RESOLVENTS, POISSON OPERATORS AND SCATTERING MATRICES ON ASYMPTOTICALLY HYPERBOLIC AND DE SITTER SPACES

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Abstract. We describe how the global operator induced on the boundary of an asymptotically Minkowski space links two even asymptotically hyperbolic spaces and an even asymptotically de Sitter space, and compute the scattering operator of the linked problem in terms of the scattering operator of the constituent pieces.

1. Introduction

In [11] and [10] new methods were introduced to study the spectral and scattering theory of the Laplacian on asymptotically hyperbolic spaces and of the d'Alembertian on asymptotically de Sitter spaces (X, g). Concretely, examples of these spaces showed up as boundary values of a one higher dimensional space $\tilde{M}$ equipped with a Lorentzian metric $\tilde{g}$, which was either a blown-up version of de Sitter space, or a Kerr-de Sitter type space (which is a generalization of the former), or a Minkowski space. However, the analysis could be done (as long as g was a so-called even metric) without introducing a one higher dimensional space, by extending across the boundary of the conformal compactification $\overline{X}$, with a new smooth structure (the defining function of the boundary replaced by its square, hence the relevance of evenness) in a suitable manner. This was done systematically and in full generality in [10] for the case of an asymptotically hyperbolic space, with complex absorption introduced in the de Sitter region, and was extended to differential forms in [9].

Here we recall that a compact n-dimensional manifold with boundary, $\overline{X}$, with interior X equipped with a metric g, is asymptotically hyperbolic, resp. de Sitter, if $g = \frac{\hat{g}}{x^2}$ where $\hat{g}$ is a $\mathcal{C}^\infty$ Riemannian, resp. Lorentzian (of signature $(1, n-1)$), metric on $\overline{X}$, with $\hat{g}(dx, dx)|_{x=0} = 1$, for a boundary defining function x. In the Lorentzian setting one also assumes that the boundary Y of $\overline{X}$ is of the form $Y = Y_+ \cup Y_-$, with $Y_\pm$ unions of connected components, and all (null-)bicharacteristics[1] $\gamma(t)$, or equivalently null-geodesics, of g defined over $\mathbb{R}$ in X tend to Y_+ as the parameter $t \to +\infty$ and to Y_- as $t \to -\infty$, or vice versa. (This implies global hyperbolicity

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[1] By bicharacteristics we always mean null-bicharacteristics.

and that $\overline{X}$ is diffeomorphic to $[-1,1]\times Y_+$. Further, the null-bicharacteristics, and hence the null-geodesics, are simply reparameterized by a conformal factor, such as x^2, away from where it is singular/vanishes, i.e. away from the boundary. Correspondingly, the requirement on the bicharacteristics is equivalent to maximally extended bicharacteristics of $\hat{g}$ being defined over compact intervals, taking values over Y_+ at one endpoint and Y_- at the other.)

As shown by Graham and Lee [3] in the Riemannian case, and by a similar argument in the Lorentzian case, there is then a product decomposition near the boundary Y_y of $\overline{X}$ such that

$$g = \frac{dx^2 + \tilde{h}(x,y,dy)}{x^2}.$$

If this decomposition can be chosen so that $\tilde{h}$ is even in x, i.e. $\tilde{h} = h(x^2,y,dy)$, following Guillarmou [5] we call g *even*; see [5, Definition 1.2] for a more natural way of phrasing the evenness condition. This is equivalent to saying that h is $\mathcal{C}^\infty$ on $\overline{X}_{\mathrm{even}}$, the even version of $\overline{X}$, which is $\overline{X}$ as a topological manifold, but the $\mathcal{C}^\infty$ structure is changed so that $\mu = x^2$ is the new defining function of the boundary. We recall here that the class of even metrics was introduced by Guillarmou in order to strengthen the statement of the Mazzeo-Melrose theorem [7] on the nature of the analytic continuation of the resolvent on asymptotically hyperbolic spaces, namely to eliminate potential essential singularities at pure imaginary half-integers (which was achieved using the work of Graham and Zworski [4]).

Returning to the general discussion, there are natural settings, namely asymptotically Minkowski spaces, in which combinations of *even* asymptotically de Sitter and asymptotically hyperbolic spaces appear linked in interesting ways. A class of asymptotically Minkowski spaces $(\tilde{M},\tilde{g})$, with $\overline{\tilde{M}}$ being the compactification of $\tilde{M}$ with respect to which $\tilde{g}$ has appropriate properties, was introduced by Baskin, Vasy and Wunsch in [1], but as here we think of $\tilde{M}$ as a motivation for linking two copies (X_+,g_+) and (X_-,g_-) of asymptotically hyperbolic spaces (in case of Minkowski space, the quotient of the interior of the future and past light cones by the $\mathbb{R}^+$-action) and an asymptotically de Sitter space (X_0,g_0) (in case of Minkowski space, the quotient of the exterior of the light cones by the $\mathbb{R}^+$-action) rather than the main object of interest, this general class is not directly important here; the important aspect is the asymptotic behavior of its elements at infinity. In particular, we may assume that $\tilde{M}$ is replaced by a new manifold equipped with an $\mathbb{R}_+$-action, denoted by M, indeed is of the form $\mathbb{R}^+_\rho \times \tilde{X}$, with $\tilde{X} = \partial\overline{\tilde{M}}$; here $\tilde{\rho} = \rho^{-1}$ is a boundary defining function of $\overline{\tilde{M}}$ (thus the boundary of $\overline{\tilde{M}}$ is where ρ is infinite). Within

$$\tilde{X} = \overline{X_+} \cup \overline{X_-} \cup \overline{X_0},$$

the boundaries of $\overline{X_+}$, resp. $\overline{X_-}$, and the future, resp. past boundaries, $\partial_+\overline{X_0}$, resp. $\partial_-\overline{X_0}$, are identified. Mellin transforming (the conjugate by $\rho^{(n-1)/2}$ of) $\rho^2\Box_{\tilde{g}}$ induces a family of operators $\tilde{P}_\sigma$ on $\tilde{X}$; we refer to this as the family of *global* operators (on $\tilde{X}$). On the other hand, a differently normalized Mellin transform over the smaller domains $X_\pm$ and X_0 (which becomes singular at the boundary of these domains) induces the spectral families of asymptotically hyperbolic ($X_\pm$) Laplacians and asymptotically de Sitter (X_0) d'Alembertians; we call these the *constituent* operators. Starting with [11] and [10], continued in [9] and [1], some aspects of the connection being the global and constituent operators were explored. In this paper we show how the global operator on $\tilde{X}$ links the three constituent operators explicitly. In particular, we relate the scattering operators (or matrices) of the constituent operators to the global scattering operator. We remark here that given either an even asymptotically hyperbolic space or an even asymptotically de Sitter space, the spaces $\tilde{X}$ and M can always be constructed (after possibly taking two copies of the asymptotically de Sitter space); see Section 3.

To make this concrete, the relationship between the scattering operators

$$\begin{aligned}
&\mathcal{S}_{\tilde{X},\mathrm{past}}(\sigma) : \mathcal{C}^\infty(\partial X_+) \oplus \mathcal{C}^\infty(\partial X_+) \to \mathcal{C}^\infty(\partial X_-) \oplus \mathcal{C}^\infty(\partial X_-) \text{ on } \tilde{X},\\
&\mathcal{S}_{X_+}(\sigma) : \mathcal{C}^\infty(\partial X_+) \to \mathcal{C}^\infty(\partial X_+) \text{ on } X_+,\\
&\mathcal{S}_{X_-}(\sigma) : \mathcal{C}^\infty(\partial X_-) \to \mathcal{C}^\infty(\partial X_-) \text{ on } X_-, \text{ and}\\
&\mathcal{S}_{X_0,\mathrm{past}}(\sigma) : \mathcal{C}^\infty(\partial_+ X_0) \oplus \mathcal{C}^\infty(\partial_+ X_0) \to \mathcal{C}^\infty(\partial_- X_0) \oplus \mathcal{C}^\infty(\partial_- X_0) \text{ on } X_0,
\end{aligned}$$

(recall that $\partial_+ X_0 = \partial X_+$ and $\partial_- X_0 = \partial X_-$), defined in Definitions 4.12, 4.5 and 4.9 respectively, is given by the following theorem:

Theorem 1.1. *(See Theorem 4.13 and Corollary 4.14.) For $\sigma \notin \imath\mathbb{Z}$, if σ is not a pole of the inverse $\tilde{P}_{\sigma,\mathrm{past}}^{-1}$ of the global operator $\tilde{P}_\sigma$ on $\tilde{X}$ (acting between function spaces discussed at the end of Section 3, which amounts to solving the backwards, or past-oriented problem, propagating regularity towards $\partial_- X_0$) then*

$$\mathcal{S}_{\tilde{X},\mathrm{past}}(\sigma) = \begin{bmatrix} e^{-\pi\sigma} & e^{\pi\sigma} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{S}_{X_-}(-\sigma) \end{bmatrix} \mathcal{S}_{X_0,\mathrm{past}}(\sigma) \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{S}_{X_+}(\sigma) \end{bmatrix} \begin{bmatrix} e^{-\pi\sigma} & e^{\pi\sigma} \\ 1 & 1 \end{bmatrix},$$

i.e. apart from integer issues corresponding to the matrices with $e^{\pi\sigma}$ terms, $\mathcal{S}_{\tilde{X},\mathrm{past}}(\sigma)$ is essentially the product of $\mathcal{S}_{X_\pm}(\pm\sigma)$ and $\mathcal{S}_{X_0,\mathrm{past}}(\sigma)$.

Furthermore, $\mathcal{S}_{\tilde{X},\mathrm{past}}(\sigma)$ is an elliptic Fourier integral operator of order 0 associated to the (rescaled or limiting) null-geodesic flow on X_0, from $\partial_+ X_0$ to $\partial_- X_0$, with principal symbol as stated in Corollary 4.14.

The Fourier integral operator statement is proved using results of Joshi and Sá Barreto [6] (using results of Mazzeo and Melrose [7]) on the scattering matrix on asymptotically hyperbolic spaces being a pseudodifferential operator, and of the author that the scattering operator on asymptotically de Sitter spaces is a Fourier integral operator associated to the null-geodesic flow [12]. Proving the FIO property of $\mathcal{S}_{\tilde{X},\mathrm{past}}(\sigma)$ intrinsically on $\tilde{X}$ is a subject of current work with Nick Haber.

We also describe $\tilde{P}_{\sigma,\mathrm{past}}^{-1}$ in terms of the resolvents and Poisson operators in terms of the constituent pieces, see Theorem 4.16.

In the whole paper we consider the operators acting on functions to simplify the notation. In [9] the setup was translated to differential forms, and at the cost of somewhat more complicated notation/asymptotics (distinguishing closed and coclosed forms), one could work with the form bundles. However, while the methods of [6] and [12] work on the form bundles, the analysis there was not carried out in that setting, so the extension of the FIO statement would require additional work.

The plan of this paper is the following. In Section 2 we recall how the spaces are linked via the Mellin transform in the case of Minkowski space. Motivated by this, in Section 3 we show that given an asymptotically de Sitter or asymptotically hyperbolic space, one can construct an asymptotically Minkowski space so that via the Mellin transform one obtains a family of operators related to the spectral family of the individual spaces which links them together. In Section 4 we establish the relationship between these operators as well as their Poisson operators and scattering operators.

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2. Minkowski space, hyperbolic space and de Sitter space

In this section we connect the analysis of the Laplacians/d'Alembertians on Minkowski, hyperbolic and de Sitter spaces. This connection has a direct extension, with simple modifications, to the general asymptotically hyperbolic/de Sitter setting, considered in the next section. Here we follow [9], which considered differential forms, in the setup, but for the sake of the simplicity of notation we work in the scalar setting (but this is completely unimportant).

The starting point of analysis is the manifold $\mathbb{R}^{n+1}$, or rather $\mathbb{R}^{n+1}\setminus o$, which is equipped with an $\mathbb{R}^+$-action given by dilations: $(\lambda,z)\mapsto\lambda z$. A transversal to this action is, as a differentiable manifold, $\mathbb{S}^n$, which may be considered as the unit sphere with respect to the Euclidean metric, though the metric properties are not important here (since we are interested in the Minkowski metric after all). Thus, writing $(z_1,\ldots,z_{n+1})$ as the coordinates, let

$$dz_1^2+\ldots+dz_n^2+dz_{n+1}^2,$$

be the *Euclidean* metric, and let ρ be the Euclidean distance function on $\mathbb{R}^{n+1}$ from the origin, namely

$$\rho=(z_1^2+\ldots+z_n^2+z_{n+1}^2)^{1/2}.$$

Then $\mathbb{S}^n$ is the 1-level set of ρ. One can identify $\mathbb{R}^{n+1}\setminus\{0\}$ via the Euclidean polar coordinate map with $\mathbb{R}^+_\rho\times\mathbb{S}^n$, namely the map is $\mathbb{R}^+_\rho\times\mathbb{S}^n\ni(\rho,y)\mapsto\rho y\in\mathbb{R}^{n+1}\setminus\{0\}$.

The Minkowski metric is given by

$$\tilde g=dz_{n+1}^2-(dz_1^2+\ldots+dz_n^2),$$

and we also consider the Minkowski distance function r. Thus, away from the light cone, where $z_{n+1}^2=z_1^2+\ldots+z_n^2$, let

$$r=|z_{n+1}^2-(z_1^2+\ldots+z_n^2)|^{1/2}.$$

To analyze $\Box_{\tilde g}$, we conjugate $\rho^2\Box_{\tilde g}$ by the Mellin transform $\mathcal{M}_\rho$ on $\mathbb{R}^+_\rho\times\mathbb{S}^n$, identified with $\mathbb{R}^{n+1}\setminus\{0\}$ as above. The so-obtained operator,

$$\tilde P_{0,\tilde\sigma}=\mathcal{M}_\rho\rho^2\Box_{\tilde g}\mathcal{M}_\rho^{-1}\in\mathrm{Diff}^2(\mathbb{S}^n),$$

with $\tilde\sigma$ the Mellin dual parameter, fits into the framework of [11] and [10], see [11, Section 5]. As an aside, we remark that it will be convenient to shift the Mellin parameter, or equivalently conjugate $\Box_{\tilde g}$ by a power of ρ; this is the reason for adding the cumbersome subscript 0 to $\tilde P_{0,\tilde\sigma}$ presently.

While so far we explained why the Minkowski wave operator can be analyzed by means of [11] and [10], we still need to connect this to asymptotically hyperbolic and de Sitter spaces. But in the region in $\mathbb{S}^n$ corresponding to the interior of the future light cone, which can be identified with the hyperboloid

$$\mathbb{H}^n:\ z_{n+1}^2-(z_1^2+\ldots+z_n^2)=1,\ z_{n+1}>0,$$

via the $\mathbb{R}^+$-quotient, one can also consider the Mellin transform of $r^2\Box_{\tilde{g}}$ with respect to the decomposition $\mathbb{R}^+_r \times \mathbb{H}^n$, to get

$$P_{\tilde{\sigma}} = \mathcal{M}_r r^2 \Box_{\tilde{g}} \mathcal{M}_r^{-1} \in \mathrm{Diff}^2(\mathbb{H}^n).$$

(There is a similar setup for the second copy of $\mathbb{H}^n$ in the past light cone, where $z_{n+1} < 0$.) Now, $P_{\tilde{\sigma}}$ is not well-behaved at the boundary of the future light cone, but it is closely related to $\tilde{P}_{\tilde{\sigma}}$. Namely, if we use coordinates

$$y_j = \frac{z_j}{z_{n+1}},\ j = 1, \ldots, n,$$

on the sphere away from the equator $z_{n+1} = 0$,

$$r = F(y)\rho,\ F(y) = \sqrt{\frac{1-|y|^2}{1+|y|^2}}.$$

Note that F^2 is a smooth function on $\mathbb{S}^n$ near (its intersection with) the light cone which vanishes non-degenerately at the light cone. On the other hand, the Poincaré ball model $\overline{\mathbb{H}^n}$ of $\mathbb{H}^n$ arises by regarding it as a graph over $\mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}$, and compactifying $\mathbb{R}^n$ radially (or geodesically) to a ball, with boundary defining function, say, $(z_1^2 + \ldots + z_n^2)^{-1/2}$, or, ρ^{-1} – these two differ by a smooth positive multiple on $\overline{\mathbb{H}^n}$. As $r = 1$ on $\mathbb{H}^n$, this means that F is a valid boundary defining function in the Poincaré model, in contrast with the natural F^2 defining function of the light cone. In particular, with $\hat{y}_j$, $j = 1, \ldots, n-1$, denoting local coordinates on $\mathbb{S}^{n-1}$, identified with $\partial\overline{\mathbb{H}^n}$, hence the light cone at infinity is identified with $\mathbb{S}^{n-1}$, pulling back the Minkowski metric to $\mathbb{H}^n$, which by definition yields the hyperbolic metric, a straightforward calculation yields that that

$$g = \frac{(dF)^2}{F^2(1-F^2)} + \frac{1-F^2}{2F^2} h(\hat{y}, d\hat{y}), \tag{2.1}$$

with h the round metric on the sphere; this satisfies $F^2 g$ being a smooth metric up to the boundary, $F = 0$ (with a polar coordinate singularity at $F = 1$; F and $\hat{y}$ are not valid coordinates there, though F is still $\mathcal{C}^\infty$ near $F = 1$, and the metric is still $\mathcal{C}^\infty$ there as well, as can be seen by using valid coordinates), with the coefficients even functions of F. The metric g can be put in the normal form $g = \frac{dx^2+h}{x^2}$ by letting $x = \frac{F}{1+\sqrt{1-F^2}}$, which is an equivalent boundary defining function, but this is not necessary here.

Since

$$\mathcal{M}_\rho f(\tilde{\sigma}, y) = \int_0^\infty \rho^{-\imath\tilde{\sigma}} f\, \frac{d\rho}{\rho},$$

with a similar formula for $\mathcal{M}_r$, we have, if we identify $\mathbb{H}^n$ with an open subset of $\mathbb{S}^n$ (the interior of the future light cone),

$$\mathcal{M}_\rho \rho^2 \Box_{\tilde{g}} \mathcal{M}_\rho^{-1}(\tilde{\sigma}) = F^{\imath\tilde{\sigma}-2} \mathcal{M}_r r^2 \Box_{\tilde{g}} \mathcal{M}_r^{-1} F^{-\imath\tilde{\sigma}}. \tag{2.2}$$

We next compute $\mathcal{M}_r r^2 \Box_{\tilde{g}} \mathcal{M}_r^{-1}$; this is feasible since $\mathbb{R}^+ \times \mathbb{H}^n$ is an orthogonal decomposition relative to $\tilde{g}$. Concretely, the Minkowski metric is

$$\tilde{g} = dr^2 - r^2 g,$$

where g is the hyperbolic metric, since by definition the hyperbolic metric *is* the *negative* of the restriction of the Minkowski metric to the hyperboloid $\mathbb{H}^n$. This is a[2] conic metric, whose Laplacian is

$$\Box_{\tilde g} = -r^{-2}\Delta_X - r^{-n}\partial_r r^n \partial_r, \tag{2.3}$$

(cf. [2, Equation (3.8)] for the form version of the computation). Rewriting this as

$$r^2\Box_{\tilde g} = -\Delta_X - r^{-n+1}(r\partial_r)r^{n-1}(r\partial_r) = -\Delta_X - (r\partial_r + n - 1)(r\partial_r),$$

the Mellin transform of $r^2\Box_{\tilde g}$ with respect to r is

$$\begin{aligned}\mathcal{M}_r r^2\Box_{\tilde g}\mathcal{M}_r^{-1}(\tilde\sigma) &= -\Delta_X - (\imath\tilde\sigma + n - 1)(\imath\tilde\sigma)\\ &= -\Delta_X + (\tilde\sigma - \imath(n-1))\tilde\sigma = -\Delta_X + (\tilde\sigma - \imath(n-1)/2)^2 + (n-1)^2/4,\end{aligned}$$

which shows that it is useful to introduce $\sigma = \tilde\sigma - \imath(n-1)/2$, corresponding to the conjugation

$$\mathcal{M}_r r^{(n-1)/2} r^2\Box_{\tilde g} r^{-(n-1)/2}\mathcal{M}_r^{-1}(\sigma) = -\Delta_X + \sigma^2 + (n-1)^2/4.$$

We remark that (2.2) becomes

$$\begin{aligned}&\mathcal{M}_\rho \rho^2 \rho^{(n-1)/2}\Box_{\tilde g}\rho^{-(n-1)/2}\mathcal{M}_\rho^{-1}(\sigma)\\ &\qquad = F^{\imath\sigma-(n-1)/2-2}\mathcal{M}_r r^{(n-1)/2} r^2\Box_{\tilde g} r^{-(n-1)/2}\mathcal{M}_r^{-1}F^{-\imath\sigma+(n-1)/2}\\ &\qquad = F^{\imath\sigma-(n-1)/2-2}(-\Delta_X + \sigma^2 + (n-1)^2/4)F^{-\imath\sigma+(n-1)/2}.\end{aligned} \tag{2.4}$$

We now replace $\mathbb{H}^n$ with dS^n in our considerations. Thus, we work in the region in $\mathbb{S}^n$ corresponding to the exterior of the future and past light cones (the 'equatorial belt'), which can be identified with the hyperboloid

$$\mathrm{dS}^n: \ z_{n+1}^2 - (z_1^2 + \ldots + z_n^2) = -1,$$

via the $\mathbb{R}^+$-quotient. Now

$$\tilde g = -dr^2 + g,$$

where g is the de Sitter metric. We next consider the Mellin transform of $r^2\Box_{\tilde g}$ with respect to the decomposition $\mathbb{R}^+_r \times \mathrm{dS}^n$, to get

$$P_{\tilde\sigma} = \mathcal{M}_r r^2\Box_{\tilde g}\mathcal{M}_r^{-1} \in \mathrm{Diff}^2(\mathrm{dS}^n).$$

Note that

$$\Box_{\tilde g} = r^{-2}\Box_X + r^{-n}\partial_r r^n\partial_r,$$

in analogy with (2.3), so the Mellin transform of $r^{(n-1)/2}r^2\Box_{\tilde g}r^{-(n-1)/2}$ with respect to r is

$$\mathcal{M}_r r^{(n-1)/2} r^2\Box_{\tilde g} r^{-(n-1)/2}\mathcal{M}_r^{-1}(\sigma) = \Box_X - \sigma^2 - (n-1)^2/4.$$

We can relate this to the spherical Mellin transform by completely analogous arguments as in the case of $\mathbb{H}^n$, except that F is replaced by

$$\tilde F = \sqrt{\frac{|y|^2-1}{|y|^2+1}} = \sqrt{\frac{1-|y|^{-2}}{1+|y|^{-2}}}.$$

In principle this works only away from the equator (where one could use y as coordinates); to see that this in fact works globally, one should use Euclidean polar

[2] Lorentzian, but this does not affect these computations.

coordinates $|z'|$ and $\hat{y} = \frac{z'}{|z'|}$ in $\mathbb{R}^n_{z'}$, and use $|y|^{-1} = \frac{z_{n+1}}{|z'|}$ and $\hat{y}$ in $(-1,1) \times \mathbb{S}^{n-1}$; the second expression for $\tilde{F}$ now shows the desired smooth behavior on dS^n. Thus,

$$\begin{aligned} &\mathcal{M}_\rho \rho^{(n-1)/2} \rho^2 \Box_{\tilde{g}} \rho^{-(n-1)/2} \mathcal{M}_\rho^{-1}(\sigma) \\ &= \tilde{F}^{\imath\sigma-(n-1)/2-2} \mathcal{M}_r r^2 \Box_{\tilde{g}} \mathcal{M}_r^{-1} \tilde{F}^{-\imath\sigma+(n-1)/2} \\ &= \tilde{F}^{\imath\sigma-(n-1)/2-2} (\Box_X - \sigma^2 - (n-1)^2/4) \tilde{F}^{-\imath\sigma+(n-1)/2} \end{aligned} \tag{2.5}$$

3. Asymptotically Minkowski spaces

We now extend the results to the operators induced on the boundary at infinity of general asymptotically Minkowski spaces; we further show below how these spaces arise from asymptotically hyperbolic or de Sitter spaces in a natural way. Since for us it is the boundary behavior that matters (rather than the potentially complicated bicharacteristic flow in the interior), it is convenient to set this up as a homogeneous metric (of degree 2) on $\mathbb{R}^+ \times \tilde{X}$, where $\tilde{X}$ is a compact manifold; for general Lorentzian scattering metrics in the sense of [1] this is the model at the boundary of the compactified Lorentzian manifold (thus, we do not need the full Lorentzian scattering metric setup of [1]). Thus, as in [1], but using the product structure, consider Lorentzian metrics of the form

$$\tilde{g} = v \frac{d\tilde{\rho}^2}{\tilde{\rho}^4} - \Big(\frac{d\tilde{\rho}}{\tilde{\rho}^2} \otimes \frac{\alpha}{\tilde{\rho}} + \frac{\alpha}{\tilde{\rho}} \otimes \frac{d\tilde{\rho}}{\tilde{\rho}^2} \Big) - \frac{\check{g}}{\tilde{\rho}^2}$$

where $\tilde{\rho} = \rho^{-1}$ is the defining function of the boundary at infinity (so is homogeneous of degree -1), $v \in \mathcal{C}^\infty(\tilde{X})$, α a $\mathcal{C}^\infty$ one-form on $\tilde{X}$, $\alpha|_{v=0} = \frac{1}{2}\, dv$, $\check{g}$ a symmetric $\mathcal{C}^\infty$ 2-cotensor on $\tilde{X}$ which is positive definite on the annihilator of dv; in terms of ρ this takes the form

$$\tilde{g} = v\, d\rho^2 + \rho \Big(d\rho \otimes \alpha + \alpha \otimes d\rho \Big) - \rho^2 \check{g}. \tag{3.1}$$

Such a metric gives rise to an asymptotically hyperbolic manifold (with multiple connected components under the further assumptions we make below) in $v > 0$, and an asymptotically de-Sitter manifold in $v < 0$ (without the full dynamical hypotheses on these).

To see how the spectral family of the Laplacian, resp. the d'Alembertian, of an *even* metric $g = g_\bullet$ on $X = X_\bullet$ (with compactification $\overline{X_\bullet}$), fits into an asymptotically Minkowski framework, first consider the operator

$$P_\sigma = -\Delta_{X_\bullet} + \sigma^2 + \Big(\frac{n-1}{2} \Big)^2, \tag{3.2}$$

resp.

$$P_\sigma = \Box_{X_\bullet} - \sigma^2 - \Big(\frac{n-1}{2} \Big)^2, \tag{3.3}$$

on the space $X_\bullet$, where $\bullet$ denotes a subscript, such as $+$ or 0 below. With $\overline{X_{\bullet,\mathrm{even}}}$ the even version of $\overline{X_\bullet}$, and with $x_{X_\bullet}$ a boundary defining function of $\overline{X_\bullet}$, we modify this to the operator

$$\tilde{P}_\sigma|_{X_{\bullet,\mathrm{even}}} = x_{X_\bullet}^{\imath\sigma-(n-1)/2-2} P_\sigma x_{X_\bullet}^{-\imath\sigma+(n-1)/2}, \tag{3.4}$$

which one now checks is the restriction of an operator $\tilde{P}_\sigma$ defined on an extension $\tilde{X}$ of $X_{\bullet,\mathrm{even}}$ across $Y = \partial X_{\bullet,\mathrm{even}}$, and satisfying the requirements of [11] and [10]. This was checked explicitly in [10]. Note that at the level of the principal symbol, given by the dual metric *function*, this means that $x^{-2}G$ extends smoothly to $T^*\tilde{X}$, which is automatic for an even asymptotically hyperbolic metric. One does need to check the behavior of the lower order terms (which *would* be singular without the conjugation by $x_{X_\bullet}^{-\imath\sigma+(n-1)/2}$, while for the principal symbol the latter does not matter), but this was again done in [10].

A different way of proceeding is via extending the metric $g = g_\bullet$ to an ambient metric, playing the role of the Minkowski metric, which is homogeneous of degree 2. Thus, one considers $M = \mathbb{R}^+_\rho \times \tilde{X}$, as well as $\mathbb{R}^+_r \times X_\bullet$, with $\bullet = \pm$ for the asymptotically hyperbolic spaces, and with $r = x_{X_\pm}\rho$, so $F = x_{X_\pm}$ in the Minkowski setting. We note, however, that while with F defined above in the Minkowski setting, the hyperbolic metric has some higher order (in $x = x_{X_\pm}$) $dx^2 = dx^2_{X_\pm}$ terms in view of (2.1), these do not affect properties of the extension across $x_{X_\pm} = 0$. On $\mathbb{R}^+_r \times X_\bullet$ the analogue of the Minkowski metric is

$$\tilde{g} = dr^2 - r^2 g = r^2\Big(\frac{dr^2}{r^2} - g\Big) = \rho^2\Big(x^2_{X_\pm}\Big(\frac{d\rho}{\rho} + \frac{dx_{X_\pm}}{x_{X_\pm}}\Big)^2 - x^2_{X_\pm} g\Big).$$

Substituting the form of g and writing $x^2_{X_\pm} = \mu$,

$$\tilde{g} = \rho^2\Big(\mu\frac{d\rho^2}{\rho^2} + \frac{1}{2}\Big(\frac{d\rho}{\rho}\otimes d\mu + d\mu\otimes\frac{d\rho}{\rho}\Big) - h(\mu,\hat{y},d\hat{y})\Big). \tag{3.5}$$

But now the desired extension is immediate to a neighborhood of $X_{\bullet,\mathrm{even}}$ in $\tilde{X}$ (which is all that is required for the analysis if one uses complex absorption as in [11, 10, 9]), by simply extending h smoothly to a neighborhood (i.e. from $\mu \geq 0$ to μ near 0). This is easily checked to be Lorentzian, and indeed a special case[3] of the scattering metrics of [1] in view of (3.1). Notice that the metric in $\mu < 0$ takes the form, with $\mu = -x^2_{X_0}$,

$$\begin{aligned}\tilde{g} &= \rho^2\Big(-x^2_{X_0}\frac{d\rho^2}{\rho^2} - x^2_{X_0}\Big(\frac{d\rho}{\rho}\otimes\frac{dx_{X_0}}{x_{X_0}} + \frac{dx_{X_0}}{x_{X_0}}\otimes\frac{d\rho}{\rho}\Big) - h(-x^2_{X_0},\hat{y},d\hat{y})\Big)\\ &= \rho^2\Big(-x^2_{X_0}\Big(\frac{d\rho}{\rho} + \frac{dx_{X_0}}{x_{X_0}}\Big)^2 + x^2_{X_0} g_{X_0}\Big),\end{aligned}$$

with

$$g_{X_0} = \frac{dx^2_{X_0} - h(-x^2_{X_0},\hat{y},d\hat{y})}{x^2_{X_0}}, \tag{3.6}$$

i.e. g_{X_0} is asymptotically de Sitter, with cross-section metric given by $h(-x^2_{X_0},\hat{y},d\hat{y})$ rather than $h(x^2_{X_0},\hat{y},d\hat{y})$, i.e. it is the extension of h in the first argument across 0 that enters into g_{X_0}.

The analogous construction also works on asymptotically de Sitter spaces (X_0, g), $g = g_{X_0}$; one lets

$$\tilde{g} = -dr^2 + r^2 g = r^2\Big(-\frac{dr^2}{r^2} + g\Big) = \rho^2\Big(-x^2_{X_0}\Big(\frac{d\rho}{\rho} + \frac{dx_{X_0}}{x_{X_0}}\Big)^2 + x^2_{X_0} g\Big),$$

[3]This assumes that one ignores the interior of the space carrying a Lorentzian scattering metric; more precisely it is a special case of the restriction of a Lorentzian scattering metric to a neighborhood of the boundary of the compactification of the space.

which now gives, with $x_{X_0}^2 = -\mu$,

$$\tilde{g} = \rho^2\Big(\mu\frac{d\rho^2}{\rho^2} + \frac{1}{2}\Big(\frac{d\rho}{\rho}\otimes d\mu + d\mu\otimes\frac{d\rho}{\rho}\Big) - h(-\mu,\hat{y},d\hat{y})\Big), \tag{3.7}$$

which is the same formula as (3.5), except the appearance of $-\mu$ in the argument of h, corresponding to the relationship between g_{X_+} and g_{X_0} when one started with $g = g_{X_+}$, as expressed by (3.6).

Thus, suppose we have an asymptotically de Sitter metric on a manifold $(\overline{X_0}, g_{X_0})$ with two boundary hypersurfaces $Y_\pm$ and a family of metrics $\tilde{h}_\pm$ on $Y_\pm$ depending smoothly in an even fashion on the boundary defining function x_{X_0} (i.e. smoothly on $x_{X_0}^2$), and that $Y_\pm$ bound[4] manifolds with boundary $\overline{X_\pm}$. Then one can put an asymptotically hyperbolic metric $g_\pm$ of the form

$$\frac{dx_{X_\pm}^2 + h_\pm(-x_{X_\pm}^2,\hat{y},d\hat{y})}{x_{X_\pm}^2}$$

near $Y_\pm = \partial X_\pm$ (relative to a chosen product decomposition, with a factor $[0,\epsilon)_{x_{X_\pm}}$ corresponding to the boundary defining function $x_{X_\pm}$) on $\overline{X_\pm}$, and let $\mu = x_{X_\pm}^2$ on $\overline{X_\pm}$. Further, we define a compact manifold with boundary by

$$\tilde{X} = \overline{X_{+,\mathrm{even}}} \cup \overline{X_{0,\mathrm{even}}} \cup \overline{X_{-,\mathrm{even}}}, \tag{3.8}$$

with the summands smoothly identified at the boundaries using the product decomposition used in transferring the metric. Then we define a Lorentzian metric $\tilde{g}$ on $\mathbb{R}^+_\rho \times \tilde{X}$ by the respective form (3.5)-(3.7) with h understood as $x_{X_\pm}^2 g_\pm - dx_{X_\pm}^2$, resp. $-x_{X_0}^2 g_0 + dx_{X_0}^2$ away from a neighborhood of $Y_\pm$; these definitions extend smoothly and consistently to $\mu = 0$ (i.e. $\mathbb{R}^+ \times Y_\pm$).

Returning to the previous discussion, when we started out with $\overline{X_+}$, we can construct a global space $\tilde{X}$ by taking two copies of $\overline{X_+}$, denoting the second copy by $\overline{X_-}$, letting $Y_\pm = \partial\overline{X_\pm}$, and $\overline{X_0} = Y_+ \times [0,1]_s$, and defining $\tilde{X}$ as in (3.8), with the corresponding identifications. This defines asymptotically de Sitter metrics near the boundaries of $\overline{X_0}$. Using the product structure on $\overline{X_0}$ this can be extended to a Lorentzian metric on X_0 of a warped product form $f(s)\,ds^2 - h_0(s,\hat{y},d\hat{y})$ on $(0,1)_s \times Y_+$ with $f > 0$, h_0 positive definite; note that this matches the metric near $Y_\pm$ if h_0 is appropriately chosen, and all null-geodesics indeed tend to $Y_\pm$ as the parameter along them approaches infinity, so indeed this fits into the asymptotically de Sitter framework described in the introduction.

Now the Mellin transform of $\Box_{\tilde{g}}$ gives rise to a smooth family of operators $\tilde{P}_\sigma$ on $\tilde{X}$, related to P_σ in (3.2)-(3.3) via the same procedure as in the Minkowski setting. In summary, we have shown:

Proposition 3.1. *Given an even asymptotically hyperbolic space (X_+, g_{X_+}), resp. an even asymptotically de Sitter space (X_0, g_{X_0}), after possibly replacing (X_0, g_{X_0}) by two copies of the same space, there is a 'global' space $\tilde{X}$, of the form* (3.8) *with the not already given constituent pieces asymptotically hyperbolic in case of $(X_\pm, g_{X_\pm})$ and asymptotically de Sitter in case of (X_0, g_{X_0}), and there is an operator $\tilde{P}_\sigma \in \mathrm{Diff}^2(\tilde{X})$ on $\tilde{X}$, such that the restriction of $\tilde{P}_\sigma$ to $X_\pm$, resp. X_0, is given by* (3.4)*, with P_σ as in* (3.2)*, resp.* (3.3).

[4] If one starts with an $\overline{X_0}$ for which this is not the case, one can take two copies of it; the two copies of Y_+ bound now the manifold $Y_+ \times [0,1]$ and similarly with Y_-.

The requirements for the analysis of $\tilde{P}_\sigma$ in [11] involve the principal symbol globally as well as the imaginary part of the subprincipal symbol at $N^*Y_\pm$, with the latter entering since they determine the threshold regularity at radial points. Further, if one wants to obtain high energy estimates, letting $|\sigma|\to\infty$ in strips $|\operatorname{Im}\sigma|<C$, one also needs information on the principal symbol in the high energy/large parameter sense. Here we do not address the latter (it involves e.g. the non-trapping nature of the asymptotically hyperbolic spaces), but mention that these are encoded in the b-principal symbol of $\Box_{\tilde{g}}$ (which is the dual metric function), and indeed even the σ-dependence of the subprincipal symbol can be read off from the b-principal symbol of $\Box_{\tilde{g}}$.

The requirements on the principal symbol are satisfied in view of the limiting behavior of the null-geodesics on the asymptotically de Sitter space; apart from the behavior of the latter, the other requirements were all checked in [11, Section 4] and [10, Section 3]; the complex absorption added there is not needed as we regard one of the radial sets N^*Y_+ and N^*Y_- as the region from which we start propagating estimates, the other as the region towards which we propagate estimates, as was done in the recent work [1, Section 5]. Thus, what is left is finding the subprincipal symbol at $N^*Y_\pm$, and what is left in this is finding a σ-independent constant, which again, at most shifts by a constant what function spaces should be used in the Fredholm analysis. In turn, this constant can be found by formal self-adjointness considerations as it is the principal symbol of $\frac{1}{2\imath}(\tilde{P}_\sigma-\tilde{P}_\sigma^*)$ at the radial set. The latter vanishes for σ real, as $\rho^2\rho^{(n-1)/2}\Box_{\tilde{g}}\rho^{-(n-1)/2}$ is formally self-adjoint with respect to the $\mathbb{R}^+$-invariant b-density $\rho^{-(n+1)}\,d\tilde{g}$, hence the Mellin transform is formally self-adjoint for σ real with respect to a density ω on $\tilde{X}$ such that $\rho^{-(n+1)}\,d\tilde{g}=\frac{d\rho}{\rho}\omega$ (cf. [11, Section 3.3]). It is actually instructive to compute this subprincipal symbol (rather than just its imaginary part) at N^*Y, $Y=Y_+\cup Y_-$, cf. [9, Section 3] for the general setting of differential forms; one obtains that, with $\mathcal{V}_b(\tilde{X};Y)$ denoting set of vector fields on $\tilde{X}$ tangent to Y,

$$\mathcal{M}_\rho\rho^2\Box_{\tilde{g}}\mathcal{M}_\rho^{-1}=(4\partial_\mu\mu\partial_\mu-4(\imath\tilde{\sigma}+(n-1)/2)\partial_\mu)+Q,\ Q\in\mathcal{V}_b^2(\tilde{X};Y),$$

or

$$\begin{aligned}\tilde{P}_\sigma&=\mathcal{M}_\rho\rho^2\rho^{(n-1)/2}\Box_{\tilde{g}}\rho^{-(n-1)/2}\mathcal{M}_\rho^{-1}\\&=(4\partial_\mu\mu\partial_\mu-4\imath\sigma\partial_\mu)+Q,\ Q\in\mathcal{V}_b^2(\tilde{X};Y).\end{aligned}\tag{3.9}$$

This means $(\mu\pm\imath0)^{\imath\sigma}$ are approximate elements of the distributional kernel of $\tilde{P}_\sigma$ (in that they solve $\tilde{P}_\sigma u=0$ modulo two orders better, namely smooth multiples of $(\mu\pm\imath0)^{\imath\sigma}$, than a priori expected in view of the second order nature of $\tilde{P}_\sigma$: one order of gain comes from N^*Y being characteristic for the operator and $(\mu\pm\imath0)^{\imath\sigma}$ is conormal to this, but the second order gain encodes the correct behavior of the subprincipal symbol. Note that these distributions lie in H^s for $s<-\operatorname{Im}\sigma+1/2$. Since in our global problem we are interested in solutions of $\tilde{P}_\sigma u=f$ which are smooth at the future light cone, $Y_+=\partial_+X_0$, if f is smooth, we need to propagate estimates from $Y_+=\partial_+X_0$ to $Y_-=\partial_-X_0$, and thus we need to use Sobolev spaces which are stronger than the above threshold regularity, $-\operatorname{Im}\sigma+1/2$, at $Y_+=\partial_+X_0$, but are weaker than it at $Y_-=\partial_-X_0$. Thus, as in [1, Section 5], see also the Appendix of that paper, we need variable order Sobolev spaces H^s, where s is a $\mathcal{C}^\infty$ function on $S^*\tilde{X}$ (though in this case one can take it to be a function simply on $\tilde{X}$), corresponding to s_{past} of [1, Section 5], so

(i) $s|_{N^*\partial_+ X_0} > 1/2 - \operatorname{Im}\sigma$, constant near $N^*\partial_+ X_0$,
(ii) $s|_{N^*\partial_- X_0} < 1/2 - \operatorname{Im}\sigma$, constant near $N^*\partial_- X_0$,
(iii) s is monotone along the null-bicharacteristics (which all go from $N^*\partial_+ X_0$ to $N^*\partial_- X_0$ or vice versa).

Then the spaces for Fredholm analysis are

$$\tilde{P}_\sigma : \mathcal{X}^s \to \mathcal{Y}^{s-1},\ \mathcal{X}^s = \{u \in H^s :\ \tilde{P}_\sigma u \in H^{s-1}\},\ \mathcal{Y}^{s-1} = H^{s-1}, \tag{3.10}$$

thus $\tilde{P}^{-1}_{\sigma,\mathrm{past}} : \mathcal{Y}^{s-1} \to \mathcal{X}^s$ is a meromorphic Fredholm family; see [1, Section 5] for details. Here the subscript 'past' is added to denote the function spaces we are using, which amounts to propagating regularity towards the past, i.e. $\partial_- X_0$: reversing the roles of $\partial_+ X_0$ and $\partial_- X_0$ in the definition of the function spaces would result in the the future solution operator $\tilde{P}^{-1}_{\sigma,\mathrm{future}}$.

4. The global operator and the conformally compact spaces

The solution operator $\tilde{P}^{-1}_{\sigma,\mathrm{past}}$ considered above now gives the solution operator for the backward Cauchy problem for the spectral family of $\Box_{X_0}$ as well as the resolvent for $\Delta_{X_\pm}$. This connection has been explored in [11] and [10] in the asymptotically hyperbolic and de Sitter setting (the two setting considered separately), and in [1] in this generality (except that a compact M was taken satisfying various additional non-trapping conditions, but for the purposes of the discussion here the latter are irrelevant). Here we expand this discussion and include the Poisson operators and scattering operators in it; the latter enter in perhaps surprising ways.

Sometimes we write $x^{\pm}_{X_0}$ for the boundary defining function when we work near the future and past boundaries $\partial_\pm X_0$ of de Sitter space to emphasize the local nature of the expansion; these are understood to be equal to x_{X_0} near the relevant boundary $\partial_\pm X_0$. *Further, as the only smooth structure used below is the even one (corresponding to the restriction of the smooth structure of $\tilde{X}$), below $\mathcal{C}^\infty(\overline{X_\bullet})$ stands for $\mathcal{C}^\infty(\overline{X_{\bullet,\mathrm{even}}})$, $\bullet = +,-,0$, unless otherwise noted.*

To elaborate on the connection mentioned above, concretely one has, e.g. on $\mathcal{C}^\infty_c(X_+)$, for $\operatorname{Im}\sigma \gg 0$,

$$\begin{aligned}\mathcal{R}_{X_+}(\sigma) &= \Big(-\Delta_{X_+} + \sigma^2 + \Big(\frac{n-1}{2}\Big)^2\Big)^{-1}\\ &= x_{X_+}^{-\imath\sigma+(n-1)/2}\tilde{P}^{-1}_{\sigma,\mathrm{past}} x_{X_+}^{\imath\sigma-(n-1)/2-2},\end{aligned} \tag{4.1}$$

where the inverse on the left hand side is the inverse given by the essential self-adjointness (on $\mathcal{C}^\infty_c(X_+)$) and positivity of Δ_{X_+}. Notice that then the equality of the extreme left and right hand sides holds for all $\sigma \in \mathbb{C}$ as the equality of meromorphic families; alternatively, as in [10] the right hand side can be used to *define* the analytic continuation of the resolvent of Δ_{X_+}, i.e. $\mathcal{R}_{X_+}(\sigma)$. On the other hand, on $\mathcal{C}^\infty_c(X_0)$ the *backward*, or *past-oriented*, solution operator $\mathcal{R}_{X_0,\mathrm{past}}(\sigma)$ is given by

$$\begin{aligned}\mathcal{R}_{X_0,\mathrm{past}}(\sigma) &= \Big(\Box_{X_0} - \sigma^2 - \Big(\frac{n-1}{2}\Big)^2\Big)^{-1}\\ &= x_{X_0}^{-\imath\sigma+(n-1)/2}\tilde{P}^{-1}_{\sigma,\mathrm{past}} x_{X_0}^{\imath\sigma-(n-1)/2-2}.\end{aligned} \tag{4.2}$$

The former, (4.1), was extensively discussed in [11] and [10]: applied to $f \in \mathcal{C}^\infty_c(X_+)$, both sides give an element of $L^2(X_+, dg_+)$ when $\operatorname{Im}\sigma \gg 0$ since $\tilde{P}^{-1}_{\sigma,\mathrm{past}}$

maps into $\mathcal{C}^\infty(\overline{X_+})$, and in view of (2.4) both sides satisfy that $-\Delta_{X_+}+\sigma^2+\left(\frac{n-1}{2}\right)^2$ applied to them yields f; since there is a unique element of $L^2(X_+,dg_+)$ with this property, the claim follows.

To check the latter claim, (4.2), we first note that

$$f\in\mathcal{C}_c^\infty(X_0)\Rightarrow \operatorname{supp}\tilde P_{\sigma,\mathrm{past}}^{-1}x_{X_0}^{\imath\sigma-(n-1)/2-2}f\cap\overline{X_+}=\emptyset. \tag{4.3}$$

We give two different arguments for this. One is essentially a direct application of Proposition 3.9 of [11]. This proposition uses complex absorption, but in a way that makes the proof go through without changes in our setting: Q_σ enters there only to make the P_σ into a Fredholm family, which we have here through control of the global dynamics. The conclusion is that, using $-\mu$ as the time function $\mathfrak{t}$ of [11] near ∂_+X_0 (where it is time-like in X_0), $\tilde P_{\sigma,\mathrm{past}}^{-1}$ propagates supports forward in $\mathfrak{t}$, i.e. backwards in μ, giving the desired conclusion. For an alternative proof of (4.3) note that for $f\in\mathcal{C}_c^\infty(X_0)$, $x_{X_0}^{\imath\sigma-(n-1)/2-2}f$ vanishes in X_+. Thus, $\tilde P_{\sigma,\mathrm{past}}^{-1}x_{X_0}^{\imath\sigma-(n-1)/2-2}f$ also vanishes there since this restriction is given by $\mathcal{R}_{X_+}(\sigma)$ (the analytic continuation of the resolvent of Δ_{X_+}, with argument as in (4.1)) applied to the function 0 by what we have shown. But $\tilde P_{\sigma,\mathrm{past}}^{-1}x_{X_0}^{\imath\sigma-(n-1)/2-2}f$ is $\mathcal{C}^\infty$ near $\partial X_+=\partial_+X_0$ (the future boundary of de Sitter space), and thus the restriction to $\overline{X_0}$ vanishes to infinite order at ∂_+X_0, so the same remains true after multiplication by $x_{X_0}^{-\imath\sigma+(n-1)/2}$. Calling the result u, which thus satisfies $(\Box_{X_0}-\sigma^2-(\frac{n-1}{2})^2)u=f$, a slight modification of [12, Proposition 5.3] gives that (for f compactly supported in X_0) u vanishes identically near ∂_+X_0. The slight modification we are referring to is that as stated, [12, Proposition 5.3] applies only for real σ, but as the spectral variable is semiclassically two orders below the principal term, it does not affect the Carleman estimate argument presented there (it affects the error term R_2 in the proof by a term in $h^2\mathrm{Diff}_{0,h}^0(X_0)$ with the notation of that paper, which does not change the fact that R_2 is in the class stated there). Note that the notion of semiclassicality is very different in this Carleman estimate of [12] from that of [11] since it is semiclassicality with respect to an exponential conjugation parameter, not $|\sigma|^{-1}$. Returning to u, this proves that u is the backward solution for the de Sitter Klein-Gordon equation.

To complete the picture, consider also when f is supported in X_-. To be clear we write μ_- for its boundary defining function (which is positive in X_-), and we similarly write x_{X_-}, etc. Then by our argument thus far, $\tilde P_{\sigma,\mathrm{past}}^{-1}x_{X_-}^{\imath\sigma-(n-1)/2-2}f$ vanishes outside $\overline{X_-}$, i.e. is supported in $\overline{X_-}$. Further, just under the assumption that $f\in\mathcal{C}^\infty(\tilde X)$ (i.e. without support assumptions), $u=\tilde P_{\sigma,\mathrm{past}}^{-1}f$ has $\mathrm{WF}(u)\subset N^*\partial X_-$, and indeed has an expansion there, see [1, Corollary 6.9], namely if $\imath\sigma\notin\mathbb{Z}$ then

$$\begin{aligned} &u=v_{\tilde X,\mathrm{past}}^{+}+v_{\tilde X,\mathrm{past}}^{-}+v_{\tilde X,\mathrm{past}}^{0},\\ &v_{\tilde X,\mathrm{past}}^{\pm}=a_{\tilde X,\mathrm{past}}^{\pm}(\mu_-\pm\imath 0)^{\imath\sigma},\qquad a_{\tilde X,\mathrm{past}}^{\pm},v_{\tilde X,\mathrm{past}}^{0}\in\mathcal{C}^\infty(\tilde X). \end{aligned} \tag{4.4}$$

Note that there is a sign switch in [1, Corollary 6.9] in σ compared to the setting here; this is due to the use of a homogeneous degree 1 function in defining the Mellin transform here and its reciprocal, i.e. a homogeneous degree -1 function (thus a defining function of the boundary of the radial compactification of the space-time), being used in [1] to perform the Mellin transform. Also, if $\imath\sigma\in\mathbb{Z}$, logarithmic terms appear in the expression corresponding to the fact that $(\mu_- \pm \imath 0)^{\imath\sigma+k}$ is $\mathcal{C}^\infty$ if $\imath\sigma+k$ is a non-negative integer; this property of being $\mathcal{C}^\infty$ shows up as an obstacle in the construction of [1] for $k \geq 0$ integer, hence the restriction $\imath\sigma \notin \mathbb{Z}$ here (though the general case can also be treated). Again with $\imath\sigma \notin \mathbb{Z}$, the first two terms can be rewritten in terms of the distributions $(\mu_-)^{\imath\sigma}_{\pm}$, of which $(\mu_-)^{\imath\sigma}_{+}$ is supported in $\overline{X_-}$. Thus, for f supported in X_-, the fact that u is supported in $\overline{X_-}$ implies, apart from integer coincidences, that[5] $u = b(\mu_-)^{\imath\sigma}_{+}$. Correspondingly, $\tilde{u} = x_{X_-}^{-\imath\sigma+(n-1)/2} u|_{X_-}$ satisfies

$$\Big(-\Delta_{X_-} + \sigma^2 + \Big(\frac{n-1}{2}\Big)^2 \Big)\tilde{u} = f,$$

and

$$\tilde{u} = x_{X_-}^{\imath\sigma+(n-1)/2}\tilde{a},\ \tilde{a} \in \mathcal{C}^\infty(\overline{X_-}).$$

Now, for $\operatorname{Im}\sigma \ll 0$ this gives that

$$\mathcal{R}_{X_-}(-\sigma)f = x_{X_-}^{-\imath\sigma+(n-1)/2}\tilde{P}^{-1}_{\sigma,\mathrm{past}}x_{X_-}^{\imath\sigma-(n-1)/2-2}f; \tag{4.5}$$

this then holds in general in the sense of meromorphic Banach space valued operators, even near $\imath\sigma \in \mathbb{Z}$. Notice that the right hand side gives an independent way of analytically continuing $\mathcal{R}_{X_-}(-\sigma)$, similarly to how (4.1) gives the analytic continuation of $\mathcal{R}_{X_+}(\sigma)$ from $\operatorname{Im}\sigma \gg 0$. In summary, we have shown:

Proposition 4.1. *(See* [1, Proposition 7.3]*.) For any σ for which $\tilde{P}(\sigma)$ is invertible, the resolvents $\mathcal{R}_{X_+}(\sigma)$, $\mathcal{R}_{X_-}(-\sigma)$ and the backward solution operator $\mathcal{R}_{X_0,\mathrm{past}}(\sigma)$ are determined by $\tilde{P}(\sigma)$; in particular they are regular at these points.*

We want to have a converse result as well, namely that the poles of $\tilde{P}(\sigma)$ are a subset of poles associated to operators on $X_\pm$ and X_0 apart from possible issues when $\imath\sigma \in \mathbb{Z}$. In order to do this, it is useful to consider solution operators for the homogeneous PDE, i.e. where non-trivial boundary data are specified – these are the so-called Poisson operators. We recall that

$$\partial_+ X_0 = \partial X_+,\ \partial_- X_0 = \partial X_-.$$

First, given $a^{\pm}_{\tilde{X},0} \in \mathcal{C}^\infty(\partial X_0)$ and $\imath\sigma \notin \mathbb{Z}$ one can easily write down approximate solutions of the form

$$v^{\pm}_{\tilde{X}} = a^{\pm}_{\tilde{X}}(\mu \pm \imath 0)^{\imath\sigma},\ a^{\pm}_{\tilde{X}}|_{\partial X_0} = a^{\pm}_{\tilde{X},0},\ a^{\pm}_{\tilde{X}} \in \mathcal{C}^\infty(\tilde{X}), \tag{4.6}$$

i.e. such that

$$\tilde{P}_\sigma v^{\pm}_{\tilde{X}} \in \mathcal{C}^\infty(\tilde{X});$$

see [1, Lemma 6.4] for details (which in turn essentially follows [8]); the Taylor series of $a^{\pm}_{\tilde{X}}$ at ∂X_0 are determined by $a^{\pm}_{\tilde{X},0}$. Note that the Taylor series of $a^{\pm}_{\tilde{X}}$ at ∂X_0 is determined locally (in the strong sense that any Taylor coefficient depends only on finitely many derivatives of $a^{\pm}_{\tilde{X},0}$ evaluated at the same point), so in particular if $a^{\pm}_{\tilde{X},0}|_{\partial_- X_0} = 0$ then $a^{\pm}_{\tilde{X}}$ vanishes to infinite order at $\partial_- X_0$.

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[5] Indeed, the $(\mu_-)^{\imath\sigma}_{+}$ term has the desired support property, so one is reduced to observing that the sum of a $\mathcal{C}^\infty$ multiple, say ϕ, of $(\mu_-)^{\imath\sigma}_{-}$ and a $\mathcal{C}^\infty$ function, say ψ, is actually $\mathcal{C}^\infty$ if it is supported in $\overline{X_-}$, and thus can be written as a multiple (with vanishing derivatives at ∂X_-) of $(\mu_-)^{\imath\sigma}_{-}$. Indeed, if the sum is so supported, the mismatch in the powers of the Taylor series of ϕ and ψ at ∂X_- due to $\imath\sigma$ non-integral shows that both Taylor series vanish at ∂X_-, so the summands are in fact both $\mathcal{C}^\infty$, and thus so is the sum, as desired.

Similarly, purely from the perspective of $X_\pm$ and X_0, given $a^\pm_{X_\pm,0} \in \mathcal{C}^\infty(\partial X_\pm)$, $a^\pm_{X_0,0} \in \mathcal{C}^\infty(\partial X_0)$ one can construct

$$\begin{aligned} v^\pm_{X_\pm} &= a^\pm_{X_\pm} x_{X_\pm}^{(n-1)/2\pm\imath\sigma}, \qquad a^\pm_{X_\pm}|_{\partial X_\pm} = a^\pm_{X_\pm,0},\ a^\pm_{X_\pm} \in \mathcal{C}^\infty(\overline{X_\pm}),\\ v^\pm_{X_0} &= a^\pm_{X_0} x_{X_0}^{(n-1)/2\pm\imath\sigma}, \qquad a^\pm_{X_0}|_{\partial X_0} = a^\pm_{X_0,0},\ a^\pm_{X_0} \in \mathcal{C}^\infty(\overline{X_0}),\end{aligned} \tag{4.7}$$

with

$$\begin{aligned}\Big(-\Delta_{X_\pm} + \sigma^2 + \Big(\frac{n-1}{2}\Big)^2\Big) v^\pm_{X_\pm} &= f^\pm_{X_\pm} \in \dot{\mathcal{C}}^\infty(\overline{X_\pm}),\\ \Big(\Box_{X_0} - \sigma^2 - \Big(\frac{n-1}{2}\Big)^2\Big) v^\pm_{X_0} &= f^\pm_{X_0} \in \dot{\mathcal{C}}^\infty(\overline{X_0}).\end{aligned}$$

Note the distinction: while on $\tilde{X}$ 'trivial' or 'residual' functions are those in $\mathcal{C}^\infty(\tilde{X})$ (with no vanishing specified anywhere), on $X_\bullet$ they are those in $\dot{\mathcal{C}}^\infty(X_\bullet)$ (i.e. with infinite order vanishing at the boundary).

We make the following observation:

Lemma 4.2. *Regarded as smooth functions on $\overline{X_+}$, resp. $\overline{X_0}$ (with the even structure, i.e. of $\mu = x^2_{X_+}$, resp. $-\mu = x^2_{X_0}$ rather than x_{X_+} and x_{X_0}), at $\partial X_+ = \partial_+ X_0$, if $a^\pm_{X_+,0} = a^\pm_{X_0,0}$ then $a^\pm_{X_+}$ and $a^\pm_{X_0}$ have the matching Taylor series as functions in $\mu \geq 0$, resp. $\mu \leq 0$ (i.e. the even coefficients are the same, the odd coefficients have opposite signs).*

Note that X_+ can be replaced by X_- in this lemma.

Proof. We consider $a^-_{X_+,0}$ and $a^-_{X_0,0}$. We notice that in view of the (modified) conjugation relating $\tilde{P}_\sigma$ to $-\Delta_{X_+} + \sigma^2 + (n-1)^2/4$ on the one hand and $\Box_{X_0} - \sigma^2 - (n-1)^2/4$ on the other, these both solve $\tilde{P}_\sigma|_{\overline{X_+}} a^-_{X_+,0} = 0$ and $\tilde{P}_\sigma|_{\overline{X_0}} a^-_{X_0,0} = 0$ in Taylor series at $\partial_+ X_0 = \partial X_+$. Since the form (3.9) of $\tilde{P}_\sigma$ shows that the Taylor series of $\mathcal{C}^\infty$ functions in the approximate nullspace (modulo functions vanishing to infinite order at ∂X_+) of $\tilde{P}_\sigma$ is determined[6] by the restriction to ∂X_+, the result follows. For $a^-_{X_+,0}$ and $a^-_{X_0,0}$ the result follows by considering $\tilde{P}_{-\sigma}$ in place of $\tilde{P}_\sigma$. □

We can now define the Poisson operators:

Proposition 4.3. *(See* [6, Section 1] *for an explicit statement, and also* [7]*.) Suppose $\imath\sigma \notin \mathbb{Z}$, and σ is not a pole of $\mathcal{R}_{X_+}$. Given $b^+_{X_+,0} \in \mathcal{C}^\infty(\partial X_+)$ there is a solution u_{X_+} of*

$$\Big(-\Delta_{X_+} + \sigma^2 + \Big(\frac{n-1}{2}\Big)^2\Big) u_{X_+} = 0$$

with $u_{X_+} = v^+_{X_+} + v^-_{X_+}$, $v^\pm_{X_+}$ of the form (4.7)*, with $a^+_{X_+,0} = b^+_{X_+,0}$.*

Further, a solution u_{X_+} of this form is unique provided $\imath\sigma \notin \mathbb{Z}$ and $\sigma^2 + \left(\frac{n-1}{2}\right)^2$ is not an L^2-eigenvalue of Δ_{X_+}.

Remark 4.4. Note that $a^-_{X_+,0}$, i.e. the renormalized boundary value of $v^-_{X_+}$, is not specified.

[6] As $\mu^j \mathcal{C}^\infty(\tilde{X})$ is mapped to $\mu^{j-1}\mathcal{C}^\infty(\tilde{X})$ for $j \geq 1$ integer by $\tilde{P}_\sigma$, with $\tilde{P}_\sigma(\mu^j b) - j(j - \imath\sigma) b \mu^{j-1} \in \mu^j \mathcal{C}^\infty(\tilde{X})$, the claim follows by induction, noting that $j(j - \imath\sigma)$ cannot vanish when $j \geq 1$ is an integer as $\imath\sigma$ is not an integer.

Definition 4.5. The Poisson operator $\mathcal{P}_{X_+}(\sigma) : \mathcal{C}^\infty(\partial X_+) \to \mathcal{C}^{-\infty}(\overline{X_+})$ is defined as the meromorphic map $b^+_{X_+,0} \mapsto u_{X_+}$ for $\imath\sigma \notin \mathbb{Z}$.

The scattering matrix on X_+ is the operator $\mathcal{S}_{X_+}(\sigma) : \mathcal{C}^\infty(\partial X_+) \to \mathcal{C}^\infty(\partial X_+)$ given by $\mathcal{S}_{X_+}(\sigma) : b^+_{X_+,0} = a^+_{X_+,0} \mapsto a^-_{X_+,0}$ with the notation of the proposition and (4.7).

Remark 4.6. We could define $\mathcal{P}^-_{X_+}(\sigma)$ similarly, in which $a^-_{X_+,0}$ is specified in place of $a^+_{X_+,0}$, but this is just $\mathcal{P}_{X_+}(-\sigma)$ as reversing the sign of σ interchanges the two functions $v^\pm_{X_+}$. In particular, this gives $\mathcal{S}_{X_+}(\sigma) = \mathcal{P}_{X_+}(-\sigma)^{-1}\mathcal{P}_{X_+}(\sigma)$.

Proof. While this result is stated in [6], we give a summary of the argument.

For existence, u_{X_+} is given by first constructing $v^+_{X_+}$ as above from $a^+_{X_+,0}$, and then for σ not a pole of $\mathcal{R}_{X_+}$,

$$u_{X_+} = v^+_{X_+} - \mathcal{R}_{X_+}(\sigma) f^+_{X_+},$$

with the second term of the form $v^-_{X_+}$ indeed.

Now consider uniqueness. The difference of two such u_{X_+} is of the form $v^-_{X_+}$ necessarily since the leading coefficient $a^+_{X_+,0}$ determines the full Taylor series of $a^+_{X_+}$ (taking into account the evenness of the Taylor series in terms of x_{X_+} to separate $v^+_{X_+}$ and $v^-_{X_+}$). If $\operatorname{Im}\sigma > 0$ and $\sigma^2 + (n-1)^2/4$ is not an L^2-eigenvalue of Δ_{X_+}, uniqueness follows since $v^-_{X_+}$ is then in $H^2_0(\overline{X_+})$ (understood relative to the non-even, i.e. standard, smooth structure). In general one can show by a pairing argument, see [6], which in turn follows [8] that in fact the leading coefficient $a^-_{X_+,0}$ vanishes and then in fact $v^-_{X_+}$ is in $\dot{\mathcal{C}}^\infty(\overline{X_+})$, and then one can finish the argument as above. $\square$

We can analogously define a Poisson operator for X_0 at $\partial_+ X_0$, but here we specify both $a^\pm_{X_0,0}|_{\partial_+ X_0}$:

Proposition 4.7. *(See* [12, Theorem 5.5]*.) Suppose $\sigma \notin \imath\mathbb{Z}$. Given $b^\pm_{X_0,0} \in \mathcal{C}^\infty(\partial_+ X_0)$ there is a unique solution u_{X_0} of*

$$\Big(\Box_{X_0} - \sigma^2 - \Big(\frac{n-1}{2}\Big)^2\Big) u_{X_0} = 0$$

with $u_{X_0} = v^+_{X_0} + v^-_{X_0}$, $v^\pm_{X_0}$ of the form (4.7)*, with $a^\pm_{X_0}|_{\partial_+ X_0} = b^\pm_{X_0,0}$.*

Remark 4.8. Note that there are two boundary hypersurfaces of X_0; we are specifying both pieces of data $a^\pm_{X_0,0}$ at $\partial_+ X_0$ and neither of them at $\partial_- X_0$.

Also, in [12] only σ^2 real was considered, but allowing general $\sigma \in \mathbb{C}$ causes only minimal changes to the arguments. See also the remarks following (4.3) in this regard.

Proof. For existence, with $v^+_{X_0}, v^-_{X_0}$ as in (4.7) corresponding to $a^\pm_{X_0,0}|_{\partial_+ X_0} = b^\pm_{X_0,0}$ and $a^\pm_{X_0,0}|_{\partial_- X_0} = 0$, let

$$u_{X_0} = v^+_{X_0} + v^-_{X_0} - \mathcal{R}_{X_0,\mathrm{past}}(\sigma)(f^+_{X_0} + f^-_{X_0}),$$

with the inverse being the backward solution of the wave equation; this has all the desired properties as shown in [12]. Uniqueness follows since the homogeneous PDE has no solutions which vanish to infinite order at $\partial_+ X_0$ as shown in [12]. $\square$

Definition 4.9. The backward Poisson operator

$$\mathcal{P}_{X_0,\mathrm{past}}(\sigma):\mathcal{C}^\infty(\partial_+X_0)\oplus\mathcal{C}^\infty(\partial_+X_0)\to\mathcal{C}^{-\infty}(\overline{X_0})$$

is given by $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)(b^+_{X_0,0},b^-_{X_0,0})=u_{X_0}$ in the notation of the proposition, while the scattering matrix

$$\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma):\mathcal{C}^\infty(\partial_+X_0)\oplus\mathcal{C}^\infty(\partial_+X_0)\to\mathcal{C}^\infty(\partial_-X_0)\oplus\mathcal{C}^\infty(\partial_-X_0)$$

is given by

$$\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)(b^+_{X_0,0},b^-_{X_0,0})=(a^+_{X_0,0}|_{\partial_-X_0},a^-_{X_0,0}|_{\partial_-X_0}).$$

Remark 4.10. Here the index 'past' of $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$ denotes that we are solving the equation backwards, from ∂_+X_0 to ∂_-X_0. The forward Poisson operator $\mathcal{P}_{X_0,\mathrm{future}}(\sigma)$ is defined similarly, with the data $(a^+_{X_0,0}|_{\partial_-X_0},a^-_{X_0,0}|_{\partial_-X_0})$ specified.

We also remark that replacing σ by $-\sigma$ simply switches the two pieces of data $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$ is applied to, i.e. if J is this exchange operator then $\mathcal{P}_{X_0,\mathrm{past}}(-\sigma)=\mathcal{P}_{X_0,\mathrm{past}}(\sigma)J$. This is in contrast to the asymptotically hyperbolic space, in which $\mathcal{P}_{X_+}(\sigma)$ and $\mathcal{P}_{X_+}(-\sigma)$ are related by the much more complicated scattering matrix $\mathcal{S}_{X_+}(\sigma)$: $\mathcal{P}_{X_+}(-\sigma)\mathcal{S}_{X_+}(\sigma)=\mathcal{P}_{X_+}(\sigma)$.

Finally, we also have a Poisson operator for the Mellin transformed global operator, specifying both $a^\pm_{X_0,0}|_{\partial_+X_0}$ again:

Proposition 4.11. *Suppose $\tilde P_\sigma$ is invertible as a map* (3.10)*. Then given $b^\pm_{\tilde X,0}\in\mathcal{C}^\infty(\partial_+X_0)$ there is a unique solution u of*

$$\tilde P_\sigma u=0$$

with

$$u_{\tilde X}=v^+_{\tilde X}+v^-_{\tilde X}+v^0_{\tilde X},\tag{4.8}$$

with $v^\pm_{\tilde X}$ of the form (4.6)*, with $a^\pm_{\tilde X,0}|_{\partial_+X_0}=b^\pm_{\tilde X,0}$, and with $v^0_{\tilde X}\in\mathcal{C}^\infty(\tilde X)$.*

Proof. Again, we let $v^\pm_{\tilde X}$ be as above with $a^\pm_{\tilde X,0}|_{\partial_+X_0}=b^\pm_{\tilde X,0}$, $a^\pm_{\tilde X,0}|_{\partial_-X_0}=0$ (so $v^\pm_{\tilde X}$ is $\mathcal{C}^\infty$ at ∂_-X_0), and then

$$u_{\tilde X}=v^+_{\tilde X}+v^-_{\tilde X}-\tilde P^{-1}_{\sigma,\mathrm{past}}(f^+_{\tilde X}+f^-_{\tilde X}),$$

is the unique distributional solution of $\tilde P_\sigma u=0$ with $u_{\tilde X}-(v^+_{\tilde X}+v^-_{\tilde X})$ having wave front set disjoint from $N^*\partial X_+$ (which properties would hold for any u of the desired form, thus giving uniqueness). Further, $u_{\tilde X}-(v^+_{\tilde X}+v^-_{\tilde X})$ has wave front set in $N^*\partial X_-$, and indeed its structure given by (4.4) at ∂X_-, so $u_{\tilde X}$ has the decomposition claimed in the proposition. □

Definition 4.12. The backward Poisson operator

$$\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma):\mathcal{C}^\infty(\partial_+X_0)\oplus\mathcal{C}^\infty(\partial_+X_0)\to\mathcal{C}^{-\infty}(\tilde X)$$

is given by $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)(b^+_{\tilde X,0},b^-_{\tilde X,0})=u_{\tilde X}$ in the notation of the proposition, while the scattering matrix

$$\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma):\mathcal{C}^\infty(\partial_+X_0)\oplus\mathcal{C}^\infty(\partial_+X_0)\to\mathcal{C}^\infty(\partial_-X_0)\oplus\mathcal{C}^\infty(\partial_-X_0)$$

is given by

$$\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)(b^+_{\tilde X,0},b^-_{\tilde X,0})=(a^+_{\tilde X,0}|_{\partial_-X_0},a^-_{\tilde X,0}|_{\partial_-X_0}).$$

We now work out the relationships between these operators. Thus, let

$$u_{\tilde{X}} = \mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)(b^+_{\tilde{X},0}, b^-_{\tilde{X},0}).$$

Keeping in mind that $\mu = x^2_{X_+}$, in view of (4.8), $u_{X_+} = x_{X_+}^{-\imath\sigma+(n-1)/2} u_{\tilde{X}}|_{X_+}$ satisfies

$$\Big(-\Delta_{X_+} + \sigma^2 + \Big(\frac{n-1}{2}\Big)^2\Big) u_{X_+} = 0,$$

with $u_{X_+} = v^+_{X_+} + v^-_{X_+}$,

$$v^\pm_{X_+} = a^\pm_{X_+} x_{X_+}^{(n-1)/2\pm\imath\sigma},\ a^\pm_{X_+} \in \mathcal{C}^\infty(\overline{X_+}),$$

with

$$a^+_{X_+}|_{\partial X_+} = b^+_{\tilde{X},0} + b^-_{\tilde{X},0}$$

since the distribution $(\mu \pm \imath 0)^s$ restricted to $\mu > 0$ is just the function μ^s, and with

$$a^-_{X_+}|_{\partial X_+} = v^0_{\tilde{X}}|_{\partial X_+}. \tag{4.9}$$

Correspondingly,

$$\Big(x_{X_+}^{-\imath\sigma+(n-1)/2} \mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)(b^+_{\tilde{X},0}, b^-_{\tilde{X},0})\Big)\Big|_{X_+} = u_{X_+} = \mathcal{P}_{X_+}(\sigma)(b^+_{\tilde{X},0} + b^-_{\tilde{X},0}). \tag{4.10}$$

As an aside, this means that

$$\mathcal{P}_{X_+}(\sigma)(b^+_{X_+,0}) = x_{X_+}^{-\imath\sigma+(n-1)/2} (\mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)(b^+_{X_+,0}, 0))|_{X_+}, \tag{4.11}$$

and one could equally well use $(0, b^+_{X_+,0})$ as the data for $\mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)$. Returning to $u_{X_+} = v^+_{X_+} + v^-_{X_+}$, we can now identify $a^-_{X_+}|_{\partial X_+}$ in terms of the scattering matrix on X_+:

$$a^-_{X_+}|_{\partial X_+} = \mathcal{S}_{X_+}(\sigma)(a^+_{X_+}|_{\partial X_+}) = \mathcal{S}_{X_+}(\sigma)(b^+_{\tilde{X},0} + b^-_{\tilde{X},0}). \tag{4.12}$$

Thus, switching to the de Sitter side, with $u_{\tilde{X}} = \mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)(b^+_{\tilde{X},0}, b^-_{\tilde{X},0})$ still, with $\mu = -x^2_{X_0}$ now, $u_{X_0} = x_{X_0}^{-\imath\sigma+(n-1)/2} u_{\tilde{X}}|_{X_0}$ satisfies

$$\Big(\Box_{X_0} - \sigma^2 - \Big(\frac{n-1}{2}\Big)^2\Big) u_{X_0} = 0,$$

with $u_{X_0} = v^+_{X_0} + v^-_{X_0}$,

$$v^\pm_{X_0} = a^\pm_{X_0} x_{X_0}^{(n-1)/2\pm\imath\sigma},\ a^\pm_{X_0} \in \mathcal{C}^\infty(\overline{X_0}),$$

with

$$a^+_{X_0}|_{\partial X_+} = e^{-\pi\sigma} b^+_{\tilde{X},0} + e^{\pi\sigma} b^-_{\tilde{X},0}$$

since the distribution $(\mu \pm \imath 0)^s$ restricted to $\mu < 0$ is just the function $e^{\pm\imath\pi s}|\mu|^s = e^{\pm\imath\pi s} x^{2s}_{X_0}$, and with

$$a^-_{X_0}|_{\partial X_+} = v^0_{\tilde{X}}|_{\partial_+ X_0} = \mathcal{S}_{X_+}(\sigma)(b^+_{\tilde{X},0} + b^-_{\tilde{X},0})$$

in view of (4.8) for the first equality and (4.9) and (4.12) for the second. Correspondingly,

$$\begin{aligned} &x_{X_0}^{-\imath\sigma+(n-1)/2} \mathcal{P}_{\tilde{X},\mathrm{past}}(\sigma)(b^+_{\tilde{X},0}, b^-_{\tilde{X},0})|_{X_0} \\ &\quad = u_{X_0} = \mathcal{P}_{X_0,\mathrm{past}}(\sigma)(e^{-\pi\sigma} b^+_{\tilde{X},0} + e^{\pi\sigma} b^-_{\tilde{X},0}, \mathcal{S}_{X_+}(\sigma)(b^+_{\tilde{X},0} + b^-_{\tilde{X},0})). \end{aligned}$$

Thus,

$$\mathcal{P}_{X_0,\mathrm{past}}(\sigma)(b^+_{X_0,0},b^-_{X_0,0})=x_{X_0}^{-\imath\sigma+(n-1)/2}\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)(b^+_{\tilde X,0},b^-_{\tilde X,0})|_{X_0} \tag{4.13}$$

with

$$\begin{bmatrix} b^+_{\tilde X,0}\\ b^-_{\tilde X,0}\end{bmatrix}=\begin{bmatrix} e^{-\pi\sigma} & e^{\pi\sigma}\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix} b^+_{X_0,0}\\ \mathcal{S}_{X_+}(\sigma)^{-1}b^-_{X_0,0}\end{bmatrix}, \tag{4.14}$$

assuming $\mathcal{S}_{X_+}(\sigma)$ is invertible and $\sigma\notin\imath\mathbb{Z}$ so that the matrix itself is invertible.

Finally we can turn to X_-. As recalled above, near $\partial_-X_0=\partial X_-$,

$$\begin{aligned} &u_{\tilde X}=v^+_{\tilde X,\mathrm{past}}+v^-_{\tilde X,\mathrm{past}}+v^0_{\tilde X},\\ &v^\pm_{\tilde X,\mathrm{past}}=a^\pm_{\tilde X,\mathrm{past}}(\mu_-\pm\imath 0)^{\imath\sigma},\qquad a^\pm_{\tilde X,\mathrm{past}}|_{\partial X_-}=a^\pm_{\tilde X,\mathrm{past},0},\\ &a^\pm_{\tilde X,\mathrm{past}},v^\pm_{\tilde X,\mathrm{past}}\in\mathcal{C}^\infty(X_0\cup\overline{X_-}).\end{aligned}$$

Thus, $u_{X_0}=x_{X_0}^{-\imath\sigma+(n-1)/2}u_{\tilde X}|_{X_0}$ has asymptotic expansion at ∂X_- given by

$$\begin{aligned} &u_{X_0}=v^+_{X_0,\mathrm{past}}+v^-_{X_0,\mathrm{past}},\\ &v^\pm_{X_0,\mathrm{past}}=(x^-_{X_0})^{(n-1)/2\pm\imath\sigma}a^\pm_{X_0,\mathrm{past}},\qquad a^\pm_{X_0,\mathrm{past}}\in\mathcal{C}^\infty(\overline{X_0}),\end{aligned}$$

and

$$a^+_{X_0,\mathrm{past}}|_{\partial X_-}=e^{-\pi\sigma}a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}+e^{\pi\sigma}a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0},\ a^-_{X_0,\mathrm{past}}|_{\partial_-X_0}=v^0_{\tilde X}|_{\partial_-X_0}. \tag{4.15}$$

Correspondingly,

$$\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\begin{bmatrix} b^+_{X_0,0}\\ b^-_{X_0,0}\end{bmatrix}=\begin{bmatrix} e^{-\pi\sigma}a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}+e^{\pi\sigma}a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0}\\ v^0_{\tilde X}|_{\partial_-X_0}\end{bmatrix}, \tag{4.16}$$

with $a^\pm_{\tilde X,0}$ and $b^\pm_{X_0,0}$ related as in (4.13)-(4.14).

Now, in X_- the resolvent is in the dual regime relative to that of the X_+ problem (cf. the appearance of $-\sigma$ vs. σ in the argument of the resolvents in Proposition 4.1), namely $u_{X_-}=(x_{X_-})^{-\imath\sigma+(n-1)/2}u_{\tilde X}|_{X_-}$ solves

$$\Big(-\Delta_{X_-}+\sigma^2+\Big(\frac{n-1}{2}\Big)^2\Big)u_{X_-}=0,$$

with asymptotics

$$\begin{aligned} &u_{X_-}=v^+_{X_-}+v^-_{X_-},\\ &v^\pm_{X_-}=(x_{X_-})^{(n-1)/2\pm\imath\sigma}a^\pm_{X_-},\qquad a^\pm_{X_-}\in\mathcal{C}^\infty(\overline{X_-}),\end{aligned}$$

and

$$a^+_{X_-}|_{\partial X_-}=a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}+a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0},\qquad a^-_{X_-}|_{\partial_-X_0}=v^0_{\tilde X}|_{\partial_-X_0}.$$

Thus, much as in the case of the resolvent considered first above, except using $\mathcal{P}_{X_-}(-\sigma)$, so the coefficient of $x_{X_-}^{(n-1)/2-\imath\sigma}$, namely $v_0^-|_{\partial X_-}$, is the input,

$$\begin{aligned}\mathcal{P}_{X_-}(-\sigma)(v_0^-|_{\partial X_-})=u_{X_-}&=(x_{X_-})^{-\imath\sigma+(n-1)/2}u_{\tilde X}|_{X_-}\\ &=(x_{X_-})^{-\imath\sigma+(n-1)/2}\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)(a^+_{\tilde X,0},a^-_{\tilde X,0})|_{X_-},\end{aligned} \tag{4.17}$$

and

$$\mathcal{S}_{X_-}(-\sigma)v^-_{X_0}|_{\partial X_-}=a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}+a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0}.$$

Thus, using (4.15),

$$\begin{bmatrix}\mathrm{Id} & 0\\ 0 & \mathcal{S}_{X_-}(-\sigma)\end{bmatrix}\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\begin{bmatrix}b^+_{X_0,0}\\ b^-_{X_0,0}\end{bmatrix}=\begin{bmatrix}e^{-\pi\sigma} & e^{\pi\sigma}\\ 1 & 1\end{bmatrix}\begin{bmatrix}a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}\\ a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0}\end{bmatrix}.$$

Combining this with (4.14) we have shown

Theorem 4.13. *For $\sigma\notin\imath\mathbb{Z}$, if σ is not a pole of $\tilde P^{-1}_{\sigma,\mathrm{past}}$ then the global Poisson operator $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ on $\tilde X$ determines those of $X_\pm$ and X_0, $\mathcal{P}_{X_+}(\sigma)$, $\mathcal{P}_{X_-}(-\sigma)$ and $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$, and conversely, $\mathcal{P}_{X_+}(\sigma)$, $\mathcal{P}_{X_-}(-\sigma)$ and $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$ determine the global Poisson operator $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$.*

Furthermore, for σ as above,

$$\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)(b^+_{\tilde X,0},b^-_{\tilde X,0})=\begin{bmatrix}a^+_{\tilde X,\mathrm{past}}|_{\partial_-X_0}\\ a^-_{\tilde X,\mathrm{past}}|_{\partial_-X_0}\end{bmatrix}$$
$$=\begin{bmatrix}e^{-\pi\sigma} & e^{\pi\sigma}\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}\mathrm{Id} & 0\\ 0 & \mathcal{S}_{X_-}(-\sigma)\end{bmatrix}\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\begin{bmatrix}\mathrm{Id} & 0\\ 0 & \mathcal{S}_{X_+}(\sigma)\end{bmatrix}\begin{bmatrix}e^{-\pi\sigma} & e^{\pi\sigma}\\ 1 & 1\end{bmatrix}\begin{bmatrix}b^+_{\tilde X,0}\\ b^-_{\tilde X,0}\end{bmatrix},$$

i.e. $\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)$ is essentially the product of $\mathcal{S}_{X_\pm}(\pm\sigma)$ and $\mathcal{S}_{X_0,\mathrm{past}}(\sigma)$.

Proof. First (4.11) shows that the global Poisson operator $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ determines $\mathcal{P}_{X_+}(\sigma)$, and in particular $\mathcal{S}_{X_+}(\sigma)$. Next, (4.13) that $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ determines $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$, and in particular $\mathcal{S}_{X_0,\mathrm{past}}(\sigma)$. Finally, (4.17) combined with (4.16) show that $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ determines $\mathcal{P}_{X_-}(\sigma)$.

For the converse, (4.10) shows that $\mathcal{P}_{X_+}(\sigma)$ determines the restriction of $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ to X_+, and in particular the Cauchy data at future infinity, $(b^+_{X_0,0},b^-_{X_0,0})$, for the de Sitter problem. Then (4.13) shows that the restriction of $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ to X_0 is determined, and in particular the data for $\mathcal{P}_{X_-}(-\sigma)$ is determined. Then (4.17) shows that the restriction of $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ to X_- is determined. Since $\imath\sigma$ is not a negative integer, the form (4.4) shows that these restrictions determine $\mathcal{P}_{\tilde X,\mathrm{past}}(\sigma)$ since there cannot be solutions of the homogeneous equation supported at $\partial X_\pm$. □

Now, $\mathcal{S}_{X_\pm}(\sigma)$ are elliptic pseudodifferential operators of (complex) order $-2\imath\sigma$, as shown by Joshi and Sá Barreto[7], [6], so $\mathcal{S}_{X_-}(-\sigma)$ has order $2\imath\sigma$. In particular, if $\Delta_{\partial X_\pm}$ is the Laplacian of a metric on $\partial X_\pm$, say of (a representative of the conformal class of) the conformal metric h, then

$$(\Delta'_{\partial X_+})^{\imath\sigma}\mathcal{S}_{X_+}(\sigma),\ \mathcal{S}_{X_-}(-\sigma)(\Delta'_{\partial X_-})^{-\imath\sigma} \tag{4.18}$$

are pseudodifferential operators of order 0, where $\Delta'_{\partial X_+}$ is the operator that is $\Delta_{\partial X_+}$ on the orthocomplement of the nullspace of $\Delta_{\partial X_+}$ and the identity on the nullspace.[8] Further, $\mathcal{S}_{X_0,\mathrm{past}}(\sigma)$ is an elliptic Fourier integral operator associated

[7]Note that Joshi and Sá Barreto use the spectral parameter $-\zeta(n-1-\zeta)$, with our notation for the dimension of X, with $\mathrm{Re}\,\zeta>(n-1)/2$ being the physical half plane, corresponding to our $\sigma^2+(n-1)^2/4$ with $\mathrm{Im}\,\sigma>0$ being the physical half plane, so $\sigma=\imath(\zeta-(n-1)/2)$ is the conversion between the two parameterizations.

[8]Other second order positive elliptic operators, bounded below by a positive constant, would do equally well; with the choice of $\Delta'_{\partial X_+}$, the principal symbol of the 0th order operators in (4.18) is a constant c_σ, resp. $c_{-\sigma}$, dependent on σ only via powers of 2 and the Γ-function, see [6, Theorem 1.1] and with $c_\sigma c_{-\sigma}=1$.

to the backward null-geodesic flow from ∂_+X_0 to ∂_-X_0 as shown by the author[9] in [12], with the property that

$$\big((\Delta'_{\partial_-X_0})^{-s_-(\lambda)/2+n/4}\oplus(\Delta'_{\partial_-X_0})^{-s_+(\lambda)/2+n/4}\big)\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\\ \big((\Delta'_{\partial_+X_0})^{s_-(\lambda)/2-n/4}\oplus(\Delta'_{\partial_+X_0})^{s_+(\lambda)/2-n/4}\big)$$

is a Fourier integral operator of order 0, where the spectral parameter is $\lambda=\sigma^2+(n-1)^2/4$, and

$$s_\pm(\lambda)=\frac{n-1}{2}\pm\sqrt{(n-1)^2/4-\lambda},$$

with the square root being the standard one in $\mathbb{C}\setminus(-\infty,0]$, which means that

$$s_\pm(\lambda)=\frac{n-1}{2}\mp\imath\sigma, \tag{4.19}$$

$\operatorname{Im}\sigma>0$ being the physical half plane. Composing with

$$(\Delta'_{\partial_-X_0})^{s_-(\lambda)/2-n/4}\oplus(\Delta'_{\partial_-X_0})^{s_-(\lambda)/2-n/4}$$

from the left and

$$(\Delta'_{\partial_+X_0})^{-s_-(\lambda)/2+n/4}\oplus(\Delta'_{\partial_+X_0})^{-s_-(\lambda)/2+n/4}$$

from the right, one still has an order 0 Fourier integral operator, i.e.

$$\big(\mathrm{Id}\oplus(\Delta'_{\partial_-X_0})^{s_-(\lambda)/2-s_+(\lambda)/2}\big)\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\big(\mathrm{Id}\oplus(\Delta'_{\partial_+X_0})^{s_+(\lambda)/2-s_-(\lambda)/2}\big) \tag{4.20}$$

is 0th order. Noting that $(s_+(\lambda)-s_-(\lambda))/2=-\imath\sigma$,

$$\begin{aligned}\begin{bmatrix}\mathrm{Id}&0\\0&\mathcal{S}_{X_-}(-\sigma)\end{bmatrix}&\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\begin{bmatrix}\mathrm{Id}&0\\0&\mathcal{S}_{X_+}(\sigma)\end{bmatrix}\\ &=\begin{bmatrix}\mathrm{Id}&0\\0&\mathcal{S}_{X_-}(-\sigma)(\Delta'_{\partial X_-})^{-\imath\sigma}\end{bmatrix}\Big(\big(\mathrm{Id}\oplus(\Delta'_{\partial_-X_0})^{\imath\sigma}\big)\mathcal{S}_{X_0,\mathrm{past}}(\sigma)\big(\mathrm{Id}\oplus(\Delta'_{\partial_+X_0})^{-\imath\sigma}\big)\Big)\\ &\qquad\begin{bmatrix}\mathrm{Id}&0\\0&(\Delta'_{\partial X_+})^{\imath\sigma}\mathcal{S}_{X_+}(\sigma)\end{bmatrix},\end{aligned}$$

it follows immediately that $\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)$ is a Fourier integral operator associated to the same flow, with principal symbol the same as that of $\mathcal{S}_{X_0,\mathrm{past}}(\sigma)$ in view of Footnote 8.

Corollary 4.14. *For $\sigma\in\mathbb{C}$ with $\imath\sigma\notin\mathbb{Z}$, and σ not a pole of $\tilde P^{-1}_{\sigma,\mathrm{past}}$, $\mathcal{S}_{\tilde X,\mathrm{past}}(\sigma)$ is an elliptic 0th order Fourier integral operator associated with the null-geodesic flow from ∂_+X_0 to ∂_-X_0 on X_0, with principal symbol the same as that of the renormalized backwards scattering operator on X_0 as in* (4.20) *conjugated by the matrix*

$$\begin{bmatrix}e^{-\pi\sigma}&e^{\pi\sigma}\\1&1\end{bmatrix}$$

as in Theorem 4.13.

[9]Note that in [12] the two summands are interchanged: the $x^{s_+(\lambda)}w^+_{X_0}$ term is put first, $x^{s_-(\lambda)}w^-_{X_0}$ is put second, $w^+_{X_0},w^-_{X_0}\in\mathcal{C}^\infty(\overline{X_0})$, which is the reverse of Definition 4.9 in view of (4.19). Further, the assumption in [12] in the stated version of Theorem 1.2 is that $2\imath\sigma$ is not an integer, but as is explained below the statement of this theorem, if the metric is even, as in our case, $\imath\sigma$ not an integer suffices.

We can now put together the *local* relationship between the resolvents of the problems on X_0 and $X_\pm$ on the one hand, and on $\tilde{X}$ on the other, namely the ingredients (4.1), (4.2) and (4.5) of Proposition 4.1, together with the global understanding of the Poisson operators to show that not only does $\tilde{P}_{\sigma,\mathrm{past}}^{-1}$ determine the local inverses, but the converse also holds. We remark that this has been partially explored in [1, Section 7], in which the diagonal elements of the matrix described in Theorem 4.16 were obtained, following [11], in a somewhat weaker sense (in terms of support properties of f to which $\tilde{P}_{\sigma,\mathrm{past}}^{-1}$ is being applied).

Thus, given $f \in \mathcal{C}^\infty(\tilde{X})$, we first define a distribution $u_{\tilde{X}}$ (which in fact will be $\mathcal{C}^\infty$ away from ∂X_-) by defining its restrictions $u_{\tilde{X},X_+}$, $u_{\tilde{X},X_0}$, resp. $u_{\tilde{X},X_-}$ to X_+, X_0 resp. X_-, checking that $u_{\tilde{X},X_+}$ and $u_{\tilde{X},X_0}$ extend smoothly to ∂X_+, hence $u_{\tilde{X}}$ can defined to be smooth across ∂X_+, and then analyzing the precise singularity of $u_{\tilde{X},X_0}$ and $u_{\tilde{X},X_-}$ at ∂X_- and using this to actually define a distribution near ∂X_- as well.

So first let

$$u_{\tilde{X},X_+} = x_{X_+}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_+}(\sigma)x_{X_+}^{-\imath\sigma+(n-1)/2+2}f|_{X_+}. \tag{4.21}$$

Then $u_{\tilde{X},X_+} \in \mathcal{C}^\infty(\overline{X_+})$ (in the even sense!) by the mapping properties of the resolvent on X_+; let $v^-_{\tilde{X},X_+,0} = u_{\tilde{X},X_+}|_{\partial X_+}$. Next, we define $u_{\tilde{X},X_0} \in \mathcal{C}^\infty(X_0)$ by

$$\begin{aligned} u_{\tilde{X},X_0} =& x_{X_0}^{\imath\sigma-(n-1)/2}\mathcal{P}_{X_0,\mathrm{past}}(\sigma)(0, v^-_{\tilde{X},X_+,0}) \\ &+ x_{X_0}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_0,\mathrm{past}}(\sigma)x_{X_0}^{-\imath\sigma+(n-1)/2+2}f|_{X_0}. \end{aligned} \tag{4.22}$$

Then $u_{\tilde{X},X_0}$ is $\mathcal{C}^\infty$ up to $\partial_+ X_0$, and it has an asymptotic expansion at $\partial_- X_0$ of the form

$$u_{\tilde{X},X_0} = v^+_{\tilde{X},X_0} + v^-_{\tilde{X},X_0}, \text{ with } v^+_{\tilde{X},X_0} = (x^-_{X_0})^{2\imath\sigma}a^+_{\tilde{X},X_0},\ v^-_{\tilde{X},X_0} = a^-_{\tilde{X},X_0},$$

with $a^\pm_{\tilde{X},X_0}$ being $\mathcal{C}^\infty$ up to $\partial_- X_0$. Here, $u_{\tilde{X},X_+}$ and $u_{\tilde{X},X_0}$ not only have the same restriction at ∂X_+ (which is automatic by the definition of the Poisson operator), but have matching Taylor series (in terms of the 'even' smooth structure, i.e. that of $\tilde{X}$) by Lemma 4.2. We next let

$$u_{\tilde{X},X_-} = x_{X_-}^{\imath\sigma-(n-1)/2}\mathcal{P}_{X_-}(-\sigma)a^-_{\tilde{X},X_0}|_{\partial_- X_0} + x_{X_-}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_-}(-\sigma)x_{X_-}^{-\imath\sigma+(n-1)/2+2}f|_{X_-}. \tag{4.23}$$

Then $u_{\tilde{X},X_-}$ has an asymptotic expansion at ∂X_- of the form

$$v^+_{\tilde{X},X_-} + v^-_{\tilde{X},X_-},\ v^+_{\tilde{X},X_-} = x_{X_-}^{2\imath\sigma}a^+_{\tilde{X},X_-},\ v^-_{\tilde{X},X_-} = a^-_{\tilde{X},X_-},$$

and $a^\pm_{\tilde{X},X_-}$ are $\mathcal{C}^\infty$ up to $\partial X_- = \partial_- X_0$. Further, again, $a^-_{\tilde{X},X_-}$ and $a^-_{\tilde{X},X_0}$ not only have the same restriction at ∂X_- (which is automatic by the definition of the Poisson operator), but have matching Taylor series by Lemma 4.2. Now notice that for $\sigma \notin \imath\mathbb{Z}$ there is a unique distribution defined near ∂X_-, of the form

$$a^+_{\tilde{X},\mathrm{past}}(\mu + \imath 0)^{\imath\sigma} + a^-_{\tilde{X},\mathrm{past}}(\mu - \imath 0)^{\imath\sigma}, \tag{4.24}$$

$a^\pm_{\tilde{X},\mathrm{past}}$ being $\mathcal{C}^\infty$ near ∂X_-, whose restriction to X_0, resp. X_- is $v^+_{\tilde{X},X_0}$, resp. $v^+_{\tilde{X},X_-}$. Indeed, the difference of any two such distributions would be a differentiated delta distribution supported on ∂X_-, which are never of this form if $\sigma \notin \imath\mathbb{Z}$, showing uniqueness, while expanding $a^+_{\tilde{X},X_0}$, $a^+_{\tilde{X},X_-}$ and the putative $a^\pm_{\tilde{X}}$ in Taylor series around ∂X_-, one is reduced to observing that one must have for the jth term in the (μ-based, i.e. even in terms of $x_{X_\bullet}$) Taylor series

$$\begin{bmatrix} a^-_{\tilde X, X_0, j} \\ a^-_{\tilde X, X_-, j}\end{bmatrix} = \begin{bmatrix} e^{-\pi(\sigma-\imath j)} & e^{\pi(\sigma-\imath j)} \\ 1 & 1\end{bmatrix}\begin{bmatrix} a^+_{\tilde X,\mathrm{past},j} \\ a^-_{\tilde X,\mathrm{past},j}\end{bmatrix},$$

and in case $\sigma\notin\imath\mathbb{Z}$, the matrix on the right hand side is invertible. Thus, there is a unique distribution $u_{\tilde X}$ on $\tilde X$ which is $\mathcal{C}^\infty$ away from ∂X_-, which is of the form

$$a^0_{\tilde X,\mathrm{past}} + a^+_{\tilde X,\mathrm{past}}(\mu+\imath 0)^{\imath\sigma} + a^-_{\tilde X,\mathrm{past}}(\mu-\imath 0)^{\imath\sigma}, \tag{4.25}$$

with $a^0_{\tilde X,\mathrm{past}}, a^\pm_{\tilde X,\mathrm{past}}$ being $\mathcal{C}^\infty$ near ∂X_-, and whose restrictions to X_+, resp. X_0, resp. X_- are $u_{\tilde X, X_+}$, resp, $u_{\tilde X,X_0}$, resp. $u_{\tilde X,X_-}$. This distribution satisfies $\tilde P_\sigma u_{\tilde X} = f$ on each of X_+, X_0 and X_-. Further, $u_{\tilde X}$ being $\mathcal{C}^\infty$ near ∂X_+, $\tilde P_\sigma u_{\tilde X} - f$ is $\mathcal{C}^\infty$ there, vanishing on $X_0\cup X_+$, thus vanishing near ∂X_+ as well, i.e. $\tilde P_\sigma u_{\tilde X} - f$ is supported at ∂X_-. But there $u_{\tilde X}$ has the form (4.25), and thus $\tilde P_\sigma u_{\tilde X}$ necessarily has a similar form with the exponents decreased by 1 (since $\tilde P_\sigma$ is second order, but is characteristic on $N^*\partial X_-$). Correspondingly, as long as $\sigma\notin\imath\mathbb{Z}$, $\tilde P_\sigma u_{\tilde X} - f$ cannot be a sum of differentiated delta distributions on ∂X_-, so the vanishing of $\tilde P_\sigma u_{\tilde X} - f$ away from ∂X_- shows that $\tilde P_\sigma u_{\tilde X} = f$. Thus, given $\sigma\notin\imath\mathbb{Z}$ which is not a pole of $\mathcal{R}_{X_\pm}(\pm\sigma)$, and given $f\in\mathcal{C}^\infty(\tilde X)$, we showed that $f = \tilde P_\sigma u_{\tilde X}$.

Proposition 4.15. *For $\sigma\notin\imath\mathbb{Z}$, if σ is not a pole of $\mathcal{R}_{X_\pm}(\pm\cdot)$, then σ is not a pole of $\tilde P^{-1}_{\cdot,\mathrm{past}}$.*

Combining Propositions 4.1 and 4.15 yields

Theorem 4.16. *(Strengthened version of* [1, Proposition 7.3]*.) The poles of $\tilde P^{-1}_{\sigma,\mathrm{past}}$ in $\mathbb{C}\setminus\imath\mathbb{Z}$ are exactly the union of the poles of $\mathcal{R}_{X_+}(\sigma)$ and $\mathcal{R}_{X_-}(-\sigma)$.*

Furthermore, with the blocks X_+, X_0 and X_- listed left-to-right and top-to-bottom, and $(.)_{jk}$ referring to the jk entry of this matrix to shorten the notation, and with $\mathcal{P}_{X_0,\mathrm{future}}(\sigma)^{-1}_j$ denoting the jth component of $\mathcal{P}_{X_0,\mathrm{future}}(\sigma)^{-1}$ ($j=1,2$, so $j=1$ corresponds to the superscript $+$, $j=2$ to the superscript $-$ in Definition 4.9), the matrix of $\tilde P^{-1}_{\sigma,\mathrm{past}}$ is, column by column, (so X_+ is the first column, etc.)

$$(\tilde P^{-1}_{\sigma,\mathrm{past}})_{.1} = \begin{bmatrix} x_{X_+}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_+}(\sigma)x_{X_+}^{-\imath\sigma+(n-1)/2+2} \\ x_{X_0}^{\imath\sigma-(n-1)/2}\mathcal{P}_{X_0,\mathrm{past}}(\sigma)(0,\mathcal{P}^{-1}_{X_+}(-\sigma)\mathcal{R}_{X_+}(\sigma)x_{X_+}^{-\imath\sigma+(n-1)/2+2}) \\ x_{X_-}^{\imath\sigma-(n-1)/2}\mathcal{P}_{X_-}(-\sigma)\mathcal{P}_{X_0,\mathrm{future}}(\sigma)_2^{-1}x_{X_0}^{-\imath\sigma+(n-1)/2}()_{21}\end{bmatrix},$$

$$(\tilde P^{-1}_{\sigma,\mathrm{past}})_{.2} = \begin{bmatrix} 0 \\ x_{X_0}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_0}(\sigma)x_{X_0}^{-\imath\sigma+(n-1)/2+2} \\ x_{X_-}^{\imath\sigma-(n-1)/2}\mathcal{P}_{X_-}(-\sigma)\mathcal{P}_{X_0,\mathrm{future}}(\sigma)_2^{-1}x_{X_0}^{-\imath\sigma+(n-1)/2}()_{22}\end{bmatrix},$$

$$(\tilde P^{-1}_{\sigma,\mathrm{past}})_{.3} = \begin{bmatrix} 0 \\ 0 \\ x_{X_-}^{\imath\sigma-(n-1)/2}\mathcal{R}_{X_-}(-\sigma)x_{X_-}^{-\imath\sigma+(n-1)/2+2}\end{bmatrix}.$$

Remark 4.17. We finally remark that excluding $\imath\sigma\in\mathbb{Z}$ in (4.24) was excessive; it suffices to rule out that $\imath\sigma$ is a negative integer if we work in terms of the distributions $\mu_\pm^{\imath\sigma}$ instead, i.e. $\operatorname{Im}\sigma<1$ suffices there. Further, for $\operatorname{Im}\sigma>-1$, all operators in the two-by-two upper left block are well-defined (and holomorphic) even if $\imath\sigma$ is an integer as long as σ is not a pole of $\mathcal{R}_{X_+}(\sigma)$. Indeed, $\mathcal{P}_{X_+}^{-1}(-\sigma)$ reads off the leading asymptotic term of $\mathcal{R}_{X_+}(\sigma)$, while, for $\operatorname{Im}\sigma > 0$, $\mathcal{P}_{X_0,\mathrm{past}}(\sigma)$ solves the de Sitter Klein-Gordon equation where the second, more decaying (here we use $\operatorname{Im}\sigma > 0$) datum is specified, which makes sense in a holomorphic manner even in the case of integer $\imath\sigma$, and if we merely assume $\operatorname{Im}\sigma > -1$, the same conclusion holds though the specified behavior, $x^{(n-1)/2-\imath\sigma}a_{X_0}^-|_{\partial_+X_0}$, is now possibly the less decaying one. (At $\operatorname{Im}\sigma = -1$, constructing $v_{X_0}^-$ near ∂_+X_0 introduces logarithmic terms and changes the construction significantly. This is still possible, as was done in [12], but this seriously affects holomorphic arguments.) Thus, when composed with restriction to $\overline{X_+} \cup X_0$ from the left and extension of compactly supported functions on $\overline{X_+} \cup X_0$ from the right, the only poles of $\tilde{P}_{\sigma,\mathrm{past}}^{-1}$ are those of $\mathcal{R}_{X_+}(\sigma)$ and possibly σ with $\imath\sigma$ an integer with $\operatorname{Im}\sigma \leq -1$. We also refer to [11, Remark 4.6], where the same conclusion is established via a different argument.

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