition in view of (4.18), we have a distribution solving the PDE, defined on $t < T_1$, with the desired support condition. In particular, using a cutoff function χ which is identically 1 for $t \in (-\infty, T_1']$, is supported for $t \in (-\infty, T_1]$, $\chi u \in H^{1,-1}_{0,\mathrm{b}}(X)$, χu vanishes for $t < t_0 + \delta_0'$ as well as $t \geq T_1$, and $Pu = f$ on $(-\infty, T_1')$, thus completing the proof. □

Proposition 4.15. *Suppose $\lambda < (n-1)^2/4$. For $f \in H^{-1}_{0,\mathrm{loc}}(X)$ supported in $t > t_0$, there exists $u \in H^{1,-1}_{0,\mathrm{b},\mathrm{loc}}(X)$, $\operatorname{supp} u \subset \{p : t(p) \geq t_0\}$ and $Pu = f$.*

Proof. We subdivide the time line into intervals $[t_j, t_{j+1}]$, each of which is sufficiently short so that energy estimates hold even on $[t_{j-2}, t_{j+3}]$; this can be done in view of the uniform estimates on the length of such intervals over compact subsets. Using a partition of unity, we may assume that f is supported in $[t_{k-1}, t_{k+2}]$, and need to construct a global solution of $Pu = f$ with u supported in $[t_{k-1}, \infty)$. First we obtain u_k as above solving the PDE on $(-\infty, t_{k+2}]$ (i.e. $Pu_k - f$ is supported in (t_{k+2}, ∞)) and supported in $[t_{k-1}, t_{k+3}]$. Let $f_{k+1} = Pu_k - f$, this is thus supported in $[t_{k+2}, t_{k+3}]$. We next solve $Pu_{k+1} = -f_{k+1}$ on $(-\infty, t_{k+3}]$ with a result supported in $[t_{k+1}, t_{k+4}]$. Then $P(u_k + u_{k+1}) - f$ is supported in $[t_{k+3}, t_{k+4}]$, etc. Proceeding inductively, and noting that the resulting sum is locally finite, we obtain the solution on all of X. □

Well-posedness of the solution will follow once we show that for solutions $u \in H^{1,s'}_{0,\mathrm{b},\mathrm{loc}}(X)$ of $Pu = f$, $f \in H^{-1,s}_{0,\mathrm{b},\mathrm{loc}}(X)$ supported in $t > t_0$, we in fact have $u \in H^{1,s-1}_{0,\mathrm{b},\mathrm{loc}}(X)$; indeed, this is a consequence of the propagation of singularities. We state this as a theorem now, recalling the standing assumptions as well:

Theorem 4.16. *Assume that (TF) and (PT) hold. Suppose $\lambda < (n-1)^2/4$. For $f \in H^{-1,1}_{0,\mathrm{b},\mathrm{loc}}(X)$ supported in $t > t_0$, there exists a unique $u \in H^1_{0,\mathrm{loc}}(X)$ such that $\operatorname{supp} u \subset \{p : t(p) \geq t_0\}$ and $Pu = f$. Moreover, for $K \subset X$ compact there is $K' \subset X$ compact, depending on K and t_0 only, such that*

$$\|u|_K\|_{H^1_0(X)} \leq \|f|_{K'}\|_{H^{-1,1}_{0,\mathrm{b}}(X)}. \tag{4.19}$$

Remark 4.17. While we used τ of Lemma 4.10 instead of t throughout, the conclusion of this theorem is invariant under this change (since $\delta_0 > 0$ is arbitrary in Lemma 4.10), and thus is actually valid for the original t as well.

Proof. Uniqueness and (4.19) follow from Corollary 4.13 and the estimate preceding it. By Proposition 4.15, this problem has a solution $u \in H^{1,-1}_{0,\mathrm{b},\mathrm{loc}}(X)$ with the desired support property. By the propagation of singularities, Theorem 8.8, $u \in H^1_{0,\mathrm{loc}}(X)$ since u vanishes for $t < t_0$. □

5. Zero-differential operators and b-pseudodifferential operators

In order to microlocalize, we need to replace $\mathrm{Diff}_\mathrm{b}(X)$ by $\Psi_\mathrm{b}(X)$ and $\Psi_\mathrm{bc}(X)$. We refer to [23] for a thorough discussion and [32, Section 2] for a concise introduction to this operator algebra including all the facts that are required here. In general, for $A \in \Psi^m_\mathrm{bc}(X)$, $B \in \Psi^{m'}_\mathrm{bc}(X)$, $AB \in \Psi^{m+m'}_\mathrm{bc}(X)$ and the commutator satisfies

$[A,B]\in\Psi_{\mathrm{bc}}^{m+m'-1}(X)$, i.e. it is one order lower than the product, but there is no gain of decay at ∂X. However, crucially, we also recall the following crucial lemma:

Lemma 5.1. *For $A\in\Psi_{\mathrm{bc}}^m(X)$, $[xD_x,A]\in x\Psi_{\mathrm{bc}}^m(X)$.*

Proof. The lemma is an immediate consequence of xD_x having a commutative normal operator; see [23] for a detailed discussion and [32, Section 2] for a brief explanation. □

The analogue of Lemma 4.4 with $\mathrm{Diff}_{\mathrm{b}}(X)$ replaced by $\Psi_{\mathrm{b}}(X)$ still holds, without the awkward restriction on positivity of b-orders (which is simply due to the lack of non-trivial negative order differential operators).

Definition 5.2. Let $\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X)$ be the (complex) vector space of operators on $\dot{\mathcal{C}}^\infty(X)$ of the form

$$\sum P_jQ_j,\ P_j\in\mathrm{Diff}_0^k(X),\ Q_j\in\Psi_{\mathrm{b}}^k(X),$$

where the sum is locally finite, and let

$$\mathrm{Diff}_0\Psi_{\mathrm{b}}(X)=\cup_{k=0}^\infty\cup_{m\in\mathbb{R}}^\infty\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X).$$

The ring structure (even with a weight x^r) of $\mathrm{Diff}_0\Psi_{\mathrm{b}}(X)$ was proved in [33, Corollary 4.4 and Lemma 4.5], which we recall here. We add to the statements of [33, Corollary 4.4 and Lemma 4.5] that $\mathrm{Diff}_0\Psi_{\mathrm{b}}(X)$ is also closed under adjoints with respect to any weighted non-degenerate b-density, in particular with respect to a non-degenerate 0-density such as $|dg|$, for both $\mathrm{Diff}_0(X)$ and $\Psi_{\mathrm{b}}(X)$ are closed under these adjoints and $(AB)^*=B^*A^*$.

Lemma 5.3. $\mathrm{Diff}_0\Psi_{\mathrm{b}}(X)$ *is a filtered *-ring under composition (and adjoints) with*

$$A\in\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X),\ B\in\mathrm{Diff}_0^{k'}\Psi_{\mathrm{b}}^{m'}(X)\Rightarrow AB\in\mathrm{Diff}_0^{k+k'}\Psi_{\mathrm{b}}^{m+m'}(X),$$

and

$$A\in\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X)\Rightarrow A^*\in\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X),$$

where the adjoint is taken with respect to a (i.e. any fixed*) non-degenerate 0-density. Moreover, composition is commutative to leading order in* Diff_0*, i.e. for A,B as above, $k+k'\geq 1$,*

$$[A,B]\in\mathrm{Diff}_0^{k+k'-1}\Psi_{\mathrm{b}}^{m+m'}(X).$$

Just like for differential operators, we again have a lemma that improves the b-order (rather than merely the 0-order) of the commutator provided one of the commutants is in $\Psi_{\mathrm{b}}(X)$. Again, it is crucial here that there are no weights on $\Psi_{\mathrm{b}}(X)$.

Lemma 5.4. *For $A\in\Psi_{\mathrm{b}}^s(X)$, $B\in\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^m(X)$, $[A,B]\in\mathrm{Diff}_0^k\Psi_{\mathrm{b}}^{s+m-1}(X)$.*

Proof. Expanding elements of $\mathrm{Diff}_0^k(X)$ as finite sums of products of vector fields and functions, and using that $\Psi_{\mathrm{b}}(X)$ is commutative to leading order, we need to consider commutators $[f,A]$, $f\in\mathcal{C}^\infty(X)$, $A\in\Psi_{\mathrm{b}}^s(X)$ and show that this is in $\Psi_{\mathrm{b}}^{s-1}(X)$, which is automatic as $\mathcal{C}^\infty(X)\subset\Psi_{\mathrm{b}}^0(X)$, as well as $[V,A]$, $V\in\mathcal{V}_0(X)$, $A\in\Psi_{\mathrm{b}}^s(X)$, and show that this is in $\mathrm{Diff}_0^1\Psi_{\mathrm{b}}^{s-1}(X)$, i.e.

$$[V,A]=\sum W_jB_j+C_j,\ B_j,C_j\in\Psi_{\mathrm{b}}^{s-1}(X),\ W_j\in\mathcal{V}_0(X).$$

But $V = xV'$, $V' \in \mathcal{V}(X)$, and

$$[V', A] = \sum_j W'_j B'_j + C'_j,\ W'_j \in \mathcal{V}(X),\ B'_j, C'_j \in \Psi_{\mathrm{b}}^{s-1}(X),$$

see [32, Lemma 2.2], while $B'' = [x, A]x^{-1} \in \Psi_{\mathrm{b}}^{s-1}(X)$, so

$$[V, A] = [x, A]V' + x[V', A] = B''(xV') + \sum_j (xW'_j)B'_j + xC'_j,$$

which is of the desired form once the first term is rearranged using Lemma 5.3, i.e. explicitly $B''(xV') = (xV')B'' + [B'', xV']$, with the last term being an element of $\Psi_{\mathrm{b}}^{s-1}(X)$. □

We also have an analogue of Lemma 4.5.

Lemma 5.5. *For any $l \geq 0$ integer,*

$$x^l \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X) \subset \mathrm{Diff}_0^{k+l} \Psi_{\mathrm{b}}^{m-l}(X).$$

Proof. It suffices to show that $x\Psi_{\mathrm{b}}^m(X) \subset \mathrm{Diff}_0^1 \Psi_{\mathrm{b}}^{m-1}(X)$, for the rest follows by induction. Also, we may localize and assume that A is supported in a coordinate patch; note that

$$\Psi_{\mathrm{b}}^{-\infty}(X) \subset \mathrm{Diff}_0^1 \Psi_{\mathrm{b}}^{-\infty}(X)$$

since $\mathcal{C}^\infty(X) \subset \mathrm{Diff}_0^1(X)$. Thus, suppose $A \in \Psi_{\mathrm{b}}^m(X)$. Then there exist $A_j \in \Psi_{\mathrm{b}}^{m-1}(X)$, $j = 0, \ldots, n-1$, and $R \in \Psi_{\mathrm{b}}^{-\infty}(X)$ such that

$$A = (xD_x)A_0 + \sum_j D_{y_j} A_j + R;$$

indeed, one simply needs to use the ellipticity of $L = (xD_x)^2 + \sum D_{y_j}^2$ to achieve this by constructing a parametrix $G \in \Psi_{\mathrm{b}}^{-2}(X)$ to it, and writing $A = LGA + EA$, $E \in \Psi_{\mathrm{b}}^{-\infty}(X)$. As $x(xD_x), xD_{y_j} \in \mathcal{V}_0(X)$, the conclusion follows. □

As a consequence of our results thus far, we deduce that $\Psi_{\mathrm{b}}^0(X)$ is bounded on $H_0^m(X)$, already stated in [33, Lemma 4.7].

Proposition 5.6. *Suppose $m \in \mathbb{Z}$. Any $A \in \Psi_{\mathrm{bc}}^0(X)$ with compact support defines a bounded operator on $H_0^m(X)$, with operator norm bounded by a seminorm of A in $\Psi_{\mathrm{bc}}^0(X)$.*

Proof. For $m \geq 0$ this is a special case of [33, Lemma 4.7], though the fact that the operator norm is bounded by a seminorm of A in $\Psi_{\mathrm{bc}}^0(X)$ was not explicitly stated there though follows from the proof; $m < 0$ follows by duality.

For the convenience of the reader we recall the proof in the case we actually use in this paper, namely $m = 1$ (then $m = -1$ follows by duality). Any A as in the statement of the proposition is bounded on $L^2(X)$ with the stated properties. Thus, we need to show that if $V \in \mathcal{V}_0(X)$, then $VA : H_0^1(X) \to L^2(X)$. But $VA = AV + [V, A]$, $[V, A] \in \mathrm{Diff}_0^1 \Psi_{\mathrm{b}}^{-1}(X) \subset \Psi_{\mathrm{b}}^0(X)$, hence $AV : H_0^1(X) \to L^2(X)$ and $[V, A] : L^2(X) \to L^2(X)$, with the claimed norm behavior. □

Recall now that points in the b-cotangent bundle, ${}^{\mathrm{b}}T^*X$, of X are of the form

$$\underline{\xi}\, \frac{dx}{x} + \sum_{j=1}^{n-1} \underline{\zeta}_j\, dy_j.$$

Thus, $(x, y, \underline{\xi}, \underline{\zeta})$ give coordinates on ${}^{\mathrm{b}}T^*X$. If q is a homogeneous function on ${}^{\mathrm{b}}T^*X \setminus o$, then we again consider the Hamilton vector field H_q associated to it on $T^*X^\circ \setminus o$. A change of coordinates calculation shows that in the b-canonical coordinates given above

$$\mathsf{H}_q = (\partial_{\underline{\xi}} q)x\partial_x - (x\partial_x q)\partial_{\underline{\xi}} + (\partial_{\underline{\zeta}} q)\partial_y - (\partial_y q)\partial_{\underline{\zeta}},$$

so H_q extends to a $\mathcal{C}^\infty$ vector field on ${}^{\mathrm{b}}T^*X \setminus o$ which is tangent to ${}^{\mathrm{b}}T^*_{\partial X}X$. If $Q \in \Psi_{\mathrm{b}}^{m'}(X)$, $P \in \Psi_{\mathrm{b}}^m(X)$, then $[Q, P] \in \Psi_{\mathrm{b}}^{m+m'-1}(X)$ has principal symbol

$$\sigma_{\mathrm{b},m+m'-1}([Q, P]) = \frac{1}{\imath}\mathsf{H}_q p.$$

Using Proposition 5.6 we can define a meaningful WF_{b} relative to $H_0^1(X)$. First we recall the definition of the corresponding global function space from [33, Section 4]:

For $k \geq 0$ we the b-Sobolev spaces relative to $H_0^r(X)$ are given by[3]

$$H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X) = \{u \in H^r_{0,\mathrm{comp}}(X) :\ \forall A \in \Psi^k_{\mathrm{b}}(X),\ Au \in H^r_{0,\mathrm{comp}}(X)\}.$$

These can be normed by taking any properly supported elliptic $A \in \Psi_{\mathrm{b}}^k(X)$ and letting

$$\|u\|^2_{H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)} = \|u\|^2_{H_0^r(X)} + \|Au\|^2_{H_0^r(X)}.$$

Although the norm depends on the choice of A, for u supported in a fixed compact set, different choices give equivalent norms, see [33, Section 4] for details in the 0-setting (where supports are not an issue), and [32, Section 3] for an analysis involving supports. We also let $H^{r,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ be the subspace of $H^r_{0,\mathrm{loc}}(X)$ consisting of $u \in H^r_{0,\mathrm{loc}}(X)$ such that for any $\phi \in \mathcal{C}^\infty_{\mathrm{comp}}(X)$, $\phi u \in H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)$.

Here it is also useful to have Sobolev spaces with a negative amount of b-regularity, in a manner completely analogous to [32, Definition 3.15]:

Definition 5.7. Let r be an integer, $k < 0$, and $A \in \Psi_{\mathrm{b}}^{-k}(X)$ be elliptic on ${}^{\mathrm{b}}S^*X$ with proper support. We let $H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)$ be the space of all $u \in \mathcal{C}^{-\infty}(X)$ of the form $u = u_1 + Au_2$ with $u_1, u_2 \in H^r_{0,\mathrm{comp}}(X)$. We let

$$\|u\|_{H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)} = \inf\{\|u_1\|_{H_0^r(X)} + \|u_2\|_{H_0^r(X)} :\ u = u_1 + Au_2\}.$$

We also let $H^{r,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ be the space of all $u \in \mathcal{C}^{-\infty}(X)$ such that $\phi u \in H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)$ for all $\phi \in \mathcal{C}^\infty_{\mathrm{comp}}(X)$.

As discussed for analogous spaces in [32] following Definition 3.15 there, this definition is independent of the particular A chosen, and different A give equivalent norms for distributions u supported in a fixed compact set K. Moreover, we have

Lemma 5.8. *Suppose $r \in \mathbb{Z}$, $k \in \mathbb{R}$. Any $B \in \Psi^0_{\mathrm{bc}}(X)$ with compact support defines a bounded operator on $H^{r,k}_{0,\mathrm{b}}(X)$, with operator norm bounded by a seminorm of B in $\Psi^0_{\mathrm{bc}}(X)$.*

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[3]We do not need weighted spaces, unlike in [33], so we only state the definition in the special case when the weight is identically 1. On the other hand, we are working on a non-compact space, so we must consider local spaces and spaces of compactly supported functions as in [32, Section 3]. Note also that we reversed the index convention (which index comes first) relative to [33], to match the notation for the wave front sets.

Proof. Suppose $k \geq 0$ first. Then for an $A \in \Psi_{\mathrm{b}}^k(X)$ as in the definition above,

$$\|Bu\|^2_{H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)} = \|Bu\|^2_{H^r_0(X)} + \|ABu\|^2_{H^r_0(X)}.$$

The first term on the right hand side is bounded in the desired manner due to Proposition 5.6. Letting $G \in \Psi_{\mathrm{b}}^{-k}(X)$ be a properly supported parametrix for A so $GA = \mathrm{Id}+E$, $E \in \Psi_{\mathrm{b}}^{-\infty}(X)$, we have $ABu = AB(GA-E)u = (ABG)Au-(ABE)u$, with $ABG \in \Psi^0_{\mathrm{bc}}(X)$, $ABE \in \Psi^{-\infty}_{\mathrm{bc}}(X) \subset \Psi^0_{\mathrm{bc}}(X)$, so

$$\|ABu\|_{H^r_0(X)} \leq C\|Au\|_{H^r_0(X)} + C\|u\|_{H^r_0(X)}$$

by Proposition 5.6, with C bounded by a seminorm of B. This completes the proof if $k \geq 0$.

For $k < 0$, let $A \in \Psi_{\mathrm{b}}^{-k}(X)$ be as in the definition. If $u = u_1 + Au_2$, and $G \in \Psi_{\mathrm{b}}^k(X)$ is a parametrix for A so $AG = \mathrm{Id} + F$, $F \in \Psi_{\mathrm{b}}^{-\infty}(X)$, hence

$$Bu = Bu_1 + BAu_2 = Bu_1 + (AG - F)BAu_2 = Bu_1 + A(GBA)u_2 - (FBA)u_2.$$

Now, $B, FBA, GBA \in \Psi^0_{\mathrm{b}}(X)$ so $Bu \in H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)$ indeed, and choosing u_1, u_2 so that $\|u_1\|_{H^r_0(X)} + \|u_2\|_{H^r_0(X)} \leq 2\|u\|_{H^{r,k}_{0,\mathrm{b},\mathrm{comp}}(X)}$ shows the desired continuity, as well as that the operator norm of B is bounded by a $\Psi^0_{\mathrm{bc}}(X)$-seminorm. □

Now we define the wave front set relative to $H^r_{0,\mathrm{loc}}(X)$. We also allow negative a priori b-regularity relative to this space.

Definition 5.9. Suppose $u \in H^{r,k}_{0,\mathrm{loc}}(X)$, $r \in \mathbb{Z}$, $k \in \mathbb{R}$. Then $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{r,\infty}(u)$ if there is an $A \in \Psi^0_{\mathrm{b}}(X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $QAu \in H^r_{0,\mathrm{loc}}(X)$ for all $Q \in \mathrm{Diff}_{\mathrm{b}}(X)$, i.e. if $Au \in H^{r,\infty}_{0,\mathrm{b},\mathrm{loc}}(X)$.

Moreover, $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{r,m}(u)$ if there is an $A \in \Psi^m_{\mathrm{b}}(X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $Au \in H^r_{0,\mathrm{loc}}(X)$.

Proposition 5.6 implies that $\Psi_{\mathrm{bc}}(X)$ acts microlocally, i.e. preserves WF_{b}; see [32, Section 3] for a similar argument. In particular, the proofs for both the qualitative and quantitative version of microlocality go through without any significant changes; one simply replaces the use of [32, Lemma 3.2] by Proposition 5.6.

Lemma 5.10. *(cf.* [32, Lemma 3.9] *Suppose that $u \in H^{r,k'}_{0,\mathrm{b},\mathrm{loc}}(X)$, $B \in \Psi^k_{\mathrm{bc}}(X)$. Then $\mathrm{WF}_{\mathrm{b}}^{r,m-k}(Bu) \subset \mathrm{WF}_{\mathrm{b}}^{r,m}(u) \cap \mathrm{WF}'_{\mathrm{b}}(B)$.*

As in [32, Section 3], the wave front set microlocalizes the 'b-singular support relative to $H^r_{0,\mathrm{loc}}(X)$', meaning:

Lemma 5.11. *(cf.* [32, Lemma 3.10]*) Suppose $u \in H^{r,k}_{0,\mathrm{b},\mathrm{loc}}(X)$, $p \in X$. If ${}^{\mathrm{b}}S^*_pX \cap \mathrm{WF}_{\mathrm{b}}^{1,m}(u) = \emptyset$, then in a neighborhood of p, u lies in $H^{1,m}_{0,\mathrm{b}}(X)$, i.e. there is $\phi \in \mathcal{C}^\infty_{\mathrm{comp}}(X)$ with $\phi \equiv 1$ near p such that $\phi u \in H^{1,m}_{0,\mathrm{b}}(X)$.*

Corollary 5.12. *(cf.* [32, Corollary 3.11]*) If $u \in H^{r,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ and $\mathrm{WF}_{\mathrm{b}}^{r,m}(u) = \emptyset$, then $u \in H^{r,m}_{0,\mathrm{b},\mathrm{loc}}(X)$.*

In particular, if $u \in H^{r,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ and $\mathrm{WF}_{\mathrm{b}}^{r,m}(u) = \emptyset$ for all m, then $u \in H^{r,\infty}_{0,\mathrm{b},\mathrm{loc}}(X)$, i.e. u is conormal in the sense that $Au \in H^r_{0,\mathrm{loc}}(X)$ for all $A \in \mathrm{Diff}_{\mathrm{b}}(X)$ (or indeed $A \in \Psi_{\mathrm{b}}(X)$).

Finally, we have the following quantitative bound for which we recall the definition of the wave front set of bounded subsets of $\Psi^k_{\mathrm{bc}}(X)$:

Definition 5.13. (cf. [32, Definition 3.12]) Suppose that $\mathcal{B}$ is a bounded subset of $\Psi^k_{\mathrm{bc}}(X)$, and $q \in {}^{\mathrm{b}}S^*X$. We say that $q \notin \mathrm{WF}'_{\mathrm{b}}(\mathcal{B})$ if there is some $A \in \Psi_{\mathrm{b}}(X)$ which is elliptic at q such that $\{AB : B \in \mathcal{B}\}$ is a bounded subset of $\Psi^{-\infty}_{\mathrm{b}}(X)$.

Lemma 5.14. *(cf. [32, Lemma 3.13, Lemma 3.18]) Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, and U a neighborhood of K in ${}^{\mathrm{b}}S^*X$. Let $\tilde{K} \subset X$ compact, and $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X with compact closure. Let $Q \in \Psi^k_{\mathrm{b}}(X)$ be elliptic on K with $\mathrm{WF}'_{\mathrm{b}}(Q) \subset U$, with Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Let $\mathcal{B}$ be a bounded subset of $\Psi^k_{\mathrm{bc}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{B}) \subset K$ and Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Then for any $s \leq 0$ there is a constant $C > 0$ such that for $B \in \mathcal{B}$, $u \in H^{r,s}_{0,\mathrm{b,loc}}(X)$ with $\mathrm{WF}^{r,k}_{\mathrm{b}}(u) \cap U = \emptyset$,*

$$\|Bu\|_{H^r_0(X)} \leq C(\|u\|_{H^{r,s}_{0,\mathrm{b}}(\tilde{U})} + \|Qu\|_{H^r_0(X)}).$$

We can use this lemma to obtain uniform bounds for pairings. We call a subset $\mathcal{B}$ of $\mathrm{Diff}^m_0 \Psi^{2k}_{\mathrm{bc}}(X)$ bounded if its elements are locally finite linear combinations of a fixed, locally finite, collection of elements of $\mathrm{Diff}^m_0(X)$ with coefficients that lie in a bounded subset of $\Psi^{2k}_{\mathrm{bc}}(X)$.

Corollary 5.15. *Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, and U a neighborhood of K in ${}^{\mathrm{b}}S^*X$. Let $\tilde{K} \subset X$ compact, and $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X with compact closure. Let $Q \in \Psi^k_{\mathrm{b}}(X)$ be elliptic on K with $\mathrm{WF}'_{\mathrm{b}}(Q) \subset U$, with Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Let $\mathcal{B}$ be a bounded subset of $\mathrm{Diff}^2_0 \Psi^{2k}_{\mathrm{bc}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{B}) \subset K$ and Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Then there is a constant $C > 0$ such that for $B \in \mathcal{B}$, $u \in H^{1,s}_{0,\mathrm{b,loc}}(X)$ with $\mathrm{WF}^{1,k}_{\mathrm{b}}(u) \cap U = \emptyset$,*

$$|\langle Bu, u\rangle| \leq C(\|u\|_{H^1_0(\tilde{U})} + \|Qu\|_{H^1_0(X)})^2.$$

Proof. Using Lemma 5.3 we can write B as $\sum B'_{ij} P_i^* R_j \Lambda$, where $P_i, R_j \in \mathrm{Diff}^1_0(X)$, $\Lambda \in \Psi^k_{\mathrm{b}}(X)$ (which we take to be elliptic on K, but such that Q is elliptic on $\mathrm{WF}'_{\mathrm{b}}(\Lambda)$), B'_{ij} lies in a bounded subset $\mathcal{B}'$ of $\Psi^k_{\mathrm{b}}(X)$ and the sum is finite. Then

$$\begin{aligned}
|\langle Bu, u\rangle| &\leq \sum_{ij} |\langle R_j \Lambda u, P_i (B'_{ij})^* u\rangle| \\
&\leq \sum_{ij} \|R_j \Lambda u\|_{L^2(X)} \, \|P_i (B'_{ij})^* u\|_{L^2(X)} \\
&\leq \sum_{ij} \|\Lambda u\|_{H^1_0(X)} \, \|P_i (B'_{ij})^* u\|_{H^1_0(X)} \\
&\leq \sum C(\|u\|_{H^{1,s}_{0,\mathrm{b}}(\tilde{U})} + \|Qu\|_{H^1_0(X)})^2,
\end{aligned}$$

where in the last step we used Lemma 5.14. □

It is useful to note that infinite order b-regularity relative to $L^2_0(X)$ and $H^1_0(X)$ are the same.

Lemma 5.16. *For $u \in H^1_{0,\mathrm{loc}}(X)$,*

$$\mathrm{WF}^{1,\infty}_{\mathrm{b}}(u) = \mathrm{WF}^{0,\infty}_{\mathrm{b}}(u).$$

Proof. The complements of the two sides are the set of points $q \in {}^{\mathrm{b}}S^*X$ for which there exist $A \in \Psi^0_{\mathrm{b}}(X)$ (with compactly supported Schwartz kernel, as one may assume) such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $LAu \in H^1_0(X)$, resp. $LAu \in L^2_0(X)$.

Since $H_0^1(X) \subset L_0^2(X)$, $\mathrm{WF}_{\mathrm{b}}^{0,\infty}(u) \subset \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ follows immediately. For the converse, if $LAu \in L_0^2(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, then in particular $\mathrm{Diff}_0(X) \subset \mathrm{Diff}_{\mathrm{b}}(X)$ shows that $QLAu \in L_0^2(X)$ for $Q \in \mathrm{Diff}_0^1(X)$ and $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, so $LAu \in H_0^1(X)$, i.e. $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \subset \mathrm{WF}_{\mathrm{b}}^{0,\infty}(u)$, completing the proof. □

We finally recall that $u \in \mathcal{A}^k(X)$, i.e. that u is conormal relative to $x^k L_{\mathrm{b}}^2(X)$, means that $Lu \in x^k L_{\mathrm{b}}^2(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, so in particular $u \in x^k L_{\mathrm{b}}^2(X)$. Thus,

$$\mathrm{WF}_{\mathrm{b}}^{0,\infty}(u) = \emptyset \text{ if and only if } u \in \mathcal{A}^{(n-1)/2}(X),$$

in view of $L_0^2(X) = x^{(n-1)/2} L_{\mathrm{b}}^2(X)$.

6. Generalized broken bicharacteristics

We recall here the structure of the compressed characteristic set and GBB from [34, Section 2]. It is often convenient to work on the cosphere bundle, here ${}^{\mathrm{b}}S^*X$, which is equivalent to working on conic subsets of ${}^{\mathrm{b}}T^*X \setminus o$. In a region where, say,

$$|\underline{\xi}| < C|\underline{\zeta}_{n-k}|,\ j = 1, \ldots, k,\ |\underline{\zeta}_j| < C|\underline{\zeta}_{n-k}|, j = 1, \ldots, n-k-1, \tag{6.1}$$

$C > 0$ fixed, we can take

$$x, y_1, \ldots, y_{n-1}, \hat{\underline{\xi}}, \hat{\underline{\zeta}}_1, \ldots, \hat{\underline{\zeta}}_{n-2}, |\underline{\zeta}_{n-1}|,$$
$$\hat{\underline{\xi}} = \frac{\underline{\xi}_j}{|\underline{\zeta}_{n-1}|},\ \hat{\underline{\zeta}}_j = \frac{\underline{\zeta}_j}{|\underline{\zeta}_{n-1}|},$$

as (projective) local coordinates on ${}^{\mathrm{b}}T^*X \setminus o$, hence

$$x, y_1, \ldots, y_{n-1}, \hat{\underline{\xi}}, \hat{\underline{\zeta}}_1, \ldots, \hat{\underline{\zeta}}_{n-k-1}$$

as local coordinates on the image of this region under the quotient map in ${}^{\mathrm{b}}S^*X$.

First, we choose local coordinates more carefully. In arbitrary local coordinates

$$(x, y_1, \ldots, y_{n-1})$$

on a neighborhood $\mathcal{U}$ of a point on $Y = \partial X$, so that Y is given by $x = 0$ inside $x \geq 0$, any symmetric bilinear form on T^*X can be written as

$$\hat{G}(x,y) = A(x,y)\,\partial_x\,\partial_x + \sum_j 2C_j(x,y)\,\partial_x\,\partial_{y_j} + \sum_{i,j} B_{ij}(x,y)\,\partial_{y_i}\,\partial_{y_j} \tag{6.2}$$

with A, B, C smooth. In view of (1.1), using x given there and coordinates y_j on Y pulled by to a collar neighborhood of Y by the product structure, we have in addition

$$A(0,y) = -1,\ C_j(0,y) = 0,$$

for all y, and $B(0,y) = (B_{ij}(0,y))$ is Lorentzian for all y. Below we write covectors as

$$\alpha = \xi\,dx + \sum_{i=1}^{n-1} \zeta_i\,dy_i. \tag{6.3}$$

Thus,

$$\hat{G}|_{x=0} = -\partial_x^2 + \sum_{i,j=1}^{n-1} B_{ij}(0,y)\,\partial_{y_i}\,\partial_{y_j}, \tag{6.4}$$

and hence the metric function,

$$p(q) = \hat{G}(q,q),\ q \in T^*X,$$

is

$$p|_{x=0} = -\xi^2 + \zeta \cdot B(y)\zeta. \tag{6.5}$$

Since $A(0,y) = -1 < 0$, Y is indeed time-like in the sense that the restriction of the dual metric $\hat{G}$ to N^*Y is negative definite, for locally the conormal bundle N^*Y is given by

$$\{(x,y,\xi,\zeta):\ x=0,\ \zeta=0\}.$$

It is sometimes convenient to improve the form of B near a particular point p_0, around which the coordinate system is centered. Namely, as B is Lorentzian, we can further arrange, by adjusting the y_j coordinates,

$$\sum B_{ij}(0,0)\partial_{y_i}\partial_{y_j} = \partial^2_{y_{n-1}} - \sum_{i<n-1} \partial^2_{y_i}. \tag{6.6}$$

We now recall from the introduction that $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ is the natural map corresponding to the identification of a section of T^*X as a section of ${}^{\mathrm{b}}T^*X$, and in local coordinates π is given by

$$\pi(x,y,\xi,\zeta) = (x,y,x\xi,\zeta).$$

Moreover, the image of the characteristic set $\Sigma \subset {}^{\mathrm{b}}T^*X \setminus o$, given by

$$\Sigma = \{q \in {}^{\mathrm{b}}T^*X :\ p(q) = 0\},$$

under π is the compressed characteristic set,

$$\dot{\Sigma} = \pi(\Sigma).$$

Note that (6.5) gives that

$$\dot{\Sigma} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_Y X = \{(0,y,0,\zeta):\ 0 \le \zeta \cdot B(y)\zeta,\ \zeta \neq 0\}. \tag{6.7}$$

In particular, in view of (6.6), $\dot{\Sigma}\cap\mathcal{U}$ lies in the region (6.1), at least after we possibly shrink $\mathcal{U}$.

In order to better understand the generalized broken bicharacteristics for $\Box$, we divide $\dot{\Sigma}$ into two subsets. We thus define the *glancing set* $\mathcal{G}$ as the set of points in $\dot{\Sigma}$ whose preimage under $\hat{\pi} = \pi|_\Sigma$ consists of a single point, and define the *hyperbolic set* $\mathcal{H}$ as its complement in $\dot{\Sigma}$. Thus, $q \in \dot{\Sigma}$ lies in $\mathcal{G}$ if and only if on $\hat{\pi}^{-1}(\{q\})$, $\xi = 0$. More explicitly, with the notation of (6.7),

$$\begin{aligned} \mathcal{G} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_Y X &= \{(0,y,0,\zeta):\ \zeta \cdot B(y)\zeta = 0,\ \zeta \neq 0\}, \\ \mathcal{H} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_Y X &= \{(0,y,0,\zeta):\ \zeta \cdot B(y)\zeta > 0,\ \zeta \neq 0\}. \end{aligned} \tag{6.8}$$

Thus, $\mathcal{G}$ corresponds to generalized broken bicharacteristics which are tangent to Y in view of the vanishing of ξ at $\hat{\pi}^{-1}(\mathcal{G})$ (recall that the ∂_x component of H_p is -2ξ), while $\mathcal{H}$ corresponds to generalized broken bicharacteristics which are normal to Y. Note that if Y is one-dimensional (hence X is 2-dimensional), then $\zeta \cdot B(y)\zeta$ necessarily implies $\zeta = 0$, so in fact $\mathcal{G} \cap {}^{\mathrm{b}}T^*_Y X = \emptyset$, hence there are no glancing rays.

We next make the role of $\mathcal{G}$ and $\mathcal{H}$ more explicit, which explains the relevant phenomena better. A characterization of GBB , which is equivalent to Definition 1.1, is

Lemma 6.1. *(See the discussion in* [29, Section 1] *after the statement of Definition 1.1.) A continuous map $\gamma : I \to \dot{\Sigma}$, where $I \subset \mathbb{R}$ is an interval, is a GBB (in the analytic sense that we use here) if and only if it satisfies the following requirements:*

(i) *If $q_0 = \gamma(s_0) \in \mathcal{G}$ then for all $f \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X)$,*

$$\frac{d}{ds}(f \circ \gamma)(s_0) = \mathsf{H}_p(\pi^* f)(\tilde{q}_0),\ \tilde{q}_0 = \hat{\pi}^{-1}(q_0). \tag{6.9}$$

(ii) *If $q_0 = \gamma(s_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_Y X$ then there exists $\epsilon > 0$ such that*

$$s \in I,\ 0 < |s - s_0| < \epsilon \Rightarrow \gamma(t) \notin {}^{\mathrm{b}}T^*_Y X. \tag{6.10}$$

7. Microlocal elliptic regularity

We first note the form of $\Box$ with commutator calculations in mind. Note that rather than thinking of the tangential terms, xD_y, as 'too degenerate', we think of xD_x as 'too singular' in that it causes the failure of $\Box$ to lie in $x^2\mathrm{Diff}^2_{\mathrm{b}}(X)$. This makes the calculations rather analogous to the conformal case, and also it facilitates the use of the symbolic machinery for b-ps.d.o's.

Proposition 7.1. *On a collar neighborhood of Y, $\Box$ has the form*

$$-(xD_x)^*\alpha(xD_x) + (xD_x)^* M' + M''(xD_x) + \tilde{P}, \tag{7.1}$$

with

$$\alpha - 1 \in x\mathcal{C}^\infty(X),\ M', M'' \in x^2\mathrm{Diff}^1_{\mathrm{b}}(X) \subset x\mathrm{Diff}^1_0(X),$$
$$\tilde{P} \in x^2\mathrm{Diff}^2_{\mathrm{b}}(X),\ \tilde{P} - x^2\Box_h \in x^3\mathrm{Diff}^2_{\mathrm{b}}(X) \subset x\mathrm{Diff}^2_0(X),$$

where $\Box_h$ is the d'Alembertian of the conformal metric on the boundary (extended to a neighborhood of Y using the collar structure).

Proof. Writing the coordinates as $(z_1, \ldots, z_n)$, the operator $\Box_g$ is given by

$$\Box_g = \sum_{ij} D^*_{z_i} G_{ij} D_{z_j},$$

with adjoints taken with respect to $dg = |\det g|^{1/2}\,|dz_1 \ldots dz_n|$. With $z_n = x$, $z_j = y_j$ for $j = 1, \ldots, n-1$, this can be rewritten as

$$\begin{aligned}\Box_g &= \sum_{ij}(xD_{z_i})^* \hat{G}_{ij}(xD_{z_j})\\ &= (xD_x)^*\hat{G}_{nn}(xD_x) + \sum_{j=1}^{n-1}(xD_x)^*\hat{G}_{nj}(xD_{y_j}) + \sum_{j=1}^{n-1}(xD_{y_j})^*\hat{G}_{jn}(xD_{y_j})\\ &\qquad + \sum_{i,j=1}^{n-1}(xD_{y_i})^*\hat{G}_{ij}(xD_{y_j}).\end{aligned}$$

As $\hat{G}_{nn} + 1 \in x\mathcal{C}^\infty(X)$, we may take $\alpha = -\hat{G}_{nn}$ and conclude that $\alpha - 1 \in x\mathcal{C}^\infty(X)$. As $\hat{G}_{jn}, \hat{G}_{nj} \in x\mathcal{C}^\infty(X)$, taking $M' = \sum_{j=1}^{n-1}\hat{G}_{nj}(xD_{y_j})$ and $M'' = \sum_{j=1}^{n-1}(xD_{y_j})^*\hat{G}_{jn}$, $M', M'' \in x^2\mathrm{Diff}^1_{\mathrm{b}}(X)$ follow. Finally,

$$\tilde{P} = \sum_{ij=1}^{n-1}(xD_{y_i})^*\hat{G}_{ij}(xD_{y_j}) \in x^2\mathrm{Diff}^2_{\mathrm{b}}(X),$$

and modulo $x^3\mathrm{Diff}^2_{\mathrm{b}}(X)$, we can pull out the factors of x and restrict $\hat{G}_{ij}$ to Y, so $\tilde{P}$ differs from $x^2\Box_h = x^2\sum D^*_{y_i}h_{ij}D_{y_j}$ by an element of $x^3\mathrm{Diff}^2_{\mathrm{b}}(X)$, completing the proof. $\square$

We next state the lemma regarding Dirichlet form which is of fundamental use in both the elliptic and hyperbolic/glancing estimates. Below the main assumption is that $P = \Box_g + \lambda$, with $\Box_g$ as in (7.1). We first recall the notation for local norms:

Remark 7.2. Since X is non-compact and our results are microlocal, we may always fix a compact set $\tilde{K} \subset X$ and assume that all ps.d.o's have Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. We also let $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X such that $\tilde{U}$ has compact closure, and use the $H^1_0(\tilde{U})$ norm in place of the $H^1_0(X)$ norm to accommodate $u \in H^1_{0,\mathrm{loc}}(X)$. (We may instead take $\phi \in \mathcal{C}^\infty_{\mathrm{comp}}(\tilde{U})$ identically 1 in a neighborhood of $\tilde{K}$, and use $\|\phi u\|_{H^1_0(X)}$.) Below we use the notation $\|.\|_{H^1_{0,\mathrm{loc}}(X)}$ for $\|.\|_{H^1_0(\tilde{U})}$ to avoid having to specify $\tilde{U}$. We also use $\|v\|_{H^{-1}_{0,\mathrm{loc}}(X)}$ for $\|\phi v\|_{H^{-1}_0(X)}$.

Lemma 7.3. *(cf.* [32, Lemma 4.2]*) Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, $U \subset {}^{\mathrm{b}}S^*X$ is open, $K \subset U$. Suppose that $\mathcal{A} = \{A_r :\ r \in (0,1]\}$ is a bounded family of ps.d.o's in $\Psi^s_{\mathrm{bc}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \subset K$, and with $A_r \in \Psi^{s-1}_{\mathrm{b}}(X)$ for $r \in (0,1]$. Then there are $G \in \Psi^{s-1/2}_{\mathrm{b}}(X)$, $\tilde{G} \in \Psi^{s+1/2}_{\mathrm{b}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(G), \mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$ and $C_0 > 0$ such that for $r \in (0,1]$, $u \in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ (here $k \leq 0$) with $\mathrm{WF}^{1,s-1/2}_{\mathrm{b}}(u) \cap U = \emptyset$, $\mathrm{WF}^{-1,s+1/2}_{\mathrm{b}}(Pu) \cap U = \emptyset$, we have*

$$\begin{aligned}&|\langle dA_r u, dA_r u\rangle_G + \lambda\|A_r u\|^2|\\ &\qquad \leq C_0(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}_0(X)}).\end{aligned}$$

Remark 7.4. The point of this lemma is G is 1/2 order lower ($s-1/2$ vs. s) than the *family* $\mathcal{A}$. We will later take a limit, $r \to 0$, which gives control of the Dirichlet form evaluated on $A_0 u$, $A_0 \in \Psi^s_{\mathrm{bc}}(X)$, in terms of lower order information.

The role of A_r, $r > 0$, is to regularize such an argument, i.e. to make sure various terms in a formal computation, in which one uses A_0 directly, actually make sense.

The main difference with [32, Lemma 4.2] is that λ is *not negligible*.

Proof. Then for $r \in (0,1]$, $A_r u \in H^1_0(X)$, so

$$\langle dA_r u, dA_r u\rangle + \lambda\|A_r u\|^2 = \langle PA_r u, A_r u\rangle.$$

Here the right hand side is the pairing of $H^{-1}_0(X)$ with $H^1_0(X)$. Writing $PA_r = A_r P + [P, A_r]$, the right hand side can be estimated by

$$|\langle A_r Pu, A_r u\rangle| + |\langle [P, A_r]u, A_r u\rangle|. \tag{7.2}$$

The lemma is thus proved if we show that the first term of (7.2) is bounded by

$$C'_0(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}_0(X)}), \tag{7.3}$$

the second term is bounded by $C''_0(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H^1_0(X)})$. *(Recall that the 'local' norms were defined in Remark 7.2.)*

The first term is straightforward to estimate. Let $\Lambda \in \Psi^{-1/2}_{\mathrm{b}}(X)$ be elliptic with $\Lambda^- \in \Psi^{1/2}_{\mathrm{b}}(X)$ a parametrix, so

$$E = \Lambda\Lambda^- - \mathrm{Id}, E' = \Lambda^-\Lambda - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{b}}(X).$$

Then

$$\begin{aligned}\langle A_r Pu, A_r u\rangle &= \langle (\Lambda\Lambda^- - E)A_r Pu, A_r u\rangle\\ &= \langle \Lambda^- A_r Pu, \Lambda^* A_r u\rangle - \langle A_r Pu, E^* A_r u\rangle.\end{aligned}$$

Since $\Lambda^- A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1/2}(X)$, and $\Lambda^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$, $\langle \Lambda^- A_r Pu, \Lambda^* A_r\rangle$ is uniformly bounded, with a bound like (7.3) using Cauchy-Schwartz and Lemma 5.14. Indeed, by Lemma 5.14, choosing any $G \in \Psi_{\mathrm{b}}^{s-1/2}(X)$ which is elliptic on K, there is a constant $C_1 > 0$ such that

$$\|\Lambda^* A_r u\|^2_{H_0^1(X)} \leq C_1(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H_0^1(X)}).$$

Similarly, by Lemma 5.14 and its analogue for $\mathrm{WF}_{\mathrm{b}}^{-1,s}$, choosing any $\tilde G \in \Psi_{\mathrm{b}}^{s+1/2}(X)$ which is elliptic on K, there is a constant $C_1' > 0$ such that $\|\Lambda^- A_r Pu\|^2_{H_0^{-1}(X)} \leq C_1'(\|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|\tilde G Pu\|^2_{H_0^{-1}(X)})$. Combining these gives, with $C_0' = C_1 + C_1'$,

$$\begin{aligned}|\langle \Lambda^- A_r Pu, \Lambda^* A_r u\rangle| &\leq \|\Lambda^- A_r Pu\|\,\|\Lambda^* A_r u\| \leq \|\Lambda^- A_r Pu\|^2 + \|\Lambda^* A_r u\|^2\\ &\leq C_0'(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H_0^1(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|\tilde G Pu\|^2_{H_0^{-1}(X)}),\end{aligned}$$

as desired.

A similar argument, using that A_r is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1/2}(X)$ (in fact in $\Psi_{\mathrm{bc}}^{s}(X)$), and $E^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$ (in fact in $\Psi_{\mathrm{bc}}^{-\infty}(X)$), shows that $\langle A_r Pu, E^* A_r u\rangle$ is uniformly bounded.

Now we turn to the second term in (7.2), whose uniform boundedness is a direct consequence of Lemma 5.4 and Corollary 5.15. Indeed, by Lemma 5.4, $[P, A_r]$ is a bounded family in $\mathrm{Diff}_0^2\Psi_{\mathrm{bc}}^{s-1}(X)$, hence $A_r^*[P, A_r]$ is a bounded family in $\mathrm{Diff}_0^2\Psi_{\mathrm{bc}}^{2s-1}(X)$. Then one can apply Corollary 5.15 to conclude that

$$\langle A_r^*[P, A_r]u, u\rangle \leq C'(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)}),$$

proving the lemma. □

A more precise version, in terms of requirements on Pu, is the following. Here, as in Section 2, we fix a positive definite inner product on the fibers of ${}^0T^*X$ (i.e. a Riemannian 0-metric) to compute $\|dv\|^2_{L^2(X;{}^0T^*X)}$; as v has support in a compact set below, the choice of the inner product is irrelevant.

Lemma 7.5. *(cf.* [32, Lemma 4.4]*) Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, $U \subset {}^{\mathrm{b}}S^*X$ is open, $K \subset U$. Suppose that $\mathcal{A} = \{A_r :\ r \in (0,1]\}$ is a bounded family of ps.d.o's in $\Psi_{\mathrm{bc}}^{s}(X)$ with $\mathrm{WF}_{\mathrm{b}}'(\mathcal{A}) \subset K$, and with $A_r \in \Psi_{\mathrm{b}}^{s-1}(X)$ for $r \in (0,1]$. Then there are $G \in \Psi_{\mathrm{b}}^{s-1/2}(X)$, $\tilde G \in \Psi_{\mathrm{b}}^{s}(X)$ with $\mathrm{WF}_{\mathrm{b}}'(G), \mathrm{WF}_{\mathrm{b}}'(\tilde G) \subset U$ and $C_0 > 0$ such that for $\epsilon > 0$, $r \in (0,1]$, $u \in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ $(k \leq 0)$ with $\mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u) \cap U = \emptyset$, $\mathrm{WF}_{\mathrm{b}}^{-1,s}(Pu) \cap U = \emptyset$, we have*

$$\begin{aligned}|\langle dA_r u, dA_r u\rangle_G &+ \lambda\|A_r u\|^2|\\ &\leq \epsilon\|dA_r u\|^2_{L^2(X;{}^0T^*X)} + C_0(\|u\|^2_{H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|Gu\|^2_{H_0^1(X)}\\ &\qquad + \epsilon^{-1}\|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \epsilon^{-1}\|\tilde G Pu\|^2_{H_0^{-1}(X)}).\end{aligned}$$

Remark 7.6. The point of this lemma is that on the one hand the new term $\epsilon\|dA_r u\|^2$ can be absorbed in the left hand side in the elliptic region, hence is negligible, on the other hand, there is a gain in the order of $\tilde{G}$ (s, versus $s+1/2$ in the previous lemma).

Proof. We only need to modify the previous proof slightly. Thus, we need to estimate the term $|\langle A_r Pu, A_r u\rangle|$ in (7.2) differently, namely

$$|\langle A_r Pu, A_r u\rangle| \leq \|A_r Pu\|_{H_0^{-1}(X)}\|A_r u\|_{H_0^1(X)} \leq \tilde{\epsilon}\|A_r u\|^2_{H_0^1(X)} + \tilde{\epsilon}^{-1}\|A_r Pu\|^2_{H_0^{-1}(X)}.$$

Now the lemma follows by using Lemma 5.14 and the remark following it, namely choosing any $\tilde{G} \in \Psi_{\mathrm{b}}^s(X)$ which is elliptic on K, there is a constant $C_1' > 0$ such that $\|A_r Pu\|^2_{H_0^{-1}(X)} \leq C_1'(\|Pu\|^2_{H^{-1,k}_{0,\mathrm{b},\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H_0^{-1}(X)})$, then using the Poincaré inequality to estimate $\|A_r u\|_{H_0^1(X)}$ by $C_2\|dA_r u\|_{L^2(X)}$, and finishing the proof exactly as for Lemma 7.3. □

We next state microlocal elliptic regularity. Note that for this result the restrictions on $\lambda \in \mathbb{C}$ are weak (only a half-line is disallowed), but on the other hand, a solution u satisfying our hypotheses may not exist for values of λ when $\lambda \notin (-\infty, (n-1)^2/4)$.

Proposition 7.7. *(Microlocal elliptic regularity.) Suppose that $P = \Box + \lambda$, $\lambda \in \mathbb{C} \setminus [(n-1)^2/4, \infty)$ and $m \in \mathbb{R}$ or $m = \infty$. Suppose $u \in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ for some $k \leq 0$. Then*

$$\mathrm{WF}_{\mathrm{b}}^{1,m}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu).$$

Proof. We first prove a slightly weaker result in which $\mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu)$ is replaced by $\mathrm{WF}_{\mathrm{b}}^{-1,m+1/2}(Pu)$ – we rely on Lemma 7.3. We then prove the original statement using Lemma 7.5.

Suppose that $q \in {}^{\mathrm{b}}T^*_Y X \setminus \dot{\Sigma}$. We may assume iteratively that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$; we need to prove then that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ provided $s \leq m + 1/2$ (note that the inductive hypothesis holds for $s = k + 1/2$ since $u \in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$). We use local coordinates (x, y) as in Section 6, centered so that $q \in {}^{\mathrm{b}}T^*_{(0,0)}X$, arranging that (6.6) holds, and further group the variables as $y = (y', y_{n-1})$, and hence the b-dual variables $(\underline{\zeta}', \underline{\zeta}_{n-1})$. We denote the Euclidean norm by $|\underline{\zeta}'|$.

Let $A \in \Psi_{\mathrm{b}}^s(X)$ be such that

$$\mathrm{WF}_{\mathrm{b}}'(A) \cap \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u) = \emptyset,\ \ \mathrm{WF}_{\mathrm{b}}'(A) \cap \mathrm{WF}_{\mathrm{b}}^{1,s+1/2}(Pu) = \emptyset,$$

and have $\mathrm{WF}_{\mathrm{b}}'(A)$ in a small conic neighborhood U of q so that for a suitable $C > 0$ or $\epsilon > 0$, in U

(i) $\underline{\zeta}_{n-1}^2 < C\underline{\xi}^2$ if $\underline{\xi}(q) \neq 0$,

(ii) $|\underline{\xi}| < \epsilon|\underline{\zeta}|$ for all j, and $\frac{|\underline{\zeta}'|}{|\underline{\zeta}_{n-1}|} > 1 + \epsilon$, if $\underline{\xi}(q) = 0$ and $\underline{\zeta}(q) \cdot B(y(q))\underline{\zeta}(q) < 0$.

Let $\Lambda_r \in \Psi_{\mathrm{b}}^{-2}(X)$ for $r > 0$, such that $\mathcal{L} = \{\Lambda_r :\ r \in (0, 1]\}$ is a bounded family in $\Psi_{\mathrm{b}}^0(X)$, and $\Lambda_r \to \mathrm{Id}$ as $r \to 0$ in $\Psi_{\mathrm{b}}^{\tilde{\epsilon}}(X)$, $\tilde{\epsilon} > 0$, e.g. the symbol of Λ_r could be taken as $(1 + r(|\underline{\zeta}|^2 + |\underline{\xi}|^2))^{-1}$. Let $A_r = \Lambda_r A$. Let a be the symbol of A, and let A_r have symbol $(1 + r(|\underline{\zeta}|^2 + |\underline{\xi}|^2))^{-1}a$, $r > 0$, so $A_r \in \Psi_{\mathrm{b}}^{s-2}(X)$ for $r > 0$, and A_r is uniformly bounded in $\Psi_{\mathrm{bc}}^s(X)$, $A_r \to A$ in $\Psi_{\mathrm{bc}}^{s+\tilde{\epsilon}}(X)$.

By Lemma 7.3,

$$\langle dA_r u, dA_r u\rangle_G + \lambda \|A_r u\|^2$$

is uniformly bounded for $r \in (0,1]$, so

$$\langle dA_r u, dA_r u\rangle_G + \operatorname{Re}\lambda \|A_r u\|^2 \text{ and } \operatorname{Im}\lambda \|A_r u\|^2$$

are uniformly bounded. If $\operatorname{Im}\lambda \neq 0$, then taking the imaginary part at once shows that $\|A_r u\|$ is in fact uniformly bounded. On the other hand, whether $\operatorname{Im}\lambda = 0$ or not,

$$\begin{aligned}\langle dA_r u, dA_r u\rangle_G = &\int_X A(x,y) xD_x A_r u\, \overline{xD_x A_r u}\, dg \\ &+ \int_X \sum B_{ij}(x,y) xD_{y_i} A_r u\, \overline{xD_{y_j} A_r u}\, dg \\ &+ \int_X \sum C_j(x,y) xD_x A_r u\, \overline{xD_{y_j} A_r u}\, dg \\ &+ \int_X \sum C_j(x,y) xD_{y_j} A_r u\, \overline{xD_x A_r u}\, dg.\end{aligned}$$

Using that $A(x,y) = -1 + xA'(x,y) + \sum (y_j - y_j(q)) A_j(x,y)$, we see that if A_r is supported in $x < \delta$, $|y_j - y_j(q)| < \delta$ for all j, then for some $C > 0$ (independent of A_r),

$$\begin{aligned}&\left| \int_X A(x,y)\, xD_x A_r u\, \overline{xD_x A_r u}\, dg - \int_X A(0,y(q))\, xD_x A_r u\, \overline{xD_x A_r u}\, dg \right| \\ &\quad \leq C\delta \|xD_x A_r u\|^2,\end{aligned} \tag{7.4}$$

with analogous estimates[4] for $B_{ij}(x,y) - B_{ij}(0,y(q))$ and for $C_j(x,y)$. Thus, there exists $\tilde{C} > 0$ and $\delta_0 > 0$ such that if $\delta < \delta_0$ and A is supported in $|x| < \delta$ and $|y - y(q)| < \delta$ then

$$\begin{aligned}&\int_X \Big((1 - \tilde{C}\delta) |xD_x A_r u|^2 - \operatorname{Re}\lambda |A_r u|^2 \Big)\, dg \\ &\qquad + \sum_{j=1}^{n-2} \int_X \Big((1 - \tilde{C}\delta) \sum_j xD_{y_j} A_r u\, \overline{xD_{y_j} A_r u} \Big)\, dg \\ &\qquad - \int_X \Big((1 + \tilde{C}\delta) \sum_j xD_{y_{n-1}} A_r u\, \overline{xD_{y_{n-1}} A_r u} \Big)\, dg \\ &\quad \leq |\langle dA_r u, dA_r u\rangle_G + \operatorname{Re}\lambda \|A_r u\|^2|.\end{aligned} \tag{7.5}$$

Now we distinguish the cases $\underline{\xi}(q) = 0$ and $\underline{\xi}(q) \neq 0$. If $\underline{\xi}(q) = 0$, we choose $\delta \in (0, \frac{1}{2\tilde{C}})$, $\delta < \delta_0$, so that

$$(1 - \tilde{C}\delta) \frac{|\underline{\zeta}'|^2}{\underline{\zeta}_{n-1}^2} > 1 + 2\tilde{C}\delta$$

---

[4] Recall that $C_j(0,y) = 0$ and $B_{ij}(0,y(q)) = 0$ if $i \neq j$, $B_{ij}(0,y(q)) = 1$ if $i = j = n-1$, $B_{ij}(0,y(q)) = -1$ if $i = j \neq n-1$.

on a neighborhood of $\mathrm{WF}'_{\mathrm{b}}(A)$, which is possible in view of (ii) at the beginning of the proof. Then the second integral on the left hand side of (7.5) can be written as $\|BxA_r u\|^2$, with the symbol of B given by

$$\Big((1-\tilde{C}\delta)|\underline{\zeta}'|^2-(1+\tilde{C}\delta)\underline{\zeta}_{n-1}^2\Big)^{1/2}$$

(which is $\geq \delta|\underline{\zeta}_{n-1}|$), modulo a term

$$\int_X FxA_r u\,\overline{xA_r u}\,dg,\ \ F\in\Psi^1_{\mathrm{b}}(X).$$

But $A_r^* xFxA_r$ is uniformly bounded in $x^2\Psi^{2s+1}_{\mathrm{bc}}(X)\subset \mathrm{Diff}^2_0\Psi^{2s-1}_{\mathrm{bc}}(X)$, so this expression is uniformly bounded as $r\to 0$ by Corollary 5.15. We thus deduce that

$$\int_X\Big((1-\tilde{C}\delta)|xD_xA_r u|^2-\mathrm{Re}\,\lambda|A_r u|^2\Big)\,dg+\|BxA_r u\|^2$$

is uniformly bounded as $r\to 0$.

If $\underline{\xi}(q)\neq 0$, and A is supported in $|x|<\delta$,

$$\tilde{C}\delta\int_X\delta^{-2}|x^2D_xA_r u|^2\,dg\leq \tilde{C}\delta\int_X|xD_xA_r u|^2\,dg.$$

On the other hand, near $\{q':\ \underline{\xi}(q')=0\}$, for $\delta>0$ sufficiently small,

$$\int_X\left(\frac{\tilde{C}\delta}{\delta^2}|x^2D_xA_r u|^2-|xD_{y_{n-1}}A_r u|^2\right)dg=\|BxA_r u\|^2+\int_X FxA_r u\,\overline{xA_r u}\,dg,$$

with the symbol of B given by $(\frac{\tilde{C}}{\delta}\underline{\xi}^2-\underline{\zeta}_{n-1}^2)^{1/2}$ (which does not vanish on U for $\delta>0$ small), while $F\in\Psi^1_{\mathrm{b}}(X)$, so the second term on the right hand side is uniformly bounded as $r\to 0$ just as above. We thus deduce in this case that

$$\int_X((1-2\tilde{C}\delta)|xD_xA_r u|^2\,dg-\mathrm{Re}\,\lambda|A_r u|^2)+\|BxA_r u\|^2$$

is uniformly bounded as $r\to 0$.

If $\mathrm{Im}\,\lambda\neq 0$ then we already saw that $\|A_r u\|_{L^2}$ is uniformly bounded, so we deduce that

$$A_r u,\ xD_xA_r u,\ BxA_r u \text{ are uniformly bounded in } L^2(X). \tag{7.6}$$

If $\mathrm{Im}\,\lambda=0$, but $\lambda<(n-1)^2/4$, then the Poincaré inequality allows us to reach the same conclusion, since on the one hand in case (ii)

$$(1-\tilde{C}\delta)\|xD_xA_r u\|^2-\mathrm{Re}\,\lambda\|A_r u\|^2,$$

resp. in case (i)

$$(1-2\tilde{C}\delta)\|xD_xA_r u\|^2-\mathrm{Re}\,\lambda\|A_r u\|^2,$$

is uniformly bounded, on the other hand by Proposition 2.3, for $\delta>0$ sufficiently small there exists $c>0$ such that

$$(1-2\tilde{C}\delta)\|xD_xA_r u\|^2-\mathrm{Re}\,\lambda\|A_r u\|^2\geq c(\|xD_xA_r u\|^2+\|A_r u\|^2).$$

Correspondingly there are sequences $A_{r_k}u$, $xD_xA_{r_k}u$, $BxA_{r_k}u$, weakly convergent in $L^2(X)$, and such that $r_k\to 0$, as $k\to\infty$. Since they converge to Au, xD_xAu, $BxAu$, respectively, in $\mathcal{C}^{-\infty}(X)$, we deduce that the weak limits are Au, xD_xAu, $BxAu$, which therefore lie in $L^2(X)$. Consequently, that $q\notin \mathrm{WF}^{1,s}_{\mathrm{b}}(u)$, hence proving the proposition with $\mathrm{WF}^{-1,m}_{\mathrm{b}}(Pu)$ replaced by $\mathrm{WF}^{-1,m+1/2}_{\mathrm{b}}(Pu)$.

To obtain the optimal result, we note that due to Lemma 7.5 we still have, for any $\epsilon > 0$, that

$$\langle dA_r u, dA_r u\rangle_G - \epsilon\|dA_r u\|^2$$

is uniformly bounded above for $r \in (0,1]$. By arguing just as above, with B as above, for sufficiently small $\epsilon > 0$, the right hand side gives an upper bound for

$$\int_X \Big((1 - 2\tilde{C}\delta - \epsilon)|xD_x A_r u|^2 - \operatorname{Re}\lambda|A_r u|^2\Big)\, dg + \|BxA_r u\|^2,$$

which is thus uniformly bounded as $r \to 0$. The proof is then finished exactly as above. □

We remark that the analogous argument works for the conformally compact elliptic problem, i.e. on asymptotically hyperbolic spaces, to give that for $\lambda \in \mathbb{C}\setminus[(n-1)^2/4, \infty)$, local solutions of $(\Delta_g - \lambda)u$ are actually conormal to Y provided they lie in $H^1_0(X)$ locally, or indeed in $H^{1,-\infty}_{0,\mathrm{b}}(X)$.

8. Propagation of singularities

We first describe the form of commutators of P with $\Psi_{\mathrm{b}}(X)$.

Proposition 8.1. *Suppose $\mathcal{A} = \{A_r :\ r \in (0,1]\}$ is a family of operators $A_r \in \Psi^0_{\mathrm{b}}(X)$ uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X)$, of the form $A_r = A\Lambda_r$, $A \in \Psi^0_{\mathrm{b}}(X)$, $a = \sigma_{\mathrm{b},0}(A)$, $w_r = \sigma_{\mathrm{b},s+1/2}(\Lambda_r)$. Then*

$$\imath[A_r^* A_r, \Box] = (xD_x)^* C_r^\sharp (xD_x) + (xD_x)^* x C_r' + x C_r''(xD_x) + x^2 C_r^\flat, \tag{8.1}$$

with

$$C_r^\sharp \in L^\infty((0,1];\Psi^{2s}_{\mathrm{bc}}(X)),\ C_r', C_r'' \in L^\infty((0,1];\Psi^{2s+1}_{\mathrm{bc}}(X)),\ C_r^\flat \in \Psi^{2s+2}_{\mathrm{bc}}(X),$$

and

$$\begin{aligned}
&\sigma_{\mathrm{b},2s}(C_r^\sharp) = 2w_r^2 a(V^\sharp a + a\tilde{c}_r^\sharp),\\
&\sigma_{\mathrm{b},2s+1}(C_r') = \sigma_{\mathrm{b},2s+1}(C_r'') = 2w_r^2 a(V' a + a\tilde{c}_r'),\\
&\sigma_{\mathrm{b},2s+2}(C_r^\flat) = 2w_r^2 a(V^\flat a + a\tilde{c}_r^\flat),
\end{aligned}$$

with $\tilde{c}_r^\sharp, \tilde{c}_r', \tilde{c}_r^\flat$ uniformly bounded in S^{-1}, S^0, S^1 respectively, $V^\sharp, V', V^\flat$ smooth and homogeneous of degree $-1, 0, 1$ respectively on ${}^{\mathrm{b}}T^*X \setminus o$, $V^\sharp|_Y$ and $V'|_Y$ annihilate $\underline{\xi}$ and

$$V^\flat|_Y = 2h\partial_{\underline{\xi}} - \mathsf{H}_h. \tag{8.2}$$

Proof. We start by observing that, in Proposition 7.1, $\Box$ is decomposed into a sum of products of weighted b-operators, so analogously expanding the commutator, all calculations can be done in $x^l\Psi_{\mathrm{b}}(X)$ for various values of l. In particular, keeping in mind Lemma 5.1 (which gives the additional order of decay),

$$\imath[A_r^* A_r, xD_x], \imath[A_r^* A_r, (xD_x)^*] \in L^\infty((0,1]_r, x\Psi^{2s+1}_{\mathrm{b}}(X)),$$

with principal symbol $-2w_r^2 a x\partial_x a - 2a^2 w_r(x\partial_x w_r)$. By this observation, all commutators with factors of xD_x or $(xD_x)^*$ in (7.1) can be absorbed into the 'next term' of (8.1), so $[A_r^* A_r, (xD_x)^*]\alpha(xD_x)$ is absorbed into $xC_r''(xD_x)$, $(xD_x)\alpha[A_r^* A_r, xD_x]$ is absorbed into $(xD_x)^* xC_r'$, $[A_r^* A_r, (xD_x)^*]M'$ and $M''[A_r^* A_r, (xD_x)]$ are absorbed into $x^2 C_r^\flat$. The principal symbols of these terms are of the desired form, i.e. after factoring out $2w_r^2 a$, they are the result of a vector field applied to a plus a multiple of a, and this vector field is $-\alpha\partial_x$ in the case of the first two terms (thus annihilates $\underline{\xi}$), $-mx^{-1}\partial_x$ in the case of the last two terms, which in view of $m=\sigma_{\mathrm{b},1}(M')=\sigma_{\mathrm{b},1}(M'')\in x^2S^1$, shows that it actually does not affect $V^\flat|_Y$.

Next, $\imath(xD_x)^*[A_r^*A_r,\alpha](xD_x)$ can be absorbed into (in fact taken equal to) $(xD_x)^*C_r^\sharp(xD_x)$ with principal symbol of $C_r^\sharp$ given by

$$-(\partial_y\alpha)\partial_{\underline{\zeta}}(a^2w_r^2)-(x\partial_x\alpha)\partial_{\underline{\xi}}(a^2w_r^2)$$

in local coordinates, thus again is of the desired form since the $\partial_{\underline{\xi}}$ term has a vanishing factor of x preceding it.

Since $[A_r^*A_r,M'],[A_r^*A_r,M'']$ are uniformly bounded in $x^2\Psi_{\mathrm{b}}^{2s+1}(X)$, the corresponding commutators can be absorbed into $(xD_x)^*xC_r'$, resp. $xC_r''(xD_x)$, without affecting the principal symbols of C_r' and C_r'' at Y, and possessing the desired form.

Next, $\tilde{P}=x^2\Box_h+R$, $R\in x^3\mathrm{Diff}_{\mathrm{b}}^2(X)$, so $[A_r^*A_r,R]$ is uniformly bounded in $x^3\Psi_{\mathrm{b}}^{2s+2}(X)$, and thus can be absorbed into $C_r^\flat$ without affecting its principal symbol at Y and possessing the desired form. Finally, $\imath[A_r^*A_r,x^2\Box_h]\in x^2\Psi_{\mathrm{b}}^{2s+2}(X)$ has principal symbol $\partial_{\underline{\xi}}(a^2w_r^2)2x^2h-x^2\mathsf{H}_h(a^2w_r^2)$, thus can be absorbed into $C_r^\flat$ yielding the stated principal symbol at Y. $\square$

With this proposition, the proof of propagation of singularities proceeds with the same commutant construction as in [32], see also [30]. We also refer to [34] for a write-up that is completely analogous to the present setting (but with values in differential forms). In order to make the argument easy to compare with [34], which was written in a more systematic way than [32], utilizing [34, Proposition 3.10] which is completely analogous to Proposition 8.1 here, we state the results in a parallel manner to those of [34], even though presently we consider the scalar wave equation.

We start with propagation of singularities at hyperbolic points. Recall from the introduction that $\underline{\xi}$ is the b-dual variable of x, $\hat{\underline{\xi}}=\underline{\xi}/|\underline{\zeta}_{n-1}|$.

Proposition 8.2. *(Normal, or hyperbolic, propagation.) Suppose that* $P=\Box_g+\lambda$, $\lambda\in\mathbb{C}\setminus[(n-1)^2/4,\infty)$. *Let* $q_0=(0,y_0,0,\underline{\zeta}_0)\in\mathcal{H}\cap{}^{\mathrm{b}}T^*_YX$, *and let*

$$\eta=-\hat{\underline{\xi}}$$

be the function defined in the local coordinates discussed above, and suppose that $u\in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$ *for some* $k\le 0$, $q_0\notin\mathrm{WF}_{\mathrm{b}}^{-1,\infty}(f)$, $f=Pu$. *If* $\operatorname{Im}\lambda\le 0$ *and there exists a conic neighborhood* U *of* q_0 *in* ${}^{\mathrm{b}}T^*X\setminus o$ *such that*

$$q\in U\ \text{and}\ \eta(q)<0\Rightarrow q\notin\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)\tag{8.3}$$

then $q_0\notin\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$.

In fact, if the wave front set assumptions are relaxed to $q_0\notin\mathrm{WF}_{\mathrm{b}}^{-1,s+1}(f)$ ($f=Pu$) *and the existence of a conic neighborhood* U *of* q_0 *in* ${}^{\mathrm{b}}T^*X\setminus o$ *such that*

$$q\in U\ \text{and}\ \eta(q)<0\Rightarrow q\notin\mathrm{WF}_{\mathrm{b}}^{1,s}(u),\tag{8.4}$$

then we can still conclude that $q_0\notin\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$.

Remark 8.3. As follows immediately from the proof given below, in (8.3) and (8.4), one can replace $\eta(q)<0$ by $\eta(q)>0$, i.e. one has the conclusion for either direction (backward or forward) of propagation, *provided one also switches the sign of* $\operatorname{Im}\lambda$, *when it is non-zero, i.e. the assumption should be* $\operatorname{Im}\lambda\ge 0$. In particular, if $\operatorname{Im}\lambda=$

0, one obtains propagation estimates both along increasing and along decreasing η. Note that η is *increasing* along the GBB of $\Box_{\hat{g}}$.

Moreover, every neighborhood U of $q_0 = (y_0, \underline{\zeta}_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$ in $\dot{\Sigma}$ contains an open set of the form

$$\{q : |x(q)|^2 + |y(q) - y_0|^2 + |\hat{\underline{\zeta}}(q) - \hat{\underline{\zeta}}_0|^2 < \delta\}, \tag{8.5}$$

see [32, Equation (5.1)]. Note also that (8.3) implies the same statement with U replaced by any smaller neighborhood of q_0; in particular, for the set (8.5), provided that δ is sufficiently small. We can also assume by the same observation that $\mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu) \cap U = \emptyset$. Furthermore, we can also arrange that $h(x, y, \underline{\xi}, \underline{\zeta}) > |(\underline{\xi}, \underline{\zeta})|^2 |\underline{\zeta}_0|^{-2} h(q_0)/2$ on U since $\underline{\zeta}_0 \cdot B(y_0)\underline{\zeta}_0 = h(0, y_0, 0, \underline{\zeta}_0) > 0$. We write

$$\hat{h} = |\underline{\zeta}_{n-1}|^{-2} h = |\underline{\zeta}_{n-1}|^{-2} \underline{\zeta} \cdot B(y)\underline{\zeta}$$

for the rehomogenized version of h, which is thus homogeneous degree zero and bounded below by a positive constant on U.

Proof. This proposition is the analogue of Proposition 6.2 in [32], and as the argument is similar, we mainly emphasize the differences. These enter by virtue of λ not being negligible and the use of the Poincaré inequality. In [32], one uses a commutant $A \in \Psi^0_{\mathrm{b}}(X)$ and weights $\Lambda_r \in \Psi^0_{\mathrm{b}}(X)$, $r \in (0, 1)$, uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X)$, $A_r = A\Lambda_r$, in order to obtain the propagation of $\mathrm{WF}^{1,s}_{\mathrm{b}}(u)$ with the notation of that paper, whose analogue is $\mathrm{WF}^{1,s}_{\mathrm{b}}(u)$ here (the difference is the space relative to which one obtains b-regularity: $H^1(X)$ in the previous paper, the zero-Sobolev space $H^1_0(X)$ here). One can use *exactly the same* commutant as in [32]. Then Proposition 8.1 lets one calculate $\imath[A_r^* A_r, P]$ to obtain a completely analogous expression to Equation (6.18)[32] in the hyperbolic case. We also refer the reader to [34] because, although it studies a more delicate problem, namely natural boundary conditions (which are not scalar), the main ingredient of the proof, the commutator calculation, is written up exactly as above in Proposition 8.1, see [34, Proposition 3.10] and the way it is used subsequently in Propositions 5.1 there.

As in [34, Proof of Proposition 5.1], we first construct a commutant by defining its scalar principal symbol, a. This completely follows the scalar case, see [32, Proof of Proposition 6.2]. Next we show how to obtain the desired estimate.

So, as in [32, Proof of Proposition 6.2], let

$$\omega(q) = |x(q)|^2 + |y(q) - y_0|^2 + |\hat{\underline{\zeta}}(q) - \hat{\underline{\zeta}}_0|^2, \tag{8.6}$$

with $|.|$ denoting the Euclidean norm. For $\epsilon > 0$, $\delta > 0$, with other restrictions to be imposed later on, let

$$\phi = \eta + \frac{1}{\epsilon^2 \delta}\omega, \tag{8.7}$$

Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be equal to 0 on $(-\infty, 0]$ and $\chi_0(t) = \exp(-1/t)$ for $t > 0$. Thus, $t^2\chi_0'(t) = \chi_0(t)$ for $t \in \mathbb{R}$. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$, with $\chi_1' \geq 0$ satisfying $\chi_1' \in \mathcal{C}^\infty_{\mathrm{comp}}((0, 1))$. Finally, let $\chi_2 \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$, identically 1 on $[-c_1, c_1]$, where c_1 is such that $|\hat{\underline{\xi}}|^2 < c_1/2$ in $\dot{\Sigma} \cap U_0$. Thus, $\chi_2(|\hat{\underline{\xi}}|^2)$ is a cutoff in $|\hat{\underline{\xi}}|$, with its support properties ensuring that $d\chi_2(|\hat{\underline{\xi}}|^2)$ is supported in $|\hat{\underline{\xi}}|^2 \in [c_1, 2c_1]$ hence outside $\dot{\Sigma}$ – it should be thought of as a factor that microlocalizes near the characteristic set but effectively commutes with

P (since we already have the microlocal elliptic result). Then, for $\digamma > 0$ large, to be determined, let

$$a = \chi_0(\digamma^{-1}(2 - \phi/\delta))\chi_1(\eta/\delta + 2)\chi_2(|\hat{\underline{\xi}}|^2); \tag{8.8}$$

so a is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$. Indeed, as we see momentarily, for any $\epsilon > 0$, a has compact support inside this neighborhood (regarded as a subset of ${}^{\mathrm{b}}S^*X$, i.e. quotienting out by the $\mathbb{R}^+$-action) for δ sufficiently small, so in fact it is globally well-defined. In fact, on $\operatorname{supp} a$ we have $\phi \leq 2\delta$ and $\eta \geq -2\delta$. Since $\omega \geq 0$, the first of these inequalities implies that $\eta \leq 2\delta$, so on $\operatorname{supp} a$

$$|\eta| \leq 2\delta. \tag{8.9}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \eta) \leq 4\delta^2\epsilon^2. \tag{8.10}$$

In view of (8.6) and (8.5), this shows that given any $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$, a is supported in U. The role that $\digamma$ large plays (in the definition of a) is that it increases the size of the first derivatives of a relative to the size of a, hence it allows us to give a bound for a in terms of a small multiple of its derivative along the Hamilton vector field, much like the stress energy tensor was used to bound other terms, by making χ' large relative to χ, in the (non-microlocal) energy estimate.

Now let $A_0 \in \Psi^0_{\mathrm{b}}(X)$ with $\sigma_{\mathrm{b},0}(A_0) = a$, supported in the coordinate chart. Also let Λ_r be scalar, have symbol

$$|\underline{\zeta}_{n-1}|^{s+1/2}(1 + r|\underline{\zeta}_{n-1}|^2)^{-s}\,\mathrm{Id}, \quad r \in [0, 1), \tag{8.11}$$

so $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}(X)$ for $r > 0$ and it is uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X)$. Then, for $r > 0$,

$$\begin{aligned} \langle \imath A_r^* A_r P u, u\rangle - \langle \imath A_r^* A_r u, P u\rangle &= \langle \imath [A_r^* A_r, P] u, u\rangle + \langle \imath (P - P^*) A_r^* A_r u, u\rangle \\ &= \langle \imath [A_r^* A_r, P] u, u\rangle - 2\operatorname{Im}\lambda \|A_r u\|^2. \end{aligned} \tag{8.12}$$

We can compute this using Proposition 8.1. We arrange the terms of the proposition so that the terms in which a vector field differentiates χ_1 are included in E_r, the terms in which a vector fields differentiates χ_2 are included in E'_r. Thus, we have

$$\begin{aligned} &\imath A_r^* A_r P - \imath P A_r^* A_r \\ &\qquad = (xD_x)^* C_r^\sharp (xD_x) + (xD_x)^* x C'_r + x C''_r (xD_x) + x^2 C_r^\flat + E_r + E'_r + F_r, \end{aligned} \tag{8.13}$$

with

$$\begin{aligned} \sigma_{\mathrm{b},2s}(C_r^\sharp) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a |\underline{\zeta}_{n-1}|^{-1}(\hat{f}^\sharp + \epsilon^{-2}\delta^{-1} f^\sharp)\chi_0'\chi_1\chi_2 + a^2 \tilde{c}_r^\sharp\Big), \\ \sigma_{\mathrm{b},2s+1}(C'_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a(\hat{f}' + \delta^{-1}\epsilon^{-2} f')\chi_0'\chi_1\chi_2 + a^2 \tilde{c}'_r\Big), \\ \sigma_{\mathrm{b},2s+1}(C''_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a(\hat{f}'' + \delta^{-1}\epsilon^{-2} f'')\chi_0'\chi_1\chi_2 + a^2 \tilde{c}''_r\Big), \\ \sigma_{\mathrm{b},2s+2}(C_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1}|\underline{\zeta}_{n-1}| a(4\hat{h} + \hat{f}^\flat + \delta^{-1}\epsilon^{-2} f^\flat)\chi_0'\chi_1\chi_2 + a^2 \tilde{c}_r^\flat\Big), \end{aligned} \tag{8.14}$$

where $f^\sharp$, f', f'' and $f^\flat$ as well as $\hat{f}^\sharp, \hat{f}', \hat{f}''$ and $\hat{f}^\flat$ are all smooth functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0 (independent of ϵ and δ), and $\hat{h} = |\underline{\zeta}_{n-1}|^{-2} h$

is the rehomogenized version of h. Moreover, $f^\sharp, f', f'', f^\flat$ arise from when ω is differentiated in $\chi(\digamma^{-1}(2-\phi/\delta))$, and thus vanish when $\omega = 0$, while $\hat{f}^\sharp, \hat{f}', \hat{f}''$ and $\hat{f}^\flat$ arise when η is differentiated in $\chi(\digamma^{-1}(2-\phi/\delta))$, and comprise all such terms with the exception of those arising from the $\partial_{\underline{\xi}}$ component of $V^\flat|_Y$ (which gives $4\hat{h} = 4|\zeta_{n-1}|^{-2}h$ on the last line above) hence are the sums of functions vanishing at $x = 0$ (corresponding to us only specifying the restrictions of the vector fields in (8.2) at Y) and functions vanishing at $\hat{\underline{\xi}} = 0$ (when $|\underline{\zeta}_{n-1}|^{-1}$ in $\eta = -\underline{\xi}|\underline{\zeta}_{n-1}|^{-1}$ is differentiated)[5].

In this formula we think of

$$4\digamma^{-1}\delta^{-1}w_r^2 a|\underline{\zeta}_{n-1}|\hat{h}\chi_0'\chi_1\chi_2 \tag{8.15}$$

as the main term; note that $\hat{h}$ is positive near q_0. Compared to this, the terms with a^2 are negligible, for they can all be bounded by

$$c\digamma^{-1}(\digamma^{-1}\delta^{-1}w_r^2 a|\underline{\zeta}_{n-1}|^{-1}\chi_0'\chi_1\chi_2)$$

(cf. (8.15)), i.e. by a small multiple of $\digamma^{-1}\delta^{-1}w_r^2 a|\underline{\zeta}_{n-1}|^{-1}\chi_0'\chi_1\chi_2$ when $\digamma$ is taken large, using that $2-\phi/\delta \leq 4$ on $\operatorname{supp} a$ and

$$\chi_0(\digamma^{-1}t) = (\digamma^{-1}t)^2\chi_0'(\digamma^{-1}t) \leq 16\digamma^{-2}\chi_0'(\digamma^{-1}t),\ t \leq 4; \tag{8.16}$$

see the discussion in [31, Section 6] and [32] following Equation (6.19).

The vanishing condition on the $f^\sharp, f', f'', f^\flat$ ensures that, on $\operatorname{supp} a$,

$$|f^\sharp|, |f'|, |f''|, |f^\flat| \leq C\omega^{1/2} \leq 2C\epsilon\delta, \tag{8.17}$$

so the corresponding terms can thus be estimated using $w_r^2\digamma^{-1}\delta^{-1}a|\underline{\zeta}_{n-1}|^{-1}\chi_0'\chi_1\chi_2$ provided ϵ^{-1} is not too large, i.e. there exists $\tilde{\epsilon}_0 > 0$ such that if $\epsilon > \tilde{\epsilon}_0$, the terms with $f^\sharp, f', f'', f^\flat$ can be treated as error terms.

On the other hand, we have

$$|\hat{f}^\sharp|, |\hat{f}'|, |\hat{f}''|, |\hat{f}^\flat| \leq C|x| + C|\hat{\underline{\xi}}| \leq C\omega^{1/2} + C|\hat{\underline{\xi}}| \leq 2C\epsilon\delta + C|\hat{\underline{\xi}}|. \tag{8.18}$$

Now, on $\dot{\Sigma}$, $|\hat{\underline{\xi}}| \leq 2|x|$ (for $|\underline{\xi}| = x|\xi| \leq 2|x||\underline{\zeta}_{n-1}|$ with U sufficiently small). Thus we can write $\hat{f}^\sharp = \hat{f}^\sharp_\sharp + \hat{f}^\sharp_\flat$ with $\hat{f}^\sharp_\flat$ supported away from $\dot{\Sigma}$ and $\hat{f}^\sharp_\sharp$ satisfying

$$|\hat{f}^\sharp_\sharp| \leq C|x| + C|\hat{\underline{\xi}}| \leq C'|x| \leq C'\omega^{1/2} \leq 2C'\epsilon\delta; \tag{8.19}$$

we can also obtain a similar decomposition for $\hat{f}', \hat{f}'', \hat{f}^\flat$.

Indeed, using (8.16) it is useful to rewrite (8.14) as

$$\begin{aligned}
&\sigma_{\mathrm{b},2s}(C_r^\sharp) = w_r^2\digamma^{-1}\delta^{-1}a|\underline{\zeta}_{n-1}|^{-1}(\hat{f}^\sharp + \epsilon^{-2}\delta^{-1}f^\sharp + \digamma^{-1}\delta\hat{c}_r^\sharp)\chi_0'\chi_1\chi_2,\\
&\sigma_{\mathrm{b},2s+1}(C_r') = w_r^2\delta^{-1}\digamma^{-1}a(\hat{f}' + \delta^{-1}\epsilon^{-2}f' + \digamma^{-1}\delta\hat{c}_r')\chi_0'\chi_1\chi_2,\\
&\sigma_{\mathrm{b},2s+1}(C_r'') = w_r^2\delta^{-1}\digamma^{-1}a(\hat{f}'' + \delta^{-1}\epsilon^{-2}f'' + \digamma^{-1}\delta\hat{c}_r'')\chi_0'\chi_1\chi_2,\\
&\sigma_{\mathrm{b},2s+2}(C_r^\flat) = w_r^2\delta^{-1}\digamma^{-1}a|\underline{\zeta}_{n-1}|(4\hat{h} + \hat{f}^\flat + \delta^{-1}\epsilon^{-2}f^\flat + \digamma^{-1}\hat{c}_r^\flat)\chi_0'\chi_1\chi_2,
\end{aligned} \tag{8.20}$$

with

- $f^\sharp$, f', f'' and $f^\flat$ are all smooth functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0, satisfying (8.17) (and are independent of $\digamma, \epsilon, \delta, r$),

[5]Terms of the latter kind did not occur in [32] as time-translation invariance was assumed, but it does occur in [31] and [34], where the Lorentzian scalar setting is considered.

- $\hat{f}^\sharp$, $\hat{f}'$, $\hat{f}''$ and $\hat{f}^\flat$ are all smooth functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0, with $\hat{f}^\sharp = \hat{f}^\sharp_\sharp + \hat{f}^\sharp_\flat$, $\hat{f}^\sharp_\sharp, \hat{f}'_\sharp, \hat{f}''_\sharp, \hat{f}^\flat_\sharp$ satisfying (8.19) (and are independent of $\digamma, \epsilon, \delta, r$), while $\hat{f}^\sharp_\flat, \hat{f}'_\flat, \hat{f}''_\flat, \hat{f}^\flat_\flat$ is supported away from $\dot{\Sigma}$,
- and $\hat{c}^\sharp_r$, $\hat{c}'_r$, $\hat{c}''_r$ and $\hat{c}^\flat_r$ are all smooth functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0, uniformly bounded in $\epsilon, \delta, r, \digamma$.

Let

$$b_r = 2w_r|\underline{\zeta}_{n-1}|^{1/2}(\digamma\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

and let $\tilde{B}_r \in \Psi^{s+1}_{\mathrm{b}}(X)$ with principal symbol b_r. Then let

$$C \in \Psi^0_{\mathrm{b}}(X),\ \sigma_{\mathrm{b},0}(C) = |\underline{\zeta}_{n-1}|^{-1}h^{1/2}\psi = \hat{h}^{1/2}\psi,$$

where $\psi \in S^0_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o)$ is identically 1 on U considered as a subset of ${}^{\mathrm{b}}S^*X$; recall from Remark 8.3 that $\hat{h}$ is bounded below by a positive quantity here.

If $\tilde{C}_r \in \Psi^{2s}_{\mathrm{b}}(X)$ with principal symbol

$$\sigma_{\mathrm{b},2s}(\tilde{C}_r) = -4w_r^2\digamma^{-1}\delta^{-1}a|\underline{\zeta}_{n-1}|^{-1}\chi_0'\chi_1\chi_2 = -|\underline{\zeta}_{n-1}|^{-2}b_r^2,$$

then we deduce from (8.13)-(8.20) that[6]

$$\begin{aligned} &\imath A_r^*A_rP - \imath PA_r^*A_r \\ &= \tilde{B}_r^*\big(C^*x^2C + xR^\flat x + (xD_x)^*\tilde{R}'x + x\tilde{R}''(xD_x) + (xD_x)^*R^\sharp(xD_x)\big)\tilde{B}_r \\ &\qquad + R_r'' + E_r + E_r' \end{aligned} \tag{8.21}$$

with

$$\begin{aligned} &R^\flat \in \Psi^0_{\mathrm{b}}(X),\ \tilde{R}', \tilde{R}'' \in \Psi^{-1}_{\mathrm{b}}(X),\ R^\sharp \in \Psi^{-2}_{\mathrm{b}}(X), \\ &R_r'' \in L^\infty((0,1); \mathrm{Diff}^2_0\Psi^{2s-1}_{\mathrm{b}}(X)),\ E_r, E_r' \in L^\infty((0,1); \mathrm{Diff}^2_0\Psi^{2s}_{\mathrm{b}}(X)), \end{aligned}$$

with $\mathrm{WF}'_{\mathrm{b}}(E) \subset \eta^{-1}((-\infty,-\delta]) \cap U$, $\mathrm{WF}'_{\mathrm{b}}(E') \cap \dot{\Sigma} = \emptyset$, and with $r^\flat = \sigma_{\mathrm{b},0}(R^\flat)$, $\tilde{r}' = \sigma_{\mathrm{b},-1}(\tilde{R}')$, $\tilde{r}'' = \sigma_{\mathrm{b},-1}(\tilde{R}'')$, $r^\sharp \in \sigma_{\mathrm{b},-2}(R^\sharp)$,

$$\begin{aligned} &|r^\flat| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}),\ |\underline{\zeta}_{n-1}\tilde{r}'| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}), \\ &|\underline{\zeta}_{n-1}\tilde{r}''| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}),\ |\underline{\zeta}^2_{n-1}r^\sharp| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}). \end{aligned}$$

This is almost completely analogous to [32, Equation (6.18)] with the understanding that each term of [32, Equation (6.18)] inside the paranthesis attains an additional factor of x^2 (corresponding to $\Box$ being in $\mathrm{Diff}^2_0(X)$ rather than $\mathrm{Diff}^2(X)$) which we partially include in xD_x (vs. D_x). The only difference is the presence of the $\delta\digamma^{-1}$ term which however is treated like the $\epsilon\delta$ term for $\digamma$ sufficiently large, hence the rest of the proof proceeds very similarly to that paper. We go through this argument to show the role that λ and the Poincaré inequality play, and in particular how the restrictions on λ arise.

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[6]The $f^\sharp_\sharp$ terms are included in $R^\sharp$, while the $f^\sharp_\flat$ terms are included in E', and similarly for the other analogous terms in f', f'', $f^\flat$. Moreover, in view of Lemma 5.4, we can freely rearrange factors, e.g. writing C^*x^2C as xC^*Cx, if we wish, with the exception of commuting powers of x with xD_x or $(xD_x)^*$ since we need to regard the latter as elements of $\mathrm{Diff}^1_0(X)$ rather than $\mathrm{Diff}^1_{\mathrm{b}}(X)$. Indeed, the difference between rearrangements has lower b-order than the product, in this case being in $x^2\Psi^{-1}_{\mathrm{b}}(X)$, which in view of Lemma 5.5, at the cost of dropping powers of x, can be translated into a gain in 0-order, $x^2\Psi^{-1}_{\mathrm{b}}(X) \subset \mathrm{Diff}^2_0\Psi^{-3}_{\mathrm{b}}(X)$, with the result that these terms can be moved to the 'error term', $R'' \in L^\infty((0,1); \mathrm{Diff}^2_0\Psi^{2s-1}_{\mathrm{b}}(X))$.

Having calculated the commutator, we proceed to estimate the 'error terms' $R^\flat$, $\tilde R'$, $\tilde R''$ and $R^\sharp$ as operators. We start with $R^\flat$. By the standard square root construction to prove the boundedness of ps.d.o's on L^2, there exists $R^\flat_\flat\in\Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|R^\flat v\|\leq 2\sup|r^\flat|\,\|v\|+\|R^\flat_\flat v\|$$

for all $v\in L^2(X)$. Here $\|\cdot\|$ is the $L^2(X)$-norm, as usual. Thus, we can estimate, for any $\gamma>0$,

$$\begin{aligned}|\langle R^\flat v,v\rangle|&\leq\|R^\flat v\|\,\|v\|\leq 2\sup|r^\flat|\,\|v\|^2+\|R^\flat_\flat v\|\,\|v\|\\&\leq 2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|v\|^2+\gamma^{-1}\|R^\flat_\flat v\|^2+\gamma\|v\|^2.\end{aligned}$$

Now we turn to $\tilde R'$. Let $T\in\Psi_{\mathrm{b}}^{-1}(X)$ be elliptic (which we use to shift the orders of ps.d.o's at our convenience), with symbol $|\underline{\zeta}_{n-1}|^{-1}$ on $\operatorname{supp}a$, $T^-\in\Psi^1_{\mathrm{b}}(X)$ a parametrix, so $T^-T=\mathrm{Id}+F$, $F\in\Psi_{\mathrm{b}}^{-\infty}(X)$. Then there exists $\tilde R'_\flat\in\Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\begin{aligned}\|(\tilde R')^*w\|&=\|(\tilde R')^*(T^-T-F)w\|\leq\|((\tilde R')^*T^-)(Tw)\|+\|(\tilde R')^*Fw\|\\&\leq 2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|Tw\|+\|\tilde R'_\flat Tw\|+\|(\tilde R')^*Fw\|\end{aligned}$$

for all w with $Tw\in L^2(X)$, and similarly, there exists $\tilde R''_\flat\in\Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|\tilde R''w\|\leq 2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|Tw\|+\|\tilde R''_\flat Tw\|+\|\tilde R''Fw\|.$$

Finally, there exists $R^\sharp_\flat\in\Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|(T^-)^*R^\sharp w\|\leq 2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|Tw\|+\|R^\sharp_\flat Tw\|+\|(T^-)^*R^\sharp Fw\|$$

for all w with $Tw\in L^2(X)$. Thus,

$$\begin{aligned}|\langle xv,(\tilde R')^*(xD_x)v\rangle|\leq&2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|TxD_xv\|\,\|xv\|\\&+2\gamma\|xv\|^2+\gamma^{-1}\|\tilde R'_\flat TxD_xv\|^2+\gamma^{-1}\|F'xD_xv\|^2,\\|\langle\tilde R''xD_xv,xv\rangle|\leq&2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|TxD_xv\|\,\|xv\|\\&+2\gamma\|xv\|^2+\gamma^{-1}\|\tilde R''_\flat TxD_xv\|^2+\gamma^{-1}\|F''xD_xv\|^2,\end{aligned}$$

and, writing $xD_xv=T^-T(xD_xv)-F(xD_xv)$ in the right factor, and taking the adjoint of T^-,

$$\begin{aligned}|\langle R^\sharp xD_xv,xD_xv\rangle|\leq&2C_2(\delta\epsilon+\epsilon^{-1}+\delta F^{-1})\|T(xD_x)v\|\,\|T(xD_x)v\|\\&+2\gamma\|T(xD_x)v\|^2+\gamma^{-1}\|R^\sharp_\flat T(xD_x)v\|^2+\gamma^{-1}\|F(xD_x)v\|^2\\&+\|R^\sharp(xD_x)v\|\,\|F^\sharp(xD_xv)\|,\end{aligned}$$

with $F',F'',F^\sharp\in\Psi_{\mathrm{b}}^{-\infty}(X)$.

Now, by (8.21),

$$\begin{aligned}\langle\imath[A_r^*A_r,P]u,u\rangle=&\|Cx\tilde B_ru\|^2+\langle R^\flat x\tilde B_ru,x\tilde B_ru\rangle\\&+\langle\tilde R''xD_x\tilde B_ru,x\tilde B_ru\rangle+\langle x\tilde B_ru,(\tilde R')^*xD_x\tilde B_ru\rangle\\&+\langle R^\sharp xD_x\tilde B_ru,xD_x\tilde B_ru\rangle\\&+\langle R''_ru,u\rangle+\langle(E_r+E'_r)u,u\rangle\end{aligned}\tag{8.22}$$

On the other hand, this commutator can be expressed as in (8.12), so

(8.23)
$$\begin{aligned}\langle \imath A_r^* A_r Pu, u\rangle - \langle \imath A_r^* A_r u, Pu\rangle = -2\operatorname{Im}\lambda \|A_r u\|^2 + \|Cx\tilde{B}_r u\|^2 + \langle R^\flat x\tilde{B}_r u, x\tilde{B}_r u\rangle \\ + \langle \tilde{R}'' xD_x \tilde{B}_r u, x\tilde{B}_r u\rangle + \langle x\tilde{B}_r u, (\tilde{R}')^* xD_x\tilde{B}_r u\rangle \\ + \langle R^\sharp xD_x\tilde{B}_r u, xD_x\tilde{B}_r u\rangle \\ + \langle R_r'' u, u\rangle + \langle (E_r + E_r')u, u\rangle,\end{aligned}$$

so the sign of the first two terms agree if $\operatorname{Im}\lambda < 0$, and the $\operatorname{Im}\lambda$ term vanishes if λ is real.

Assume for the moment that $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)\cap U = \emptyset$ – this is certainly the case in our setup if $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, but this assumption is a little stronger than $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$, which is what we need to assume for the second paragraph in the statement of the proposition. We deal with the weakened hypothesis $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$ at the end of the proof. Returning to (8.23), the utility of the commutator calculation is that we have good information about Pu (this is where we use that we have a microlocal solution of the PDE!). Namely, we estimate the left hand side as

$$\begin{aligned}|\langle A_r Pu, A_r u\rangle| &\leq |\langle (T^-)^* A_r Pu, TA_r u\rangle| + |\langle A_r Pu, FA_r u\rangle| \\ &\leq \|(T^-)^* A_r Pu\|_{H_0^{-1}(X)}\|TA_r u\|_{H_0^1(X)} \\ &\quad + \|A_r Pu\|_{H_0^{-1}(X)}\|FA_r u\|_{H_0^1(X)}.\end{aligned} \tag{8.24}$$

Since $(T^-)^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+3/2}(X)$, TA_r is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$, both with $\mathrm{WF}_{\mathrm{b}}'$ in U, with $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$, resp. $\mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$ disjoint from them, we deduce (using Lemma 5.14 and its H_0^{-1} analogue) that $|\langle (T^-)^* A_r Pu, TA_r u\rangle|$ is uniformly bounded. Similarly, taking into account that FA_r is uniformly bounded in $\Psi_{\mathrm{b}}^{-\infty}(X)$, we see that $|\langle A_r Pu, FA_r u\rangle|$ is also uniformly bounded, so $|\langle A_r Pu, A_r u\rangle|$ is uniformly bounded for $r \in (0,1]$.

Thus,

$$\begin{aligned}&\|Cx\tilde{B}_r u\|^2 - \operatorname{Im}\lambda\|A_r u\|^2 \\ &\leq 2|\langle A_r Pu, A_r u\rangle| + |\langle (E_r + E_r')u, u\rangle| \\ &\quad + \left(2C_2(\delta\epsilon + \epsilon^{-1} + \delta F^{-1}) + \gamma\right)\|x\tilde{B}_r u\|^2 + \gamma^{-1}\|R_\flat^\flat x\tilde{B}_r u\|^2 \\ &\quad + 4C_2(\delta\epsilon + \epsilon^{-1} + \delta F^{-1})\|x\tilde{B}_r u\|\|T(xD_x)\tilde{B}_r u\| \\ &\quad + \gamma^{-1}\|\tilde{R}_\flat' T(xD_x)\tilde{B}_r u\|^2 + \gamma^{-1}\|\tilde{R}_\flat'' T(xD_x)\tilde{B}_r u\|^2 + 4\gamma\|x\tilde{B}_r u\|^2 \\ &\quad + \left(2C_2(\delta\epsilon + \epsilon^{-1} + \delta F^{-1}) + 2\gamma\right)\|T(xD_x)\tilde{B}_r u\|^2 \\ &\quad + \gamma^{-1}\|R_\flat^\sharp T(xD_x)\tilde{B}_r u\|^2 + \|R^\sharp(xD_x)\tilde{B}_r u\|\,\|F(xD_x)\tilde{B}_r u\| \\ &\quad + \gamma^{-1}\|F(xD_x)\tilde{B}_r u\|^2 \\ &\quad + \gamma^{-1}\|F'(xD_x)\tilde{B}_r u\|^2 + \gamma^{-1}\|F''(xD_x)\tilde{B}_r u\|^2.\end{aligned} \tag{8.25}$$

All terms but the ones involving C_2 or γ (not γ^{-1}) remain bounded as $r \to 0$. The C_2 and γ terms can be estimated by writing $T(xD_x) = (xD_x)T' + T''$ for some $T', T'' \in \Psi_{\mathrm{b}}^{-1}(X)$, and using Lemma 7.3 and the Poincaré lemma where necessary. Namely, we use either $\operatorname{Im}\lambda \neq 0$ or $\lambda < (n-1)^2/4$ to control $xD_x L\tilde{B}_r u$ and $L\tilde{B}_r u$ in $L^2(X)$ in terms of $\|x\tilde{B}_r u\|_{L^2}$ where $L \in \Psi_{\mathrm{b}}^{-1}(X)$; this is possible by factoring

$D_{y_{n-1}}$ (which is elliptic on $\mathrm{WF}'(\tilde{B}_r)$) out of $\tilde{B}_r$ modulo an error $\tilde{F}_r$ bounded in $\Psi^s_{\mathrm{bc}}(X)$, which in turn can be incorporated into the 'error' given by the right hand side of Lemma 7.3. Thus, there exists $C_3 > 0$, $G \in \Psi_{\mathrm{b}}^{s-1/2}(X)$, $\tilde{G} \in \Psi_{\mathrm{b}}^{s+1/2}(X)$ as in Lemma 7.3 such that

$$\begin{aligned}&\|xD_xL\tilde{B}_ru\|^2 + \|L\tilde{B}_ru\|\\ &\quad\leq C_3(\|x\tilde{B}_ru\|^2 + \|u\|^2_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b,loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}_0(X)}).\end{aligned}$$

We further estimate $\|x\tilde{B}_ru\|$ in terms of $\|Cx\tilde{B}_ru\|$ and $\|u\|_{H^1_{0,\mathrm{loc}}(X)}$ using that C is elliptic on $\mathrm{WF}'_{\mathrm{b}}(B)$ and Lemma 5.14. We conclude, using $\operatorname{Im}\lambda \leq 0$, taking ϵ sufficiently large, then γ, δ_0 sufficiently small, and finally $\digamma$ sufficiently large, that there exist $\gamma > 0$, $\epsilon > 0$, $\delta_0 > 0$ and $C_4 > 0$, $C_5 > 0$ such that for $\delta \in (0, \delta_0)$,

$$\begin{aligned}C_4\|x\tilde{B}_ru\|^2 \leq &2|\langle A_rPu, A_ru\rangle| + |\langle (E_r + E'_r)u, u\rangle|\\ &+ C_5(\|Gu\|^2_{H^1_0(X)} + \|\tilde{G}Pu\|^2_{H^{-1}_0(X)})\\ &+ C_5(\|u\|_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Pu\|_{H^{-1,k}_{0,\mathrm{b,loc}}(X)}).\end{aligned}$$

Letting $r \to 0$ now keeps the right hand side bounded, proving that $\|x\tilde{B}_ru\|$ is uniformly bounded as $r \to 0$, hence $x\tilde{B}_0u \in L^2(X)$ (cf. the proof of Proposition 7.7). In view of Lemma 7.3 and the Poincaré inequality (as in the proof of Proposition 7.7) this proves that $q_0 \notin \mathrm{WF}^{1,s}_{\mathrm{b}}(u)$, and hence proves the first statement of the proposition.

In fact, recalling that we needed $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$ for the uniform boundedness in (8.24), this proves a slightly weaker version of the second statement of the proposition with $\mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$ replaced by $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$. For the more precise statement we modify (8.24) – this is the only term in (8.25) that needs modification to prove the optimal statement. Let $\tilde{T} \in \Psi_{\mathrm{b}}^{-1/2}(X)$ be elliptic, $\tilde{T}^- \in \Psi_{\mathrm{b}}^{1/2}(X)$ a parametrix, $\tilde{F} = \tilde{T}^-\tilde{T} - \mathrm{Id} \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Then, similarly to (8.24), we have for any $\gamma > 0$,

$$\begin{aligned}|\langle A_rPu, A_ru\rangle| &\leq |\langle (\tilde{T}^-)^*A_rPu, \tilde{T}A_ru\rangle| + |\langle A_rPu, \tilde{F}A_ru\rangle|\\ &\leq \gamma^{-1}\|(\tilde{T}^-)^*A_rPu\|^2_{H^{-1}_0(X)} + \gamma\|\tilde{T}A_ru\|^2_{H^1(X)}\\ &\quad + \|A_rPu\|_{H^{-1}(X)}\|\tilde{F}A_ru\|_{H^1_0(X)}.\end{aligned} \tag{8.26}$$

The last term on the right hand side can be estimated as before. As $(\tilde{T}^-)^*A_r$ is bounded in $\Psi_{\mathrm{bc}}^{s+1}(X)$ with $\mathrm{WF}'_{\mathrm{b}}$ disjoint from U, we see that $\|(\tilde{T}^-)^*A_rPu\|_{H^{-1}_0(X)}$ is uniformly bounded. Moreover, $\|\tilde{T}A\Lambda_ru\|^2_{H^1_0(X)}$ can be estimated, using Lemma 7.3 and the Poincaré inequality, by $\|xD_{y_{n-1}}\tilde{T}A\Lambda_ru\|^2_{L^2(X)}$ modulo terms that are uniformly bounded as $r \to 0$. The principal symbol of $D_{y_{n-1}}\tilde{T}A$ is $\underline{\zeta}_{n-1}\sigma_{\mathrm{b},-1/2}(\tilde{T})a$, with $a = \chi_0\chi_1\chi_2$, where χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc., so we can write:

$$\begin{aligned}|\underline{\zeta}_{n-1}|^{1/2}a = |\underline{\zeta}_{n-1}|^{1/2}\chi_0\chi_1\chi_2 &= A_0^{-1}(2 - \phi/\delta)|\underline{\zeta}_{n-1}|^{1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2\\ &= \digamma^{-1/2}\delta^{1/2}(2 - \phi/\delta)\tilde{b},\end{aligned}$$

where we used that

$$\chi_0'(\digamma^{-1}(2 - \phi/\delta)) = \digamma^2(2 - \phi/\delta)^{-2}\chi_0(\digamma^{-1}(2 - \phi/\delta))$$

when $2 - \phi/\delta > 0$, while a, $\tilde{b}$ vanish otherwise. Correspondingly, using that $|\underline{\zeta}_{n-1}|^{1/2}\sigma_{\mathrm{b},-1/2}(\tilde{T})$ is $\mathcal{C}^\infty$, homogeneous degree zero, near the support of a in ${}^{\mathrm{b}}T^*X \setminus o$, we can write $D_{y_{n-1}}\tilde{T}A = G\tilde{B} + F$, $G \in \Psi_{\mathrm{b}}^0(X)$, $F \in \Psi_{\mathrm{b}}^{-1/2}(X)$. Thus, modulo terms that are bounded as $r \to 0$, $\|xD_{y_{n-1}}\tilde{T}A\Lambda_r u\|^2$ (hence $\|\tilde{T}A\Lambda_r u\|^2_{H_0^1(X)}$) can be estimated from above by $C_6\|x\tilde{B}_r u\|^2$. Therefore, modulo terms that are bounded as $r \to 0$, for $\gamma > 0$ sufficiently small, $\gamma\|\tilde{T}A_r u\|^2_{H_0^1(X)}$ can be absorbed into $\|Cx\tilde{B}_r u\|^2$. As the treatment of the other terms on the right hand side of (8.25) requires no change, we deduce as above that $x\tilde{B}_0 u \in L^2(X)$, which (in view of Lemma 7.3) proves that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$, completing the proof of the iterative step.

We need to make one more remark to prove the proposition for $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, namely we need to show that the neighborhoods of q_0 which are disjoint from $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ do not shrink uncontrollably to $\{q_0\}$ as $s \to \infty$. This argument parallels to last paragraph of the proof of [16, Proposition 24.5.1]. In fact, note that above we have proved that the elliptic set of $\tilde{B} = \tilde{B}_s$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$. In the next step, when we are proving $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s+1/2}(u)$, we decrease $\delta > 0$ slightly (by an arbitrary small amount), thus decreasing the support of $a = a_{s+1/2}$ in (8.8), to make sure that $\operatorname{supp} a_{s+1/2}$ is a subset of the elliptic set of the union of $\tilde{B}_s$ with the region $\eta < 0$, and hence that $\mathrm{WF}_{\mathrm{b}}^{1,s}(u) \cap \operatorname{supp} a_{s+1/2} = \emptyset$. Each iterative step thus shrinks the elliptic set of $\tilde{B}_s$ by an arbitrarily small amount, which allows us to conclude that q_0 has a neighborhood U' such that $\mathrm{WF}_{\mathrm{b}}^{1,s}(u) \cap U' = \emptyset$ for all s. This proves that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, and indeed that $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \cap U' = \emptyset$, for if $A \in \Psi_{\mathrm{b}}^m(X)$ with $\mathrm{WF}_{\mathrm{b}}'(A) \subset U'$ then $Au \in H_0^1(X)$ by Lemma 5.10 and Corollary 5.12. □

Before turning to tangential propagation we need a technical lemma, roughly stating that when applied to solutions of $Pu = 0$, $u \in H_0^1(X)$, microlocally near $\mathcal{G}$, xD_x and Id are not merely bounded by $xD_{y_{n-1}}$, but it is small compared to it, provided that $\lambda \in \mathbb{C}\setminus[(n-1)^2/4, \infty)$. This result is the analogue of [32, Lemma 7.1], and is proved as there, with the only difference being that the term $\langle \lambda A_r u, A_r u\rangle$ cannot be dropped, but it is treated just as in Proposition 7.7 above. Below a δ-neighborhood refers to a δ-neighborhood with respect to the metric associated to any Riemannian metric on the manifold ${}^{\mathrm{b}}T^*X$, and we identify ${}^{\mathrm{b}}S^*X$ as the unit ball bundle with respect to some fiber metric on ${}^{\mathrm{b}}T^*X$.

Lemma 8.4. *(cf.* [32, Lemma 7.1]*.) Suppose that $P = \Box_g + \lambda$,*

$$\lambda \in \mathbb{C} \setminus [(n-1)^2/4, \infty).$$

Suppose $u \in H^{1,k}_{0,\mathrm{b},\mathrm{loc}}(X)$, and suppose that we are given $K \subset {}^{\mathrm{b}}S^*X$ compact satisfying

$$K \subset \mathcal{G} \cap T^*Y \setminus \mathrm{WF}_{\mathrm{b}}^{-1,s+1/2}(Pu).$$

Then there exist $\delta_0 > 0$ and $C_0 > 0$ with the following property. Let $\delta < \delta_0$, $U \subset {}^{\mathrm{b}}S^*X$ open in a δ-neighborhood of K, and $\mathcal{A} = \{A_r : r \in (0,1]\}$ be a bounded family of ps.d.o's in $\Psi_{\mathrm{bc}}^s(X)$ with $\mathrm{WF}_{\mathrm{b}}'(\mathcal{A}) \subset U$, and with $A_r \in \Psi_{\mathrm{b}}^{s-1}(X)$ for $r \in (0,1]$.

Then there exist $G \in \Psi_{\mathrm{b}}^{s-1/2}(X)$, $\tilde G \in \Psi_{\mathrm{b}}^{s+1/2}(X)$ *with* $\mathrm{WF}_{\mathrm{b}}'(G), \mathrm{WF}_{\mathrm{b}}'(\tilde G) \subset U$ *and* $\tilde C_0 = \tilde C_0(\delta) > 0$ *such that for all* $r > 0$,

$$\|xD_xA_ru\|^2 + \|A_ru\|^2 \leq C_0\delta\|xD_{y_{n-1}}A_ru\|^2 + \tilde C_0\Big(\|u\|^2_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b,loc}}(X)} + \|\tilde GPu\|^2_{H_0^{-1}(X)}\Big). \tag{8.27}$$

The meaning of $\|u\|_{H^{1,k}_{0,\mathrm{b,loc}}(X)}$ *and* $\|Pu\|^2_{H^{-1,k}_{0,\mathrm{b,loc}}(X)}$ *is stated in Remark 7.2.*

Remark 8.5. As K is compact, this is essentially a local result. In particular, we may assume that K is a subset of ${}^{\mathrm{b}}T^*X$ over a suitable local coordinate patch. Moreover, we may assume that $\delta_0 > 0$ is sufficiently small so that $D_{y_{n-1}}$ is elliptic on U.

Proof. By Lemma 7.3 applied with K replaced by $\mathrm{WF}_{\mathrm{b}}'(\mathcal{A})$ in the hypothesis (note that the latter is compact), we already know that

$$\begin{aligned}&|\langle dA_ru, dA_ru\rangle_G + \lambda\|A_ru\|^2|\\ &\quad\leq C_0'(\|u\|^2_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b,loc}}(X)} + \|\tilde GPu\|^2_{H_0^{-1}(X)}).\end{aligned} \tag{8.28}$$

for some $C_0' > 0$ and for some G, $\tilde G$ as in the statement of the lemma. Freezing the coefficients at Y, as in the proof of Proposition 7.7, see [32, Lemma 7.1] for details, we deduce that

$$\begin{aligned}&\big|\|xD_xA_ru\|^2 - \lambda\|A_ru\|^2\big|\\ &\quad\leq \int_X \Big(B_{ij}(0,y)(xD_{y_i})A_ru\,\overline{(xD_{y_j})A_ru}\Big)\,|dg| + C_1\delta\|xD_{y_{n-1}}A_ru\|^2\\ &\qquad + C_0''(\|u\|^2_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Gu\|^2_{H^1_0(X)} + \|Pu\|^2_{H^{-1,k}_{0,\mathrm{b,loc}}(X)} + \|\tilde GPu\|^2_{H^{-1}(X)}).\end{aligned} \tag{8.29}$$

Now, one can show that

$$\begin{aligned}&\Big|\int_X \Big(\sum D_{y_i}^*B_{ij}(0,y)D_{y_j})xA_ru\,\overline{xA_ru}\Big)\,|dg|\Big|\\ &\quad\leq C_2\delta\|D_{y_{n-1}}A_ru\|^2 + \tilde C_2(\delta)(\|u\|^2_{H^{1,k}_{0,\mathrm{b,loc}}(X)} + \|Gu\|^2_{H^1_0(X)})\end{aligned} \tag{8.30}$$

precisely as in the proof of [32, Lemma 7.1]. Equations (8.29)-(8.30) imply (8.27) with the left hand side replaced by $\big|\|xD_xA_ru\|^2 - \lambda\|A_ru\|^2\big|$. If $\mathrm{Im}\,\lambda \neq 0$, taking the imaginary part of $\|xD_xA_ru\|^2 - \lambda\|A_ru\|^2$ gives the desired bound for $\|A_ru\|^2$, hence taking the real part gives the desired bound for $\|xD_xA_ru\|^2$ as well. If $\mathrm{Im}\,\lambda = 0$ but $\lambda < (n-1)^2/4$, we finish the proof using the Poincaré inequality, cf. the proof of Proposition 7.7. □

We finally state the tangential propagation result.

Proposition 8.6. *(Tangential propagation.) Suppose that* $P = \Box_g + \lambda$, $\lambda \in \mathbb{C} \setminus [(n-1)^2/4, \infty)$. *Let* $\mathcal{U}_0$ *be a coordinate chart in* X, $\mathcal{U}$ *open with* $\overline{\mathcal{U}} \subset \mathcal{U}_0$. *Let* $u \in H^{1,k}_{0,\mathrm{b,loc}}(X)$ *for some* $k \leq 0$, *and let* $\tilde\pi : T^*X \to T^*Y$ *be the coordinate projection*

$$\tilde\pi : (x, y, \xi, \zeta) \mapsto (y, \zeta).$$

Given $K \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ compact with

$$K \subset (\mathcal{G} \cap {}^{\mathrm{b}}T^*_Y X) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(f),\ f = Pu, \tag{8.31}$$

there exist constants $C_0 > 0$, $\delta_0 > 0$ such that the following holds. If $\operatorname{Im}\lambda \leq 0$, $q_0 = (y_0, \underline{\zeta}_0) \in K$, $\alpha_0 = \hat{\pi}^{-1}(q_0)$, $W_0 = \tilde{\pi}_*|_{\alpha_0}\mathsf{H}_p$ considered as a constant vector field in local coordinates, and for some $0 < \delta < \delta_0$, $C_0\delta \leq \epsilon < 1$ and for all $\alpha = (x, y, \xi, \zeta) \in \Sigma$

$$\begin{aligned} &\alpha \in T^*X \text{ and } |\tilde{\pi}(\alpha - \alpha_0 - \delta W_0)| \leq \epsilon\delta \text{ and } |x(\alpha)| \leq \epsilon\delta \\ &\qquad \Rightarrow \pi(\alpha) \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u), \end{aligned} \tag{8.32}$$

then $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$.

Remark 8.7. One can again change the direction of propagation, i.e. replace δ by $-\delta$ in $\alpha - \alpha_0 - \delta W_0$, provided one also changes the sign of $\operatorname{Im}\lambda$ to $\operatorname{Im}\lambda \geq 0$. In particular, if $\operatorname{Im}\lambda = 0$, one obtains propagation estimates in both the forward and backward directions.

Proof. Again, the proof follows a proof in [32] closely, in this case Proposition 7.3, as corrected at a point in [30], so we merely point out the main steps. Again, one uses a commutant $A \in \Psi^0_{\mathrm{b}}(X)$ and weights $\Lambda_r \in \Psi^0_{\mathrm{b}}(X)$, $r \in (0,1)$, uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X)$, $A_r = A\Lambda_r$, in order to obtain the propagation of $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ with the notation of that paper, whose analogue is $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ here (the difference is the space relative to which one obtains b-regularity: $H^1(X)$ in the previous paper, the zero-Sobolev space $H^1_0(X)$ here). One can use *exactly the same* commutants as in [32] (with a small correction given in [30]). Then Proposition 8.1 lets one calculate $\imath[A_r^* A_r, P]$ to obtain a completely analogous expression to the formulae below Equation (7.16) of [32], as corrected in [30]). The rest of the argument is completely analogous as well. Again, we refer the reader to [34] because the commutator calculation is written up exactly as above in Proposition 8.1, see [34, Proposition 3.10] and it is used subsequently in 6.1 there the same way it needs to be used here – any modifications are analogous to those in Proposition 8.2 and arise due to the non-negligible nature of λ.

Again, we first construct the symbol a of our commutator following [32, Proof of Proposition 7.3] as corrected in [30]. Note that (with $\tilde{p} = x^{-2}\sigma_{\mathrm{b},2}(\tilde{P}) = h$)

$$W_0(q_0) = \mathsf{H}_{\tilde{p}}(q_0),$$

and let

$$W = |\underline{\zeta}_{n-1}|^{-1} W_0,$$

so W is homogeneous of degree zero (with respect to the $\mathbb{R}^+$-action on the fibers of $T^*Y \setminus o$). We use

$$\tilde{\eta} = -(\operatorname{sgn}(\underline{\zeta}_{n-1})_0)(y_{n-1} - (y_{n-1})_0)$$

now to measure propagation, since $\underline{\zeta}_{n-1}^{-1}\mathsf{H}_{\tilde{p}}(y_{n-1}) = 2 > 0$ at q_0, so $-\mathsf{H}_{\tilde{p}}\tilde{\eta}$ is $2|\underline{\zeta}_{n-1}| > 0$ at q_0.

First, we require

$$\rho_1 = \tilde{p}(y, \hat{\underline{\zeta}}) = |\underline{\zeta}_{n-1}|^{-2}\tilde{p}(y, \underline{\zeta});$$

note that $d\rho_1 \neq 0$ at q_0 for $\underline{\zeta} \neq 0$ there, but $\mathsf{H}_{\tilde{p}}\tilde{p} \equiv 0$, so

$$W\rho_1(q_0) = 0.$$

Next, since $\dim Y = n-1$, $\dim T^*Y = 2n-2$, hence $\dim S^*Y = 2n-3$. With a slight abuse of notation, we also regard q_0 as a point in S^*Y – recall that $S^*Y = (T^*Y \setminus o)/\mathbb{R}^+$. We can also regard W as a vector field on S^*Y in view of its homogeneity. As W does not vanish as a vector in $T_{q_0}S^*Y$ in view of $W\tilde{\eta}(q_0) \neq 0$, $\tilde{\eta}$ being homogeneous degree zero, hence a function on S^*Y, the kernel of W in $T^*_{q_0}S^*Y$ has dimension $2n-4$. Thus there exist ρ_j, $j = 2, \ldots, 2n-4$ be homogeneous degree zero functions on T^*Y (hence functions on S^*Y) such that

$$\begin{aligned}&\rho_j(q_0) = 0,\ j = 2, \ldots, 2n-4,\\ &W\rho_j(q_0) = 0,\ j = 2, \ldots, 2n-4,\\ &d\rho_j(q_0),\ j = 1, \ldots, 2n-4 \text{ are linearly independent at } q_0.\end{aligned} \tag{8.33}$$

By dimensional considerations, $d\rho_j(q_0)$, $j = 1, \ldots, 2n-4$, together with $d\tilde{\eta}$ span the cotangent space of S^*Y at q_0, i.e. of the quotient of T^*Y by the $\mathbb{R}^+$-action.

Hence,

$$|\underline{\zeta}_{n-1}|^{-1} W_0 \rho_j = \sum_{i=1}^{2n-4} \tilde{F}_{ji}\rho_i + \tilde{F}_{j,2n-3}\tilde{\eta},\ j = 2, \ldots, 2n-4,$$

with $\tilde{F}_{ji}$ smooth, $i = 1, \ldots, 2n-3$, $j = 2, \ldots, 2n-4$. Then we extend ρ_j to a function on ${}^{\mathrm{b}}T^*X \setminus o$ (using the coordinates $(x, y, \underline{\xi}, \underline{\zeta})$), and conclude that

$$|\underline{\zeta}_{n-1}|^{-1} \mathsf{H}_{\tilde{p}}\rho_j = \sum_{l=1}^{2n-4} \tilde{F}_{jl}\rho_l + \tilde{F}_{j,2n-3}\tilde{\eta} + \tilde{F}_{j0}x,\ j = 2, \ldots, 2n-4, \tag{8.34}$$

with $\tilde{F}_{jl}$ smooth. Similarly,

$$|\underline{\zeta}_{n-1}|^{-1} \mathsf{H}_{\tilde{p}}\tilde{\eta} = -2 + \sum_{l=1}^{2n-4} \check{F}_{l}\rho_l + \check{F}_{2n-3}\tilde{\eta} + \check{F}_0 x, \tag{8.35}$$

with $\check{F}_l$ smooth.

Let

$$\omega = |x|^2 + \sum_{j=1}^{2n-4} \rho_j^2. \tag{8.36}$$

Finally, we let

$$\phi = \tilde{\eta} + \frac{1}{\epsilon^2\delta}\omega, \tag{8.37}$$

and define a by

$$a = \chi_0(F^{-1}(2 - \phi/\delta))\chi_1((\tilde{\eta}\delta)/\epsilon\delta + 1)\chi_2(|\underline{\xi}|^2/\underline{\zeta}_{n-1}^2), \tag{8.38}$$

with χ_0, χ_1 and χ_2 as in the case of the normal propagation estimate, stated after (8.7). We always assume $\epsilon < 1$, so on $\operatorname{supp} a$ we have

$$\phi \leq 2\delta \text{ and } \tilde{\eta} \geq -\epsilon\delta - \delta \geq -2\delta.$$

Since $\omega \geq 0$, the first of these inequalities implies that $\tilde{\eta} \leq 2\delta$, so on $\operatorname{supp} a$

$$|\tilde{\eta}| \leq 2\delta. \tag{8.39}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \tilde{\eta}) \leq 4\delta^2\epsilon^2. \tag{8.40}$$

Moreover, on $\operatorname{supp} d\chi_1$,

$$\tilde\eta \in [-\delta - \epsilon\delta, -\delta],\ \omega^{1/2} \le 2\epsilon\delta, \tag{8.41}$$

so this region lies in (8.32) after ϵ and δ are both replaced by appropriate constant multiples, namely the present δ should be replaced by $\delta/(2|(\underline{\zeta}_{n-1})_0|)$.

We proceed as in the case of hyperbolic points, letting $A_0 \in \Psi^0_{\mathrm{b}}(X)$ with $\sigma_{\mathrm{b},0}(A_0) = a$, supported in the coordinate chart. Also let Λ_r be scalar, have symbol

$$|\underline{\zeta}_{n-1}|^{s+1/2}(1 + r|\underline{\zeta}_{n-1}|^2)^{-s}\,\mathrm{Id}, \quad r \in [0, 1), \tag{8.42}$$

so $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}(X)$ for $r > 0$ and it is uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X)$. Then, for $r > 0$,

$$\begin{aligned}\langle \imath A_r^* A_r P u, u\rangle - \langle \imath A_r^* A_r u, P u\rangle &= \langle \imath [A_r^* A_r, P] u, u\rangle + \langle \imath (P - P^*) A_r^* A_r u, u\rangle \\ &= \langle \imath [A_r^* A_r, P] u, u\rangle - 2\,\mathrm{Im}\,\lambda \|A_r u\|^2.\end{aligned} \tag{8.43}$$

and we compute the commutator here using Proposition 8.1. We arrange the terms of the proposition so that the terms in which a vector field differentiates χ_1 are included in E_r, the terms in which a vector fields differentiates χ_2 are included in E'_r. Thus, we have

$$\begin{aligned}&\imath A_r^* A_r P - \imath P A_r^* A_r \\ &\quad = (xD_x)^* C_r^\sharp (xD_x) + (xD_x)^* x C'_r + x C''_r (xD_x) + x^2 C_r^\flat + E_r + E'_r + F_r,\end{aligned} \tag{8.44}$$

with

$$\begin{aligned}\sigma_{\mathrm{b},2s}(C_r^\sharp) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a |\underline{\zeta}_{n-1}|^{-1}(\hat f^\sharp + \epsilon^{-2}\delta^{-1} f^\sharp)\chi_0'\chi_1\chi_2 + a^2 \tilde c_r^\sharp\Big), \\ \sigma_{\mathrm{b},2s+1}(C'_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a (\hat f' + \delta^{-1}\epsilon^{-2} f')\chi_0'\chi_1\chi_2 + a^2 \tilde c'_r\Big), \\ \sigma_{\mathrm{b},2s+1}(C''_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1} a (\hat f'' + \delta^{-1}\epsilon^{-2} f'')\chi_0'\chi_1\chi_2 + a^2 \tilde c''_r\Big), \\ \sigma_{\mathrm{b},2s+2}(C_r) &= w_r^2\Big(\digamma^{-1}\delta^{-1} |\underline{\zeta}_{n-1}| a (4\hat h + \hat f^\flat + \delta^{-1}\epsilon^{-2} f^\flat)\chi_0'\chi_1\chi_2 + a^2 \tilde c_r^\flat\Big),\end{aligned} \tag{8.45}$$

where $f^\sharp$, f', f'' and $f^\flat$ as well as $\hat f^\sharp, \hat f', \hat f''$ and $\hat f^\flat$ are all smooth functions on ${}^{\mathrm{b}}T^*X\setminus o$, homogeneous of degree 0 (independent of ϵ and δ). Moreover, $f^\sharp, f', f'', f^\flat$ arise from when ω is differentiated in $\chi_0(\digamma^{-1}(2-\phi/\delta))$, while $\hat f^\sharp, \hat f', \hat f''$ and $\hat f^\flat$ arise when $\tilde\eta$ is differentiated in $\chi_0(\digamma^{-1}(2-\phi/\delta))$, and comprise all such terms with the exception of part of that arising from the $-\mathsf{H}_h$ component of $V^\flat|_Y$ (which gives $4\hat h = 4|\zeta_{n-1}|^{-2} h$ on the last line above, modulo a term included in $\hat f^\flat$ and which vanishes at $\omega = 0$). In addition, as $V^\bullet\rho^2 = 2\rho V^\bullet\rho$ for any function ρ, the terms $f^\bullet$, $\bullet = \sharp, {}', {}'', \flat$, have vanishing factors of ρ_l, resp. x, with the structure of the remaining factor dictated by the form of $V^\bullet\rho_l$, resp. $V^\bullet x$. Thus, using (8.34) to compute $f^\flat$, (8.35) to compute $\hat{f}^\flat$, we have

$$\begin{aligned} f^\sharp &= \sum_k \rho_k f_k^\sharp + x f_0^\sharp, \\ f^\bullet &= \sum_k \rho_k f_k^\bullet + x f_0^\bullet, \ \bullet =',''\,, \\ f^\flat &= \sum_{kl} \rho_k\rho_l f_{kl}^\flat + \sum_k \rho_k x f_k^\flat + x^2 f_0 + \sum_k \rho_k \tilde\eta f_{k+}^\flat, \\ \hat f^\flat &= x\hat f_0^\flat + \sum_k \rho_k \hat f_k^\flat + \tilde\eta \hat f_+^\flat, \end{aligned}$$

with $f_k^\sharp$, etc., smooth. We deduce that

$$\epsilon^{-2}\delta^{-1}|f^\sharp| \le C\epsilon^{-1},\ |\hat f^\sharp| \le C, \tag{8.46}$$

while

$$\epsilon^{-2}\delta^{-1}|f^\bullet| \le C\epsilon^{-1},\ |\hat f^\bullet| \le C, \tag{8.47}$$

$\bullet =',''$, and

$$\epsilon^{-2}\delta^{-1}|f^\flat| \le C\epsilon^{-1}\delta,\ |\hat f^\flat| \le C\delta. \tag{8.48}$$

We remark that although thus far we worked with a single $q_0 \in K$, the same construction works with q_0 in a neighborhood $U_{q_0'}$ of a fixed $q_0' \in K$, with a *uniform* constant C. In view of the compactness of K, this suffices (by the rest of the argument we present below) to give the uniform estimate of the proposition.

Since (8.46)-(8.48) are exactly the same (with slightly different notation) as [34, Equations (6.16)-(6.18)], the rest of the proof is analogous, except that [34, Lemma 4.6] is replaced by Lemma 8.4 here. Thus, for a small constant $c_0 > 0$ to be determined, which we may assume to be less than C, we demand below that the expressions on the right hand sides of (8.46) are bounded by $c_0(\epsilon\delta)^{-1}$, those on the right hand sides of (8.47) are bounded by $c_0(\epsilon\delta)^{-1/2}$, while those on the right hand sides of (8.48) are bounded by c_0. This demand is due to the appearance of two, resp. one, resp. zero, factors of xD_x in (8.44) for the terms whose principal symbols are affected by these, taking into account that in view of Lemma 8.4 we can estimate $\|Q_i v\|$ by $C_{\mathcal{G},K}(\epsilon\delta)^{1/2}\|D_{y_{n-1}}v\|$ if v is microlocalized to a $\epsilon\delta$-neighborhood of $\mathcal{G}$, which is the case for us with $v = A_r u$ in terms of support properties of a.

Thus, recalling that $c_0 > 0$ is to be determined, we require that

$$(C/c_0)^2\delta \le \epsilon \le 1, \tag{8.49}$$

and

$$\delta < (c_0/C)^2; \tag{8.50}$$

see [34, Proposition 6.1] for motivation. Then with ϵ, δ satisfying (8.49) and (8.50), hence $\delta^{-1} > (C/c_0)^2 > C/c_0$, (8.46)-(8.48) give that

$$\epsilon^{-2}\delta^{-1}|f^\sharp| \le c_0\delta^{-1}\epsilon^{-1},\ |\hat f^\sharp| \le c_0\delta^{-1}\epsilon^{-1}, \tag{8.51}$$

while

$$\epsilon^{-2}\delta^{-1}|f^\bullet| \le c_0\delta^{-1/2}\epsilon^{-1/2},\ |\hat f^\bullet| \le c_0\delta^{-1/2}\epsilon^{-1/2}, \tag{8.52}$$

$\bullet =',''$, and

$$\epsilon^{-2}\delta^{-1}|f^\flat| \le c_0,\ |\hat f^\flat| \le c_0, \tag{8.53}$$

as desired. One deduces that

$$\begin{aligned} &\imath A_r^* A_r P - \imath P A_r^* A_r \\ &= \tilde{B}_r^* \big(C^* x^2 C + x R^\flat x + (xD_x)^* \tilde{R}' x + x \tilde{R}'' (xD_x) + (xD_x)^* R^\sharp (xD_x) \big) \tilde{B}_r \\ &\qquad + R_r'' + E_r + E_r' \end{aligned} \tag{8.54}$$

with

$$R^\flat \in \Psi_b^0(X),\ \tilde{R}', \tilde{R}'' \in \Psi_b^{-1}(X),\ R^\sharp \in \Psi_b^{-2}(X),$$
$$R_r'' \in L^\infty((0,1); \mathrm{Diff}_0^2 \Psi_b^{2s-1}(X)),\ E_r, E_r' \in L^\infty((0,1); \mathrm{Diff}_0^2 \Psi_b^{2s}(X)),$$

with $\mathrm{WF}_b'(E) \subset \eta^{-1}((-\infty, -\delta]) \cap U$, $\mathrm{WF}_b'(E') \cap \dot{\Sigma} = \emptyset$, and with $r^\flat = \sigma_{b,0}(R^\flat)$, $\tilde{r}' = \sigma_{b,-1}(\tilde{R}')$, $\tilde{r}'' = \sigma_{b,-1}(\tilde{R}'')$, $r^\sharp \in \sigma_{b,-2}(R^\sharp)$,

$$|r^\flat| \leq 2c_0 + C_2 \delta \digamma^{-1},\ |\underline{\zeta}_{n-1} \tilde{r}'| \leq 2c_0 \delta^{-1/2} \epsilon^{-1/2} + C_2 \delta \digamma^{-1},$$
$$|\underline{\zeta}_{n-1} \tilde{r}''| \leq 2c_0 \delta^{-1/2} \epsilon^{-1/2} + C_2 \delta \digamma^{-1},\ |\underline{\zeta}_{n-1}^2 r^\sharp| \leq 2c_0 \delta^{-1} \epsilon^{-1} + C_2 \delta \digamma^{-1}.$$

These are analogues of the result of the second displayed equation after [32, Equation (7.16)], as corrected in [30], with the small (at this point arbitrary) constant c_0 replacing some constants given there in terms of ϵ and δ; see [34, Equation (6.25)] for estimates stated in exactly the same form in the form-valued setting. The rest of the argument thus proceeds as in [32, Proof of Proposition 7.3], taking into account [30], and using Lemma 8.4 in place of [32, Lemma 7.1]. □

Since for λ real, $\lambda < (n-1)^2/4$, both forward and backward propagation is covered by these two results, see Remarks 8.3 and 8.7, we deduce our main result on the propagation of singularities:

Theorem 8.8. *Suppose that* $P = \Box + \lambda$, $\lambda < (n-1)^2/4$, $m \in \mathbb{R}$ *or* $m = \infty$. *Suppose* $u \in H^{1,k}_{0,b,\mathrm{loc}}(X)$ *for some* $k \leq 0$. *Then*

$$(\mathrm{WF}_b^{1,m}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}_b^{-1,m+1}(Pu)$$

is a union of maximally extended generalized broken bicharacteristics of the conformal metric $\hat{g}$ *in*

$$\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,m+1}(Pu).$$

In particular, if $Pu = 0$ *then* $\mathrm{WF}_b^{1,\infty}(u) \subset \dot{\Sigma}$ *is a union of maximally extended generalized broken bicharacteristics of* $\hat{g}$.

Proof. The proof proceeds as in [32, Proof of Theorem 8.1], since the Propositions 8.2 and 8.6 are complete analogues of [32, Proposition 6.2] and [32, Proposition 7.3]. Given the results of the previous sections, this argument itself is only a slight modification of an argument originally due to Melrose and Sjöstrand [21], as presented by Lebeau [17] (although we do not need Lebeau's treatment of corners here). □

In fact, even if $\mathrm{Im}\,\lambda \neq 0$, we get one-sided statements:

Theorem 8.9. *Suppose that* $P = \Box + \lambda$, $\mathrm{Im}\,\lambda > 0$, *resp.* $\mathrm{Im}\,\lambda < 0$, *and* $m \in \mathbb{R}$ *or* $m = \infty$. *Suppose* $u \in H^{1,k}_{0,b,\mathrm{loc}}(X)$ *for some* $k \leq 0$. *Then*

$$(\mathrm{WF}_b^{1,m}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}_b^{-1,m+1}(Pu)$$

is a union of maximally forward extended, resp. backward extended *generalized broken bicharacteristics of the conformal metric* $\hat{g}$ *in*

$$\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,m+1}(Pu).$$

In particular, if $Pu = 0$ then $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \subset \dot{\Sigma}$ is a union of maximally extended generalized broken bicharacteristics of $\hat{g}$.

Proof. The proof proceeds again as for Theorem 8.8, but now Propositions 8.2 and 8.6 only allow propagation in one direction. Thus, if $\operatorname{Im}\lambda < 0$, they allow one to conclude that if a point in $\dot{\Sigma}\setminus \mathrm{WF}_{\mathrm{b}}^{-1,m+1}(Pu)$ is in $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)$, then there is another point in $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)$ which is roughly along a *backward* GBB segment emanating from it. Then an actual backward GBB can be constructed as in the works of Melrose and Sjöstrand [21], and Lebeau [17]. □

In the absence of b-wave front set we can easily read off the actual expansion at the boundary as well.

Proposition 8.10. *Suppose that $P = \Box + \lambda$, $\lambda \in \mathbb{C}$. Let $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. Suppose $u \in H^1_{0,\mathrm{loc}}(X)$, $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) = \emptyset$ and $Pu \in \dot{\mathcal{C}}^\infty(X)$. Then*

$$u = x^{s_+(\lambda)} v_+,\ v_+ \in \mathcal{C}^\infty(X). \tag{8.55}$$

Conversely, if $\lambda < (n-1)^2/4$, given any $g_+ \in \mathcal{C}^\infty(Y)$, there exists $v_+ \in \mathcal{C}^\infty(X)$, $v_+|_Y = g_+$ such that $u = x^{s_+(\lambda)} v_+$ satisfies $Pu \in \dot{\mathcal{C}}^\infty(X)$; in particular $u \in H^1_{0,\mathrm{loc}}(X)$ and $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) = \emptyset$.

This proposition reiterates the importance of the constraint on λ in that

$$x^{(n-1)/2+i\alpha} \notin H^1_{0,\mathrm{loc}}(X)$$

for $\alpha \in \mathbb{R}$; for $\lambda \geq (n-1)^2/4$, the growth or decay relative to $H^1_{0,\mathrm{loc}}(X)$ does not distinguish between the two approximate solutions $x^{s_\pm(\lambda)} v_\pm$, $v_\pm \in \mathcal{C}^\infty(X)$.

Proof. For the first part of the lemma, by Lemma 5.16 and the subsequent remark, under our assumptions we have $u \in \mathcal{A}^{(n-1)/2}(X)$. By (7.1),

$$P + \big(((xD_x + \imath(n-1))(xD_x) - \lambda\big) \in x\mathrm{Diff}^2_{\mathrm{b}}(X). \tag{8.56}$$

This is, up to a change in overall the sign of the second summand,

$$(xD_x + \imath(n-1))(xD_x) - \lambda,$$

the same as the analogous expression in the de Sitter setting, see the first line of the proof of Lemma 4.13 of [33]. Thus, the proof of that lemma goes through without changes – the reader needs to keep in mind that $u \in \mathcal{A}^{(n-1)/2}(X)$ excludes one of the indicial roots from appearing in the argument of that lemma. (In the De Sitter setting, in Lemma 4.13 of [33], there was no a priori weight (relative to which one has conormality) specified.)

The converse again works as in Lemma 4.13 of [33] using (8.56). □

We can now state the 'inhomogeneous Dirichlet problem':

Theorem 8.11. *Assume (TF) and (PT). Suppose $\lambda < (n-1)^2/4$, and $s_+(\lambda) - s_-(\lambda) = 2\sqrt{\frac{(n-1)^2}{4} - \lambda}$ is not an integer, $P = P(\lambda) = \Box_g + \lambda$.*

Given $v_0 \in \mathcal{C}^\infty(Y)$ and $f \in \dot{\mathcal{C}}^\infty(X)$, both supported in $\{t \geq t_0\}$, the problem

$$Pu = f,\ u|_{t<t_0} = 0,\ u = x^{s_-(\lambda)} v_- + x^{s_+(\lambda)} v_+,\ v_\pm \in \mathcal{C}^\infty(X),\ v_-|_Y = v_0,$$

has a unique solution

If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in \mathcal{C}^\infty(X)$ by $v_- = \mathcal{C}^\infty(X) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$.

Proof. The proof of Lemma 4.13 of [33] shows that there exists $\tilde{u}$, supported in $t \geq t_0$, such that $\tilde{u} = x^{s_-(\lambda)}v_-$, v_- as in the statement of the theorem, and $P\tilde{u} \in \dot{\mathcal{C}}^\infty(X)$. Now let u' be the solution of $Pu' = f - P\tilde{u}$ supported in $\{t \geq t_0\}$, whose existence follows from Theorem 4.16, and which is of the form $x^{s_+(\lambda)}v_+$ by Theorem 8.8 and Proposition 8.10. Then $u = \tilde{u} + u'$ solves the PDE as stated. Uniqueness follows from the basic well-posedness theorem, Theorem 4.16. □

Finally we add well-posedness of possibly rough initial data:

Theorem 8.12. *Assume (TF) and (PT). Suppose $f \in H^{-1,m+1}_{0,\mathrm{b},\mathrm{loc}}(X)$ for some $m \in \mathbb{R}$, and let $m' \leq m$. Then (1.6) has a unique solution in $H^{1,m'}_{0,\mathrm{b},\mathrm{loc}}(X)$, which in fact lies in $H^{1,m}_{0,\mathrm{b},\mathrm{loc}}(X)$, and for all compact $K \subset X$ there exists a compact $K' \subset X$ and a constant $C > 0$ such that*

$$\|u\|_{H^{1,m}_0(K)} \leq C\|f\|_{H^{-1,m+1}_{0,\mathrm{b}}(K')}.$$

Remark 8.13. It should be emphasized that if one only wants to prove this result, without microlocal propagation, one could use more elementary energy estimates.

Proof. If $m \geq 0$, then by Theorem 4.16, (1.6) has a unique solution in $H^1_{0,\mathrm{loc}}(X)$, and by propagation of singularities it lies in $H^{1,m}_{0,\mathrm{b},\mathrm{loc}}(X)$, with the desired estimate. Moreover, again by the propagation of singularities, any solution of (1.6) in $H^{1,m'}_{0,\mathrm{b},\mathrm{loc}}(X)$ lies in $H^{1,m}_{0,\mathrm{b},\mathrm{loc}}(X)$, so the solution is indeed unique even in $H^{1,m'}_{0,\mathrm{b},\mathrm{loc}}(X)$.

If $m < 0$, uniqueness and the stability estimate follow as above. To see existence, let $T_0 < t_0$, and let $f_j \to f$ such that $f_j \in H^{-1,1}_{0,\mathrm{b},\mathrm{loc}}$ and $\operatorname{supp} f_j \subset \{t > T_0\}$. This can be achieved by taking $A_r \in \Psi^{-\infty}_{\mathrm{bc}}(X)$ with properly supported Schwartz kernel (of sufficiently small support) such that $\{A_r : \ r \in (0,1]\}$ is a bounded family in $\Psi^0_{\mathrm{bc}}(X)$, converging to Id in $\Psi^\epsilon_{\mathrm{bc}}(X)$ for $\epsilon > 0$, then with $f_j = A_{r_j} f$, $r_j \to 0$, we have the desired properties. By Theorem 4.16, (1.6) with f replaced by f_j has a unique solution $u_j \in H^1_{0,\mathrm{loc}}(X)$. Moreover, by the propagation of singularities, one has a uniform estimate

$$\|u_k - u_j\|_{H^{1,m}_0(K)} \leq C\|f_k - f_j\|_{H^{-1,m+1}_{0,\mathrm{b}}(K')},$$

with C independent of j, k. In view of the convergence of the f_j in $H^{-1,m+1}_{0,\mathrm{b}}(K')$, we deduce the convergence of the u_j in $H^{1,m}_{0,\mathrm{b}}(K)$ to some $u \in H^{1,m}_{0,\mathrm{b}}(K)$, hence (by uniqueness) we deduce the existence of $u \in H^{1,m}_{0,\mathrm{b},\mathrm{loc}}(X)$ solving $Pu = f$ with support in $\{t \geq T_0\}$. However, as $\operatorname{supp} f \subset \{t \geq t_0\}$, uniqueness shows the vanishing of u on $\{t < t_0\}$, proving the theorem. □

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