PROPAGATION OF SINGULARITIES IN MANY-BODY SCATTERING IN THE PRESENCE OF BOUND STATES

ANDRÁS VASY

Abstract. In this paper we describe the propagation of singularities of tempered distributional solutions $u \in \mathcal{S}'$ of $(H-\lambda)u = 0$, where H is a many-body Hamiltonian $H = \Delta + V$, $\Delta \geq 0$, $V = \sum_a V_a$, and λ is not a threshold of H, under the assumption that the inter-particle (e.g. two-body) interactions V_a are real-valued polyhomogeneous symbols of order -1 (e.g. Coulomb-type with the singularity at the origin removed). Here the term 'singularity' refers to a microlocal description of the lack of decay at infinity. Thus, we prove that the set of singularities of u is a union of maximally extended broken bicharacteristics of H. These are curves in the characteristic variety of $H-\lambda$, which can be quite complicated due to the existence of bound states. We use this result to describe the wave front relation of the S-matrices. We also analyze Lagrangian properties of this relation, which shows that the relation is not 'too large' in terms of its dimension.

1. Introduction

In this paper we describe the propagation of singularities of generalized eigenfunctions of a many-body Hamiltonian $H = \Delta + V$, $V = \sum_a V_a$, on $\mathbb{R}^n$ under the assumption that the inter-particle interactions V_a are real-valued polyhomogeneous symbols of order -1 (e.g. Coulomb-type with the singularity at the origin removed). More precisely, we use the 'many-body scattering wave front set' $\mathrm{WF}_{\mathrm{Sc}}(u)$ at infinity for tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$, and prove that for $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $(H-\lambda)u = 0$, $\mathrm{WF}_{\mathrm{Sc}}(u)$ is a union of maximally extended generalized broken bicharacteristics of H, broken at the collision planes. Here $\mathrm{WF}_{\mathrm{Sc}}(u)$ provides a microlocal description of the lack of decay of u modulo $\mathcal{S}(\mathbb{R}^n)$, similarly to how the usual wave front set describes distributions modulo $\mathcal{C}^\infty$ functions.

The definition of generalized broken bicharacteristics is quite technical due to the presence of bound states in the subsystems. However, if these bound states are absent, our definition becomes completely analogous to Lebeau's definition [19] for the wave equation in domains with corners. Indeed, in this case the propagation result itself, which was proved in [37], is a direct ($\mathcal{C}^\infty$-type) analogue of Lebeau's result for the propagation of analytic singularities for solutions of the wave equation in domains with corners.

If there are bound states in the subsystems, but either the set of thresholds is discrete, or H is a four-body Hamiltonian, the geometry of generalized broken

Date: May 4, 2000. Revised March 16, 2001.

1991 *Mathematics Subject Classification.* 35P25, 47A40, 58G25, 81U10.

Address: Department of Mathematics, University of California, Berkeley, CA 94720-3840. *E-mail:* andras@math.berkeley.edu.

Partially supported by NSF grant #DMS-99-70607.

bicharacteristics is not much more complicated than in Lebeau's setting. The general definition reflects that when particles collide, the total energy as well as the external momentum is preserved. The complication in the presence of bound states is that kinetic energy is not preserved, even asymptotically. In summary, our results provide a connection between quantum and classical objects, just as Lebeau's results connect the wave equation and geometric optics. Note that these results only provide the answer as to where in phase space the singularities of generalized eigenfunctions may be located; they leave open the question of what these singularities are like, i.e. we do not have FIO-type results. Such results exist at least in certain 3-body settings [33, 9], see the remarks in the next section. In addition, we show that in our setting, if there are no bound states in any subsystems, then the broken bicharacteristic relation has the correct Lagrangian geometry to give rise to such FIO results. We also obtain a similar result on a certain natural part of the broken bicharacteristic relation under the assumption that the set of thresholds is discrete.

We also prove the corresponding result in the 'limiting absorption principle' setting, namely that under certain assumptions on $\mathrm{WF}_{\mathrm{Sc}}(f)$, $R(\lambda \pm i0)f$ are defined, and $\mathrm{WF}_{\mathrm{Sc}}(R(\lambda + i0)f)$ is a subset of the image of $\mathrm{WF}_{\mathrm{Sc}}(f) \cup R_-(\lambda)$ under the forward broken bicharacteristic relation. Here $R_-(\lambda)$ is the outgoing 'radial set'. Such a result makes the 'radial-variable' propagation estimates that have been used in many-body scattering, especially as derived in the works of Gérard, Isozaki and Skibsted [6, 7], more precise.

We use this result to analyze that the wave front relation of the scattering matrices (S-matrices). These connect the incoming and outgoing data of generalized eigenfunctions of H, so one expects that their singularities are described by considering the limit points of generalized broken bicharacteristics $\gamma = \gamma(t)$ as $t \to \pm\infty$. In fact, in addition to the propagation of singularities result, the only ingredient that is required for this analysis is a good approximation for the incoming Poisson operators with incoming state α near the incoming region, and similar results for the outgoing Poisson operators with, say, outgoing state β. In general, one expects a WKB-type construction, essentially as in Hadamard's parametrix construction. Indeed, this is what Melrose and Zworski do in the geometric two-body type setting, [25]. In the Euclidean many-body setting this construction has been done by Skibsted [32] in the short-range and by Bommier [1] in the long-range setting, in the latter case by adopting the construction of Isozaki and Kitada [17], at least under the assumption that the energies of the states α, β are below the continuous spectrum of the corresponding subsystem Hamiltonians. Such a construction is unnecessary if V_c are Schwartz, for then the product decomposition is sufficiently accurate to give a good approximation for the Poisson operator. We thus obtain the following result.

Theorem. *Suppose that H is a many-body Hamiltonian, and λ is not a threshold or (L^2-)eigenvalue of H. Suppose also that either α and β are channels such that the corresponding eigenvalues ϵ_α, ϵ_β, of the subsystem Hamiltonians H^a, H^b, are in the discrete spectrum of H^a and H^b respectively, or that V_c is Schwartz for all c. Then the wave front relation of the S-matrix $S_{\beta\alpha}(\lambda)$, is given by the generalized broken bicharacteristic relation of H as stated precisely below in Theorem 2.8.*

Special cases, which have already been analyzed, include the free-to-free S-matrix in three-body scattering [33, 9, 36], or indeed in many-body scattering under the additional assumption that there are no bound states in any subsystem [37]. In these cases the wave front relation is given by the broken geodesic relation, broken at the collision planes, on $\mathbb{S}^{n-1}$ at distance π. In both cases, one can naturally extend the results to geometric many-body type problems on asymptotically Euclidean manifolds.

Also, Bommier [1] and Skibsted [32] have shown that the kernels of the 2-cluster to free cluster and 2-cluster to 2-cluster S-matrices are smooth (except for the diagonal singularity if the 2-clusters are the same), and previously Isozaki had showed this in the three-body setting [14]. We remark that (under our polyhomogeneous assumption) the proofs of Bommier and Skibsted in fact show that the 2-cluster to same 2-cluster S-matrices are (non-classical, if the potentials are long-range) pseudo-differential operators, which differ from the identity operator by operators of order -1 if the potentials are short range. In our geometric normalization this means that they are Fourier integral operators associated to the geodesic flow on the sphere at infinity to distance π (along the cluster).

It may seem that the (generalized) broken bicharacteristic relation is rather large since a single bicharacteristic can be continued in many ways after it hits a collision plane. In fact, this is not the case, since many bicharacteristics do not hit these collision planes at all. A more precise statement would be the that the wave front relation of the S-matrices is given by a union of Lagrangian submanifolds, so at least it has the same structure as if the S-matrices were (sums of) Fourier integral operators. Indeed, we prove that if there are no bound states in any subsystems, then the wave front relations of the S-matrices are given by *finite* unions of smooth Lagrangian relations, and we obtain a similar result on a part of the wave front relation under the assumption that the set of thresholds is discrete.

We remark that the results of this paper would remain valid if we assumed only that $V_a \in S^{-\rho}(X^a)$, $\rho > 0$, as customary. In fact, the proof of the propagation of singularities for generalized eigenfunctions remains essentially unchanged, and the only difference in the above Theorem is that the parametrix for the Poisson operators is not as explicit, cf. [37]; instead, one needs to use the constructions of Isozaki-Kitada [17] (as presented by Skibsted and Bommier) directly. The reason for the polyhomogeneous assumption is that the proofs are somewhat nicer, especially in notation, and it is a particularly natural assumption to make in the compactification approach we adopt.

Our main tool in proving the propagation of singularities results consists of microlocally positive commutator estimates, i.e. on the construction of operators which have a positive commutator with H in the part of phase space, say U, where we wish to conclude that a generalized eigenfunction u has no scattering wave front set. These commutators are usually negative in another region of phase space, namely backwards (or forwards, depending on the construction) along generalized broken bicharacteristics through U. We thus assume the absence of this region from $\mathrm{WF}_{\mathrm{Sc}}(u)$, and conclude that the positive commutator region, U, is also missing from $\mathrm{WF}_{\mathrm{Sc}}(u)$. Such techniques have been used by Hörmander, Melrose and Sjöstrand [12, 24] to show the propagation of singularities for hyperbolic equations (real principal type propagation) such as the wave equation, including in regions with smooth boundaries. Indeed, the best way to interpret our results is to say that $H - \lambda$ is hyperbolic at infinity. In two-body scattering the analogy with the wave equation in domains without boundary is rather complete; this was the basis of Melrose's proof of propagation estimates for scattering theory for 'scattering metrics' in [23]. In many-body scattering, the lack of commutativity of the appropriate pseudo-differential algebra, even to top order, makes the estimates (and their proofs) more delicate. We remark that, as can be seen directly from the approach we take, the wave front set estimates can be easily turned into microlocal estimates on the resolvent considered as an operator between weighted Sobolev spaces; wave front set statements are a particularly convenient way of describing propagation.

Indeed, there is some freedom in the precise definition of the wave front set; see the remarks preceeding the statement of Theorem 2.2. The alternative definitions differ slightly, but agree for generalized eigenfunctions, and the corresponding propagation results are based on the same positive commutator estimates. Thus, the reader may find the explicit microlocal estimates of Section 7 particularly clear. However, piecing together these estimates to describe propagation is more cumbersome than making simple geometric statements based on wave front sets; for this reason we emphasize the latter.

Positive commutator estimates have also played a major role in many-body scattering starting with the work of Mourre [26], Perry, Sigal and Simon [27], Froese and Herbst [5], Jensen [18], Gérard, Isozaki and Skibsted [6, 7] and Wang [38]. In particular, the Mourre estimate is one of them; it estimates $i[H, w \cdot D_w + D_w \cdot w]$. This and some other *global* positive commutator results have been used to prove the global results mentioned in the first paragraph about some of the S-matrices with initial state in a two-cluster. They also give the basis for the existence, uniqueness and equivalence statements in our definition of the S-matrix by asymptotic expansions; these statements are discussed in [15, 16, 35] in more detail.

More delicate (and often time-dependent) commutator estimates have been used in the proof of asymptotic completeness. This completeness property of many-body Hamiltonians was proved by Sigal and Soffer, Graf, Dereziński and Yafaev under different assumptions on the potentials and by different techniques [28, 29, 31, 30, 8, 2, 39]. In particular, Yafaev's paper [39] shows quite explicitly the importance of the special structure of the Euclidean Hamiltonian. This structure enables him to obtain a (time-independent) positive commutator estimate, which would not follow from the indicial operator arguments of [36, 37] and the present paper, and which is then used to prove asymptotic completeness.

We briefly outline of the positive commutator proofs. They consist of two parts: first, the construction of a symbol (and an associated pseudo-differential operator) that we claim has a microlocally positive commutator with H, when localized in energy, and second, the proof that the commutator is indeed positive in the appropriate part of the phase space. The first part has much in common with the analysis of generalized broken bicharacteristics, since both are intimately connected to various Hamilton vector fields associated to the subsystems. The second part is essentially a microlocal version of the proof of the Mourre estimate that was obtained by Froese and Herbst [5]. Indeed, we could follow the full indicial operator version of this proof, as was done in [36] and [37]. However, in the presence of bound states in the proper subsystems we would have to rely more heavily on the approximate product structure of the Hamiltonian at each cluster. Indeed, in the full geometric problem considered in [37], the L^2 eigenvalues and eigenfunctions of the indicial operator of H can vary as one moves along the collision planes, which makes even the description of the bicharacteristics more complicated. However, it turns out that in the Euclidean many-body setting, after an explicit calculation of the indicial operators, we can use the Mourre estimate explicitly in the normal (non-commutative) variables at each cluster, and use the standard Poisson bracket formula for the commutator in the tangential (commutative) variables. Since it eliminates the need to present arguments that are essentially simple (microlocal) modifications of the Froese-Herbst proof of the Mourre estimate, we adopt the second approach.

The structure of the paper is the following. In the next section we set up the framework of many-body scattering, mostly following the book of Dereziński and Gérard, [3], and we state the precise results on the propagation of singularities. In Section 3 we relate this to the Melrose's approach to scattering via compactification [23]. We also include in this section the outline of the resolved space construction and the definition of the algebra of many-body scattering differential operators from [37]. As we indicate in this section, from an algebraic point of view, it is the lack of commutativity of this algebra to 'top weight' (at infinity) that gives rise to the breaks in the generalized broken bicharacteristics along which singularities propagate. In Section 4 we recall the definition and basic properties of many-body scattering pseudo-differential operators as well as the definition of the corresponding wave front set from [37], and we analyze the characteristic variety of many-body Hamiltonians. Microlocal elliptic regularity is stated here in Corollary 4.3. In Section 5 we describe generalized broken bicharacteristics, and in Section 6 we explain the positive commutator argument that is the key to our propagation results. Propagation of singularities itself, stated in Theorem 2.2, is proved in Section 7. Sections 8-10 turn this result into theorems on the resolvent and the scattering matrices, stated in Theorems 2.4-2.10. Finally, in Appendix A we show that if the set of thresholds of H is discrete, or if H is a four-body Hamiltonian, generalized broken bicharacteristics are piecewise integral curves of the Hamilton vector fields, with only a finite number of breaks, and then in Appendix B we analyze the Lagrangian structure of the broken bicharacteristic relation.

Most results of this paper were announced in [34]; this paper contains the detailed proofs.

I am very grateful to Andrew Hassell, Richard Melrose and Maciej Zworski for numerous very fruitful discussions; in particular, I would like to thank Richard Melrose for his comments on this paper. I am grateful to Maciej Zworski for introducing me to the work of Gilles Lebeau [19]. If there are no bound states in any subsystems, many-body scattering is 'philosophically' and, to a certain extent, technically (e.g. the structure of generalized broken bicharacteristics) is very similar to the wave equation in domain with corners. Thus, Lebeau's paper played an important direct role in my paper [37], and remained philosophically important while working on the present manuscript. My joint projects with Andrew Hassell, as well as our discussions in general, provided very valuable insights into the broken bicharacteristic geometry, especially towards understanding their Lagrangian structure. It is my pleasure to acknowledge that the careful reading of the referee of my paper [37] initiated numerous improvements in that paper, which in turn influenced the present one. I would also like to thank Rafe Mazzeo, Erik Skibsted and Jared Wunsch for helpful discussions, their encouragement and for their interest in this research.

2. Notation and detailed statement of results

Before we can state the precise definitions, we need to introduce some basic (and mostly standard) notation. We refer to [3] for a very detailed discussion of the setup and the basic results. We consider the Euclidean space $\mathbb{R}^n$, and let g be the standard Euclidean metric on it. We assume also that we are given a (finite) family $\mathcal{X}$ of linear subspaces X_a, $a \in I$, of $\mathbb{R}^n$ which is closed under intersections and includes the subspace $X_1 = \{0\}$ consisting of the origin, and the whole space $X_0 = \mathbb{R}^n$. Let X^a be the orthocomplement of X_a. We write g_a and g^a for the induced metrics on X_a and X^a respectively. We let π^a be the orthogonal projection to X^a, π_a to X_a. A many-body Hamiltonian is an operator of the form

$$H = \Delta + \sum_{a \in I} (\pi^a)^* V_a; \tag{2.1}$$

here Δ is the positive Laplacian, $V_0 = 0$, and the V_a are real-valued functions in an appropriate class which we take here to be polyhomogeneous symbols of order -1 on the vector space X_a to simplify the problem:

$$V_a \in S^{-1}_{\mathrm{phg}}(X^a). \tag{2.2}$$

In particular, smooth potentials V_a which behave at infinity like the Coulomb potential are allowed. Since $(\pi^a)^* V_a$ is bounded and self-adjoint and Δ is self-adjoint with domain $H^2(\mathbb{R}^n)$ on $L^2 = L^2(\mathbb{R}^n)$, H is also a self-adjoint operator on L^2 with domain $H^2(\mathbb{R}^n)$. We let $R(\lambda) = (H - \lambda)^{-1}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be the resolvent of H.

There is a natural partial ordering on I induced by the ordering of X^a by inclusion. (Though the ordering based on inclusion of the X_a would be sometimes more natural, here we use the conventional ordering, we simply write $X_a \subset X_b$ if the opposite ordering is required.) Let $I_1 = \{1\}$ (recall that $X_1 = \{0\}$); 1 is the maximal element of I. A maximal element of $I \setminus I_1$ is called a 2-cluster; I_2 denotes the set of 2-clusters. In general, once I_k has been defined for $k = 1, \ldots, m-1$, we let I_m (the set of m-clusters) be the set of maximal elements of $I'_m = I \setminus \cup_{k=1}^{m-1} I_k$, if I'_m is not empty. If $I'_m = \{0\}$ (so I'_{m+1} is empty), we call H an m-body Hamiltonian. For example, if $I \neq \{0, 1\}$, and for all $a, b \notin \{0, 1\}$ with $a \neq b$ we have $X_a \cap X_b = \{0\}$, then H is a 3-body Hamiltonian. The N-cluster of an N-body Hamiltonian is also called the free cluster, since it corresponds to the particles which are asymptotically free.

Our goal is to study generalized eigenfunctions of H, i.e. solutions $u \in \mathcal{S}'(\mathbb{R}^n)$ of $(H - \lambda)u = 0$. Since $H - \lambda$ is an elliptic partial differential operator with smooth coefficients, $(H - \lambda)u \in \mathcal{C}^\infty(\mathbb{R}^n)$ implies that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus, the place where such u can have interesting behavior is at infinity. Analysis at infinity can be viewed either as analysis of uniform properties, or as that of properties in the appropriate compactification of $\mathbb{R}^n$. We adopt the second point of view by compactifying $\mathbb{R}^n$ as in [23]. Thus, we let

$$\bar{X} = \bar{X}_0 = \mathbb{S}^n_+ \tag{2.3}$$

to be the radial compactification of $\mathbb{R}^n$ (also called the geodesic compactification) to a closed hemisphere, i.e. a ball, and $\mathbb{S}^{n-1} = \partial \mathbb{S}^n_+$. Recall from [23] that RC : $\mathbb{R}^n \to \mathbb{S}^n_+$ is given by

$$\mathrm{RC}(w) = (1/(1 + |w|^2)^{1/2}, w/(1 + |w|^2)^{1/2}) \in \mathbb{S}^n_+ \subset \mathbb{R}^{n+1}, \quad w \in \mathbb{R}^n. \tag{2.4}$$

Here we use the notation RC instead of SP, used in [23], to avoid confusion with the standard stereographic projection giving a one-point compactification of $\mathbb{R}^n$. We write the coordinates on $\mathbb{R}^n = X_a \oplus X^a$ as (w_a, w^a). We let

$$\bar{X}_a = \mathrm{cl}(\mathrm{RC}(X_a)), \quad C_a = \bar{X}_a \cap \partial \mathbb{S}^n_+. \tag{2.5}$$

Hence, C_a is a sphere of dimension $n_a - 1$ where $n_a = \dim X_a$. We also let

$$\mathcal{C} = \{C_a : \ a \in I\}. \tag{2.6}$$

Thus, $C_0 = \partial \mathbb{S}^n_+ = \mathbb{S}^{n-1}$, and $a \le b$ if and only if $C_b \subset C_a$. Since throughout this paper we work in the Euclidean setting, where the notation X, X_a, etc., has been used for the (non-compact) vector spaces, we always use a bar, as in $\bar{X}$, $\bar{X}_a$, etc., to denote the corresponding compact spaces. In particular, even when talking about general compact manifolds with boundary in the following sections, recalling the results of [37], we will write them as $\bar{X}$.

We note that if a is a 2-cluster then $C_a \cap C_b = \emptyset$ unless $C_a \subset C_b$, i.e. $b \le a$. We also define the 'singular part' of C_a as the set

$$C_{a,\mathrm{sing}} = \cup_{b \not\le a} (C_b \cap C_a), \tag{2.7}$$

and its 'regular part' as the set

$$C'_a = C_a \setminus \cup_{b \not\le a} C_b = C_a \setminus C_{a,\mathrm{sing}}. \tag{2.8}$$

For example, if a is a 2-cluster then $C_{a,\mathrm{sing}} = \emptyset$ and $C'_a = C_a$.

We usually identify (the interior of) $\mathbb{S}^n_+$ with $\mathbb{R}^n$. A particularly useful boundary defining function of $\mathbb{S}^n_+$ is given by $x \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ defined as $x = r^{-1} = |w|^{-1}$ (for $r \ge 1$, say, smoothed out near the origin); so $\mathbb{S}^{n-1} = \partial \mathbb{S}^n_+$ is given by $x = 0$, $x > 0$ elsewhere, and $dx \ne 0$ at $\mathbb{S}^{n-1}$. We write $S^m_{\mathrm{phg}}(\mathbb{S}^n_+)$ and $S^m_{\mathrm{phg}}(\mathbb{R}^n)$ interchangeably. We also remark that

$$S^m_{\mathrm{phg}}(\mathbb{S}^n_+) = x^{-m} \mathcal{C}^\infty(\mathbb{S}^n_+). \tag{2.9}$$

We recall that under RC, $\dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$, the space of smooth functions on $\mathbb{S}^n_+$ vanishing to infinite order at the boundary, corresponds to the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, and its dual, $\mathcal{C}^{-\infty}(\mathbb{S}^n_+)$, to tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. We also have the following correspondence of weighted Sobolev spaces

$$H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+) = H^{k,l} = H^{k,l}(\mathbb{R}^n) = \langle w \rangle^{-l} H^k(\mathbb{R}^n) \tag{2.10}$$

where $\langle w \rangle = (1 + |w|^2)^{1/2}$.

Corresponding to each cluster a we introduce the cluster Hamiltonian H^a as an operator on $L^2(X^a)$ given by

$$H^a = \Delta + \sum_{b \le a} V_b, \tag{2.11}$$

Δ being the Laplacian of the induced metric on X^a. Thus, if H is a N-body Hamiltonian and a is a k-cluster, then H^a is a $(N + 1 - k)$-body Hamiltonian. The L^2 eigenfunctions of H^a (also called bound states) play an important role in many-body scattering; we remark that by a result of Froese and Herbst, [4], $\mathrm{spec}_{pp}(H^a) \subset (-\infty, 0]$ (there are no positive eigenvalues). Moreover, $\mathrm{spec}_{pp}(H^a)$ is bounded below since H^a differs from Δ by a bounded operator. Note that $X^0 = \{0\}$, $H^0 = 0$, so the unique eigenvalue of H^0 is 0.

The eigenvalues of H^a can be used to define the set of thresholds of H^b. Namely, we let

$$\Lambda_a = \cup_{b<a} \operatorname{spec}_{pp}(H^b) \tag{2.12}$$

be the set of thresholds of H^a, and we also let

$$\Lambda'_a = \Lambda_a \cup \operatorname{spec}_{pp}(H^a) = \cup_{b\leq a} \operatorname{spec}_{pp}(H^b). \tag{2.13}$$

Thus, $0 \in \Lambda_a$ for $a \neq 0$ and $\Lambda_a \subset (-\infty, 0]$. It follows from the Mourre theory (see e.g. [5, 27]) that Λ_a is closed, countable, and $\operatorname{spec}_{pp}(H^a)$ can only accumulate at Λ_a. Moreover, $R(\lambda)$, considered as an operator on weighted Sobolev spaces, has a limit

$$R(\lambda \pm i0) : H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+) \to H^{k+2,l'}_{\mathrm{sc}}(\mathbb{S}^n_+) \tag{2.14}$$

for $l > 1/2$, $l' < -1/2$, from either half of the complex plane away from

$$\Lambda = \Lambda_1 \cup \operatorname{spec}_{pp}(H). \tag{2.15}$$

In addition, L^2 eigenfunctions of H^a with eigenvalues which are not thresholds are necessarily Schwartz functions on X^a (in fact, they decay exponentially, see [4]). We also label the eigenvalues of H^a, counted with multiplicities, by integers m, and we call the pairs $\alpha = (a, m)$ channels. We denote the eigenvalue of the channel α by ϵ_α, write ψ_α for a corresponding normalized eigenfunction, and let e_α be the orthogonal projection to ψ_α in $L^2(X^a)$.

The phase space in scattering theory is the cotangent bundle $T^*\mathbb{R}^n$. Again, it is convenient to consider its appropriate partial compactification, i.e. to consider it as a vector bundle over $\mathbb{S}^n_+$. Thus, consider the set of all one-forms on $\mathbb{R}^n$ of the form

$$\sum_{j=1}^n a_j\, dw_j \tag{2.16}$$

where $a_j \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ (we drop RC from the notation as usual). This is then the set of all smooth sections of a trivial vector bundle over $\mathbb{S}^n_+$, with basis $dw_1, \ldots, dw_n$. Following Melrose's geometric approach to scattering theory, see [23], we consider this as the (dual) structure bundle, and call it the scattering cotangent bundle of $\mathbb{S}^n_+$, denoted by ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$. Note that $T^*\mathbb{R}^n$ can be identified with $\mathbb{R}^n \times \mathbb{R}^n$ via the metric g; correspondingly ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$ is identified with $\mathbb{S}^n_+ \times \mathbb{R}^n$, i.e. we simply compactified the base of the standard cotangent bundle. We remark that the construction of ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$ is completely natural and geometric, just like the following ones, see [23], or Section 3 for a summary.

However, in many-body scattering ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$ is *not* the natural place for microlocal analysis for the very same reason that introduces the compressed cotangent bundle in the study of the wave equation on bounded domains. We can see what causes trouble from both the dynamical and the quantum point of view. Regarding dynamics, the issue is that only the external part of the momentum is preserved in a collision, the internal part is not; while from the quantum point of view the problem is that there is only partial commutativity in the algebra of the associated pseudo-differential operators, even to top order. To rectify this, we replace the full bundle ${}^{\mathrm{sc}}T^*_{C'_a}\mathbb{S}^n_+ = C'_a \times \mathbb{R}^n$ over $C'_a \subset \mathbb{S}^{n-1}$ by ${}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a = C'_a \times X_a$, i.e. we consider

$${}^{\mathrm{sc}}\dot{T}^*\mathbb{S}^n_+ = \cup_a {}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a. \tag{2.17}$$

Over C_a', there is a natural projection $\pi_a : {}^{sc}T^*_{C_a'}\mathbb{S}^n_+ \to {}^{sc}T^*_{C_a'}\bar{X}_a$ corresponding to the pull-back of one-forms; in the trivialization given by the metric it is induced by the orthogonal projection to X_a in the fibers. By putting the π_a together, we obtain a projection $\pi : {}^{sc}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+ \to {}^{sc}\dot{T}^*\mathbb{S}^n_+$. We put the topology induced by π on ${}^{sc}\dot{T}^*\mathbb{S}^n_+$. This definition is analogous to that of the compressed cotangent bundle in the works of Melrose, Sjöstrand [24] and Lebeau [19] on the wave equation in domains with smooth boundaries or corners, respectively.

We also recall from [23] that the characteristic variety $\Sigma_0(\lambda)$ of $\Delta-\lambda$ is simply the subset of ${}^{sc}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+$ where $g-\lambda$ vanishes; g being the metric function. If $\Lambda_1 = \{0\}$, the compressed characteristic set of $H-\lambda$ will be simply $\pi(\Sigma_0(\lambda)) \subset {}^{sc}\dot{T}^*\mathbb{S}^n_+$. In general, all the bound states contribute to the characteristic variety. Thus, we let

$$\Sigma_b(\lambda) = \{\xi_b \in {}^{sc}T^*_{C_b}\bar{X}_b : \ \lambda - |\xi_b|^2 \in \operatorname{spec}_{pp} H^b\} \subset {}^{sc}T^*_{C_b}\bar{X}_b; \tag{2.18}$$

note that $|\xi_b|^2$ is the kinetic energy of a particle in a bound state of H^b. If $C_a \subset C_b$, there is also a natural projection $\pi_{ba} : {}^{sc}T^*_{C_a'}\bar{X}_b \to {}^{sc}T^*_{C_a'}\bar{X}_a$ (in the metric trivialization we can use the orthogonal projection $X_b \to X_a$ as above), and then we define the characteristic set of $H-\lambda$ to be

$$\dot{\Sigma}(\lambda) = \cup_a \dot{\Sigma}_a(\lambda), \qquad \dot{\Sigma}_a(\lambda) = \cup_{C_b \supset C_a} \pi_{ba}(\Sigma_b(\lambda)) \cap {}^{sc}T^*_{C_a'}\bar{X}_a, \tag{2.19}$$

so $\dot{\Sigma}(\lambda) \subset {}^{sc}\dot{T}^*\mathbb{S}^n_+$. We let $\hat{\pi}_b$ be the restriction of $\pi_b : {}^{sc}T^*_{C_a'}\bar{X}_b \to \dot{\Sigma}(\lambda)$ to $\Sigma_b(\lambda)$.

We next recall from [37] the definition of generalized broken bicharacteristics in case there are no bound states in any of the subsystems. In fact, in this case the word 'generalized' can be dropped; for the generalized broken bicharacteristics have a simple geometry as stated below. First, note that the rescaled Hamilton vector field of the metric function g, i.e.

$$2\langle w\rangle\xi \cdot \partial_w \in \mathcal{V}(T^*\mathbb{R}^n) = \mathcal{V}(\mathbb{R}^n \times \mathbb{R}^n) \tag{2.20}$$

extends to a smooth vector field, ${}^{sc}H_g \in \mathcal{V}({}^{sc}T^*\mathbb{S}^n_+) = \mathcal{V}(\mathbb{S}^n_+ \times \mathbb{R}^n)$, with $\mathbb{S}^n_+$ considered as the radial compactification of $\mathbb{R}^n$; in fact, ${}^{sc}H_g$ is tangent to the boundary ${}^{sc}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+ = \mathbb{S}^{n-1} \times \mathbb{R}^n$.

Definition. Suppose $\Lambda_1 = \{0\}$, and $I = [\alpha, \beta]$ is an interval. We say that a continuous map $\gamma : I \to \dot{\Sigma}(\lambda)$ is a broken bicharacteristic of $H-\lambda$ if there exists a finite set of points $t_j \in I$, $\alpha = t_0 < t_1 < \ldots < t_{k-1} < t_k = \beta$ such that for each j, $\gamma|_{(t_j,t_{j+1})}$ is the image of an integral curve of ${}^{sc}H_g$ in $\Sigma_0(\lambda)$ under π. If I is an interval (possibly $\mathbb{R}$), we say that $\gamma : I \to \dot{\Sigma}(\lambda)$ is a broken bicharacteristic of $H-\lambda$, if the restriction of γ to every compact subinterval of I is a broken bicharacteristic in the above sense.

Here $\gamma(I) \subset \dot{\Sigma}(\lambda) = \pi(\Sigma_0(\lambda))$ corresponds to the conservation of kinetic energy in collisions (since there are no bound states), and the use of the compressed space ${}^{sc}\dot{T}^*\mathbb{S}^n_+$ shows that external momentum is conserved in the collisions. It turns out, see [37], that γ is essentially the lift of a broken geodesic on $\mathbb{S}^n_+$ of length π (if $I = \mathbb{R}$, otherwise shorter), broken at the collision planes, i.e. at $\mathcal{C}$. In particular, even if $I = \mathbb{R}$, it has only a finite number of breaks, and in fact, there is a uniform bound on the number of such breaks (depending only on the geometry, i.e. on $\mathcal{C}$, not on γ).

The definitions are less explicit if $\Lambda_1 \neq \{0\}$, but they essentially still state that the total energy and the external momentum are preserved in collisions. Thus, generalized broken bicharacteristics will be continuous maps γ defined on intervals I, $\gamma : I \to \dot{\Sigma}(\lambda)$ with certain appropriate generalization of the integral curve condition described above. In order to take the bound states into consideration, we also need to consider the rescaled Hamilton vector fields ${}^{\mathrm{sc}}H_g^b$ of the metric g_b in the subsystem b. Thus, under the inclusion map

$$\imath_b : {}^{\mathrm{sc}}T^*_{C_b}\bar{X}_b \hookrightarrow {}^{\mathrm{sc}}T^*_{C_b}\mathbb{S}^n_+ \tag{2.21}$$

induced by the inclusion $X_b \hookrightarrow \mathbb{R}^n$ in the fibers, $(\imath_b)_* {}^{\mathrm{sc}}H_g^b = {}^{\mathrm{sc}}H_g$ (i.e. the restriction of the vector field ${}^{\mathrm{sc}}H_g$ to ${}^{\mathrm{sc}}T^*_{C_b}\bar{X}_b$, considered as a subset of ${}^{\mathrm{sc}}T^*_{C_b}\mathbb{S}^n_+$). Thus, we require that lower bounds on the Hamilton vector fields ${}^{\mathrm{sc}}H_g^b$ applied to π-invariant functions, i.e. to functions $f \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+)$ such that $f(\xi) = f(\xi')$ if $\pi(\xi) = \pi(\xi')$, imply lower bounds on the derivatives of f_π along γ. Here f_π is the function induced by f on ${}^{\mathrm{sc}}\dot{T}^*\mathbb{S}^n_+$, so $f = f_\pi \circ \pi$.

Definition 2.1. A generalized broken bicharacteristic of $H - \lambda$ is a continuous map $\gamma : I \to \dot{\Sigma}(\lambda)$, where $I \subset \mathbb{R}$ is an interval, such that for all $t_0 \in I$ and for each sign $+$ and $-$ the following holds. Let $\xi_0 = \gamma(t_0)$, suppose that $\xi_0 \in {}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a$. Then for all π-invariant functions $f \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+)$,

$$D_\pm(f_\pi \circ \gamma)(t_0) \geq \inf\{{}^{\mathrm{sc}}H_g^b f(\tilde{\xi}_0) : \ \tilde{\xi}_0 \in \hat{\pi}_b^{-1}(\xi_0),\ C_a \subset C_b\}. \tag{2.22}$$

Here $D_\pm$ are the one-sided lower derivatives: if g is defined on an interval I, $(D_\pm g)(t_0) = \liminf_{t \to t_0\pm}(g(t) - g(t_0))/(t - t_0)$.

Although it is not apparent, this definition is equivalent to the previous one if $\Lambda_1 = \{0\}$. Moreover, in four-body scattering, even if $\Lambda_1 \neq \{0\}$, one can describe the generalized broken bicharacteristics piecewise as projections of integral curves of ${}^{\mathrm{sc}}H_g$. In general many-body scattering, the lack of conservation of kinetic energy makes such a description harder, but if Λ_1 is discrete, we obtain a description that parallels the one above. More precisely, in Theorem 5.10 we prove the following. Suppose that Λ_1 is discrete and $\gamma : \mathbb{R} \to \dot{\Sigma}(\lambda)$ is a continuous curve. Then γ is a generalized broken bicharacteristic of $H - \lambda$ if and only if there exist $t_0 < t_1 < t_2 < \ldots < t_k$ such that $\gamma|_{[t_j, t_{j+1}]}$, as well as $\gamma|_{(-\infty, t_0]}$ and $\gamma|_{[t_k, +\infty)}$, are the projections of integral curves of the Hamilton vector field ${}^{\mathrm{sc}}H_g^a$ for some a. In addition, there is a uniform bound on k (independent of γ), depending only on $\mathcal{C}$ and Λ_1. Similar results hold if the interval of definition, $\mathbb{R}$, is replaced by any interval.

As mentioned in the introduction, 'singularities' (i.e. lack of decay at infinity) of $u \in \mathcal{S}'$ are described by the many-body scattering wave front set, $\mathrm{WF}_{\mathrm{Sc}}(u)$, which was introduced in [37], and which describes u modulo Schwartz functions, similarly to how the usual wave front set describes distributions modulo smooth functions. Just as for the image of the bicharacteristics, ${}^{\mathrm{sc}}\dot{T}^*\mathbb{S}^n_+$ provides the natural setting in which $\mathrm{WF}_{\mathrm{Sc}}$ is defined: $\mathrm{WF}_{\mathrm{Sc}}(u)$ is a closed subset of ${}^{\mathrm{sc}}\dot{T}^*\mathbb{S}^n_+$. The definition of $\mathrm{WF}_{\mathrm{Sc}}(u)$ relies on the algebra of many-body scattering pseudo-differential operators, also introduced in [37]. There are several possible definitions of $\mathrm{WF}_{\mathrm{Sc}}$, all of which agree for generalized eigenfunctions of H, but the one given in [37] that is modelled on the fibred-cusp wave front set of Mazzeo and Melrose [20] enjoys many properties of the usual wave front set. A slightly different definition, for which the crucial property in the last line of (4.35) still holds, was discussed in [36]; this is the only essential property for the positive commutator estimates in this paper. Also, the discussion after (9.13) indicates why 'finite order' versions of the wave front set, i.e. versions in which we only require a fixed (though possibly high) number of the symbol estimates for ps.d.o.'s, somewhat akin to the constructions of [6], would be helpful; nonetheless, the Mazzeo-Melrose definition appears to be the most natural one from the point of view of general microlocal analysis.

We recall the precise definitions in Section 4; here we also translate our results into statements on the S-matrices where the usual wave front set can be used. We remark that in the two-body setting, when ${}^{\mathrm{sc}}\dot{T}^*\mathbb{S}^n_+ = {}^{\mathrm{sc}}T^*\mathbb{S}^n_+$, $\mathrm{WF}_{\mathrm{Sc}}$ is just the scattering wave front set $\mathrm{WF}_{\mathrm{sc}}$ introduced by Melrose, [23], which in turn is closely related to the usual wave front set via the Fourier transform. Thus, for $(\omega,\xi) \in {}^{\mathrm{sc}}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+$, considered as $\mathbb{S}^{n-1}\times\mathbb{R}^n = \partial\mathbb{S}^n_+ \times \mathbb{R}^n$, $(\omega,\xi)\notin \mathrm{WF}_{\mathrm{sc}}(u)$ means that there exists $\phi \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ such that $\phi(\omega)\neq 0$ and $\mathcal{F}(\phi u)$ is $\mathcal{C}^\infty$ near ξ. If we employed the usual conic terminology instead of the compactified one, we would think of ϕ as a conic cut-off function in the direction ω. Thus, $\mathrm{WF}_{\mathrm{sc}}$ at infinity is analogous to WF with the role of position and momentum reversed. The definition of $\mathrm{WF}_{\mathrm{Sc}}(u)$ is more complicated, but if $u = \psi(H)v$ for some $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ (any other operator in $\Psi^{-\infty,0}_{\mathrm{Sc}}(\bar{X},\mathcal{C})$ would do instead of $\psi(H)$), then the following is a sufficient condition for $(\omega,\xi_a) \in {}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a$, considered as $C'_a \times X_a$, not to be in $\mathrm{WF}_{\mathrm{Sc}}(u)$. Suppose that there exists $\phi \in \mathcal{C}^\infty(\mathbb{S}^n_+)$, $\phi(\omega)\neq 0$, and $\rho \in \mathcal{C}^\infty_c(X_a)$, $\rho(\xi_a)\neq 0$, and $((\pi^a)^*\rho)\mathcal{F}(\phi u) \in \mathcal{S}(\mathbb{R}^n) = \mathcal{S}(X_0)$. Then $(\omega,\xi_a)\notin \mathrm{WF}_{\mathrm{Sc}}(u)$. We also remark that we state all of the following results for the absolute wave front sets (i.e. we work modulo Schwartz functions), but they have complete analogues for the relative wave front sets (working modulo weighted Sobolev spaces); indeed, it is the latter that is used to prove the results on the former.

Our main result is then the following theorem, in which we allow arbitrary thresholds, and which describes the relationship between $\mathrm{WF}_{\mathrm{Sc}}(u)$ and generalized broken bicharacteristics, if, for example, $(H-\lambda)u = 0$. Note that if $(H-\lambda)u=0$, then $u = \psi(H)u$ for $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\psi \equiv 1$ near λ, so the above description of $\mathrm{WF}_{\mathrm{Sc}}(u)$ is applicable.

Theorem 2.2. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $\lambda \notin \Lambda_1$. Then*

$$\mathrm{WF}_{Sc}(u) \setminus \mathrm{WF}_{Sc}((H-\lambda)u) \tag{2.23}$$

is a union of maximally extended generalized broken bicharacteristics of $H-\lambda$ in $\dot{\Sigma}(\lambda)\setminus \mathrm{WF}_{Sc}((H-\lambda)u)$.

We remark that the statement of the theorem is empty at points $\xi_0 \in {}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a$ at which ${}^{\mathrm{sc}}H^b_g(\tilde{\xi}_0) = 0$ for some $\tilde{\xi}_0 \in \hat{\pi}_b^{-1}(\xi_0)$ and some b with $C_a \subset C_b$. Indeed, at such points the constant curve ($\gamma(t) = \tilde{\xi}_0$ for all t in some interval) is a generalized broken bicharacteristic. A simple calculation shows that the set of these points ξ_0 is $R_+(\lambda)\cup R_-(\lambda)$, where

$$R_\pm(\lambda) = \{\xi = (\omega,\xi_a) \in {}^{\mathrm{sc}}T^*_{C'_a}\bar{X}_a :\ \exists b,\ C_a \subset C_b,\ \lambda - \tau(\xi)^2 \in \mathrm{spec}_{pp}(H^b),\ \pm\tau(\xi) = |\xi_a|\} \tag{2.24}$$

are the incoming $(+)$ and outgoing $(-)$ radial sets respectively, and τ is the sc-dual variable of the boundary defining function x, so in terms of the Euclidean variables

$$\tau = -\frac{w\cdot\xi}{|w|}; \tag{2.25}$$

see the next section for further details. Hence, the theorem permits singularities to emerge 'out of nowhere' at the radial sets. Although we do not prove that this indeed does happen, based on general principles, this appears fairly likely.

Moreover, the optimality of Theorem 2.2 if $\Lambda_1 = \{0\}$ follows from [9, 33], see the remarks about this in [37]; the amplitude of the reflected 'wave' is given (to top order) by the appropriate subsystem S-matrix.

There is a similar result for $\mathrm{WF}_{\mathrm{Sc}}(u)$, $u = R(\lambda + i0)f$; namely that $\mathrm{WF}_{\mathrm{Sc}}(u) \setminus \mathrm{WF}_{\mathrm{Sc}}(f)$ is the image of $\mathrm{WF}_{\mathrm{Sc}}(f) \cup R_-(\lambda)$ under forward propagation, if e.g. $f \in H^{r,s}$, $s > 1/2$. The set $R_-(\lambda)$ appears here since there can be maximally extended generalized broken bicharacteristics which are either not disjoint from $R_-(\lambda)$, or simply whose closure is not disjoint from $R_-(\lambda)$. In particular, even if f is Schwartz, $\mathrm{WF}_{\mathrm{Sc}}(u)$ is not necessarily a subset of $R_-(\lambda)$, rather a subset of its image under forward propagation. Indeed, by duality, this is exactly what gives rise to the conditions on $\mathrm{WF}_{\mathrm{Sc}}(f)$ under which $u = R(\lambda + i0)f$ can be defined. To make it easier to state these results, we make the following definition.

Definition 2.3. Suppose $K \subset \dot{\Sigma}(\lambda)$. The image $\Phi_+(K)$ of K under the forward broken bicharacteristic relation is defined as

(2.26)
$$\begin{aligned}\Phi_+(K) =& \{\xi_0 \in \dot{\Sigma}(\lambda) : \ \exists \text{ a generalized broken bicharacteristic } \gamma : (-\infty, t_0] \to \dot{\Sigma}(\lambda) \\ & \text{s.t. } \gamma(t_0) = \xi_0,\ \overline{\gamma((-\infty, t_0])} \cap K \neq \emptyset\}.\end{aligned}$$

The image $\Phi_-(K)$ of K under the backward broken bicharacteristic relation is defined similarly, with $[t_0, +\infty)$ in place of $(-\infty, t_0]$.

Note that $\Phi_+(K) = \cup_{\xi \in K} \Phi_+(\{\xi\})$ directly from the definition. The result on the boundary values of the resolvent is then:

Theorem 2.4. *Suppose that $\lambda \notin \Lambda$, $f \in \mathcal{S}(\mathbb{R}^n)$, and let $u = R(\lambda + i0)f$. Then $\mathrm{WF}_{Sc}(u) \subset \Phi_+(R_-(\lambda))$. Moreover, $R(\lambda + i0)$ extends by continuity to $v \in \mathcal{S}'(\mathbb{R}^n)$ with $\mathrm{WF}_{Sc}(v) \cap \Phi_-(R_+(\lambda)) = \emptyset$, and for such v,*

$$\mathrm{WF}_{Sc}(R(\lambda + i0)v) \subset \Phi_+(\mathrm{WF}_{Sc}(v)) \cup \Phi_+(R_-(\lambda)). \tag{2.27}$$

The scattering matrices $S_{\beta\alpha}(\lambda)$ of H with incoming channel α, outgoing channel β can be defined either via the wave operators, or via the asymptotic behavior of generalized eigenfunctions. It was shown in [35] that the two are the same, up to normalization (free motion is factored out in the wave operator definition); here we briefly recall the second definition. We first state it for short-range V_c (V_c polyhomogeneous of order -2 for all c). Thus, for $\lambda \in (\epsilon_\alpha, \infty) \setminus \Lambda$ and $g \in \mathcal{C}_c^\infty(C'_a)$, there is a unique $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $(H - \lambda)u = 0$, and u has the form

$$u = e^{-i\sqrt{\lambda - \epsilon_\alpha}\, r} r^{-\dim C_a/2}((\pi^a)^* \psi_\alpha) v_- + R(\lambda + i0)f, \tag{2.28}$$

where $v_- \in \mathcal{C}^\infty(\mathbb{S}^n_+)$, $v_-|_{C_a} = g$, and $f \in H_{\mathrm{sc}}^{\infty, 1/2 + \epsilon'}(\mathbb{S}^n_+)$, $\epsilon' > 0$. The Poisson operator $P_{\alpha,+}(\lambda)$ is the map

$$P_{\alpha,+}(\lambda) : \mathcal{C}_c^\infty(C'_a) \to \mathcal{S}'(\mathbb{R}^n) \text{ defined by } P_{\alpha,+}(\lambda)g = u. \tag{2.29}$$

The term $R(\lambda + i0)f$ has distributional asymptotics of a similar form 'at the channel β', i.e. of the form $e^{i\sqrt{\lambda - \epsilon_\beta}\, r} r^{-\dim C_b/2}((\pi^b)^* \psi_\beta) v_{\beta,+}$, see [35] for the precise definitions.

Only minor modifications are necessary for $V_c \in S^{-1}_{\mathrm{phg}}(X^c)$. Namely, write

$$I_a = \sum_{b \not\leq a} V_b,\ \tilde{I}_a = (r_a I_a)|_{C_a} \in \mathcal{C}^\infty(C_a), \tag{2.30}$$

I_a is $\mathcal{C}^\infty$ near C_a' with simple vanishing at C_a' (since $b \not\leq a$ means $C_a \not\subset C_b$, hence $C_a \cap C_b \subset C_{a,\mathrm{sing}}$), so $r_a I_a$ is $\mathcal{C}^\infty$ there. Then the asymptotics in (2.28) must be replaced by

$$e^{-i\sqrt{\lambda-\epsilon_\beta}r} r^{-\dim C_a/2} r^{i\tilde{I}_a/2\sqrt{\lambda-\epsilon_\alpha}}((\pi^a)^*\psi_\alpha)v_- \tag{2.31}$$

plus lower order terms; see Section 9 for details.

The scattering matrix $S_{\beta\alpha}(\lambda)$ maps $g = v_-|_{C_a}$ to $v_{\beta,+}|_{C_b'}$. It is also given by the formula

$$S_{\beta\alpha}(\lambda) = \frac{1}{2i\sqrt{\lambda - \epsilon_\beta}}((H-\lambda)\tilde{P}_{\beta,-}(\lambda))^* P_{\alpha,+}(\lambda), \tag{2.32}$$

$\lambda > \max(\epsilon_\alpha, \epsilon_\beta)$, $\lambda \notin \Lambda$. Here $\tilde{P}_{\beta,-}(\lambda)$ is a microlocalized version of the outgoing Poisson operator, microlocalized near the outgoing region for β, i.e. where τ is near $-\sqrt{\lambda-\epsilon_\beta}$, see [35]. In fact, we can simply take $\tilde{P}_{\beta,-}(\lambda)$ to be a microlocal (cut-off) parametrix for $P_{\beta,-}(\lambda)$. This formula is closely related to that of Isozaki and Kitada [17].

A very good parametrix, $\tilde{P}_{\alpha,+}(\lambda)$, for $P_{\alpha,+}(\lambda)$ in the region of phase space where τ is close to $\sqrt{\lambda-\epsilon_\alpha}$ has been constructed by Skibsted [32] in the short-range and by Bommier [1] in the long-range setting, under the assumption that $\epsilon_\alpha \in \mathrm{spec}_d(H^a)$. If we instead assume that the V_c are all Schwartz, then the trivial (product type) construction gives the desired parametrix. Their constructions enable us to deduce the structure of the S-matrices immediately from our propagation theorem, Theorem 2.4 via (2.32) and

$$P_{\alpha,+}(\lambda) = \tilde{P}_{\alpha,+}(\lambda) - R(\lambda + i0)(H-\lambda)\tilde{P}_{\alpha,+}(\lambda). \tag{2.33}$$

Since the parametrix (near the incoming or outgoing sets) is important for turning the results on the propagation of singularities to wave front set results, in all our results on the Poisson operators and S-matrices $S_{\beta\alpha}(\lambda)$ in this paper we make the following assumption:

$$\text{either } \epsilon_\alpha \in \mathrm{spec}_d(H^a) \text{ and } \epsilon_\beta \in \mathrm{spec}_d(H^b), \text{ or } V_c \in \mathcal{S}(X^c) \text{ for all } c. \tag{2.34}$$

It is easy to describe the wave front set of $\tilde{P}_{\alpha,+}(\lambda)g$, $g \in \mathcal{C}_c^{-\infty}(C_a')$, near its 'beginning point', i.e. near the $(\alpha,+)$-incoming set. Namely, it is the union of integral curves of ${}^{\mathrm{sc}}H_g^a$ (in ${}^{\mathrm{sc}}T^*_{C_a'}\bar{X}_a$) (which are in particular bicharacteristics of $H-\lambda$, hence broken bicharacteristics), one integral curve for each $\zeta \in \mathrm{WF}(g) \subset S^*C_a'$; we denote these integral curves by $\gamma_{\alpha,-}(\zeta)$. It is actually convenient to replace the parameter t of the integral curve by s, the arclength parameter of its projection to C_a. The relationship between these two is that if we write $s = S(t)$, then S solves the ODE $dS/dt = 2(\lambda - \epsilon_\alpha - \tau(\gamma(t))^2)^{1/2}$. The reparameterized integral curves are then given by

$$\tau_a = \sqrt{\lambda-\epsilon_\alpha}\cos(s-s_0),\ (y_a,\mu_a) = \sqrt{\lambda-\epsilon_\alpha}\sin(s-s_0)\exp((s-s_0)^{\mathrm{sc}}H_g^a)(\zeta) \tag{2.35}$$

where $s \in (s_0, s_0+\pi)$. This defines $\gamma_{\alpha,-}(\zeta)$ up to replacing t by $t - t_1$ for any fixed $t_1 \in \mathbb{R}$, so we are abusing the notation slightly. Due to (2.33), Theorem 2.4 describes $\mathrm{WF}_{\mathrm{Sc}}(P_{\alpha,+}(\lambda)g)$ elsewhere. A similar result also applies for $\tilde{P}_{\beta,-}(\lambda)$; in this case one simply has to replace the range of the arclength parameter by $s \in (s_0 - \pi, s_0)$. We denote the corresponding integral curves by $\gamma_{\beta,+}(\zeta)$.

Definition 2.5. The forward broken bicharacteristic relation with initial channel α is defined to be the relation $\mathcal{R}_{\alpha,+} \subset \dot\Sigma(\lambda) \times S^*C'_a$ given by

$$\mathcal{R}_{\alpha,+} = \{(\zeta,\xi) \in \dot\Sigma(\lambda) \times S^*C'_a :\ \Phi_-(\{\xi\}) \cap \gamma_{\alpha,-}(\zeta) \neq \emptyset\}. \tag{2.36}$$

The backward broken bicharacteristic relation with initial channel β, denoted by $\mathcal{R}_{\beta,-}$ is defined similarly, with Φ_+ in place of Φ_- and $\gamma_{\beta,+}$ in place of $\gamma_{\alpha,-}$. Finally, the forward broken bicharacteristic relation with initial channel α, final channel β, $\mathcal{R}_{\beta\alpha} \subset S^*C'_b \times S^*C'_a$ is defined as the composite relation of $\mathcal{R}_{\alpha,+}$ and $\mathcal{R}_{\beta,-}^{-1}$:

$$\mathcal{R}_{\beta\alpha} = \{(\zeta,\zeta') \in S^*C'_b \times S^*C'_a :\ \exists \xi \in \dot\Sigma(\lambda), (\zeta,\xi) \in \mathcal{R}_{\alpha,+},\ (\zeta',\xi) \in \mathcal{R}_{\beta,-}\}. \tag{2.37}$$

Note that $(\zeta,\xi) \in \mathcal{R}_{\alpha,+}$ thus means that there exists a generalized broken bicharacteristic $\gamma : \mathbb{R} \to \dot\Sigma(\lambda)$ and $t_0 \in \mathbb{R}$ such that $\gamma|_{(-\infty,t_0]}$ is given by $\gamma_{\alpha,-}(\zeta)$, and $\xi \in \overline{\gamma([t_0,+\infty))}$. Thus, $\mathcal{R}_{\alpha,+}$ should be thought of as the relation induced by Φ_- 'at channel α' as time goes to $-\infty$.

If $\mathcal{R} \subset B \times A$ is a relation, $K \subset A$, by $\mathcal{R}(K)$ we mean $\{\xi \in B :\ \exists \zeta \in K,\ (\xi,\zeta) \in \mathcal{R}\}$. Similarly, if $U \subset B$, by $\mathcal{R}^{-1}(U)$ we mean $\{\zeta \in A :\ \exists \xi \in U,\ (\xi,\zeta) \in \mathcal{R}\}$. We call $\mathcal{R}(K)$ the image of K under $\mathcal{R}$. Thus, if $K \subset S^*C'_a$,

$$\mathcal{R}_{\alpha,+}(K) = \{\xi \in \dot\Sigma(\lambda) :\ \exists \zeta \in K,\ \Phi_-(\{\xi\}) \cap \gamma_{\alpha,-}(\zeta) \neq \emptyset\},$$

and if $U \subset \dot\Sigma(\lambda)$, then

$$\mathcal{R}_{\alpha,+}^{-1}(U) = \{\zeta \in S^*C'_a :\ \exists \xi \in U,\ \Phi_-(\{\xi\}) \cap \gamma_{\alpha,-}(\zeta) \neq \emptyset\}.$$

This definition, (2.33) and Theorem 2.4 immediately prove the following proposition.

Proposition 2.6. *Suppose that H is a many-body Hamiltonian, $\lambda \notin \Lambda$, and* (2.34) *holds. Suppose also that $g \in \mathcal{C}_c^\infty(C'_a)$. Then*

$$\mathrm{WF}_{Sc}(P_{\alpha,+}(\lambda)g) \setminus R_+(\lambda) \subset \Phi_+(R_-(\lambda)). \tag{2.38}$$

In addition, $P_{\alpha,+}(\lambda)$ extends by continuity from $\mathcal{C}_c^\infty(C'_a)$ to distributions $g \in \mathcal{C}_c^{-\infty}(C'_a)$ with $(R_+(\lambda) \times \mathrm{WF}(g)) \cap \mathcal{R}_{\alpha,+} = \emptyset$ (i.e. $\mathcal{R}_{\alpha,+}(\mathrm{WF}(g)) \cap R_+(\lambda) = \emptyset$). If g is such a distribution, then

$$\mathrm{WF}_{Sc}(P_{\alpha,+}(\lambda)g) \setminus R_+(\lambda) \subset \Phi_+(R_-(\lambda)) \cup \mathcal{R}_{\alpha,+}(\mathrm{WF}(g)). \tag{2.39}$$

One of the main features of (2.38) is that in general $\mathrm{WF}_{\mathrm{Sc}}(P_{\alpha,+}(\lambda)g)$ cannot be expected to be contained in the radial sets; one also has to include the image of the outgoing radial set under forward propagation in the statement. As a corollary, (2.32) shows that in general $S_{\beta\alpha}(\lambda)$ does not map smooth incoming data to smooth outgoing data. However, if β is a two-cluster channel, every generalized broken bicharacteristic γ such that for some $t_0 \in \mathbb{R}$, $\gamma|_{(-\infty,t_0]}$ is given by $\gamma_{\beta,+}(\zeta)$, $\zeta \in S^*C'_b$, is actually equal to $\gamma_{\beta,+}(\zeta)$ for all times, and, b being a 2-cluster, $\gamma_{\beta,+}(\zeta)$ never intersects the radial sets, and as $t \to \pm\infty$, $\gamma_{\beta,+}(\zeta)(t)$ goes to $R_\mp(\lambda)$. Thus, if β is a 2-cluster, α is any cluster, $S_{\beta\alpha}(\lambda)$ maps smooth functions to smooth functions. On the other hand, if β is the free channel 0, then the absence of positive thresholds gives a similar conclusion.

Corollary 2.7. *Suppose that H is a many-body Hamiltonian, $\lambda \notin \Lambda$, and* (2.34) *holds. Suppose $g \in \mathcal{C}_c^\infty(C'_a)$. Then $\mathrm{WF}(S_{\beta\alpha}(\lambda)g) \subset \mathcal{R}_{\beta,-}^{-1}(R_-(\lambda))$. Thus, if β is the free channel or it is a two-cluster channel, then $S_{\beta\alpha}(\lambda)g$ is $\mathcal{C}^\infty$.*

Our theorem on the wave front relation of the S-matrix is then the following.

Theorem 2.8. *Suppose that H is a many-body Hamiltonian, $\lambda \notin \Lambda$, and (2.34) holds. Then $S_{\beta\alpha}(\lambda)$ extends by continuity from $\mathcal{C}_c^\infty(C'_a)$ to distributions $g \in \mathcal{C}_c^{-\infty}(C'_a)$ with $\mathcal{R}_{\alpha,+}(\mathrm{WF}(g)) \cap R_+(\lambda) = \emptyset$. If g is such a distribution, then*

$$\mathrm{WF}(S_{\beta\alpha}(\lambda)g) \subset \mathcal{R}_{\beta,-}^{-1}(R_-(\lambda)) \cup \mathcal{R}_{\beta\alpha}(\mathrm{WF}(g)). \tag{2.40}$$

If $\Lambda_1 = \{0\}$, then maximally extended generalized broken bicharacteristics are essentially the lift of generalized broken geodesics on $\mathbb{S}^{n-1}$ of length π, so in this case we recover the following result of [37].

Corollary 2.9. *If no subsystem of H has bound states and $\lambda > 0$, then the wave front relation of $S_{00}(\lambda)$ is given by the broken geodesic relation on $\mathbb{S}^{n-1}$, broken at $\mathcal{C}$, at distance π.*

It may seem that our results are too weak in the sense that some broken bicharacteristics can be continued in many ways when they hit a collision plane. However, it should be noted that not every broken bicharacteristic hits a collision plane; indeed, if the dimension of a collision plane increases, more broken bicharacteristics will hit it, but each will generate a lower dimensional family of continuations. We can make this precise in terms of a Lagrangian characterization of the wave front relation of the S-matrices. To do so, recall first that the wave front relation of operators mapping $\mathcal{C}_c^\infty(C'_a)$ to $\mathcal{C}^{-\infty}(C'_b)$ is often understood as a subset of $T^*C'_b \times T^*C'_a \setminus 0$ corresponding to the wave front set of its Schwartz kernel, with the sign of the second component, the one in $T^*C'_a$, switched. For example, in this sense, the wave front relation of an FIO, say P, is given by a homogeneous canonical relation, i.e. by a Lagrangian submanifold $\tilde{\Lambda}$ of $T^*C'_b \times T^*C'_a$, Lagrangian with respect to the usual twisted symplectic form $\omega_b - \omega_a$, that is conic with respect to the diagonal $\mathbb{R}^+$ action in $T^*C'_b \times T^*C'_a$. Here ω_a denotes the canonical symplectic form on T^*C_a, etc. The wave front set mapping properties of such an operator are that Pu is defined if $(0 \times \mathrm{WF}(u)) \cap \tilde{\Lambda} = \emptyset$, and if Pu is thus defined,

$$\begin{aligned}\mathrm{WF}(Pu) \subset &\tilde{\Lambda}(\mathrm{WF}(u)) \cup \{\xi \in T^*C'_b \setminus 0 :\ (\xi, 0) \in \tilde{\Lambda}\} \\ &= \{\xi \in T^*C'_b \setminus 0 :\ (\xi, 0) \in \tilde{\Lambda} \text{ or } \exists \xi' \in \mathrm{WF}(u),\ (\xi, \xi') \in \tilde{\Lambda}\}.\end{aligned} \tag{2.41}$$

Note that $\mathrm{WF}(u)$ is a conic subset of $T^*C'_a$, so it is better to regard it as a subset of $S^*C'_a$ (quotient out by the $\mathbb{R}^+$ action on $T^*C'_a \setminus 0$); indeed, this is what we have done. But every element ζ of $S^*C'_a$ (thought of as the quotient bundle) has several representatives ξ in $T^*C'_a$. Correspondingly, the wave front relation, understood as a relation connecting $S^*C'_a$ and $S^*C'_b$, i.e. as a subset of $S^*C'_b \times S^*C'_a$, only determines the corresponding conic relation on $(T^*C'_b \setminus 0) \times (T^*C'_a \setminus 0)$ up to the rescaling of one factor with respect to the other. Thus, our results by themselves cannot pinpoint the wave front set of the Schwartz kernel of $S_{\beta\alpha}(\lambda)$ itself, only its image under the quotient map with respect to the $\mathbb{R}^+ \times \mathbb{R}^+$ action. Note, however, that $\tilde{\Lambda} \cap (T^*C'_b \times 0)$ and $\tilde{\Lambda} \cap (0 \times T^*C'_a)$ show up in the mapping properties of P, namely whether it can be applied to all distributions, and whether it maps smooth functions to smooth functions. Thus, by Theorem 2.8 we would expect, if $S_{\beta\alpha}(\lambda)$ were an FIO, that the (twisted) wave front set of its kernel lies in

$$(\mathcal{R}_{\beta,-}^{-1}(R_-(\lambda)) \times 0) \cup (0 \times \mathcal{R}_{\alpha,+}^{-1}(R_+(\lambda)) \cup \tilde{\Lambda}_0, \tag{2.42}$$

where $\tilde{\Lambda}_0 \subset (T^*C'_b \setminus 0) \times (T^*C'_a \setminus 0)$ is conic Lagrangian and projects to $\mathcal{R}_{\beta\alpha}$ under the quotient map in both factors. While, as explained, our results as stated do not prove that the wave front set of the kernel of $S_{\beta\alpha}(\lambda)$ lies in (2.42), we can prove its 'quotiented' version, at least if 0 is the only threshold of H, and obtain partial results if the set of thresholds is discrete.

To see what the expected result is, recall that if the set of thresholds is discrete, the generalized broken bicharacteristics are actually broken bicharacteristics with a finite number of breaks (and a uniform bound on this finite number). Correspondingly, we can associate to each generalized broken bicharacteristic one of a finite number of collision patterns which describe from which clusters in which sequence the bicharacteristics reflect, and along which collision plane at which kinetic energy they travel meanwhile. Each of these collision patterns can be described as a composition of smooth Lagrangian relations. If all the compositions are transversal, it follows automatically that the composite relation is also smooth and Lagrangian. (The other criteria for such a conclusion are automatically satisfied in our setting.) In Appendix B we show transversality if 0 is the only threshold, and discuss other cases when it holds. We briefly outline these below. First, however, we mention that even if the intersection fails to be transversal, and presumably fails to be clean as well, the composite relation can probably be described as a stratified space, since each of the elementary Lagrangians is algebraic. Such a treatment would, however, go beyond the scope of this paper, where the Lagrangian property is only used to illustrate the size of the broken bicharacteristic relation and to connect it to FIOs.

An important property of broken bicharacteristics is whether, upon breaking at C'_c, they become tangential to C'_c. We will say that a broken bicharacteristic has the *non-tangential breaking property* if it never becomes tangential to C'_c upon hitting it. Thus, a broken bicharacteristic γ having this property means that for all $t_0 \in \mathbb{R}$, $\gamma(t_0) \in {}^{\mathrm{sc}}T^*_{C'_c}\bar{X}_c$ implies that there exists a (small) $\epsilon > 0$, such that the inverse image of ${}^{\mathrm{sc}}T^*_{C'_c}\bar{X}_c$ under $\gamma|_{(t_0-\epsilon,t_0+\epsilon)}$ is either a single point, t_0, or it is the whole interval $(t_0-\epsilon, t_0+\epsilon)$ (hence $\gamma|_{(t_0-\epsilon,t_0+\epsilon)}$ maps into ${}^{\mathrm{sc}}T^*_{C'_c}\bar{X}_c$, is an integral curve of ${}^{\mathrm{sc}}H^c_g$, and in particular does not break). In the former case the broken bicharacteristic is said to be normal to C'_c, in the latter case it is said to be tangential. Note that if we replaced $(t_0-\epsilon, t_0+\epsilon)$ by $(t_0-\epsilon, t_0]$ or $[t_0, t_0+\epsilon)$, the conclusion would follow automatically since broken bicharacteristics are piecewise (projections of) integral curves of ${}^{\mathrm{sc}}H_g$. Thus, the non-tangential breaking property amounts to stating that broken bicharacteristics are normal both immediately prior to and immediately following a collision. Below, when we refer to the *non-tangential part of the broken bicharacteristic relation*, we mean the subset of $\mathcal{R}_{\beta\alpha}$, etc., which replaces 'generalized broken bicharacteristic' by 'non-tangentially breaking broken bicharacteristics' in Definition 2.3.

If there are no bound states in the proper subsystems, or indeed if $\Lambda_1 = \{0\}$, then by the conservation of kinetic energy, any broken bicharacteristic that has a break point ξ at C'_c, must either have been tangential both before and after the collision, and hence had actually no break at ξ, or both the incoming and the outgoing bicharacteristic segments are normal to C'_c. Thus, $\Lambda_1 = \{0\}$ implies that every broken bicharacteristic has the non-tangential breaking property. We prove in Appendix B that for each of the collision patterns, the non-tangentially breaking part of the corresponding relation, the composition of the smooth Lagrangian relations is indeed transversal, hence the composite gives a (typically not closed) conic

Lagrangian submanifold of $T^*(C'_b \times C'_a) \setminus 0$. We remark that under the absence of bound states assumption, the proof of the result is essentially the same as in the three-body setting [33, 9]; Hassell's proof employing Jacobi vector fields on the sphere is particularly easy (and transparent!) to adapt.

We thus prove the following theorem.

Theorem 2.10. *Suppose that H is a many-body Hamiltonian, $\lambda \notin \Lambda$, and (2.34) holds. Suppose also that Λ_1 is discrete. The part of the wave front relation of $S_{\beta\alpha}(\lambda)$ induced by non-tangential breaking broken bicharacteristics is a subset of the projection of a finite union of conic Lagrangian submanifolds of $T^*(C'_a \times C'_b) \setminus 0$, given by the broken bicharacteristic relation of H. Some of the Lagrangians may lie in $T^*C'_a \times 0$ or $0 \times T^*C'_b$ where 0 denotes the zero section.*

Corollary 2.11. *If $\Lambda_1 = \{0\}$, $\lambda > 0$, then the whole wave front relation of $S_{00}(\lambda)$ is a subset of the projection of a finite union of conic Lagrangian submanifolds of $T^*(C'_0 \times C'_0) \setminus 0$ as above.*

The statements of this theorem and corollary show that, e.g. if $\Lambda_1 = \{0\}$, the wave front relation of the S-matrix has the correct structure to be the wave front relation of a Fourier integral operator (FIO). However, the construction of such an FIO would require a much better understanding of the symplectic geometry of, and related analysis on, manifolds with corners that are equipped with certain boundary fibrations (corresponding to the resolved space $[\bar{X}; \mathcal{C}]$ and the fibration given by the blow-down map which are discussed in the next section). In relatively simple settings this has been discussed in [20, 10], but much progress is needed for dealing with the general case.

3. Scattering geometry and analysis

Next, we recall from [23] Melrose's definition of the Lie algebra of 'scattering vector fields' $\mathcal{V}_{\mathrm{sc}}(\bar{X})$, defined for every manifold with boundary $\bar{X}$. Thus,

$$\mathcal{V}_{\mathrm{sc}}(\bar{X}) = x\mathcal{V}_{\mathrm{b}}(\bar{X}) \tag{3.1}$$

where $\mathcal{V}_{\mathrm{b}}(\bar{X})$ is the set of smooth vector fields on $\bar{X}$ which are tangent to $\partial\bar{X}$. If $(x, y_1, \dots, y_{n-1})$ are coordinates on $\bar{X}$ where x is a boundary defining function, then locally a basis of $\mathcal{V}_{\mathrm{sc}}(\bar{X})$ is given by

$$x^2\partial_x,\ x\partial_{y_j},\ j = 1, \dots, n-1. \tag{3.2}$$

Correspondingly, there is a vector bundle ${}^{\mathrm{sc}}T\bar{X}$ over $\bar{X}$, called the scattering tangent bundle of $\bar{X}$, such that $\mathcal{V}_{\mathrm{sc}}(\bar{X})$ is the set of all smooth sections of ${}^{\mathrm{sc}}T\bar{X}$:

$$\mathcal{V}_{\mathrm{sc}}(\bar{X}) = \{v \in \mathcal{C}^\infty(\bar{X}; {}^{\mathrm{sc}}T\bar{X}) :\ \forall p \in \bar{X},\ v_p \in {}^{\mathrm{sc}}T_p\bar{X}\}. \tag{3.3}$$

The dual bundle of ${}^{\mathrm{sc}}T\bar{X}$ (called the scattering cotangent bundle) is denoted by ${}^{\mathrm{sc}}T^*\bar{X}$. Thus, covectors $v \in {}^{\mathrm{sc}}T^*_p\bar{X}$, p near $\partial\bar{X}$, can be written as

$$v = \tau\,\frac{dx}{x^2} + \mu \cdot \frac{dy}{x}. \tag{3.4}$$

Hence, we have local coordinates (x, y, τ, μ) on ${}^{\mathrm{sc}}T^*\bar{X}$ near $\partial\bar{X}$. The scattering density bundle ${}^{\mathrm{sc}}\Omega\bar{X}$ is the density bundle associated to ${}^{\mathrm{sc}}T^*\bar{X}$, so locally near $\partial\bar{X}$ it is spanned by $x^{-n-1}\,dx\,dy$ over $\mathcal{C}^\infty(\bar{X})$. Finally, $\mathrm{Diff}_{\mathrm{sc}}(\bar{X})$ is the algebra of differential operators generated by the vector fields in $\mathcal{V}_{\mathrm{sc}}(\bar{X})$; $\mathrm{Diff}^m_{\mathrm{sc}}(\bar{X})$ stands for scattering differential operators of order (at most) m.