THE WAVE EQUATION ON ASYMPTOTICALLY DE SITTER-LIKE SPACES

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ABSTRACT. In this paper we obtain the asymptotic behavior of solutions of the Klein-Gordon equation on Lorentzian manifolds (X°, g) which are de Sitter-like at infinity. Such manifolds are Lorentzian analogues of the so-called Riemannian conformally compact (or asymptotically hyperbolic) spaces. Under global assumptions on the (null)bicharacteristic flow, namely that the boundary of the compactification X is a union of two disjoint manifolds, $Y_\pm$, and each bicharacteristic converges to one of these two manifolds as the parameter along the bicharacteristic goes to $+\infty$, and to the other manifold as the parameter goes to $-\infty$, we also define the scattering operator, and show that it is a Fourier integral operator associated to the bicharacteristic flow from Y_+ to Y_-.

1. INTRODUCTION

Consider a de Sitter-like pseudo-Riemannian metric g of signature $(1, n-1)$ on an n-dimensional ($n \geq 2$) manifold X, with boundary Y, which near Y is of the form

$$g = \frac{dx^2 - h}{x^2}, \tag{1.1}$$

h a smooth symmetric 2-cotensor on X such that with respect to some product decomposition of X near Y, $X = Y \times [0, \epsilon)_x$, $h|_Y$ is a section of $T^*Y \otimes T^*Y$ (rather than merely $T^*_Y X \otimes T^*_Y X$) and is a Riemannian metric on Y. Let the wave operator $\Box$ be the Laplace-Beltrami operator associated to this metric, and let $P = P(\lambda) = \Box - \lambda$ be the Klein-Gordon operator, $\lambda \in \mathbb{R}$.

Such metrics are Lorentzian analogues of Riemannian asymptotically hyperbolic (or so-called conformally compact) metrics, studied by Mazzeo and Melrose [23] and others extensively. Recall that these conformally compact metrics are of the form $x^{-2}(dx^2 + h')$, with $h'|_Y$ Riemannian as above. These have been of great interest in mathematical physics, see e.g. the works of Fefferman and Graham [10], Graham and Lee [14] and Anderson [3], due to their relation to (Riemannian) solutions of Einstein's equations; note that hyperbolic space actually solves Einstein's equations. De Sitter space itself also satisfies these equations (with positive cosmological constant), and Anderson and Chruściel also studied the geometry of asymptotically

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de Sitter spaces [1, 2, 4]. This was done from the point of view of solutions of Einstein's equations, which are reduced to the study of the conformal metric $x^2 g$ – such a reduction cannot be performed for our problem. We postpone the discussion of actual de Sitter space and its relation to our generalized setting until after the statement of our main results; for these results the asymptotics stated above is much more relevant than the connection of the geometry to that of a particular Lorentzian symmetric space.

Returning to asympotically de Sitter spaces, below we consider solutions of $Pu = 0$. The *bicharacteristics* of P over X° are the integral curves of the Hamilton vector field of the principal symbol $\sigma_2(P)$ (given by the dual metric function) *inside the characteristic set* of P. As g is conformal to $dx^2 - h$, bicharacteristics of P are reparameterizations of bicharacteristics of $dx^2 - h$ (near Y, that is). Since g is complete, this means that the bicharacteristics γ of P have limits $\lim_{t\to\pm\infty} \gamma(t)$ in $S^*_Y X$, provided that they approach Y. While many of the results below are local in character, it is simpler to state a global result, for which we need to assume that

- (A1) $Y = Y_+ \cup Y_-$ with Y_+ and Y_- a union of connected components of Y
- (A2) each bicharacteristic γ of P converges to Y_+ as $t \to +\infty$ and to Y_- as $t \to -\infty$, or vice versa

Due to the conformality, the characteristic set $\Sigma(P)$ of P can be identified with a smooth submanifold of S^*X, transversal to ∂X, so $S^*_Y X \cap \Sigma(P)$ can be identified with two copies $S^*_\pm Y$ of S^*Y, one for each sign of the dual variable of x. Under our assumptions we thus have a classical scattering map $\mathcal{S}_{\mathrm{cl}} : S^*_+ Y_+ \to S^*_- Y_-$.

It is well-known, cf. [13], that (A1) and (A2) imply the existence of a global compactified 'time' function T, with $T \in \mathcal{C}^\infty(X)$, $T|_{Y_\pm} = \pm 1$, and the pullback of T to S^*X having positive/negative derivative along the Hamilton vector field inside the characteristic set $\Sigma(p)$ depending on whether the corresponding bicharacterstics tend to Y_+ or Y_-. Notice that $1 - x$ resp. $x - 1$ has the desired properties near Y_+ resp. Y_-, so the point is that a function like these can be extended to all of X. Moreover, such a function gives a fibration $T : X \to [-1, 1]$, hence X is in fact diffeomorphic to $[-1, 1] \times S$ for a compact manifold S. In particular, Y_+ and Y_- are both diffeomorphic to S. Denote the level set $T = t_0$ by S_{t_0}. With any choice of such a function T, a constant $t_0 \in (-1, 1)$, and a vector field V transversal to S_{t_0} (e.g. take the vector field corresponding to dT under the metric identification of TX° and T^*X°), P is strictly hyperbolic, and the Cauchy problem $Pu = 0$ in X°, $u|_{S_{t_0}} = \psi_0$, $Vu|_{S_{t_0}} = \psi_1$, $\psi_0, \psi_1 \in \mathcal{C}^\infty(S_{t_0})$ is well posed.

Theorem 1.1. *(See Theorem 6.1.) Let $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. Assuming (A1) and (A2), the solution u of the Cauchy problem has the form*

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-,\ v_\pm \in \mathcal{C}^\infty(X), \tag{1.2}$$

if $s_+(\lambda) - s_-(\lambda) = 2\sqrt{\frac{(n-1)^2}{4} - \lambda}$ is not an integer. If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in \mathcal{C}^\infty(X)$ by $v_- = \mathcal{C}^\infty(X) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$.

Conversely, the asymptotic behavior of $v_\pm$ either at Y_+ or at Y_- can be prescribed arbitrarily; see Theorem 5.5. Thus, assuming A1 and A2, if $s_+(\lambda) - s_-(\lambda)$ is not an integer, we show that given $g_\pm \in \mathcal{C}^\infty(Y_+)$ there exists a unique $u \in \mathcal{C}^\infty(X^\circ)$ such that $Pu = 0$ and which is of the form (1.2) and such that

$$v_+|_{Y_+} = g_+,\ v_-|_{Y_+} = g_-. \tag{1.3}$$

If $s_+(\lambda) - s_-(\lambda)$ is a non-zero integer, the same conclusion holds if we replace $v_- \in \mathcal{C}^\infty(X)$ by $v_- = \sum_{j=0}^{s_+(\lambda)-s_-(\lambda)-1} a_j x^j + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$, $a_j \in \mathcal{C}^\infty(Y)$; see Theorem 5.5. For $\lambda = \frac{(n-1)^2}{4}$, a similar results holds, with

$$u = x^{(n-1)/2} v_+ + x^{(n-1)/2} \log x\, v_-,\ v_\pm \in \mathcal{C}^\infty(X),\ v_\pm|_{Y_+} = g_\pm. \tag{1.4}$$

That is, for all $\lambda \in \mathbb{R}$, there is a unique solution of $Pu = 0$ with two pieces of 'Cauchy data' specified at Y_+. Note the contrast with the elliptic asymptotically hyperbolic problem (conformally compact Riemannian metrics): there one specifies one of the two pieces of the Cauchy data, but over all of Y (not only at Y_+), see [23]. (This elliptic behavior also shows up in other elliptic scattering problems, such as asymptotically Euclidean or conic spaces, see [27, 29].) The quantum scattering map is the map:

$$\mathcal{S} : \mathcal{C}^\infty(Y_+) \oplus \mathcal{C}^\infty(Y_+) \to \mathcal{C}^\infty(Y_-) \oplus \mathcal{C}^\infty(Y_-),\ \mathcal{S}(g_+, g_-) = (v_+|_{Y_-}, v_-|_{Y_-}).$$

Of course, the labelling of Y_+ and Y_- can be reversed, so $\mathcal{S}$ is invertible. In fact, it is useful to renormalize $\mathcal{S} = \mathcal{S}(\lambda)$ somewhat so that the two pieces of Cauchy data at infinity carry the same 'weight'. Let Δ'_h denote the operator which is Δ_h on the orthocomplement of the nullspace of Δ_h and is the identity on the nullspace, so Δ'_h is positive and invertible. Then the renormalization is

$$\begin{aligned}&\tilde{\mathcal{S}}(\lambda)\\ &= ((\Delta'_h)^{-s_+(\lambda)/2+n/4} \oplus (\Delta'_h)^{-s_-(\lambda)+n/4})\mathcal{S}(\lambda)((\Delta'_h)^{s_+(\lambda)/2-n/4} \oplus (\Delta'_h)^{s_-(\lambda)/2-n/4});\end{aligned}$$

this is analogous to using $A\psi_0$ in place of ψ_0 for the finite time Cauchy data, where $A \in \Psi^1(S_{t_0})$ elliptic, invertible. We show that:

Theorem 1.2. *(See Theorem 7.21.) Suppose that $s_+(\lambda) - s_-(\lambda)$ is not an integer, i.e. $\lambda \neq \frac{(n-1)^2 - m^2}{4}$, $m \in \mathbb{N}$. $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(\lambda)$ is an invertible elliptic 0th order Fourier integral operator with canonical relation given by $\mathcal{S}_{\mathrm{cl}}$, and $\mathcal{S}$ is a Fourier integral operator.*

Remark 1.3. The somewhat strange powers in the normalization correspond to making the map from Cauchy data at infinity to Cauchy data at time $t_0 \in (-1, 1)$ a FIO of order 0; see Proposition 7.20.

Note that the canonical relation is independent of λ. While our parametrix construction for $\mathcal{S}(\lambda)$ does not work apparently if $s_+(\lambda) - s_-(\lambda)$ is an integer due to the possible non-solvability of a model problem with the prescribed ansatz, it is expected that with more detailed analysis (changing the ansatz slightly to allow logarithmic terms in x) one can prove the theorem in this case as well. Moreover, we actually construct a parametrix for the solution operator $(g_+, g_-) \mapsto u$, and even if $s_+(\lambda) - s_-(\lambda)$ is an integer, the part of the operator corresponding to g_+ (i.e. with $g_- = 0$) can be constructed as a Fourier integral operator.

In addition, if g is *even*, i.e. there is a boundary defining function x such that only even powers of x appear in the Taylor series of g at ∂X expressed in geodesic normal coordinates (see [15] for the Riemannian case), then the $\log x$ terms in v_- disappear and our parametrix construction for $\mathcal{S}(\lambda)$ goes through provided that

$s_+(\lambda) - s_-(\lambda)$ is odd. In particular, this covers the actual d'Alembertian ($\lambda = 0$) if n is even.

For the Cauchy problem, we similarly have:

Theorem 1.4. *For $t_0 \in (-1,1)$ and for all $(\psi_0, \psi_1) \in \mathcal{C}^\infty(S_{t_0})^2$, let $u \in \mathcal{C}^\infty(X^\circ)$ denote the unique solution of the Cauchy problem $Pu = 0$ in X°, $u|_{S_{t_0}} = \psi_0$, $Vu|_{S_{t_0}} = \psi_1$. This solution u has asymptotic expansion as in* (1.2). *If $\lambda \neq \frac{(n-1)^2 - m^2}{4}$, $m \in \mathbb{N}$, the operators*

$$(\psi_0, \psi_1) \mapsto (v_+|_{Y_+}, v_-|_{Y_+}) \text{ and } (\psi_0, \psi_1) \mapsto (v_+|_{Y_-}, v_-|_{Y_-})$$

are both Fourier integral operators associated to the bicharacteristic flow.

The form of our results is quite similar to certain previous results in scattering theory, but the former are usually for a product type problem, namely the product of a Riemannian manifold with the real line, 'time', and for either the Schrödinger or the wave equation on these, and in this sense are less natural. Note that while the wave and Schrödinger equations behave very differently at high energies, at finite energies the behavior is quite similar, in the sense that it fits into the same framework (though the form of the results is different). This is particularly apparent in asymptotically Euclidean or conic settings. Indeed, physically scattering theory has arisen as the study of asymptotics of solutions to the Schrödinger equation, which itself is considered as a low energy limit of Klein-Gordon-type equations, and has been studied extensively both in the 2-body and in the general N-body settings, see e.g. [36, 8]. (In our problem, there are no 'finite energy' results of interest; 'finite energy' initial data are $\mathcal{C}^\infty$ on a compact manifold; on asymptotically Euclidean spaces, say, they are not necessarily Schwartz, which would be the trivial class there.) For product problems, one can use spectral analysis of the spatial Laplacian and the functional calculus to analyze either of these problems; indeed most of the existing results are such stationary results. Examples of results in product spaces include Friedlander's treatment of the radiation field in product spaces in which the spatial factor is Euclidean or asymptotically Euclidean [11, 12] and Sá Barreto's and Wunsch's improvement of these results (to a Fourier integral operator statement) and extension of the case where the spatial factor is asymptotically hyperbolic [34], as well as early work of Guillemin and Majda on sojourn times [16, 22]. It is worthwhile pointing out, however, that even in the elliptic settings, while there are no propagation phenomena in the interior of the manifold, at infinity (or at the boundary, if we compactified it), there are propagation phenomena, see e.g. [27]. Indeed, often the scattering matrix is a Fourier integral operator, as in the case of asymptotically Euclidean or conic spaces, see [29], or indeed in N-body settings, see [37, 17], although this is *not* the case in asymptotically hyperbolic spaces, where the scattering matrix is a pseudodifferential operator [23, 21].

To justify our terminology of asymptotically de Sitter spaces, we recall that de Sitter space is given by the hyperboloid $z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1$ in $\mathbb{R}^{n+1}$ equipped with the pull-back of the Lorentzian metric $dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2$. Introducing polar coordinates (r, θ) in the first n variables and writing $t = z_{n+1}$, the hyperboloid can be identified with $\mathbb{R}_t \times \mathbb{S}_\theta^{n-1}$ with the Lorentzian metric

$$\frac{dt^2}{t^2+1} - (t^2+1)\,d\theta^2,$$

with $d\theta^2$ being the standard Riemannian metric on the sphere. For $t > 1$, say, we let $x = t^{-1}$, and note that the metric becomes $\frac{(1+x^2)^{-1}\,dx^2-(1+x^2)\,d\theta^2}{x^2}$, which is of the form required by (1.1). An analogous formula holds for $t < -1$, so compactifying the real line as an interval $[-1,1]_s$ (with $s = 1 - x$ for $x < \frac{1}{2}$, say), we see that de Sitter space indeed fits into our framework. (Thus, one can take $T = s$ for the global compactified time function.) We also note that another, perhaps more familiar, form of the metric can be obtained by letting $t = \sinh\rho$; the metric becomes $d\rho^2 - \cosh^2\rho\,d\theta^2$. (One can take e.g. $T = \tanh\rho$ here.)

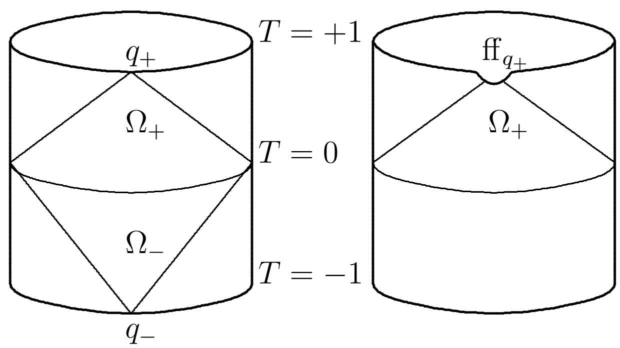

FIGURE 1. On the left, the compactification of de Sitter space with the backward light cone from q_+ and forward light cone from q_- are shown. Ω_+, resp. Ω_-, denotes the intersection of these light cones with $T > 0$, resp. $T < 0$. On the right, the blow up of de Sitter space at q_+ is shown. The interior of the light cone inside the front face ff_{q_+} can be identified with the spatial part of the static model of de Sitter space.

We also use this occasion to explain the connection with the static model of de Sitter space. Thus, on a *subset* of de Sitter space, one has a product structure, 'space' being $\mathbb{B}^{n-1}$, and time τ being in $\mathbb{R}$, such that ∂_τ is Killing, and indeed $\Box$ is conformal to an operator of the form $D_\tau^2 - L$, where L is an elliptic operator on the spatial slice, independent of τ. Thus, in such a region the wave equation can be analyzed by product-type techniques mentioned above. However, this product structure is quite singular in several ways.

Now, the static model corresponds to singling out a point on $\mathbb{S}_\theta^{n-1}$, e.g. $q_0 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. The static model of de Sitter space then is the intersection of the backward lightcone from q_0 considered as a point q_+ on Y_+ (so $T(q_+) = 1$) and the forward light cone from q_0 considered as a point q_- on Y_- (so $T(q_-) = -1$). These happen to intersect the equator $T = 0$ (here $t = 0$) in the same set, and altogether form a 'diamond', see Figure 1. Explicitly this region is given by $z_2^2 + \dots + z_n^2 \leq 1$ inside the hyperboloid. The standard static coordinates

(τ, r, ω) on the 'diamond' are given by

$$r = \sqrt{z_2^2 + \ldots + z_n^2} = \sqrt{1 + z_{n+1}^2 - z_1^2},$$
$$\sinh\tau = \frac{z_{n+1}}{\sqrt{z_1^2 - z_{n+1}^2}},$$
$$\omega = r^{-1}(z_2, \ldots, z_n) \in \mathbb{S}^{n-2}.$$

In these coordinates, in which 'time' is τ, and 'space' is the ball $\mathbb{B}^{n-1}$, here expressed in polar coordinates $[0,1)_r \times \mathbb{S}^{n-2}_\omega$, the metric becomes

$$(1 - r^2)\,d\tau^2 - (1 - r^2)^{-1}dr^2 - r^2\,d\omega^2.$$

Note that the singularity at $r = 1$ is completely artificial (is due to the coordinates), the metric is incomplete, but is conformal to a complete Lorentzian metric, of product type, with $\Box$ also of product type. While one *can* analyze the solutions of the wave equations on de Sitter space at points inside the 'diamond' by considering the diamond only (in view of the finite propagation speed for the wave equation), the resulting picture does include rather artificial limitations. For instance, the asymptotics at the sides of the diamond are automatically smooth in de Sitter space (as we have a standard wave equation there), which is not obvious if one's world consists of the diamond, and the local static asymptotics, corresponding to the tip of the diamond at Y_+, describes only a small part of the asymptotics of solutions of the Cauchy problem on de Sitter space. However, the 'spatial' part of the static operator (or modifications of it) do show up in our analysis as models for the Poisson operator $(g_+, g_-) \mapsto u$; the proper place for its existence is on the interior of the light cone in the blow up of the distinguished point q_+ in de Sitter space.

It should be pointed out that the de Sitter-Schwarzschild metric in fact has many similar features, and the analogous result is the subject of an ongoing project with Antônio Sá Barreto and Richard Melrose. Weaker results on the asymptotics in that case are contained in the part of works of Dafermos and Rodnianski [6, 7] (they also study a non-linear problem), and local energy decay was studied by Bony and Häfner [5], in part based on the stationary resonance analysis of Sá Barreto and Zworski [35].

We also note that on de Sitter space itself, one can solve the wave equation explicitly, see [33], but even the 'smooth asymptotics' result, Theorem 6.1, is not apparent from such a solution.

There are two rather different techniques used to prove the results here. The 'rough' results yielding the existence of the asymptotics, Theorems 5.5 and 6.1, are proved using positive commutator estimates, which roughly speaking describe the microlocal (i.e. phase space) propagation of L^2 (or Sobolev) mass ('energy'). Such methods are very robust, but (unless they are used in a more sophisticated form as in [18]) give less precise results. The Fourier integral operator results are proved by a parametrix construction which is significantly more delicate (taking up two fifth of this paper), but is very instructive. It is at this stage that the static de Sitter model shows up on the front face of $[X \times Y_+; \mathrm{diag}_{Y_+}]$; see P_σ in Section 7. One should think of this as analogous to the way the hyperbolic Laplacian shows up as a model on the front face of the 0-double space for conformally compact Riemannian manifolds, see [23].

The plan of the paper is the following. In Section 2 we adopt a 0-microlocal point of view, and analyze propagation of singularities in the 0-cotangent bundle introduced by Mazzeo and Melrose [23] two decades ago. The proof uses positive commutator estimates, and is quite similar to propagation of singularities for manifolds with boundary equipped with a so-called (incomplete) edge metric, which includes e.g. manifolds with conic points – see [30] and [28] and references therein. In the following sections we analyze local solvability near the boundary as well as conormal regularity of the solutions there. *We emphasize that the results of Sections 2-4 do not need the global assumptions (A1)-(A2).* In Section 5 we prove a unique continuation theorem at ∂X (i.e. at 'infinity') by a Carleman-type estimate, and use it to prove that the asymptotic behavior of the solutions in fact determines the solutions, i.e. we can talk about a 'Cauchy problem at infinity', hence also about the scattering map. In the final section we construct a parametrix for the scattering map, and use it to show that it is indeed a Fourier integral operator.

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2. 0-geometry and propagation of 0-singularities

For the purposes of analysis, we need a good understanding of bicharacteristic geometry. Thus, note that $P \in \mathrm{Diff}_0^2(X)$, in the zero-calculus of Mazzeo and Melrose [23]. This is defined as follows. First, $\mathcal{V}_0(X)$ is the set of $\mathcal{C}^\infty$ vector fields on X vanishing at Y; thus, $\mathcal{V}_0(X) = x\mathcal{V}(X)$, with $\mathcal{V}(X)$ the Lie algebra of $\mathcal{C}^\infty$ vector fields on X. Then elements of $\mathrm{Diff}_0(X)$ are differential operators acting on $\mathcal{C}^\infty(X)$ which are locally finite sums of products of vector fields in $\mathcal{V}_0(X)$ and smooth functions, i.e. elements of $\mathcal{C}^\infty(X)$. In local coordinates $(x, y_1, \ldots, y_{n-1})$, elements of $\mathcal{V}_0(X)$ have the form

$$a_0(x\partial_x) + \sum_{j=1}^{n-1} a_j(x\partial_{y_j}), \tag{2.1}$$

with a_j in $\mathcal{C}^\infty$ for $j = 0, 1, \ldots, n-1$, and correspondingly they are exactly the set of smooth sections of a vector bundle, ${}^0TX$, the zero-tangent bundle. Let ${}^0T^*X$ denote the zero-cotangent bundle of X; this is the dual bundle of ${}^0TX$. Its elements are covectors of the form

$$\xi\,\frac{dx}{x} + \eta\,\frac{dy}{x},$$

as can be seen from (2.1). Then the principal symbol $p = \sigma(P)$ is a homogeneous degree 2 polynomial on ${}^0T^*X$; explicitly at Y, $p|_Y = \xi^2 - H|_Y$, $H|_Y$ the metric function corresponding to h, and p itself is the metric function of the dual pseudo-Riemannian metric g. We refer to [23, 35] for nice descriptions of the basic setup, and [30, 28] for analysis of a hyperbolic problem in the related edge setting.

If a is a homogeneous function on ${}^0T^*X \setminus o$, then there is a (homogeneous) Hamilton vector field H_a associated to it on $T^*X^\circ \setminus o$. A change of coordinates calculation shows that in the 0-canonical coordinates given above

$$H_a = (\partial_\xi a)(\xi\partial_\xi + \eta\partial_\eta + x\partial_x) + x(\partial_\eta a\partial_y - \partial_y a\partial_\eta) - ((x\partial_x + \xi\partial_\xi + \eta\partial_\eta)a)\partial_\xi,$$

so H_a in fact extends to a $\mathcal{C}^\infty$ vector field on ${}^0T^*X \setminus o$ which is tangent to ${}^0T^*_{\partial X}X$. At $x = 0$ this gives $H_a = (\partial_\xi a)R - (Ra)\partial_\xi$, where R is the radial vector field $\xi\partial_\xi + \eta\partial_\eta$ on ${}^0T^*X$. Since a is homogeneous of degree, say, k, $Ra = ka$, and $H_a = (\partial_\xi)aR - ka\partial_\xi$, on the characteristic set $\Sigma(a) = a^{-1}(\{0\})$ of a, at $x = 0$, H_a is radial. It is thus rather convenient to consider the cosphere bundle ${}^0S^*X$ which is the boundary at fiber infinity of the fiber radial compactification ${}^0\bar{T}^*X$ of ${}^0T^*X$.

As we work with p, so that near Y, $\xi \neq 0$ on the characteristic set, we use projective coordinates $\hat\eta = \eta/|\xi|$, $\rho = |\xi|^{-1}$ valid near $\Sigma(p)$. Then

$$\begin{aligned}(\operatorname{sign}\xi)\rho^{-1}H_a = &- ((\rho\partial_\rho + \hat\eta\partial_{\hat\eta})a)\,(-\rho\partial_\rho + x\partial_x) + x(\operatorname{sign}\xi)(\partial_{\hat\eta} a\partial_y - \partial_y a\partial_{\hat\eta})\\ &+ ((x\partial_x - \rho\partial_\rho)a)\,(\rho\partial_\rho + \hat\eta\partial_{\hat\eta}).\end{aligned}$$

Thus, for $a \in \rho^{-k}\mathcal{C}^\infty({}^0\bar{T}^*X)$, $W_a = \rho^{k-1}H_a$ is a smooth vector field on ${}^0\bar{T}^*X$, whose restriction to ${}^0S^*_YX$ is $(\operatorname{sign}\xi)ka\hat\eta\partial_{\hat\eta}$, i.e. it vanishes at $a = 0$. Thus, if da is not conormal to ${}^0S^*_YX$ in ${}^0S^*X$, so $\Sigma(a)$ is transversal to ${}^0S^*_YX$, then W_a is a smooth vector field on $\Sigma(a)$ that vanishes at $x = 0$, and hence is of the form $W_a = xW'_a$, $W'_a \in \mathcal{V}(\Sigma(a))$.

Applying this with $a = p = \rho^{-2}\mathcal{C}^\infty({}^0\bar{T}^*X)$ yields that inside $\Sigma(p)$, $W_p = xW'_p$,

$$W'_p|_{x=0} = 2(\operatorname{sign}\xi)\partial_x + H_h,$$

H_h the Hamilton vector field of h (evaluated at $(y, \hat\eta)$). In particular, W'_p is transversal to Y. Also, W_p is complete, and γ is an integral curve of W_p, then a reparameterized version of γ is an integral curve of W'_p, hence $\lim_{t\to-(\operatorname{sign}\xi)\infty}\gamma(t)$ exists in ${}^0S^*_YX \cap \Sigma(p)$. (Note that the map switching the sign of covectors preserves even functions, such as p, while transforms H_p to $-H_p$.) Conversely, for any $q \in {}^0S^*_YX \cap \Sigma(p)$ there is a unique (up to translation of the parameterization) integral curve of W_p with limit q as $t \to -(\operatorname{sign}\xi)\infty$, namely this is just a reparameterization of the unique integral curve of W'_p through q. Note also that ${}^0S^*_YX \cap \Sigma(p)$ can be identified with two copies of S^*Y, one for each sign of ξ; we write these as S^*_+Y and S^*_-Y.

Suppose now that $Y = Y_+ \cup Y_-$, where $Y_\pm$ are unions of connected components of Y, and this decomposition satisfies that all bicharacteristics $t \mapsto \gamma(t)$ of P satisfy $\lim_{t\to+\infty}\gamma(t) \in S^*Y_+$, $\lim_{t\to-\infty}\gamma(t) \in S^*Y_-$, or vice versa, i.e. that (A1) and (A2) of the introduction hold. For $q \in S^*_+Y_+$ there is a unique bicharacteristic with $\lim_{t\to-\infty}\gamma(t) = q$. By (A1) and (A2), $\lim_{t\to+\infty}\gamma(t) = q' \in S^*Y_-$ exists; as we saw above, it necessarily lies in $S^*_-Y_-$. The classical scattering map is the map $\mathcal{S}_{\mathrm{cl}} : S^*Y_+ \to S^*Y_-$ with $\mathcal{S}_{\mathrm{cl}} : q \mapsto q'$. Fixing a homogeneous degree 1 function on $T^*Y \setminus o$, we can extend these to maps $T^*Y_+ \setminus o \to T^*Y_- \setminus o$ – we can use $h^{1/2}$, for instance. The induced relation on $(T^*Y_+ \setminus o) \times (T^*Y_- \setminus o)$ is Lagrangian with respect to the twisted symplectic form (i.e. with a negative sign on one of the factors).

As follows easily from the results of [13], (A1) and (A2) imply the existence of a global compactified 'time' function T, with $T \in \mathcal{C}^\infty(X)$, $T|_{Y_\pm} = \pm 1$, and the pullback π^*T of T to S^*X having positive/negative derivative along the Hamilton vector field inside the characteristic set $\Sigma(p)$ depending on whether the corresponding bicharacterstics tend to Y_+ or Y_-. Notice that $1 - x$ resp. $x - 1$ has the desired properties near Y_+ resp. Y_-, so the point is dealing with the interior of X, namely that these can be extended to all of X.

With any choice of such a function T, a constant $t_0 \in (-1,1)$, and a vector field V transversal to S_{t_0} P is strictly hyperbolic, and the Cauchy problem $Pu = 0$ in X°, $u|_{S_{t_0}} = \psi_0$, $Vu|_{S_{t_0}} = \psi_1$, $\psi_0, \psi_1 \in \mathcal{C}^\infty(S_{t_0})$ is well posed.

Our first result is that of 0-regularity of solutions of $Pu = 0$ with a weight given by a space u a priori lies in. There is a dichotomy between solutions depending on the a priori regularity relative to this weighted space. If the a priori regularity is low, we only obtain regularity up to a limit implied by the weight, but we do so without having to assume any interior regularity for u. If the a priori regularity is high, then we obtain additional regularity up to the limit corresponding to the smoothness of u in X°.

In order to state the proposition, we recall the zero Sobolev spaces. These are a special case of the edge Sobolev spaces (corresponding to a fibration whose fibers are points) of Mazzeo [24, Section 3]; see also [30]. They are spaces possessing (for positive orders) additional regularity under the application of elements of $\mathcal{V}_0(X)$, relative to $L^2_0(X)$. Here $L^2_0(X)$ is the L^2 space relative to the density $|dg|$; as a Banach space it is thus equivalent to $x^{n/2}L^2(X)$, where $L^2(X)$ is defined (up to equivalence) with respect to *any* $\mathcal{C}^\infty$ Riemannian or Lorentzian metric on X (only non-degeneracy of the metric matters, not its signature). For $r \geq 0$ integer,

$$H^r_0(X) = \{u \in L^2_0(X) :\ Au \in L^2_0(X)\ \forall A \in \mathrm{Diff}^r_0(X)\}; \tag{2.2}$$

in general it can be defined by interpolation and duality. We also let

$$H^{r,s}_0(X) = \{u \in \mathcal{C}^{-\infty}(X) :\ x^{-s}u \in H^r_0(X)\}$$

be the corresponding weighted Sobolev spaces.

These spaces can be defined and microlocalized using the zero pseudo-differential operators, $\Psi^{m,l}_0(X)$, of Mazzeo and Melrose [23]. Here we do not discuss $\Psi^{m,l}_0(X)$ in detail (we again refer the reader to [23, 35]), but its elements have principal symbols $a = \sigma_m(A) \in x^l S^m_{\mathrm{hom}}({}^0T^*X \setminus o)$, with hom denoting homogeneous functions. Then (2.2) holds with $A \in \mathrm{Diff}^r_0(X)$ replaced by $A \in \Psi^r_0(X)$ if $r \geq 0$ is real.

Furthermore, one has a wave front set corresponding to $\Psi_0(X)$. Thus, for $q \in {}^0S^*X$, $u \in H^{r_0,s_0}_0(X)$, $r \in \mathbb{R}$, one says that $q \notin \mathrm{WF}^{r,s_0}_0(u)$ if there exists $A \in \Psi^{0,0}_0(X)$ elliptic at q (i.e. $\sigma_0(A)(q) \neq 0$) such that $Au \in H^{r,s_0}_0(X)$, i.e. u is 'in $H^{r,s_0}_0(X)$ microlocally at q'. Equivalently, $q \notin \mathrm{WF}^{r,s_0}_0(u)$ if there exists $A \in \Psi^{r,-s_0}_0(X)$ elliptic at q such that $Au \in L^2_0(X)$.

With this background, we state our first result.

Proposition 2.1. *Suppose that $q \in Y$, and suppose that u is in $H^{r_0,s_0}_0(X)$ in a neighborhood of q and $Pu = 0$. Then:*

(i) *If $r_0 < s_0 + 1/2$ then u is in $H^{r,s_0}_0(X)$ near q for all $r < s_0 + 1/2$.*

(ii) *If $r_0 > s_0 + 1/2$ and $r > r_0$, $\alpha \in {}^0S^*_qX \cap \Sigma(p)$ then $\alpha \notin \mathrm{WF}^{r,s_0}_0(u)$ provided that the bicharacteristic γ approaching α is disjoint from $\mathrm{WF}^r(u) \subset S^*X^\circ$. The same conclusion holds if $r_0 \leq s_0 + 1/2$, but $\alpha \notin \mathrm{WF}^{r_1,s_0}_0(u)$ for some $r_1 > s_0 + 1/2$.*

(iii) *In particular, if $r_0 > s_0 + 1/2$ and $r > r_0$, then u is in $H^{r,s_0}_0(X)$ near q provided that all bicharacteristics approaching ${}^0S^*_qX$ are disjoint from $\mathrm{WF}^r(u) \subset S^*X^\circ$.*

Proof. This proof is very similar to the proofs of propagation of 'edge regularity' for the wave equation with *incomplete* metrics in [30] and [28], so we shall be brief.

While ρH_p restricts to a smooth vector field on $\Sigma(p)$ with vanishing restriction at Y, if we evaluate ρH_p as a section of the b-tangent bundle of ${}^0\bar{T}^*X$ at $\Sigma(p)\cap {}^0S^*_YX$, the result is more interesting: $\rho H_p = 2(-\rho\partial_\rho + x\partial_x)$ in this sense. Correspondingly, for $A \in \Psi_0^{m,l}(X)$, the symbol of $i[P,A] \in \Psi_0^{m+1,l}(X)$ is $H_p a = 2(m+l)\rho^{-1}a$, $a = \sigma(A)$, at $\Sigma(p)\cap {}^0S^*_YX$. Thus, much as [30] and [28], one can show propagation of zero-regularity into the boundary for $m+l\neq 0$. Unlike in the setting of [30], the characteristic set of P only intersects the boundary Y in radial points, i.e. there is no propagation inside Y, which explains why there is no requirement for $m+l$ having a particular sign (as long as it is non-zero), although the results are different depending on the sign: (i) has no wave front set assumptions on u. This corresponds to the presence of a cutoff χ, identically 1 near Y, such that $\partial_x\chi\leq 0$, the sign of the commutator with χ agrees with the sign arising from the weights if $m+l<0$. Moreover, one can microlocalize in S^*_YX by pulling back functions from $S^*_YX\cap\Sigma(p)$ using the flow of W'_p, extending them to a neighborhood of the characteristic set in an arbitrary smooth fashion.

Thus, let $\psi_0\in\mathcal{C}^\infty(S^*_YX\cap\Sigma(p))$, and for any integral curve $\tilde\gamma$ of W'_p with $\tilde\gamma(0)\in S^*_Y\cap\Sigma(p))$, we let $\psi(\tilde\gamma(t)) = \psi_0(\gamma(0))$. Note that this defines a $\mathcal{C}^\infty$ function on $\Sigma(p)$ near Y, for the map $\Phi: S^*_YX\cap\Sigma(p)\times[0,\epsilon)\to\Sigma(p)$ given by $\Phi(q,t) = \exp(tW'_p)q$ is a local diffeomorphism near $t=0$. As $\Sigma(p)$ is a $\mathcal{C}^\infty$ submanifold of S^*X, we can extend ψ to a $\mathcal{C}^\infty$ function on S^*X, still denoted by ψ, hence further to an element of $\mathcal{C}^\infty({}^0\bar{T}^*X)$, at least near Y. Now let $\chi\in\mathcal{C}^\infty_{\mathrm{c}}([0,\epsilon))$ be such that $\chi' = -\chi_0^2$, $\chi\equiv 1$ near 0, $\chi\geq 0$, $\chi^{1/2}$ is $\mathcal{C}^\infty$, and let

$$a = \rho^{-m}x^l\chi(x)\psi,$$

and note that $W_p\chi(x) = b^2x\chi'(x)$ with $b>0$ near Y. As $W'_p\psi$ vanishes at $p=0$, we deduce that

$$H_pa = 2(m+l)\rho^{-m-1}x^l\chi(x)\psi + 2\rho^{-m-1}x^{l+1}b^2\chi'(x) + p\rho^{-m+1}x^le + \rho^{-m}x^lf,$$

with $b,e,f\in\mathcal{C}^\infty({}^0\bar{T}^*X)$.

Now the standard positive commutator argument finishes the proof of the proposition, see e.g. [30]. For the reader's convenience, we sketch the argument, skipping the (necessary but straightforward) regularization part of the argument. Let $A\in\Psi_0^{m,l}(X)$ be a quantization of a, so $\sigma_m(A) = a$, and $\mathrm{WF}'(A)\subset\mathrm{supp}(a)$. Let $\Lambda\in\Psi_0^{1/2,0}(X)$ be elliptic formally self-adjoint with positive principal symbol, $\rho^{-1/2}$. Thus, $\sigma_{-m-1}(i[P,A]) = H_pa$ shows that there exist $\tilde A, B, E, F$ such that

$$\begin{aligned}
&i[P,A] = 2(m+l)\Lambda\tilde A^*\tilde A\Lambda - B^*B + PE + F,\\
&\tilde A\in\Psi_0^{m/2,l/2}(X),\ \sigma(\tilde A) = \sigma(A)^{1/2},\\
&B\in\Psi_0^{(m+1)/2,l/2}(X),\ \mathrm{WF}'(B)\subset\mathrm{supp}\,\chi_0\cap\mathrm{supp}\,\psi,\\
&\qquad\sigma(B) = b\chi_0(2\rho^{-m-1}x^{l+1})^{1/2}\\
&E\in\Psi_0^{m-1,l}(X),\ F\in\Psi_0^{m,l}(X).
\end{aligned}\tag{2.3}$$

Proceeding as in [30] shows that for u with $Pu=0$,

$$\Big|\pm\|\tilde A\Lambda u\|^2_{H_0^{0,0}(X)} - \|Bu\|^2_{H_0^{0,0}(X)}\Big| \leq C\|u\|^2_{H_0^{m/2,-l/2}(X)},$$

provided that the right hand side is finite, with the $-$ sign applying if $m+l<0$, and the $+$ sign applying if $m+l>0$. In the first case, the second term on the left hand side can simply be dropped, so we do not need to make any assumptions on the $H^{(m+1)/2}$ norm of u. In the second case we need to assume that $\mathrm{WF}^{(m+1)/2}(u)$ is disjoint from $\operatorname{supp}\chi_0$ (hence from $\mathrm{WF}'(B)$, due to (2.3)), in order to conclude that $\|\tilde{A}\Lambda u\|_{H_0^{0,0}(X)}$ is finite, i.e. $\mathrm{WF}_0^{(m+1)/2,-l/2}(u)$ is disjoint from the elliptic set of A, i.e. from the interior of $\operatorname{supp}\psi$ near $x = 0$.

The standard iteration argument now proves the proposition. □

The approximation process prevents us from crossing the line $r = s_0 + 1/2$, which is why we cannot directly obtain information about u in $H_0^{r,s_0}(X)$ with $r > s_0 + 1/2$ unless we know u is in $H_0^{r_0,s_0}(X)$ for $r_0 > s_0 + 1/2$. However, if $u \in H^{r_0,s_0}(X)$ with $r_0 = s_0 + 1/2 - \epsilon/2$, so $r_0 < s_0 + 1/2$, then $u \in H^{r_0,s_0-\epsilon}(X)$, and $r_0 > (s_0 - \epsilon) + 1/2$ now. We thus deduce:

Corollary 2.2. *Suppose that $q \in Y$, and suppose that u is in $H_0^{r_0,s_0}(X)$ in a neighborhood of q and $Pu = 0$. If $r > r_0$ and $s < s_0$, $\alpha \in {}^0S_q^*X \cap \Sigma(p)$ then $\alpha \notin \mathrm{WF}_0^{r,s}(u)$ provided that the bicharacteristic γ approaching α is disjoint from $\mathrm{WF}^r(u) \subset S^*X^\circ$.*

In particular, u is in $H_0^{r,s}(X)$ near q provided that all bicharacteristics approaching ${}^0S_q^*X$ are disjoint from $\mathrm{WF}^r(u) \subset S^*X^\circ$.

Remark 2.3. Thus, we gain *full* 0-regularity for solutions if we are willing to give up some (arbitrarily little) decay. Note that (ii) of the Proposition states that one can take $s = s_0$ if $r_0 > s_0 + 1/2$, so the present corollary is only interesting if $r_0 \le s_0 + 1/2$.

Proof. Let $s < s_0$ be given, and let $\epsilon = s_0 - s > 0$. As remarked, we may assume $r_0 \le s_0 + 1/2$, and if needed, we can decrease r_0 so that $r_0 < s_0 + 1/2$. By (i) of Proposition 2.1, $\alpha \notin \mathrm{WF}_0^{r,s_0}(u)$ for all $r < s_0 + 1/2$. Then $\alpha \notin \mathrm{WF}_0^{s_0+1/2-\epsilon/2,s_0}(u)$, and hence $\alpha \notin \mathrm{WF}_0^{s_0+1/2-\epsilon/2,s_0-\epsilon}(u)$. By (ii) of Proposition 2.1, $\alpha \notin \mathrm{WF}_0^{r,s_0-\epsilon}(u) = \mathrm{WF}_0^{r,s}(u)$ for all r, proving the corollary. □

3. Local solvability near ∂X

Let $P = \Box - \lambda$. In this section we show the solvability of $Pu = 0$ near ∂X in suitable senses. This relies on a positive commutator estimate with *compact* error term, so we need to control the normal operator of our commutator in the 0-calculus. Recall from [23] that the normal operator map on $\mathrm{Diff}_0^k(X)$ (or $\Psi_0^k(X)$) captures $Q \in \mathrm{Diff}_0^k(X)$ modulo $x\,\mathrm{Diff}_0^k(X)$, as opposed to the principal symbol map, which captures it modulo $\mathrm{Diff}_0^{k-1}(X)$. The compactness referred to above then is that of the inclusion map for the associated Sobolev spaces, $H_0^{r,s}(X)$ to $H_0^{r',s'}(X)$, with $r > r'$, $s > s'$; note that compactness requires improvements in both the regularity and decay orders, hence control of both the principal symbols (described in the previous section) and normal operators.

We thus start by calculating the normal operator of P, as well as that of its commutator with another operator A. Thus, we calculate the the commutator modulo terms with an additional order of vanishing. As $P \in \mathrm{Diff}_0^2(X)$, and our commutant will be an operator $A_r \in x^{r-1}\,\mathrm{Diff}_0^1(X)$, $[P, A_r] \in x^{r-1}\,\mathrm{Diff}_0^2(X)$, so we need to compute $[P, A_r]$ modulo $x^r\,\mathrm{Diff}_0^2(X)$. This is computation is thus unaffected if P is changed by addition of a term in $x\,\mathrm{Diff}_0^2(X)$, or A_r is changed by a term in $x^r\,\mathrm{Diff}_0^1(X)$. This means that effectively we may assume that X has a product decomposition near Y and h is actually a Riemannian metric on Y. The wave operator is the Laplace-Beltrami operator associated to this metric:

$$\Box = (xD_x)^2 + i(n-1)(xD_x) - x^2\Delta_Y = (xD_x)^*(xD_x) - x^2\Delta_Y,$$

with the adjoint taken with respect to the pseudo-Riemannian density $x^{-n}\,|dx\,dy|$.

We remark here that the actual normal operator in the 0-calculus (which results from restricting the Schwartz kernels to the 0-front face) is even simpler than this model, for it localizes in Y. Thus, one could simply compute with the Euclidean Laplacian in Y, but as this has absolutely no impact on our considerations, we use our more global model.

We let $A_r = x^r D_x + i\frac{n-r}{2}\,x^{r-1}$, which is symmetric, and compute

$$\begin{aligned}[P,A] &= [(xD_x)^2 + i(n-1)(xD_x), x^r D_x + i\frac{n-r}{2}\,x^{r-1}]\\ &\qquad - [x^2, x^r D_x + i\frac{n-r}{2}\,x^{r-1}]\Delta_Y\\ &= -2i\left\{(r-1)(xD_x + i\frac{n-r}{2})^* x^{r-1}(xD_x + i\frac{n-r}{2}) + x^{r+1}\Delta_Y\right\}.\end{aligned}$$

Thus, up to the factor $-2i$, this is clearly a positive operator for $r \geq 1$. We would like to improve this statement, and in particular show that this is greater than Cx^{r-1} for suitable C, at least in a range of r, and at least modulo terms of the form $PB + B^*P$.

The flexibility we have here in arranging this positivity is the choice of the coefficient B of P. Thus, we convert part of the tangential Laplacian term, $x^{r+1}\Delta_Y$ into P by writing $x^{r+1}\Delta_Y = \gamma x^{r+1}\Delta_Y + (1-\gamma)x^{r+1}\Delta_Y$, with γ to be determined, and writing

$$x^{r+1}\Delta_Y = \frac{1}{2}\{x^{r-1}((xD_x)^*(xD_x) - \lambda - P) + ((xD_x)^*(xD_x) - \lambda - P)x^{r-1}\}$$

in the first term. We deduce with $B = -\frac{\gamma}{2}\,x^{r-1}$,

$$\begin{aligned}\frac{i}{2}[P,A] =&(r-1)(xD_x + i\frac{n-r}{2})^* x^{r-1}(xD_x + i\frac{n-r}{2}) + (1-\gamma)x^{r+1}\Delta_Y\\ &+ \frac{\gamma}{2}x^{r-1}(xD_x)^*(xD_x) + \frac{\gamma}{2}(xD_x)^*(xD_x)x^{r-1} - \gamma\lambda x^{r-1} + PB + B^*P.\end{aligned}$$

Now, the form of the first term is quite convenient to us in view of the factor x^{r-1}, corresponding to a weighted estimate on $x^{-(r-1)/2}L^2$ relative to $x^{-n}\,dx$, since its null-space consists of $x^{(n-r)/2}$, which just misses being in $x^{-(r-1)/2}L^2$ (i.e. is in $x^{-(r-1)/2-\delta}L^2$ for all $\delta > 0$), so it will give us optimal zeroth order terms below, and saves us having to use that for all s,

$$\frac{(2s-n-1)^2}{4}\|x^{s-1}u\|^2 \leq \|x^s D_x u\|^2. \tag{3.1}$$

Note, however, that the first term can easily be written in a simpler looking form,

$$(xD_x + i\frac{n-r}{2})^* x^{r-1}(xD_x + i\frac{n-r}{2}) = (xD_x)^* x^{r-1}(xD_x) - \frac{(n-r)^2}{4}x^{r-1}.$$

This can be checked easily as the two sides have the same principal symbol, so their difference is first order, moreover both sides are real and self-adjoint, hence actually zeroth order, i.e. multiplication by a smooth function. Their equality can be checked by evaluating them on 1. Moreover, a similar calculation yields

$$\begin{aligned}&\frac{1}{2}(x^{r-1}(xD_x)^*(xD_x)+(xD_x)^*(xD_x)x^{r-1})\\&\qquad=(xD_x+i\frac{n-r}{2})^*x^{r-1}(xD_x+i\frac{n-r}{2})+\frac{(n+r-2)(n-r)}{4}x^{r-1}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{i}{2}[P,A]=&(r-1+\gamma)(xD_x+i\frac{n-r}{2})^*x^{r-1}(xD_x+i\frac{n-r}{2}))+(1-\gamma)x^{r+1}\Delta_Y\\&+\gamma\left(\frac{(n-r)(n+r-2)}{4}-\lambda\right)x^{r-1}+PB+B^*P.\end{aligned}\tag{3.2}$$

In order to obtain a 'positive commutator', modulo the terms involving P, we thus need that

$$r-1+\gamma,\ 1-\gamma\ \text{ and }\ \gamma\left(\frac{(n-r)(n+r-2)}{4}-\lambda\right)\tag{3.3}$$

have the same sign. As $\frac{(n-r)(n+r-2)}{4}-\lambda=0$ gives

$$\frac{r-1}{2}=\pm\sqrt{\left(\frac{n-1}{2}\right)^2-\lambda},$$

we introduce

$$l(\lambda)=\operatorname{Re}\sqrt{\left(\frac{n-1}{2}\right)^2-\lambda},\tag{3.4}$$

so $l(\lambda)=0$ for $\lambda\geq\frac{(n-1)^2}{4}$, $l(\lambda)>0$ for $\lambda<\frac{(n-1)^2}{4}$.

Lemma 3.1. *The quantities listed in* (3.3) *have the same (non-zero) sign for an appropriate choice of* γ *if:*

- *if* $r>\max(0,1-2l(\lambda))$, $r\neq 1+2l(\lambda)$, *in which case they are all positive, or*
- *if* $r<\min(0,1-2l(\lambda))$, *in which case they are all negative.*

Proof. First, note that for $\frac{r-1}{2}\in(-l(\lambda),l(\lambda))$, i.e. $r\in(1-2l(\lambda),1+2l(\lambda))$, $\frac{(n-r)(n+r-2)}{4}-\lambda>0$, while for $\frac{r-1}{2}\notin[-l(\lambda),l(\lambda)]$, $\frac{(n-r)(n+r-2)}{4}-\lambda<0$.

For $r>1$, $r\neq 1+2l(\lambda)$ it is easy to arrange that all three quantities in (3.3) have the same sign since the first two terms are positive if $|\gamma|$ is sufficiently small, so choosing the sign of γ correctly, the last term can also be made positive as long as $r\neq 1+2l(\lambda)$ ($r>1$ rules out $r=1-2l(\lambda)$).

In general, the first two terms have the same sign if $\gamma\in(1,1-r)$, resp. $\gamma\in(1-r,1)$, depending on whether $r<0$, resp. $r>0$, and this sign is negative, resp. positive in the two cases.

Suppose first that $\lambda\leq\frac{(n-1)^2}{4}$.

If $r<0$, we have $\gamma>1$ by the previous remark, so we need $(n+r-2)(n-r)-\lambda<0$, i.e. $r\notin[1-2l(\lambda),1+2l(\lambda)]$, which in view of $r<0$ amounts to $r<1-2l(\lambda)$ (and $r<0$). In the latter case, if $r\in(0,1]$, $\gamma>0$ still, but now we need $(n+r-2)(n-r)-\lambda>0$, i.e. $r\in(1-2l(\lambda),1+2l(\lambda))$. As $r\in(0,1]$, this means $r\in(\max(0,1-2l(\lambda)),1]$. On the other hand, if $r>1$, we have already seen that $\gamma\left(\frac{(n-r)(n+r-2)}{4}-\lambda\right)$ can be made positive as well as long as $r \neq 1+2l(\lambda)$. This completes the proof of the lemma if $\lambda \leq \frac{(n-1)^2}{4}$.

For $\lambda > \left(\frac{n-1}{2}\right)^2$, $\frac{(n-r)(n+r-2)}{4}-\lambda<0$ for all values of r. The 'positive' commutator criterion thus becomes that $r-1+\gamma$, $1-\gamma$ and $-\gamma$ must have the same sign. The first two give $\gamma \in (1,1-r)$, resp. $\gamma \in (1-r,1)$ depending on $r<0$ or $r>0$, as beforehand, while the last two give $\gamma \notin [0,1]$. As $(1,1-r)$ or $(1-r,1)$ intersects the complement of $[0,1]$ in a non-empty set if $r<0$ or $r>1$, we get exactly the range stated in the lemma, taking into account that $\max(0,1-2l(\lambda))=1$, $\min(0,1-2l(\lambda))=0$. □

If the conditions of Lemma 3.1 are satisfied, the right hand side of (3.2), applied to v supported near Y, is, modulo the terms involving P, bounded below a positive multiple (if all quantities in (3.3) are positive), resp. bounded above by a negative multiple (if all quantities in (3.3) are negative), of the squared $x^l H^1_0$ norm of v, $l=-\frac{r-1}{2}$. We thus have:

Lemma 3.2. *Suppose*

$$l \in (-\infty, \min(\frac{1}{2}, l(\lambda)),\ l \neq -l(\lambda)\ \text{ or }\ l \in (\max(\frac{1}{2}, l(\lambda)), +\infty). \tag{3.5}$$

Then there exists $C>0$ and $\delta>0$ such that

$$\|x^{-l}v\|_{H^1_0} \leq C\|x^{-l}Pv\|_{L^2}. \tag{3.6}$$

for all $v \in \dot{\mathcal{C}}^\infty(X)$ with $\operatorname{supp} v \subset \{x<\delta\}$.

Remark 3.3. Note that (near $x=0$) $x^s \in x^l L^2$ if $l<s-(n-1)/2$, so (neglecting the $\frac{1}{2}$ above) the two critical values $l=-l(\lambda)$ and $l=l(\lambda)$ arise from the monomials $x^{-l(\lambda)+\frac{n-1}{2}}$, resp. $x^{l(\lambda)+\frac{n-1}{2}}$, which are exactly the monomial solutions of $Pv=0$.

Proof. Note that (3.5) holds if and only if one of the conditions in Lemma 3.1 holds with $l=-\frac{r-1}{2}$.

First, suppose that $v \in \dot{\mathcal{C}}^\infty(X)$ supported in $x<\delta$ and g is an exact warped product Lorentzian metric for $x<2\delta$. Then

$$\begin{aligned}\langle \frac{i}{2}Av, Pv\rangle - \langle \frac{i}{2}Pv, Av\rangle &= \langle \frac{i}{2}[P,A]v, v\rangle \\ &= (r-1+\gamma)\|x^{\frac{r-1}{2}}(xD_x + i\frac{n-r}{2})v\|^2 + (1-\gamma)\|x^{\frac{r+1}{2}}d_Y v\|^2 \\ &\quad + \gamma\left(\frac{(n-r)(n+r-2)}{4}-\lambda\right)\|x^{\frac{r-1}{2}}v\|^2 + \langle Bv, Pv\rangle + \langle Pv, Bv\rangle,\end{aligned}$$

so as the three squares on the right hand side have coefficients with the same sign,

$$\begin{aligned}\|x^{-l}v\|^2_{H^1_0} &\leq C\|x^{-l}Pv\|_{L^2}(\|x^l Av\|_{L^2} + \|x^l Bv\|_{L^2}) \\ &\leq C\epsilon^{-1}\|x^{-l}Pv\|^2_{L^2} + C\epsilon(\|x^l Av\|^2_{L^2} + \|x^l Bv\|^2_{L^2}).\end{aligned}$$

As $\|x^l Av\|^2_{L^2} + \|x^l Bv\|^2_{L^2} \leq C'\|x^{-l}v\|^2_{H^1_0}$, for $B=-\frac{\gamma}{2}x^{-2l}$, $A=x^{-2l}(xD_x+i\frac{n-r}{2})$, for $\epsilon>0$ small we deduce that (with a new $C>0$)

$$\|x^{-l}v\|_{H^1_0} \leq C\|x^{-l}Pv\|_{L^2}.$$

This proves the lemma for warped product g (with $\delta>0$ arbitrary, as long as on $x<2\delta$ the metric is warped product).

If we do not consider an exact warped product metric near Y, then $P = P_0 + P_1$, $P_0 = \Box_0$ is the wave operator for the warped product metric and $P_1 \in x\,\mathrm{Diff}_0^2(X)$. Moreover, making A self-adjoint with respect to the new metric, $A = A_0 + A_1$, $A_1 \in x^r\,\mathrm{Diff}_0^1(X)$. Thus,

$$[P, A] = [P_0, A_0] + R',\ \ R' \in x^r\,\mathrm{Diff}_0^2(X).$$

Taking into account that $l = -\frac{r-1}{2}$, for functions v supported in $x < \delta$ this gives

$$|\langle v, R'v\rangle| \leq C\delta \|x^{-l}v\|^2_{H_0^1}$$

with C depending on R' only (i.e. independent of $\delta \in (0, 1]$), so for sufficiently small $\delta > 0$, (3.6) still holds. □

The estimate (3.6) gives, by duality, an existence result. As the argument is local near each connected component of Y, we have:

Proposition 3.4. *Suppose g is asymptotically de Sitter like, $P = \Box - \lambda$, $l(\lambda)$ is given by* (3.4)*, and*

$$l \in (-\infty, -\max(\frac{1}{2}, l(\lambda))), \quad \text{or} \ \ l \in (-\min(\frac{1}{2}, l(\lambda)), +\infty),\ \ l \neq l(\lambda). \tag{3.7}$$

For every $f \in x^l L^2(X)$ there exists $u \in x^l H_0^1(X)$ such that $Pu = f$ near Y. Moreover, if Y_j is a connected component of Y, and $\mathrm{supp}\, f$ *is disjoint from other components of Y, then* $\mathrm{supp}\, u$ *may be taken disjoint from other components of Y.*

Proof. Note that $P = P^*$ (formal adjoint). The result is standard then, see [20, Proof of Theorem 26.1.7]. Indeed, (3.6) shows that for $f \in x^{-l}H_0^1$, $v \in \dot{\mathcal{C}}^\infty(X)$ supported in $x < \delta$,

$$|\langle f, v\rangle| \leq C\|x^{-l}Pv\|_{L^2}.$$

Thus, $Pv \mapsto \langle f, v\rangle$ is an anti-linear functional on elements of $\dot{\mathcal{C}}^\infty(X)$ supported in $x < \delta$, continuous with respect to the $x^l L^2$-norm. By the Hahn-Banach theorem it can be extended to a continuous conjugate-linear functional on $x^l L^2$, so there exists $u \in x^{-l}L^2$ such that $\langle f, v\rangle = \langle u, Pv\rangle$, and u is now the desired solution for l as above. □

In order to use the positive commutator argument with v not supported near Y, we need a cutoff χ, so instead of $A = A_r$, we would really use $A = \chi(x)^2 A_r + A_r\chi(x)^2$, $\chi \equiv 1$ near 0, $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$. We can also localize at any given connected component of Y; as this can be done by a locally constant function on $\mathrm{supp}\,\chi$, we do not indicate this in the notation as it leaves the commutator unchanged. Then

$$\begin{aligned}\frac{i}{2}[P, A] =&(r - 1 + \gamma)(xD_x + i\frac{n-r}{2})^* x^{r-1}\chi^2(xD_x + i\frac{n-r}{2})\\ &+ (1-\gamma)x^{r+1}\chi^2\Delta_Y + \gamma\left(\frac{(n-r)(n+r-2)}{4} - \lambda\right)x^{r-1}\chi^2\\ &+ (xD_x)^*(\chi^2)'(xD_x) + R + PB + B^*P,\end{aligned} \tag{3.8}$$

where $R = R(x)$, $R \in \mathcal{C}_c^\infty(\mathbb{R})$, supported away from 0. (Again, this comes from a principal symbol computation, which has to be carried out away from ∂X, and reality plus self-adjointness shows that R is 0th order.) Thus, modulo the 0th order term supported in the interior and terms involving P we have a *global* 'positive commutator' estimate (all terms have the same sign) if $r < \min(0, 1 - 2l(\lambda))$; if

$r > \max(0, 1 - 2l(\lambda))$ but $r \neq 1 + 2l(\lambda)$, the term arising from commuting with χ^2 has opposite sign compared to the 'main' terms.

One can also add a regularizing factor, $\left(\frac{x}{x+\epsilon}\right)^s = (1 + \epsilon x^{-1})^{-s}$ with $s > 0$ small. For $\epsilon > 0$, this is a symbol of order $-s$ (i.e. decaying as $x \to 0$), and is uniformly bounded as a symbol of order 0. Moreover,

$$(x\partial_x)^k (1 + \epsilon x^{-1})^{-s} = s(1 + \epsilon x^{-1})^{-s} f_{k,\epsilon,s},$$

where $f_{k,\epsilon,s}$ is a symbol of order 0, and is uniformly bounded as such a symbol. Consequently, as long as one has a positive normal operator for the commutator of P with some operator A, one will also have a positive normal operator for the commutator of P with $(1 + \epsilon x^{-1})^{-s} A (1 + \epsilon x^{-1})^{-s}$ if s is small. It is actually even easier to simply apply our previous estimate, (3.6), to a regularized version $v_\epsilon = (1+\epsilon x^{-1})^{-s} v$ of v, for $Pv_\epsilon = (1+\epsilon x^{-1})^{-s} Pv + [P, (1+\epsilon x^{-1})^{-s}]v$, noting that $(1 + \epsilon x^{-1})^s [P, (1 + \epsilon x^{-1})^{-s}]$ is bounded by $C's$ in $\mathrm{Diff}^1_{0,\mathrm{c}}(X)$ (c denotes conormal coefficients), so the L^2 norm of $[P, (1+\epsilon x^{-1})^{-s}]v$ can be absorbed into the left-hand side of (3.6) for $s > 0$ small. Applying this iteratively, we deduce the following:

Proposition 3.5. *Suppose g is asymptotically de Sitter like, $P = \Box - \lambda$, $\lambda \in \mathbb{R}$. Suppose that $u \in x^{l_0} H^1_0(X)$, $Pu \in x^l L^2(X)$, $l > l_0$. Suppose also that one of the following conditions holds:*

(i) $l < -l(\lambda)$,
(ii) $l_0 > \max(\frac{1}{2}, l(\lambda))$,
(iii) $l_0 > -l(\lambda)$, $l < \min(\frac{1}{2}, l(\lambda))$.

Then $u \in x^l H^1_0(X)$.

Moreover, the result is local near each connected component of Y.

This immediately gives that if a solution of $Pu = 0$ decays faster than a borderline rate, given by $x^{l(\lambda)} L^2$, then it is Schwartz. In fact, later in Proposition 5.3, we show that such u is necessarily identically 0.

Corollary 3.6. *Suppose that $u \in x^l H^k_0(X)$, $k \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $l > \max(\frac{1}{2}, l(\lambda))$, $Pu \in \dot{\mathcal{C}}^\infty(X)$. Then $u \in \dot{\mathcal{C}}^\infty(X)$.*

If the assumptions hold near a connected component of Y only, so does the conclusion.

Remark 3.7. The assumption $l > \max(\frac{1}{2}, l(\lambda))$ is probably not optimal if $l(\lambda) < \frac{1}{2}$, cf. Remark 5.2; one expects $l > l(\lambda)$ simply. However, this makes no difference in the present paper. Moreover, for $\Box$ itself this is not a restriction as $n \geq 2$ so $l(\lambda) \geq \frac{1}{2}$.

This corollary also states in particular that for $f \in \dot{\mathcal{C}}^\infty(X)$ the solution $u \in x^l H^1_0(X)$ of $Pu = f$ near Y, whose existence is guaranteed by Proposition 3.4, is in fact in $\dot{\mathcal{C}}^\infty(X)$.

Proof. First, we may assume $k = 1$. Indeed, if $k < 1$, then $l > 1/2$ gives $k < 1 < l + 1/2$, so (i) of Proposition 2.1 applies and gives $u \in H^{1,l}_0(X)$.

By Proposition 3.5, $u \in x^l H^1_0(X)$ for all l. Thus, by Proposition 2.1, part (i), $u \in H^{r,s}_0(X)$ for all r and s with $r < s + 1/2$, hence for all (r, s). (Given (r, s), consider (r, s') with $s' > \max(s, r - 1/2)$ to see that $u \in H^{r,s'}_0(X)$ hence $u \in H^{r,s}_0(X)$.) In particular, $x^m Qu \in L^2(X)$ for all m and all $Q \in \mathrm{Diff}(X)$, proving the corollary. ∎

4. Conormal regularity

While Proposition 3.4 gives the correct critical rates of growth or decay for solutions of $Pu = 0$, and Corollary 2.2 gives their optimal smoothness in the 0-sense, this is not optimal: solutions of $Pu = 0$ which are $\mathcal{C}^\infty$ in X° are conormal to the boundary, i.e. stable (in terms of weighted L^2-spaces) under the application of b-differential operators – a notion that we recall below. In fact, as usual, cf. [39] and [28], it is convenient to work relative to 0-Sobolev spaces, i.e. to work with $\mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ defined below in Definition 4.1. However, rather than using positive commutator estimates as in these papers, we rely on an 'exact' commutator argument (exact at the level of normal operators), much like in [27, Section 12]. Although it was not discussed explicitly in [27] for reasons of brevity, the analogous space of operators in that setting would be $\mathrm{Diff}_{\mathrm{sc}}^k \Psi_{\mathrm{c}}(X)$, with $\Psi_{\mathrm{c}}(X)$ standing for *cusp* pseudodifferential operators. (Instead, in [27] 'tangential elliptic regularity' was used.)

We first recall the basic definitions and properties of b-differential and pseudo-differential operators. We refer to [32] for a more thorough description, or [39] for a short summary. Thus, $\mathcal{V}_{\mathrm{b}}(X)$ is the set of $\mathcal{C}^\infty$ vector fields on X tangent to Y, so $\mathcal{V}_0(X) \subset \mathcal{V}_{\mathrm{b}}(X) \subset \mathcal{V}(X)$; note that $\mathcal{V}_{\mathrm{b}}(X)$ is a Lie algebra. Elements of $\mathrm{Diff}_{\mathrm{b}}(X)$ are differential operators acting on $\mathcal{C}^\infty(X)$ which are locally finite sums of products of vector fields in $\mathcal{V}_{\mathrm{b}}(X)$ and smooth functions, i.e. elements of $\mathcal{C}^\infty(X)$. In local coordinates $(x, y_1, \ldots, y_{n-1})$, elements of $\mathcal{V}_{\mathrm{b}}(X)$ have the form

$$a_0(x\partial_x) + \sum_{j=1}^{n-1} a_j \partial_{y_j}, \tag{4.1}$$

with a_j in $\mathcal{C}^\infty$ for $j = 0, 1, \ldots, n-1$, and correspondingly they are exactly the set of smooth sections of a vector bundle, ${}^{\mathrm{b}}TX$, the b-tangent bundle. Let ${}^{\mathrm{b}}T^*X$ denote the b-cotangent bundle of X; this is the dual bundle of ${}^{\mathrm{b}}TX$. Its elements are covectors of the form

$$\sigma \frac{dx}{x} + \sum \zeta_j \, dy_j,$$

as can be seen from (4.1). We suggest that the reader compares these statements to (2.1) and the surrounding discussion.

There is also a corresponding space of pseudo-differential operators, $\Psi_{\mathrm{b}}^m(X)$, introduced by Melrose in [31]; see [32] for a thorough description and [39] for a summary. Elements of $\Psi_{\mathrm{b}}^m(X)$ have principal symbols

$$a = \sigma_{\mathrm{b},m}(A) \in S^m_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o),$$

with hom again denoting homogeneous functions.

Definition 4.1. Elements of $\mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ are finite sums of terms QA, $Q \in \mathrm{Diff}_0^k(X)$, $A \in \Psi_{\mathrm{b}}^m(X)$. We also let $x^r \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ be the space of operators of the form $x^r B$, $B \in \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$.

Remark 4.2. Directly from the definition, $\mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ is a $\mathcal{C}^\infty(X)$-bimodule (under left and right multiplication), so in particular $x^r \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ is well-defined independent of the choice of a boundary defining function x.

The key lemma is:

Lemma 4.3. *For $Q \in \mathrm{Diff}_0^k(X)$, $A \in \Psi_{\mathrm{b}}^m(X)$, there exist $Q_j \in \mathrm{Diff}_0^k(X)$, $A_j \in \Psi_{\mathrm{b}}^m(X)$, $j = 1, \ldots, l$, such that $QA = \sum A_j Q_j$. (With a similar conclusion holding, with different A_j, Q_j, for AQ.)*

Proof. It suffices to prove the statement for $Q \in \mathcal{V}_0(X)$; the general case then follows by an inductive argument. As $\mathcal{V}_0(X) \subset \mathcal{V}_{\mathrm{b}}(X)$, $[Q, A] \in \Psi_{\mathrm{b}}^m(X)$, so $QA = AQ + [Q, A]$ gives the desired result. □

Corollary 4.4. $\mathrm{Diff}_0 \Psi_{\mathrm{b}}(X)$ *is closed under composition: if $A \in \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^m(X)$ and $B \in \mathrm{Diff}_0^{k'} \Psi_{\mathrm{b}}^{m'}(X)$ then $AB \in \mathrm{Diff}_0^{k+k'} \Psi_{\mathrm{b}}^{m+m'}(X)$.*

We also need the corresponding result about commutators.

Lemma 4.5. *Moreover, if $A \in x^r \Psi_{\mathrm{b}}^m(X)$, $Q \in \mathrm{Diff}_0^k(X)$ then*

$$[Q, A] \in x^r \mathrm{Diff}_0^{k-1} \Psi_{\mathrm{b}}^m(X).$$

If in addition $\sigma_{\mathrm{b},m}(A)|_{{}^{\mathrm{b}}T^*\partial X} = 0$ then $[Q, A] \in x^r \mathrm{Diff}_0^k \Psi_{\mathrm{b}}^{m-1}(X)$.

Remark 4.6. ${}^{\mathrm{b}}T^*\partial X$ is a well-defined subbundle of ${}^{\mathrm{b}}T^*_{\partial X}X$. If we write b-covectors as $\sigma \, \frac{dx}{x} + \zeta \cdot dy$, then ${}^{\mathrm{b}}T^*\partial X$ is given by $x = 0$, $\sigma = 0$ in ${}^{\mathrm{b}}T^*X$.

Proof. Again, it suffices to prove the first statement for $Q \in \mathcal{V}_0(X)$. As $\mathcal{V}_0(X) \subset \mathcal{V}_{\mathrm{b}}(X)$, $[Q, A] \in \Psi_{\mathrm{b}}^m(X)$, giving the result for such Q. Iterating this also proves that for $Q \in \mathrm{Diff}_0^k(X)$, $[Q, A] \in \mathrm{Diff}_0^{k-1} \Psi_{\mathrm{b}}^m(X)$.

To have the better conclusion, it again suffices to consider $Q \in \mathcal{V}_0(X)$. As above, $[Q, A] \in \Psi_{\mathrm{b}}^m(X)$. But, with $a = \sigma_{\mathrm{b},m}(A)$, $q = \sigma_{\mathrm{b},1}(Q)$,

$$\begin{aligned} i\sigma_{\mathrm{b},m}([A, Q]) &= H_a q \\ &= (\partial_\sigma a)(x\partial_x q) - (x\partial_x a)(\partial_\sigma q) + \sum \left((\partial_{\zeta_j} a)(\partial_{y_j} q) - (\partial_{y_j} a)(\partial_{\zeta_j} q) \right). \end{aligned}$$

This vanishes at ${}^{\mathrm{b}}T^*\partial X$ for a vanishes there, hence so do all terms but the first one; for the first term the differentiation of a in σ means that the vanishing at $x = 0$, $\sigma = 0$ (see Remark 4.6) is insufficient to guarantee its vanishing. Nonetheless, the first term vanishes as $x\partial_x q$ vanishes at $x = 0$. Thus, $\sigma_{\mathrm{b},m}([A, Q]) = \sigma b + xe$ for some $b \in S^{m-1}_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o)$, $e \in S^m_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o)$. We deduce that there exists $B \in \Psi_{\mathrm{b}}^{m-1}(X)$, $E \in \Psi_{\mathrm{b}}^m(X)$, $R \in \Psi_{\mathrm{b}}^{m-1}(X)$ such that $[Q, A] = B(xD_x) + Ex + R$. As one can write $E = E_0(xD_x) + \sum E_j D_{y_j} + R'$ with $E_j, R' \in \Psi_{\mathrm{b}}^{m-1}(X)$, and as $x(xD_x), xD_{y_j} \in \mathcal{V}_0(X)$, the second claim is proved. □

Lemma 4.7. *Suppose $m \geq 0$ is an integer. Any $A \in \Psi_{\mathrm{b}}^0(X)$ defines a continuous linear map on $H_0^{m,l}(X)$ by extension from $\dot{\mathcal{C}}^\infty(X)$.*

Proof. We can use any collection $B^{(i)} \in \mathrm{Diff}_0^m(X)$, $i = 1, \ldots, N$, such that at each point of ${}^0S^*X$ at least one of the $B^{(i)}$ is elliptic, to put a norm on $H_0^{m,l}(X)$:

$$\|u\|^2_{H_0^{m,l}(X)} = \sum_i \|x^{-l} B^{(i)} u\|^2_{L^2(X)} + \|x^{-l} u\|^2_{L^2(X)}.$$

We need to show then that for A as above, $\|Au\|_{H_0^{m,l}(X)} \leq C\|u\|_{H_0^{m,l}(X)}$. Since A is bounded on $x^{-l}L^2(X)$, we only need to prove that for each i, $\|x^{-l}B^{(i)}Au\| \leq C'\|u\|_{H_0^{m,l}(X)}$. But $x^{-l}B^{(i)}A = \sum A_j x^{-l} B_j$ with $A_j \in \Psi_{\mathrm{b}}^0(X)$ and $B_j \in \mathrm{Diff}_0^m(X)$ by Lemma 4.3, so $\|x^{-l}B^{(i)}Au\| \leq \sum C_j \|x^{-l}B_j u\|$ as A_j are bounded on $L^2(X)$. This proves the corollary. □

As we work relative to $x^l H_0^r(X) = H_0^{r,l}(X)$, for $k \geq 0$ we use the Sobolev spaces

$$x^l H_{\mathrm{b},0}^{k,r}(X) = \{u \in x^l H_0^r(X) :\ \forall A \in \Psi_{\mathrm{b}}^k(X),\ Au \in x^l H_0^r(X)\}.$$

These can be normed by taking any elliptic $A \in \Psi_{\mathrm{b}}^k(X)$ and letting

$$\|u\|^2_{x^l H_{\mathrm{b},0}^{k,r}(X)} = \|u\|^2_{x^l H_0^r(X)} + \|Au\|^2_{x^l H_0^r(X)}.$$

Although the norm depends on the choice of A, different choices give equivalent norms. Indeed, if $\tilde{A} \in \Psi_{\mathrm{b}}^k(X)$, then let $G \in \Psi_{\mathrm{b}}^{-k}(X)$ be a parametrix for A, so $GA = \mathrm{Id} + E$, $AG = \mathrm{Id} + F$, $E, F \in \Psi_{\mathrm{b}}^{-\infty}(X)$, and note that

$$\begin{aligned} \|\tilde{A}u\|_{x^l H_0^r(X)} &\leq \|\tilde{A}GAu\|_{x^l H_0^r(X)} + \|\tilde{A}Eu\|_{x^l H_0^r(X)} \\ &\leq C(\|Au\|_{x^l H_0^r(X)} + \|u\|_{x^l H_0^r(X)}), \end{aligned} \tag{4.2}$$

where we used that $\tilde{A}G \in \Psi_{\mathrm{b}}^0(X)$ and $AE \in \Psi_{\mathrm{b}}^{-\infty}(X) \subset \Psi_{\mathrm{b}}^0(X)$ are bounded on $x^l H_0^r(X)$ by Lemma 4.7. If $\tilde{A}$ is elliptic, there is a similar estimate with the role of A and $\tilde{A}$ interchanged, which shows the claimed equivalence.

Lemma 4.8. *If $Q \in \Psi_{\mathrm{b}}^0(X)$, then Q is bounded on $x^l H_{\mathrm{b},0}^{k,r}(X)$.*

Proof. As Q is bounded on $x^l H_0^r(X)$, we only need to prove that for $A \in \Psi_{\mathrm{b}}^k(X)$, $\|AQu\|_{x^l H_0^r(X)} \leq C(\|u\|_{x^l H_0^r(X)} + \|Au\|_{x^l H_0^r(X)})$. But $\tilde{A} = AQ \in \Psi_{\mathrm{b}}^k(X)$, though not necessarily elliptic, so by (4.2), this estimate holds. □

Lemma 4.9. *If $L \in \mathrm{Diff}_{\mathrm{b}}^k(X)$ is elliptic, $u \in x^l H_{\mathrm{b},0}^{s,\infty}(X)$, $Lu \in x^l H_{\mathrm{b},0}^{s,\infty}(X)$, then $u \in x^l H_{\mathrm{b},0}^{s+k,\infty}(X)$.*

Proof. Let $G \in \Psi_{\mathrm{b}}^{-k}(X)$ be a parametrix for L so that $GL = \mathrm{Id} + R$, $R \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Then $u = G(Lu) - Ru$. Now, if $A \in \Psi_{\mathrm{b}}^k(X)$ then $Au = (AG)(Lu) - (AR)u \in x^l H_{\mathrm{b},0}^{s,\infty}(X)$ by Lemma 4.8 since $AG, AR \in \Psi_{\mathrm{b}}^0(X)$. This proves the lemma. □

The conormal regularity theorem is *global* in each connected component of Y. It uses the following lemma, which shows that the boundary Laplacian commutes with P one order better (in terms of decay) than a priori expected:

Lemma 4.10. *Let $\tilde{\Delta}_Y \in \mathrm{Diff}_{\mathrm{b}}^2(X)$ have normal operator given by Δ_Y. Then $[P, \tilde{\Delta}_Y] \in x\,\mathrm{Diff}_0^1\,\mathrm{Diff}_{\mathrm{b}}^2(X)$.*

Proof. Changing $\tilde{\Delta}_Y$ by $Q \in x\,\mathrm{Diff}_{\mathrm{b}}^2(X)$ changes the commutator by an element of $x\,\mathrm{Diff}_0^1\,\mathrm{Diff}_{\mathrm{b}}^2(X)$ due to Lemma 4.5, so the statement only depends on the normal operator of $\tilde{\Delta}_Y$. Similarly, it only depends on the normal operator of P. Thus, we may work on the model space $[0, \epsilon)_x \times Y$, replace P by $(xD_x)^2 + i(n-1)(xD_x) - x^2\Delta_Y$, $\tilde{\Delta}_Y$ by Δ_Y, and then the result is immediate. □

Proposition 4.11. *Suppose $l \in \mathbb{R}$, $u \in x^l H_0^{-\infty}(X)$, $Pu \in \dot{\mathcal{C}}^\infty(X)$ and $u \in \mathcal{C}^\infty(X^\circ)$. Then for all $\epsilon > 0$, $u \in x^{l-\epsilon} H_{\mathrm{b},0}^{\infty,0}(X) = x^{l-\epsilon} H_{\mathrm{b},0}^{\infty,\infty}(X)$.*

Remark 4.12. The proposition states that once one knows that u is smooth in X° and is in some weighted L^2-space, one gets b-regularity relative to that space.

Also, the proposition can be restated in terms of the standard b-spaces: $u \in x^{l+\frac{n-1}{2}-\epsilon} H_{\mathrm{b}}^\infty(X)$. The shift $\frac{n-1}{2}$ in the exponent is simply due to $H_{\mathrm{b}}^s(X)$ being defined relative to $L_{\mathrm{b}}^2(X)$, the L^2-space relative to a non-vanishing b-measure.

Proof. Assume first that $l < -l(\lambda)$. We prove that $u \in x^{l-\epsilon} H^{\infty,\infty}_{\mathrm{b},0}(X)$. We first note that by Corollary 2.2, $u \in H_0^{\infty,l-\epsilon}(X)$ for all $\epsilon > 0$, i.e. we have full 0-regularity. Let $\tilde\Delta_Y$ be as above.

As $u \in H_0^{\infty,l-\epsilon}(X)$, $\tilde\Delta_Y \in x^{-2}\,\mathrm{Diff}^2_0(X)$, we see that $\tilde\Delta_Y u \in H_0^{\infty,l-2-\epsilon}(X)$. Then

$$P\tilde\Delta_Y u = \tilde\Delta_Y Pu + [P,\tilde\Delta_Y]u \in H_0^{\infty,l-1-\epsilon}(X) \tag{4.3}$$

since $[P,\tilde\Delta_Y] \in x\,\mathrm{Diff}^1_0\,\mathrm{Diff}^2_{\mathrm{b}}(X) \subset x^{-1}\,\mathrm{Diff}^3_0(X)$. (In fact, this can be phrased by saying that $N(\tilde\Delta_Y)$ and $N(P)$ commute.) Thus, by Proposition 3.5, $\tilde\Delta_Y u \in H_0^{\infty,l-1-\epsilon}(X)$. As $(xD_x)^2 u \in H_0^{\infty,l-\epsilon}(X)$, $((xD_x)^2+\tilde\Delta_Y)u \in H_0^{\infty,l-1-\epsilon}(X)$. Since $(xD_x)^2+\tilde\Delta_Y$ is elliptic in $\mathrm{Diff}^2_{\mathrm{b}}(X)$, Lemma 4.9 shows that $u \in x^{l-1-\epsilon}H^{2,\infty}_{\mathrm{b},0}(X)$.

Thus, (4.3) and $[P,\tilde\Delta_Y] \in x\,\mathrm{Diff}^1_0\,\mathrm{Diff}^2_{\mathrm{b}}(X)$ gives $P\tilde\Delta_Y u \in x^{l-\epsilon}H_0^{\infty}(X)$, so by Proposition 3.5, $\tilde\Delta_Y u \in H_0^{\infty,l-\epsilon}(X)$. Proceeding as above, we deduce that $u \in x^{l-\epsilon}H^{2,\infty}_{\mathrm{b},0}(X)$.

We now iterate this argument for $\tilde\Delta_Y^k u$ in place of $\tilde\Delta_Y u$. So suppose we already know that $u \in x^{l-\epsilon}H^{2(k-1),\infty}_{\mathrm{b},0}(X)$ for all $\epsilon > 0$. Then $[P,\tilde\Delta_Y^k] \in x\,\mathrm{Diff}^1_0\,\mathrm{Diff}^{2k}_{\mathrm{b}} \subset x^{-1}\,\mathrm{Diff}^3_0\,\mathrm{Diff}^{2(k-1)}_{\mathrm{b}}(X)$, so

$$P\tilde\Delta_Y^k u = \tilde\Delta_Y^k Pu + [P,\tilde\Delta_Y^k]u \in H_0^{\infty,l-1-\epsilon}(X)$$

Again, by Proposition 3.5, $\tilde\Delta_Y^k u \in H_0^{\infty,l-1-\epsilon}(X)$. As $(xD_x)^{2k}u \in H_0^{\infty,l-\epsilon}(X)$, $((xD_x)^{2k}+\tilde\Delta_Y^k)u \in H_0^{\infty,l-1-\epsilon}(X)$. Using Lemma 4.9, we conclude that $u \in x^{l-1-\epsilon}H^{2k,\infty}_{\mathrm{b},0}(X)$.

Equipped with this additional knowledge, we deduce that $[P,\tilde\Delta_Y^k]u \in H_0^{\infty,l-\epsilon}(X)$, hence $P\Delta_Y^k u$ is in the same space. Applying Proposition 3.5, we see that $\tilde\Delta_Y u \in H_0^{\infty,l-\epsilon}(X)$. Proceeding as above, we deduce that $u \in x^{l-\epsilon}H^{2k,\infty}_{\mathrm{b},0}(X)$. This proves the proposition if $l < -l(\lambda)$.

In general, if $l \geq -l(\lambda)$, we may apply the previous argument with l replaced by any $l' < -l(\lambda)$ to conclude that $u \in x^{l'}H^\infty_{\mathrm{b}}(X)$ for all $l' < -l(\lambda)$. Since $u \in x^l L^2(X)$, interpolation gives $u \in x^{l-\epsilon}H^\infty_{\mathrm{b}}(X)$ as stated. □

We now consider $P = \Box - \lambda$ acting on polyhomogeneous functions, or more generally symbols. Recall that $u \in \mathcal{A}^k(X)$ means that $Lu \in x^k L^2_{\mathrm{b}}(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, so in particular $u \in x^k L^2_{\mathrm{b}}(X)$.

We remark that if $s_+, s_- \in \mathbb{C}$ with $s_+ - s_- \notin \mathbb{Z}$, and a function u has the form $x^{s_+}v_+ + x^{s_-}v_-$, $v_\pm \in \mathcal{C}^\infty(X)$, then the leading terms $v_\pm|Y$ (in fact, the full Taylor series of $v_\pm$) is well-defined. However, if $s_+ - s_-$ is an integer, this is no longer true, which explains some of the complications we face in stating the converse direction of the following lemma.

Lemma 4.13. *Suppose $\lambda \in \mathbb{R}$, $\lambda \neq \frac{(n-1)^2}{4}$. Let*

$$s = s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda},$$

be the (not necessarily real) indicial roots of $(xD_x + i(n-1))(xD_x) - \lambda$. If $u \in \mathcal{A}^k(X)$ for some k and $Pu \in \dot{\mathcal{C}}^\infty(X)$ and $s_+(\lambda) - s_-(\lambda)$ is not an integer then there exists $v_\pm \in \mathcal{C}^\infty(X)$, such that

$$u = x^{s_+(\lambda)}v_+ + x^{s_-(\lambda)}v_-.$$

If $s_+(\lambda)-s_-(\lambda)$ is an integer (in which case both $s_\pm(\lambda)$ are real) then the analogous statement holds with $v_-\in\mathcal{C}^\infty(X)$ replaced by

$$v_-\in\mathcal{C}^\infty(X)+x^{s_+(\lambda)-s_-(\lambda)}\log x\,\mathcal{C}^\infty(X).$$

In either case, if $v_\pm|_Y$ vanish, then $u\in\dot{\mathcal{C}}^\infty(X)$.

Conversely, given $g_+,g_-\in\mathcal{C}^\infty(Y)$, there exist

(i) *$v_\pm\in\mathcal{C}^\infty(X)$ if $s_+(\lambda)-s_-(\lambda)$ is not an integer,*

(ii)

$$v_+\in\mathcal{C}^\infty(X),\ v_- -\sum_{j=0}^{s_+(\lambda)-s_-(\lambda)-1} a_jx^j\in x^{s_+(\lambda)-s_-(\lambda)}\log x\,\mathcal{C}^\infty(X),\ a_j\in\mathcal{C}^\infty(Y),$$

if $s_+(\lambda)-s_-(\lambda)$ is an integer,

such that

$$u=x^{s_+(\lambda)}v_++x^{s_-(\lambda)}v_-,\ v_\pm|_Y=g_\pm,$$

satisfies $Pu\in\dot{\mathcal{C}}^\infty(X)$.

Proof. We start with the converse direction. As $P=(xD_x+i(n-1))(xD_x)-\lambda+Q$, $Q\in x\operatorname{Diff}^2_{\mathrm{b}}(X)$, for $v\in\mathcal{C}^\infty(X)$,

$$P(x^sv)=(s(n-1-s)-\lambda)x^sv+w,\ w\in x^{s+1}\mathcal{C}^\infty(X). \tag{4.4}$$

Thus, when s is an indicial root, $P(x^sv)\in x^{s+1}\mathcal{C}^\infty(X)$ automatically, and otherwise given $f\in x^s\mathcal{C}^\infty(X)$, $P(x^sv)=f$ can be solved *uniquely*, modulo $x^{s+1}\mathcal{C}^\infty(X)$, with $v\in\mathcal{C}^\infty(X)$, via dividing $x^{-s}f\in\mathcal{C}^\infty(X)$ by the non-zero quantity $s(n-1-s)-\lambda$. Iterating this argument, and using Borel summation, we deduce that unless the two indicial roots differ by an integer, given $g_+,g_-\in\mathcal{C}^\infty(Y)$, there exists $v_+,v_-\in\mathcal{C}^\infty(X)$ such that

$$u=x^{s_+(\lambda)}v_++x^{s_-(\lambda)}v_-,\ v_\pm|_Y=g_\pm,$$

satisfies $Pu\in\dot{\mathcal{C}}^\infty(X)$.

If the two indicial roots differ by an integer (but are distinct, i.e. not equal to $\frac{n-1}{2}$), only a minor modification is needed in that we need to allow logarithmic factors. Thus, for $v\in\mathcal{C}^\infty(X)$,

$$\begin{aligned}P(x^s\log xv)=&(s(n-1-s)-\lambda)\log x\,x^sv+(n-1-2s)x^sv+w,\\ &w\in x^{s+1}\log x\,\mathcal{C}^\infty(X)+x^{s+1}\mathcal{C}^\infty(X),\end{aligned} \tag{4.5}$$

so if $s=s_\pm(\lambda)$, $Pu=f$, $f\in x^s\mathcal{C}^\infty(X)$, has a solution modulo $x^{s+1}\log x\,\mathcal{C}^\infty(X)+x^{s+1}\mathcal{C}^\infty(X)$, of the form $u\in x^s\log x\,\mathcal{C}^\infty(X)$, so applying this with $s=s_+(\lambda)$, the error term arising from $s_-(\lambda)$ of the form x^s times a smooth function, can be solved away to leading order. Moreover, for $s\neq s_\pm(\lambda)$, $Pu=f$, $f\in x^s\log x\,\mathcal{C}^\infty(X)$ has a solution, modulo $x^{s+1}\log x\,\mathcal{C}^\infty(X)+x^{s+1}\mathcal{C}^\infty(X)$, of the form $u\in x^s\log x\,\mathcal{C}^\infty(X)$, so again iteration gives infinite order solvability, in this case of the form: given $g_+,g_-\in\mathcal{C}^\infty(Y)$, there exists $v_+\in\mathcal{C}^\infty(X)$, $v_-\in\mathcal{C}^\infty(X)+x^{s_+(\lambda)-s_-(\lambda)}\log x\,\mathcal{C}^\infty(X)$ such that

$$u=x^{s_+(\lambda)}v_++x^{s_-(\lambda)}v_-,\ v_\pm|_Y=g_\pm,$$

satisfies $Pu\in\dot{\mathcal{C}}^\infty(X)$.

On the other hand, suppose that $u\in\mathcal{A}^k(X)$ and $Pu\in\dot{\mathcal{C}}^\infty(X)$. As $Qu\in\mathcal{A}^{k+1}(X)$, we have $((xD_x+i(n-1))(xD_x)-\lambda)u\in\mathcal{A}^{k+1}$. Since near Y, using an product decomposition of a neighborhood of Y, $\mathcal{A}^r(X)$ can be identified with

$\mathcal{C}^\infty(Y;\mathcal{A}^r([0,\epsilon)))$, we can treat Y as a parameter and solve this ODE. If there is no indicial root in $(k,k+1]$, one deduces that $u\in\mathcal{A}^{k+1}(X)$; otherwise $u=\sum_j x^{s_j}g_j+u'$ where the s_j are the indicial roots in the interval, g_j are smooth and $u'\in\mathcal{A}^{k+1}$. By the first part of the proof one can choose v_j as in the statement of the lemma (denoted by $v_\pm$ there) to get $u_j=x^{s_j}v_j\in\mathcal{A}^k$ with $Pu_j\in\dot{\mathcal{C}}^\infty(X)$ and $u_j-x^{s_j}g\in\mathcal{A}^{k+1}$. Thus, $u-\sum u_j\in\mathcal{A}^{k+1}$ with $P(u-\sum u_j)\in\dot{\mathcal{C}}^\infty(X)$, so one can proceed iteratively to finish the existence argument. Note that if $g_j|Y$ vanish, one concludes $u\in\mathcal{A}^{k+1}$, which by iteration gives the uniqueness. □

In fact, the same argument also deals with the case $\lambda=(n-1)^2/4$, but as the result is of a slightly different form, we state it separately:

Lemma 4.14. *Suppose $\lambda=\frac{(n-1)^2}{4}$, so $s_\pm(\lambda)=\frac{n-1}{2}$. If $u\in\mathcal{A}^k(X)$ for some k and $Pu\in\dot{\mathcal{C}}^\infty(X)$ then there exists $v_\pm\in\mathcal{C}^\infty(X)$, such that*

$$u=x^{s_+(\lambda)}v_++x^{s_-(\lambda)}\log x\,v_-.$$

Conversely, given $g_+,g_-\in\mathcal{C}^\infty(Y)$, there exists $v_\pm\in\mathcal{C}^\infty(X)$, such that

$$u=x^{s_+(\lambda)}v_++x^{s_-(\lambda)}\log x\,v_-,\ v_\pm|_Y=g_\pm,$$

satisfies $Pu\in\dot{\mathcal{C}}^\infty(X)$.

Proof. $s=s_\pm(\lambda)=(n-1)/2$ now satisfies $s(n-1-s)-\lambda=0$ as $n-1-2s=0$, so (4.4) and (4.5) imply that $P(x^s v_1+x^s\log x\,v_2)\in x^{s+1}\mathcal{C}^\infty(X)+x^{s+1}\log x\mathcal{C}^\infty(X)$. The argument of the previous lemma then shows the second claim.

For the first claim, we need to observe that if $u\in\mathcal{A}^k(X)$ and $Pu\in\dot{\mathcal{C}}^\infty(X)$ then $Qu\in\mathcal{A}^{k+1}(X)$, so $((xD_x+i(n-1))(xD_x)-\lambda)u\in\mathcal{A}^{k+1}$, i.e. $(xD_x+i(n-1)/2)^2u\in\mathcal{A}^{k+1}$. Proceeding as above, the only difference is that if $s=\frac{n-1}{2}\in(k,k+1]$, one deduces that $u=x^sg_1+x^s\log x\,g_2+u'$, g_j smooth, $u'\in\mathcal{A}^{k+1}$. One finishes the proof exactly as above. □

Since we already know (by virtue of Proposition 3.4 and Remark 3.7) that we can solve $Pu'=f$, $f\in\dot{\mathcal{C}}^\infty(X)$, with $u'\in\dot{\mathcal{C}}^\infty(X)$, modulo $\mathcal{C}^\infty_{\mathrm{c}}(X^\circ)$, we deduce that these u can be further extended to be exact solutions near ∂X.

5. Global solvability

For global solvability, i.e. solvability on all of X rather than just near ∂X, of $Pu=0$ we need the additional assumptions (A1)-(A2). We thus assume that $Y=Y_+\cup Y_-$, where $Y_\pm$ are unions of connected components of Y, and this decomposition satisfies that all bicharacteristics $t\mapsto\gamma(t)$ of P (i.e. those of $\Box$, independent of λ) satisfy $\lim_{t\to+\infty}\gamma(t)\in Y_+$, $\lim_{t\to-\infty}\gamma(t)\in Y_-$, or vice versa. In this case, noting that the sign of the χ' term agrees with the others if $r<\min(0,1-2l(\lambda))$ (for they are all negative; recall $l(\lambda)=\frac{n-1}{2}$ for the wave operator itself), one can easily 'cut and paste' the estimates with

- near Y_+, $r=r_+>1+2l(\lambda)$ (or just $r=r_+>\max(0,1-2l(\lambda))$, $r_+\neq 1+2l(\lambda)$),
- near Y_-, $r=r_-<\min(0,1-2l(\lambda))$, and
- standard microlocal propagation estimates in the interior of X

to deduce that for a partition of unity $\chi_+ + \chi_- + \chi_0 = 1$ with χ_+ supported near Y_+, identically 1 in a smaller neighborhood of Y_+, analogously with χ_-, $\chi_0 \in \mathcal{C}^\infty_c(X^\circ)$, there exists $\tilde\chi_0 \in \mathcal{C}^\infty_c(X^\circ)$ such that

$$\|x^{(r_+-1)/2}\chi_+ v\|^2_{H^1_0} + \|x^{(r_--1)/2}\chi_- v\|^2_{H^1_0} + \|\chi_0 v\|^2_{H^1_0} \leq C(\|\tilde\chi_0 v\|^2_{H_0^{1/2}} + \|Pv\|^2). \tag{5.1}$$

Let $H_0^{m,q_+,q_-}(X)$ be the space $x_+^{q_+} x_-^{q_-} H_0^m(X)$, where $x_\pm$ are defining functions of $Y_\pm$. We equip it with the norm

$$\|v\|^2_{H_0^{m,q_+,q_-}(X)} = \|x^{-q_+}\chi_+ v\|^2_{H^m_0} + \|x^{-q_-}\chi_- v_-\|^2_{H^m_0} + \|\chi_0 v\|^2_{H^m_0}.$$

(Note that it is the completion of $\dot{\mathcal{C}}^\infty(X)$ with respect to this norm.) This is just $x^{q_\pm} H^m_0(X)$ near $Y_\pm$, $H^m(X^\circ)$ in the interior. Let $l_\pm = (r_\pm - 1)/2$. The argument of [20, Proof of Theorem 26.1.7] shows the following:

Proposition 5.1. *Suppose that $\lambda \in \mathbb{R}$, $l_+ > \max(\frac{1}{2}, l(\lambda))$, $l_- < -\max(\frac{1}{2}, l(\lambda))$. Then*

$$N_{l_+,l_-} = \{v \in H_0^{1,-l_+,-l_-}(X) : \ Pv = 0\}$$

is finite dimensional, and for $f \in H_0^{0,l_+,l_-}(X)$, f orthogonal to N_{l_+,l_-}, $Pu = f$ has a solution $u \in H_0^{1,l_+,l_-}(X)$.

Moreover, elements of N_{l_+,l_-} are in $H_0^{\infty,l,-l_-}(X)$ for all $l < -l_+$, are Schwartz at Y_-, and have an expansion as in Lemma 4.13 at Y_+.

Remark 5.2. Note that the expansion of Lemma 4.13 implies that N_{l_+,l_-} are in $H_0^{\infty,l,\infty}(X)$ for all $l < -l(\lambda)$, not merely $l < -l_+$.

Proof. We first prove the last statement. For $v \in N_{l_+,l_-}$, by Corollary 3.6, v is Schwartz at Y_-. In particular, v is $\mathcal{C}^\infty$ near Y_-, so by the standard propagation of singularities for P, $v \in \mathcal{C}^\infty(X^\circ)$. Then, by Corollary 2.2, $v \in H_0^{\infty,l,-l_-}$ for all $l < -l_+$. By Proposition 4.11 and the remark following it, $u \in x^{l+\frac{n-1}{2}} H^\infty_b(X) = \mathcal{A}^{l+\frac{n-1}{2}}(X)$ for all $l < -l_+$. Thus, by Lemma 4.13, it has an expansion at Y_+ of the form given by Lemma 4.13.

This in particular implies that the commutator calculations giving rise to (5.1) can be applied directly (without mollification) to all $v \in N_{l_+,l_-}$ The proof of the first part is finished as in [20], and the second part can then be proved exactly as in [20]. □

Note that the role of $Y_\pm$ is reversible, so the estimates, hence the proposition, also hold with $l_\pm$ interchanged. Correspondingly, we deduce that the solution u of $Pu = f$ above is unique modulo the finite dimensional space $N_{-l_+,-l_-}$.

One can also get uniqueness, namely that

Proposition 5.3. *Suppose $u \in \dot{\mathcal{C}}^\infty(X)$ and $Pu = 0$. Then $u = 0$.*

In fact, it suffices to assume that u is Schwartz at Y_+.

If we merely assume that u is Schwartz at a connected component Y_j of Y, and $Pu = 0$ near Y_j, then we can still conclude that $u = 0$ near Y_j.

Proof. The proof is very similar to [38, Section 4] and to [40]. Consider $P_h = x^{-1/h} h^2 P x^{1/h}$. The basic claim is that the semiclassical symbols of $\operatorname{Re} P_h \in$

$\mathrm{Diff}^2_{0,h}(X)$ and $\operatorname{Im} P_h \in \mathrm{Diff}^1_{h,0}(X)$ never vanish at the same place at Y. In fact, as P is formally self-adjoint, one has

$$\begin{aligned} P_h &= h^2 P + x^{-1/h}[h^2 P, x^{1/h}],\\ \operatorname{Re} P_h &= h^2 P + \frac{1}{2}[x^{-1/h},[h^2P,x^{1/h}]],\\ \operatorname{Im} P_h &= \frac{1}{2i}(x^{-1/h}[h^2P,x^{1/h}] + [h^2P,x^{1/h}]x^{-1/h}).\end{aligned}$$

Now, for $Q \in x^l\,\mathrm{Diff}^k_{0,h}(X)$, $x^{-1/h}[Q,x^{1/h}] \in x^l\,\mathrm{Diff}^{k-1}_{0,h}(X)$, so if we only want to compute the commutators modulo higher order terms in x, we can work with the normal operator of P instead of P. Also, modulo higher order terms in h, only the principal symbol of P matters in the calculations, as we are considering h^2P, and changing P by a first order term changes h^2P by an element of $h\,\mathrm{Diff}^1_{0,h}(X)$. Thus, a straightforward computation gives

$$\begin{aligned} \operatorname{Re} P_h &= (hxD_x)^2 - x^2\Delta_Y + \frac{1}{2}[x^{-1/h},[h^2(xD_x)^2,x^{1/h}]] + R_1\\ &= (hxD_x)^2 - h^2x^2\Delta_Y - 1 + R_1,\\ \operatorname{Im} P_h &= \frac{1}{2i}(x^{-1/h}[(hxD_x)^2,x^{1/h}] + [(hxD_x)^2,x^{1/h}]x^{-1/h}) + R_2 = -2hxD_x + R_2,\end{aligned}$$

with $R_1 \in h\,\mathrm{Diff}^2_{0,h}(X) + x\,\mathrm{Diff}^2_h(X)$, $R_2 \in h\,\mathrm{Diff}^1_{0,h}(X) + x\,\mathrm{Diff}^1_{0,h}(X)$. Moreover,

$$i[\operatorname{Re} P_h, \operatorname{Im} P_h] = i[-h^2x^2\Delta_Y, -2hxD_x] + R_3 = -4h^3x^2\Delta_Y + hR_3,$$

$R_3 \in h\,\mathrm{Diff}^2_{0,h}(X) + x\,\mathrm{Diff}^2_{0,h}(X)$. Thus,

$$i[\operatorname{Re} P_h, \operatorname{Im} P_h] = h + 4h\operatorname{Re} P_h - h(\operatorname{Im} P_h)^2 + hR_4,$$

with R_4 having the same properties as R_3.

Now let $u_h = x^{-1/h}u \in \dot{\mathcal{C}}^\infty(X)$, so $P_hu_h = 0$ and

$$\begin{aligned} 0 = \|P_hu_h\|^2 &= \|\operatorname{Re} P_hu_h\|^2 + \|\operatorname{Im} P_hu_h\|^2 + \langle i[\operatorname{Re} P_h,\operatorname{Im} P_h]u_h,u_h\rangle\\ &= \|\operatorname{Re} P_hu_h\|^2 + (1-h)\|\operatorname{Im} P_hu_h\|^2 + h\|u_h\|^2 + 4h\langle \operatorname{Re} P_hu_h,u_h\rangle + h\langle R_4u_h,u_h\rangle.\end{aligned}$$

This is the analogue of Equations (4.2) and (4.3) of [38], except that here terms arising from the commutator $i[\operatorname{Re} P_h, \operatorname{Im} P_h]$ do not have an additional factor of x compared to the first two squares on the right hand side. The proof can be finished exactly as in [38], writing $R_4 = hR_5 + x^{1/2}R_6x^{1/2}$, $R_5, R_6 \in \mathrm{Diff}^2_{0,h}(X)$, and noting that $-\operatorname{Re} P_h + (\operatorname{Im} P_h)^2$ is elliptic second order, so

$$\begin{aligned} &|\langle hR_5u_h,u_h\rangle| \le Ch(\|\operatorname{Re} P_hu_h\|\,\|u_h\| + \|\operatorname{Im} P_hu_h\|^2 + \|u_h\|^2),\\ &|\langle x^{1/2}R_6x^{1/2}u_h,u_h\rangle|\\ &\qquad \le Ch(\|\operatorname{Re} P_hx^{1/2}u_h\|\,\|x^{1/2}u_h\| + \|\operatorname{Im} P_hx^{1/2}u_h\|^2 + \|x^{1/2}u_h\|^2)\\ &\qquad \le C'h(\|\operatorname{Re} P_hu_h\|\,\|x^{1/2}u_h\| + \|\operatorname{Im} P_hu_h\|^2 + \|x^{1/2}u_h\|^2).\end{aligned}$$

Indeed, for $\delta > 0$ one writes

$$\|x^{1/2}u_h\|^2 = \|x^{1/2}u_h\|^2_{x\le\delta} + \|x^{1/2}u_h\|^2_{x\ge\delta} \le \delta\|u_h\|^2 + \delta^{1-2/h}\|u\|^2,$$

so

$$\begin{aligned} 0 \ge (1-C_1h)\|\operatorname{Re} P_hu\|^2 + (1-C_2h)\|\operatorname{Im} P_hu\|^2 + h(1-C_3h-C_4\delta)\|u_h\|^2\\ - C_5\delta^{1-2/h}\|u\|^2.\end{aligned}$$

Thus, there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$hC_5\delta^{1-2/h}\|u\|^2 \geq h(\frac{1}{2} - C_4\delta)\|u_h\|^2.$$

Suppose $\delta \in (0, \min(\frac{1}{4C_4}, \frac{1}{h_0}))$ and $\operatorname{supp} u \cap \{x \leq \frac{\delta}{4}\}$ is non-empty. Then $\|u_h\|^2 \geq C_6(\delta/4)^{-2/h}$ with $C_6 > 0$. Thus,

$$C_5\delta\|u\|^2 \geq \frac{C_6}{4}4^{2/h}.$$

As the right hand side goes to $+\infty$ as $h \to 0$, this provides a contradiction.

Thus, u vanishes for $x \leq \delta/4$, and then the usual hyperbolic uniqueness (well-posedness of the non-characteristic Cauchy problem) gives that it vanishes on X. □

Combined with Proposition 5.1 this gives:

Theorem 5.4. *Suppose that $\lambda \in \mathbb{R}$, $l_+ > \max(\frac{1}{2}, l(\lambda))$, $l_- < -\max(\frac{1}{2}, l(\lambda))$. Then for $f \in H_0^{0,l_+,l_-}(X)$, $Pu = f$ has a unique solution $u \in H_0^{1,l_+,l_-}(X)$.*

Proof. With the notation of Proposition 5.1, we want to prove $N_{l_+,l_-} = \{0\}$. But for $v \in N_{l_+,l_-}$, by Corollary 3.6, v is Schwartz at Y_-. Thus, by Proposition 5.3, $v = 0$. Thus, by Proposition 5.1, the required u exists.

Conversely, if $u \in H_0^{1,l_+,l_-}(X)$ and $Pu = 0$ then by Corollary 3.6, u is Schwartz at Y_+, so by Proposition 5.3, $u = 0$. □

We also deduce:

Theorem 5.5. *Suppose $\lambda \neq \frac{(n-1)^2}{4}$. Given $g_\pm \in \mathcal{C}^\infty(Y_+)$ there exists a unique $u \in \mathcal{C}^\infty(X^\circ)$ such that $Pu = 0$ and which is of the form*

$$u = x^{s_+(\lambda)}v_+ + x^{s_-(\lambda)}v_-,\ v_\pm|_{Y_+} = g_\pm,\ v_+ \in \mathcal{C}^\infty(X),$$

$$v_- - \sum_{j=0}^{s_+(\lambda)-s_-(\lambda)-1} a_j x^j \in x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X),\ a_j \in \mathcal{C}^\infty(Y_\pm).$$

If $s_+(\lambda) - s_-(\lambda)$ is not an integer, then $v_- \in \mathcal{C}^\infty(X)$.

On the other hand, if $\lambda = \frac{(n-1)^2}{4}$, then given $g_\pm \in \mathcal{C}^\infty(Y_+)$ there exists a unique $u \in \mathcal{C}^\infty(X^\circ)$ such that $Pu = 0$ and which is of the form

$$u = x^{(n-1)/2}v_+ + x^{(n-1)/2}\log x\, v_-,\ v_\pm|_{Y_+} = g_\pm,\ v_\pm \in \mathcal{C}^\infty(X).$$

Proof. Suppose $\lambda \neq \frac{(n-1)^2}{4}$. As shown in Lemma 4.13, there exists u_0 supported near Y_+ and of the desired form there, such that $Pu_0 \in \dot{\mathcal{C}}^\infty(X)$. By Theorem 5.4, for any $l_+ > \max(\frac{1}{2}, l(\lambda))$ and $l_- < -\max(\frac{1}{2}, l(\lambda))$ there exists a unique $u_1 \in H_0^{1,l_+,l_-}(X)$ such that $Pu_1 = -Pu_0 \in \dot{\mathcal{C}}^\infty(X)$. As $l_\pm$ are arbitrary subject to the constraints, and u_1 is unique, $u_1 \in H_0^{1,l_+,l_-}(X)$ *for all* $l_+ > \max(\frac{1}{2}, l(\lambda))$ $l_- < -\max(\frac{1}{2}, l(\lambda))$. By Corollary 3.6, u_1 is Schwartz at Y_+. Thus, $u = u_0 + u_1$ satisfies $Pu = 0$, and is smooth near Y_+, so by the standard propagation of singularities $u \in \mathcal{C}^\infty(X^\circ)$. As $u \in H_0^{1,l_-}(X)$ for all $l_- < -\max(\frac{1}{2}, l(\lambda))$ near Y_-, Corollary 2.2 gives $u \in H_0^{\infty,l_-}(X)$ for all such l_-. By Proposition 4.11 and the remark following it, $u \in x^{l_- + \frac{n-1}{2}}H_{\mathrm{b}}^\infty(X) = \mathcal{A}^{l_- + \frac{n-1}{2}}(X)$ for all such l_-. Thus, by Lemma 4.13 it has the stated form near Y_-.

Conversely, if u has the stated properties and $g_\pm = 0$, then $v_\pm$ are Schwartz at Y_+ by Lemma 4.13, so u is Schwartz at Y_+. Then $u = 0$ by Proposition 5.3.

If $\lambda = \frac{(n-1)^2}{4}$, the same argument, but using Lemma 4.14 instead of Lemma 4.13, completes the proof of the theorem. □

6. The Cauchy problem

We now consider global solutions for the Cauchy problem posed near $Y_\pm$.

Let T be a compactified time function, as in the introduction. For any constant $t_0 \in (-1, 1)$, and a vector field V transversal to S_{t_0}, P is strictly hyperbolic, and the Cauchy problem

$$\begin{aligned} &Pu = 0 \text{ in } X^\circ,\\ &u|_{S_{t_0}} = \psi_0,\\ &Vu|_{S_{t_0}} = \psi_1, \end{aligned} \tag{6.1}$$

$\psi_0, \psi_1 \in \mathcal{C}^\infty(S_{t_0})$ is well posed.

Theorem 6.1. *Let $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. Assuming (A1) and (A2), the solution u of the Cauchy problem* (6.1) *has the form*

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-,\ v_\pm \in \mathcal{C}^\infty(X), \tag{6.2}$$

if $s_+(\lambda) - s_-(\lambda) = 2\sqrt{\frac{(n-1)^2}{4} - \lambda}$ is not an integer. If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in \mathcal{C}^\infty(X)$ by $v_- \in \mathcal{C}^\infty(X) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$.

Proof. As P is strictly hyperbolic with respect to S_{t_0}, [20, Theorem 23.2.4] guarantees the existence of $u_0 \in \mathcal{C}^\infty_{\mathrm{c}}(X^\circ)$ with $Pu_0 = 0$ in a neighborhood of S_{t_0} and having the required Cauchy data. We may choose $t_1 < t_0 < t_2$ so that $Pu_0 = 0$ for $T \in (t_1, t_2)$. Let $\chi_1, \chi_2 \in \mathcal{C}^\infty(X)$ be such that $\chi_1 \equiv 1$ in a neighborhood of $T \geq t_0$, χ_1 is supported in $T > t_1$, while $\chi_2 \equiv 1$ in a neighborhood of $T \leq t_0$, supported in $T < t_2$. In particular, $\chi_1\chi_2$ is supported where $T \in (t_1, t_2)$, is identically 1 near S_{t_0}, and each χ_i is identically 1 on the support of the $d\chi_j$, $j \neq i$. Then $P(\chi_1\chi_2 u_0) = [P, \chi_1]u_0 + [P, \chi_2]u_0$. Denoting these two terms by f_1, resp. f_2, we use Theorem 5.4 to solve away f_1 towards Y_+ and f_2 towards Y_- so that the Cauchy data are unchanged.

First, by Theorem 5.4, with any $l > \max(\frac{1}{2}, l(\lambda))$, there exists $u_2 \in H_0^{1,l,-l}(X)$ such that $Pu_2 = f_2$. By Corollary 3.6, u_2 is Schwartz at Y_+, and then by Proposition 5.3, $u_2 \equiv 0$ near Y_+. Hyperbolic propagation then shows that $\operatorname{supp} u_2 \subset \{T > t_0\}$ as f_2 is supported in this set, so $u_2 \equiv 0$ near S_{t_0}. In addition, as in the argument of Theorem 5.5 we deduce that $u_2 \in \mathcal{C}^\infty(X^\circ)$ has an expansion as in Theorem 5.5.

Interchanging the weights at $Y_\pm$, we can similarly show the existence of $u_1 \in H_0^{1,-l,l}(X)$ such that $Pu_1 = f_1$, $\operatorname{supp} u_1 \subset \{T < t_0\}$, and u_1 having an expansion at Y_+. Thus, $u = \chi_1\chi_2 u_0 - u_1 - u_2 \in \mathcal{C}^\infty(X^\circ)$ satisfies $Pu = 0$, $u|_{S_{t_0}} = \psi_0$, $Vu|_{S_{t_0}} = \psi_1$, and u has an asymptotic expansion as in Theorem 5.5, proving the existence part.

Uniqueness follows easily, for if u solves the Cauchy problem with $\psi_0 = 0$, $\psi_1 = 0$, then $u = 0$ near S_{t_0}, hence vanishes globally. □

It is useful to relate the Cauchy data at different hypersurfaces to each other, particularly for hypersurfaces near Y_+, resp, Y_-. This is very easy using the standard FIO result. We renormalize this operator in order to make all entries in the FIO matrix have the same order. Namely, let Δ_{t_j} be the Laplacian of the restriction of g to S_{t_j}, $j = 1, 2$, so $\Delta_{t_j} \geq 0$ as S_{t_j} is space like. Let Δ'_{t_j} denote the operator which is Δ_{t_j} on the orthocomplement of the nullspace of Δ_{t_j} and is the identity on the nullspace, so Δ'_{t_j} is positive and invertible.

Proposition 6.2. *([9]) For any $t_1, t_2 \in (-1, 1)$, the map C_{t_1,t_2} sending Cauchy data of global smooth solutions of $Pu = 0$ at S_{t_1} to Cauchy data at S_{t_2}:*

$$C_{t_1,t_2} : ((\Delta'_{t_1})^{1/2} u|_{S_{t_1}}, Vu|_{S_{t_1}}) \mapsto ((\Delta'_{t_2})^{1/2} u|_{S_{t_2}}, Vu|_{S_{t_2}})$$

is an invertible Fourier integral operator of order 0 corresponding to the bicharacteristic flow.

7. The scattering operator

In order to prove that the scattering operator is a Fourier integral operator, we construct a parametrix as a conormal distribution on a resolution of $X \times Y_+$ for the solution operator, also called the 'Poisson operator', $(g_+, g_-) \mapsto u$ with notation as in (1.2) and (1.3).

7.1. **The geometry.** Near Y_+, the parametrix construction can be done by considering $[X \times Y_+; \mathrm{diag}_{Y_+}]$. On this space the parametrix is a conormal distribution near Y_+ associated to the 'flowout' of points in Y_+. That is, for $q' \in Y_+$, consider the bicharacteristics approaching ${}^0S^*_{q'}X$. These form a Lagrangian submanifold of T^*X°, which near Y_+ has constant rank projection (since the rank at the front face is maximal, namely $n - 1$), and is thus the conormal bundle of a submanifold $F_{q'}$ of X. These $F_{q'}$ depend smoothly on q' so that

$$F = \cup_{q'} (F_{q'} \times \{q'\})$$

is a smooth submanifold of $X^\circ \times Y_+$ near $Y_+ \times Y_+$, and indeed it extends to be smooth to a neighborhood of the inverse image of $Y_+ \times Y_+$ in $[X \times Y_+; \mathrm{diag}_{Y_+}]$. *To avoid overburdening notation, in our constructions below (which are local near the inverse image of $Y_+ \times Y_+$ in $[X \times Y_+; \mathrm{diag}_{Y_+}]$) we often call F an embedded submanifold of $[X \times Y_+; \mathrm{diag}_{Y_+}]$, without explicitly restricting to a neighborhood of the inverse image of $Y_+ \times Y_+$.*

We recall here the static picture of De Sitter space, see Figure 1. For a fixed q', $[X; \{q'\}]$ is the resolved space shown on the picture on the right. The model operator P_σ that we study below, see (7.4), is an operator on the front face of the blow-up.

In order to orient ourselves and see what needs to be done, we first make some remarks regarding distributions conormal to F. First, recall that if M is a manifold with corners of dimension m, and Z is an interior p-submanifold, $I^p(M, Z)$ is the space of distributions on M conormal to Z, see [25, 26]. Here we only need the case where Z meets all boundary faces transversally; in fact, in this case, Z only meets a (codimension one) boundary hypersurface. Thus, in local coordinates (x, y), $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_{m-k})$ in which M is locally given by $x_j \geq 0$ for all

j, and Z is given by $y_1 = \ldots = y_N = 0$, elements of $I^p(M,Z)$ have the form

$$(2\pi)^{-(m+2N)/4} \int_{\mathbb{R}^N} e^{iy'\cdot\xi}\, a(x,y,\xi)\, d\xi,$$

with $a \in S^{p+(m-2N)/4}(M;\mathbb{R}^N)$, $y' = (y_1,\ldots,y_N)$. Note that x behaves as a parameter, i.e. the presence of boundaries does not cause any complications, hence the standard treatment in the boundaryless case [19, 20] actually suffices. Note that if $A \in \mathrm{Diff}^r(M)$ and $u \in I^p(M,Z)$ then $Au \in I^{p+m}(M,Z)$, and if A is characteristic in Z, i.e. its principal symbol vanishes on N^*Z, then $Au \in I^{p+m-1}(M,Z)$, with $\sigma_{p+m-1}(Au) = H_a\sigma_m(u) + bu$, where b depends on A only. This equation is an ODE along the bicharacteristics of A, and is called a transport equation.

We also need to allow weights, i.e. consider the spaces $x^s I^p(M,Z)$. $\mathrm{Diff}(M;Z)$ is not well-behaved on these spaces (because of derivatives possibly falling on x^s) but $\mathrm{Diff}_{\mathrm{b}}(M)$ is.

Lemma 7.1. *(see [20][Section 18.2] and [25]) Suppose that $A \in \mathrm{Diff}^m_{\mathrm{b}}(M)$. Then*

$$A : x^s I^p(M,Z) \to x^s I^{p+m}(M,Z). \tag{7.1}$$

If A is characteristic on Z, then

$$A : x^s I^p(M,Z) \to x^s I^{p+m-1}(M,Z), \tag{7.2}$$

and there is function b depending on A only such that $\sigma_{p+m-1}(Au) = H_a\sigma_m(u)+bu$.

Proof. As $x^{-s}Ax^s \in \mathrm{Diff}^m_{\mathrm{b}}(M) \subset \mathrm{Diff}^m(M)$, (7.1) follows immediately from the remarks above. Next, if A is characteristic on Z, then so is $x^{-s}Ax^s$, so the remarks above prove (7.2). As the principal symbol of $x^{-s}Ax^s$ is the same as that of A, $\sigma_{p+m-1}(Au) = H_a\sigma_m(u) + bu$ follows. □

In our case, $M = [X\times Y_+; \mathrm{diag}_{Y_+}]$, and $Z = F$. The transport equation will allow us to solve away errors modulo smooth terms in our construction of the 'Poisson operator', $(g_+, g_-) \mapsto u$. However, we need to see first what the 'errors' are errors of, i.e. where the Schwartz kernel of the Poisson operator comes from, which will also give a relationship between the orders s and p above.

Even for arbitrary Y, the model on the front face is the same as when Y is Euclidean space with a translation-invariant metric. Let y denote local coordinates on Y, as well as their extension to X, so (x,y) are local coordinates on X. On $X\times Y_+$ then we have local coordinates (x,y,y'), where y' is the pull-back of y from the second factor. (The pull-back of y from the first factor, X, is still denoted by y.) Using projective coordinates

$$X = x,\ Y = \frac{y-y'}{x},\ y',$$

the $P = \Box - \lambda$ becomes

$$(XD_X - YD_Y + i(n-1))(XD_X - YD_Y) - \sum_{i,j} h_{ij}(y')D_{Y_i}D_{Y_j} - \lambda,$$

modulo $X\,\mathrm{Diff}^2_{\mathrm{b}}([X\times Y_+;\mathrm{diag}_{Y_+}])$. To analyze this operator for fixed y', we may arrange that $h_{ij}(y') = \delta_{ij}$, so the operator becomes

$$(XD_X - YD_Y + i(n-1))(XD_X - YD_Y) - \Delta_Y - \lambda. \tag{7.3}$$

When acting on functions of the form $u = x^s v$, v a function of Y, XD_X becomes a multiplication operator, and the operator we arrive at after this substitution is a degenerate PDE with radial points over $|Y| = 1$, i.e. where F hits the front face. This is indeed what enables us to find solutions supported in $|Y| \leq 1$, with singularities carried away by F.

7.2. **The model.** While (7.3) is helpful in seeing the big picture, we need to solve this exactly at $X = 0$ to leading order, for which it is useful to view $\Box$ on the warped product model as the analytic continuation of the Laplacian on hyperbolic space, which is arrived at by complex rotation in x (replacing x by ix), i.e. considering the Laplacian of $\frac{dx^2+h}{x^2}$. Correspondingly, the explicit solutions we are interested in are analytic continuations of the Eisenstein functions (Poisson kernel) on hyperbolic space, i.e. they take the form

$$X^s(|Y|^2 - 1 \pm i0)^s,\ -s(n+s-1) = \lambda.$$

Recall that the Eisenstein functions are obtained using 'upper half space' (rather than Poincaré) local coordinates on hyperbolic spaces, so the boundary metric is *Euclidean*. The values of s stated above are different from the usual indicial roots; these give

$$s = \hat{s}_\pm(\lambda) = -\frac{n-1}{2} \pm \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda} = s_\pm(\lambda) - (n-1).$$

We in fact have two interesting solutions corresponding to branches of the analytic continuation. As we are interested in solutions supported inside $|Y| \leq 1$, we take their difference,

$$X^s[(|Y|^2 - 1 + i0)^s - (|Y|^2 - 1 - i0)^s] = c_s X^s (|Y|^2 - 1)_-^s,$$

with $c_s = e^{i\pi s} - e^{-i\pi s}$ if s is not a negative integer, and

$$X^s[(|Y|^2 - 1 + i0)^s - (|Y|^2 - 1 - i0)^s] = c_s X^s \delta_0^{(-s-1)}(|Y|^2 - 1),$$

with $c_s = \frac{2\pi i(-1)^{-s}}{(-s-1)!}$ if s is a negative integer. Here the notation is that if f is a distribution on $\mathbb{R}$ which is conormal to the origin, then $f(|Y|^2 - 1)$ denotes $T^* f$, where $T : \mathbb{R}^{n-1} \to \mathbb{R}$ is the map $T(Y) = |Y|^2 - 1$. The preimage of the origin under T is the unit sphere, and on the unit sphere the differential of T is surjective, so the pull-back of these conormal distributions indeed makes sense.

If the boundary is actually (locally) Euclidean, i.e. points in Y_+ have neighborhoods which are subsets of $\mathbb{R}^{n-1}$ and in which h is the Euclidean metric, then near $Y_+ \times Y_+$ we thus obtain an exact solution with singularities on F,

$$E_{0,\pm}(x, y, y', \lambda) = C_s x^s (1 - \frac{|y - y'|^2}{x^2})_+^s,$$

with C_s to be determined and $s = \hat{s}_\pm(\lambda)$, if s is not a negative integer, and

$$E_{0,\pm}(x, y, y', \lambda) = C_s x^s \delta_0^{(-s-1)}(1 - \frac{|y - y'|^2}{x^2})$$

if s is a negative integer. Note that for each λ,

$$E_{0,\pm} = E_{0,\pm}(\lambda) \in x^s I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F),\ m(s) = -s - \frac{2n+1}{4},\ s = \hat{s}_\pm(\lambda).$$

Lemma 7.2. *Suppose that* Y_+ *is locally Euclidean, i.e.* $h|_{Y_+}$ *is the Euclidean metric. Suppose that* $\hat{s}_\pm(\lambda) \notin -\frac{n-1}{2} - \mathbb{N}_+$*. Then there is a constant* $C_s \neq 0$ *such that for all* $\phi \in \mathcal{C}^\infty(Y_+)$ *the operator* $E_{0,\pm}(\lambda)$ *with Schwartz kernel* $E_{0,\pm}\,dh$*:*

$$E_{0,\pm}\phi = \int E_{0,\pm}(x,y,y',\lambda)\phi(y')\,dh(y')$$

satisfies

$$E_{0,\pm}\phi = x^{s(\lambda)}v,\ v \in \mathcal{C}^\infty(X),\ v|_{Y_+} = \phi.$$

Remark 7.3. Note that for $\lambda > \frac{(n-1)^2}{4} - 1$, the condition $\hat{s}_\pm(\lambda) \notin -\frac{n-1}{2} - \mathbb{N}_+$ automatically holds. For $\Box$ itself (i.e. $\lambda = 0$) the condition holds if n is even. In addition, the condition always holds for *one* of the two indicial roots, namely the larger one (i.e. the one with more decay/less growth at Y_+).

Proof. Suppose first that $\hat{s}_\pm(\lambda)$ is not a negative integer.

Changing variables in the integral we deduce that for $\phi \in \mathcal{C}^\infty_c(Y_+)$, and $s = \hat{s}_\pm(\lambda)$ still,

$$\int E_{0,\pm}(x,y,y',\lambda)\phi(y')\,dy' = x^{n-1+\hat{s}_\pm(\lambda)}\int (1-|Y|^2)^s_+\phi(y-xY)\,dY$$
$$= x^{s_\pm(\lambda)}v,\ v \in \mathcal{C}^\infty(X),\ v(0,y) = C_s((1-|Y|^2)^s_+,1)\phi(y),$$

where the second factor in the expression for $v(0,y)$ is the evaluation of the distribution $(1-|Y|^2)^s_+$ on 1, and where we used that $s_\pm(\lambda) = \hat{s}_\pm(\lambda) + (n-1)$. We need to check for which values of s does C_s vanish, so we compute this pairing.

For $\operatorname{Re} s > -1$, the distributional pairing is an absolutely convergent integral, which in polar coordinates becomes

$$c_{n-2}\int (1-\rho^2)^s\rho^{n-1}\,d\rho = \frac{c_{n-2}}{2}B(\frac{n-1}{2}, s+1) = \frac{c_{n-2}\Gamma(\frac{n-1}{2})\Gamma(s+1)}{2\Gamma(\frac{n-1}{2}+s+1)},$$

where c_{n-2} is the volume of the $(n-2)$-sphere and B is the beta-function. As both the distributional pairing and the Γ function are meromorphic in s (indeed analytic away from $-\mathbb{N}$), we deduce that

$$((1-|Y|^2)^s_+,1) = \frac{c_{n-2}\Gamma(\frac{n-1}{2})\Gamma(s+1)}{2\Gamma(\frac{n-1}{2}+s+1)}$$

for all s which are not negative integers. This vanishes only if $s \in -\frac{n-1}{2} - \mathbb{N}_+$ and n is even (so s is not a negative integer).

If $s = \hat{s}_\pm(\lambda)$ is a negative integer, say $s = -k$,

$$\int E_{0,\pm}(x,y,y',\lambda)\phi(y')\,dy' = x^{n-1+\hat{s}_\pm(\lambda)}\int (1-|Y|^2)^s_+\phi(y-xY)\,dY$$
$$= x^{s_\pm(\lambda)}v,\ v \in \mathcal{C}^\infty(X),\ v(0,y) = C_s(\delta_0^{(-s-1)}(1-|Y|^2),1)\phi(y).$$

The distributional pairing now becomes

$$\frac{c_{n-2}}{2}(\delta_0^{(k-1)}(z),(1-z)^{(n-3)/2}) = \frac{c_{n-2}}{2}\frac{d^{k-1}}{(dz)^{k-1}}(1-z)^{(n-3)/2}|_{z=0}.$$

If n is even, all derivatives of $(1-z)^{(n-3)/2}$ at $z=0$ are non-zero, while if n is odd, the derivatives of order $< \frac{n-1}{2}$ are non-zero, so this pairing vanishes only if $s = -k \in -\frac{n-1}{2} - \mathbb{N}_+$.

Combining these two cases, $s = \hat{s}_\pm(\lambda) \notin -\frac{n-1}{2} - \mathbb{N}_+$ implies that the respective distributional pairings are non-zero. Letting C_s to be their reciprocal yield $E_{0,\pm}(\lambda)$ satisfying the lemma. □

7.3. **The strategy.** If the metric is not exact warped product, then $E_{0,\pm}$ will play the role of the model at the front face of $[X \times Y_+; \mathrm{diag}_{Y_+}]$, which then will need to be 'extended' into the interior. First, let $\mathcal{Y} : Y_+ \times Y_+ \to \mathbb{R}^{n-1}$ be local coordinates on the first factor of Y_+ centered at the diagonal so that at the diagonal, the metric h lifted from the first factor is the standard Euclidean metric $d\mathcal{Y}^2$. That is, informally, $\mathcal{Y} = \mathcal{Y}(y')$ is a family of local coordinates on Y_+, parameterized by $y' \in Y_+$, so that for fixed y', $\mathcal{Y}(y')$ gives local coordinates centered at y' in which h is $d\mathcal{Y}^2$ at the center, $\mathcal{Y}(y') = 0$. Thus, with the notation considered above in the Euclidean setting, we can take $\mathcal{Y} = y - y'$. Let $Y = \frac{\mathcal{Y}(y')}{x}$, so (x, Y, y') form a local coordinate system in a neighborhood of the interior of the front face of $[X \times Y_+; \mathrm{diag}_{Y_+}]$.

As F is a $\mathcal{C}^\infty$ codimension 1 submanifold of $[X \times Y_+; \mathrm{diag}_{Y_+}]$ transversal to the front face, intersecting it in the sphere $|Y| = 1$, there exists a $\mathcal{C}^\infty$ function ρ on $[X \times Y_+; \mathrm{diag}_{Y_+}]$ such that ρ defines F (i.e. ρ vanishes exactly on F, and $d\rho$ does not vanish there), and $\rho|_{\mathrm{ff}} = 1 - |Y|^2$. We let $r \geq 0$ be defined by $r = (1-\rho)^{1/2}$, so $r = |Y|$ at ff, and for convenience we often write (slightly imprecisely) $(1-r^2)_+^s$, etc., for ρ_+^s. Our model is then

$$E_{0,\pm}(x, y, y', \lambda) = C_s x^s (1-r^2)_+^s = C_s x^s \rho_+^s,$$

if $s = \hat{s}_\pm(\lambda)$ is not a negative integer, and

$$E_{0,\pm}(x, y, y', \lambda) = C_s x^s \delta_0^{(-s-1)}(1-r^2) = C_s x^s \delta_0^{(-s-1)}(\rho)$$

if s is a negative integer, with C_s as in Lemma 7.2.

Then we want to find

$$E_\pm \in x^s I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F),\ m(s) = -s - \frac{2n+1}{4},\ s = \hat{s}_\pm(\lambda),$$

with $PE_\pm = 0$, $E_\pm - E_{0,\pm} \in x^{s+1} I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F)$, and $E_\pm$ vanishing to infinite order off the front face at Y_+. The equation $PE_\pm \in \dot{\mathcal{C}}^\infty(X \times Y_+)$ becomes a degenerate transport equation at the level of principal symbols and can be solved to leading order. In fact, in order to simplify the transport equation, which is an equation for the principal symbol of $E_\pm$, given by an ODE along the Lagrangian, N^*F, it is convenient to notice that we want

$$\begin{aligned} E_\pm = a x^s (1-r^2)_+^s + E'_\pm,\ & a \in \mathcal{C}^\infty([X \times Y_+; \mathrm{diag}_{Y_+}]), \\ & E'_\pm \in x^s I^{m(s)-1+\epsilon}([X \times Y_+; \mathrm{diag}_{Y_+}], F), \end{aligned}$$

$\epsilon \in (0,1)$ arbitrarily small, so the principal symbol of $E_\pm$ can be identified with $a|_F$, and the transport equation is an ODE for $a|_F$. Namely,

$$PE_\pm = (Qa) x^s (1-r^2)_+^{s-1} + \tilde{E}_\pm,\ \tilde{E}_\pm \in x^s I^{m(s)+\epsilon}([X \times Y_+; \mathrm{diag}_{Y_+}], F),$$

where Q is a first order differential operator of the form $Q = xV + b$, V a vector field tangent to F transversal to ∂F. Indeed, $xV(q)$ is a non-vanishing multiple of the push-forward of the Hamilton vector field H_p evaluated at the one-dimensional space $N_q^* F_{q'} \setminus 0$. (This vector field is homogeneous, so the choice of $\alpha \in N_q^* F_{q'}$ only changes the push forward by a non-vanishing factor.)

Solving the transport equation below and iterating the construction gives a new $E_\pm \in x^s I^{m(s)}([X\times Y_+;\mathrm{diag}_{Y_+}],F)$ vanishing to infinite order off the front face with $PE_\pm \in x^{s+1}\mathcal{C}^\infty([X\times Y_+;\mathrm{diag}_{Y_+}])$; we show this in Proposition 7.8 below. In fact, we can do better: we can ensure that near Y_+ (where this makes sense) $E_\pm$ is supported *in the interior of the light cone*; this is important as we show momentarily. However, as the transport equation is degenerate, in view of the x factor in front of V, it will take a series of lemmas to achieve this result by Proposition 7.8.

Then, in order to remove the leading term at the front face (i.e. to improve the error, $PE_\pm$, to $x^{s+2}\mathcal{C}^\infty([X\times Y_+;\mathrm{diag}_{Y_+}])$, which can then be further iterated away), we need to study P acting on functions of the form $x^\sigma v$, $v\in\mathcal{C}^\infty([X\times Y_+;\mathrm{diag}_{Y_+}])$, modulo $x^{\sigma+1}\mathcal{C}^\infty([X\times Y_+;\mathrm{diag}_{Y_+}])$. This only uses the model at ff. But (7.3) gives

$$x^{-\sigma}Px^\sigma v = P_\sigma v,\ P_\sigma = (YD_Y - i(n-1-\sigma))(YD_Y+i\sigma) - \Delta_Y - \lambda, \tag{7.4}$$

with P_σ on operator on Euclidean space identified with the fiber of the front face over y'. This is of course a differential operator with smooth coefficients, but it is not elliptic. To see its precise behavior, it is convenient to introduce polar coordinates (r,ω) in Y. (This agrees with our preceding definition of r at the front face.) In such coordinates,

$$P_\sigma = (rD_r - i(n-1-\sigma))(rD_r+i\sigma) - D_r^2 + i\,\frac{n-2}{r}\,D_r - \frac{1}{r^2}\,\Delta_\omega - \lambda,$$

with Δ_ω the positive Laplacian on the standard $(n-2)$-sphere. The principal symbol of P_σ is $(r^2-1)|\xi|^2 - r^{-2}|\eta|^2_\omega$, with (ξ,η) denoting the dual variables of (r,ω). Thus, P_σ is elliptic for $r<1$, i.e. *inside the light cone.* A straightforward calculation shows that P_σ is microhyperbolic for $r>1$; it has some radial points at $r=1$. There are two slightly different (but related) aspects of P_σ to address: the solvability of the transport equations, i.e. the removability of singularities at $r=1$, which we address first in Subsection 7.4, and the solvability of smooth terms which we address in Subsection 7.5.

To sum up the steps of the parametrix construction are the following.

(i) Use the model from a Euclidean boundary model to get started; this was done in Subsection 7.2 and the beginning of the present subsection.
(ii) Solve the transport equation to obtain a smooth error; this is done in Subsection 7.4.
(iii) Solve away the errors at the front face to obtain a rapidly decreasing error (rapidly decreasing at the front face); this is done in Subsection 7.5.
(iv) Solve away the remaining 'trivial' error; this is done in Subsection 7.6.

7.4. **Solving away singularities.** We start with the transport equations. It is convenient to consider the conjugate $(1-r^2)^{-s}P_\sigma(1-r^2)^s$, more precisely, in view of the singularity of the conjugating factor, $(1-r^2\pm i0)^{-s}P_\sigma(1-r^2\pm i0)^s$, considered on all of the front face, i.e. as an operator from $\mathcal{C}^\infty(\mathrm{ff})$ to $\mathcal{C}^{-\infty}(\mathrm{ff})$. The following lemma is the result of a straightforward calculation when replacing $\pm i0$ by $\pm i\epsilon$, and the lemma then follows by taking the limit.

Lemma 7.4. *For all* $s \in \mathbb{R}$, P_σ *satisfies*

$$\begin{aligned}(1-r^2\pm i0)^{-s}&P_\sigma(1-r^2\pm i0)^s\\ &= 4s(s-\sigma)(1-r^2\pm i0)^{-1} + (P_\sigma - 4s(r\partial_r + s - \sigma + \frac{n-1}{2}))\\ &= 4s(s-\sigma)(1-r^2\pm i0)^{-1} + P_{\sigma-2s}\end{aligned}$$

as operators from $\mathcal{C}^\infty(\mathrm{ff})$ *to* $\mathcal{C}^{-\infty}(\mathrm{ff})$.

We in fact always need logarithmic terms to solve away singularities because there are automatic integer coincidences between the powers of x we need in the Taylor series, denoted by σ above, and the orders of the singularities along F, denoted by s above.

Lemma 7.5. *For all* $s \in \mathbb{R}$, $k \in \mathbb{N}$, P_σ *satisfies*

$$\begin{aligned}&P_\sigma(1-r^2\pm i0)^s\log(1-r^2\pm i0)^k\\ &=4k(2s-\sigma)(1-r^2\pm i0)^{s-1}\log(1-r^2\pm i0)^{k-1}\\ &\quad+4s(s-\sigma)(1-r^2\pm i0)^{s-1}\log(1-r^2\pm i0)^k\\ &\quad+(1-r^2\pm i0)^s\log(1-r^2\pm i0)^kP_{\sigma-2s}\\ &\quad+\sum_{j=0}^{k-2}(1-r^2\pm i0)^{s-1}\log(1-r^2\pm i0)^jQ_j\\ &\quad+(1-r^2\pm i0)^s\log(1-r^2\pm i0)^{k-1}Q_{k-1},\end{aligned} \tag{7.5}$$

as operators from $\mathcal{C}^\infty(\mathrm{ff})$ *to* $\mathcal{C}^{-\infty}(\mathrm{ff})$, *where the* Q_j, $j = 0, \dots, k-1$ *are first order differential operators with smooth coefficients on* ff *(depending smoothly on* s, σ, k*).*

Remark 7.6. The principal utility of allowing logarithmic singularities arises if $s = \sigma$, in which case the second term on the right hand side is missing, hence the first term can be used to remove error terms with a lower power of logarithm (that could not be removed without logarithms, i.e. by the preceding lemma).

Proof. The case $k = 0$ follows from the preceding lemma. We then proceed by induction. If $k \geq 1$, and the result has been proved for k replaced by $k-1$, then for $a \in \mathcal{C}^\infty(\mathrm{ff})$,

$$P_\sigma(1-r^2\pm i0)^s\log(1-r^2\pm i0)^ka = \frac{d}{ds}P_\sigma(1-r^2\pm i0)^s\log(1-r^2\pm i0)^{k-1}a$$

shows that we simply need to differentiate (7.5) (with $k-1$ in place of k) with respect to s. The only terms giving rise to additional factors of logarithms are the ones in which $(1-r^2\pm i0)^{s-1}$ or $(1-r^2\pm i0)^s$ is differentiated. As we are applying the result with k replaced by $k-1$, the last two (residual) terms of (7.5) (for $k-1$) give rise to residual terms (for k). Also, the only term that is not negligible even though it has a power of logarithm less than k is the first one, with a factor of $(1-r^2\pm i0)^{s-1}$. Thus, the first three terms of (7.5) (for $k-1$) will contribute to the last two residual terms (for k) except when $(1-r^2\pm i0)^{s-1}$ or $(1-r^2\pm i0)^s$ is differentiated, or when the coefficient of the second term is differentiated. The latter gives $4(2s-\sigma)(1-r^2\pm i0)^{s-1}\log(1-r^2\pm i0)^{k-1}$, so altogether we have $4(2s-\sigma)+4(2s-\sigma)(k-1) = 4k(2s-\sigma)$ of $(1-r^2\pm i0)^{s-1}\log(1-r^2\pm i0)^{k-1}$, giving the desired result. □

Corollary 7.7. *If $s \neq -1$ and $s+1 \neq \sigma$ then for b a smooth function on $\mathbb{S}^{n-2}$, there exists a unique smooth function q on $\mathbb{S}^{n-2}$ such that*

$$P(x^\sigma (1-r^2)_+^{s+1} q) = x^\sigma (1-r^2)_+^s (b + (1-r^2)e)$$

holds near $\mathbb{S}^{n-2}$, with e smooth near $\mathbb{S}^{n-2}$.

More generally, under the same assumptions on s, for b a smooth function on $\mathbb{S}^{n-2}$, there exists a unique smooth function q on $\mathbb{S}^{n-2}$ such that

$$\begin{aligned}&P(x^\sigma (1-r^2)_+^{s+1} \log(1-r^2)_+^k q)\\&= x^\sigma (1-r^2)_+^s (\log(1-r^2)_+^k b + \sum_{j=0}^{k-1} \log(1-r^2)_+^j e_j + (1-r^2)\log(1-r^2)_+^k e)\end{aligned}$$

holds near $\mathbb{S}^{n-2}$, with e, e_j smooth near $\mathbb{S}^{n-2}$, $j = 0, 1, \ldots, k-1$.

If $s = -1$ or $s + 1 = \sigma$, but $\sigma \neq 0$, then for b a smooth function on $\mathbb{S}^{n-2}$, there exists a unique smooth function q on $\mathbb{S}^{n-2}$ such that

$$\begin{aligned}&P(x^\sigma (1-r^2)_+^{s+1} \log(1-r^2)_+^{k+1} q)\\&= x^\sigma (1-r^2)_+^s (\log(1-r^2)_+^k b + \sum_{j=0}^{k} \log(1-r^2)_+^j e_j + (1-r^2)\log(1-r^2)_+^{k+1} e)\end{aligned}$$

holds near $\mathbb{S}^{n-2}$, with e, e_j smooth near $\mathbb{S}^{n-2}$, $j = 0, 1, \ldots, k$.

Proof. In the first case, let $q = 4s^{-1}(s-\sigma)^{-1} b$, and apply Lemma 7.4, expressing $(1-r^2)_+^{s+1}$ as a difference of $(1-r^2 \pm i0)^{s+1}$. Uniqueness is clear.

In the second case, proceed the same way, applying Lemma 7.5. □

Proposition 7.8. *The transport equations can be solved near the front face, i.e. there exist*

$$E_\pm \in x^s I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F),\ m(s) = -s - \frac{2n+1}{4},\ s = \hat{s}_\pm(\lambda),$$

with

$$E_\pm - E_{0,\pm} \in x^{s+1} I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F), \tag{7.6}$$

$$PE_\pm \in x^{s+1} \mathcal{C}^\infty([X \times Y_+; \mathrm{diag}_{Y_+}]),$$

and $E_\pm$ vanishing to infinite order off the front face.

Proof. First, with any $E_\pm \in x^s I^{m(s)}([X \times Y_+; \mathrm{diag}_{Y_+}], F)$ extending $E_{0,\pm}$ in the sense of (7.6), having an expansion in terms of $(1-r^2)_+^\beta$, so $E = x^{\hat{s}_\pm(\lambda)}(1-r^2)_+^{\hat{s}_\pm(\lambda)} a$, a smooth, $a|_{x=0} = 1$, one has

$$PE_\pm = x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)-1} b,$$

b smooth. By the corollary (if $\hat{s}_\pm(\lambda) \neq 0$), one can find $E_{1,\pm} = x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)} b$ such that

$$PE_{1,\pm} - PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty + x^{\hat{s}_\pm(\lambda)+2}(1-r^2)_+^{\hat{s}_\pm(\lambda)-1}\mathcal{C}^\infty,$$

so replacing $E_\pm$ by $E_\pm - E_{1,\pm}$, one has an extension of $E_{0,\pm}$ of the same form as the original $E_\pm$, but with

$$PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty + x^{\hat{s}_\pm(\lambda)+2}(1-r^2)_+^{\hat{s}_\pm(\lambda)-1}\mathcal{C}^\infty.$$

Leaving the first term unchanged, one iterates the second term away, using $E_{j,\pm} = x^{\hat{s}_\pm(\lambda)+j}(1-r^2)_+^{\hat{s}_\pm(\lambda)} b$ to remove errors in $x^{\hat{s}_\pm(\lambda)+j}(1-r^2)_+^{\hat{s}_\pm(\lambda)-1}\mathcal{C}^\infty$, with the result that the new $E_\pm$ satisfies

$$PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty + x^{\hat{s}_\pm(\lambda)+j+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)-1}\mathcal{C}^\infty.$$

Note that there is no obstacle for this procedure as long as $\hat{s}_\pm(\lambda) \neq 0$. By an asymptotic summation argument one gets an E with

$$PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty + (1-r^2)_+^{\hat{s}_\pm(\lambda)-1}\dot{\mathcal{C}}^\infty.$$

For the last term the singular transport equations are now easily solvable, due to the infinite order vanishing (this is the statement that regular singular ODEs can be solved when the inhomogeneity vanishes to infinite order at the regular singular point, with a result that still vanishes to infinite order there), so one obtains near the front face

$$PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty.$$

Now using the corollary, we can find $E_1 = x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)+1}\log(1-r^2)_+$ such that

$$\begin{aligned} PE_{1,\pm} - PE_\pm \in & x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)+1}\mathcal{C}^\infty \\ & + x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)+1}\log(1-r^2)_+\mathcal{C}^\infty \\ & + x^{\hat{s}_\pm(\lambda)+2}(1-r^2)_+^{\hat{s}_\pm(\lambda)}\mathcal{C}^\infty. \end{aligned}$$

Replacing $E_\pm$ by $E_\pm - E_{1,\pm}$, leaving the first two terms unchanged, we can iterate away the last term exactly as above to obtain

$$PE_\pm \in x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)+1}\mathcal{C}^\infty + x^{\hat{s}_\pm(\lambda)+1}(1-r^2)_+^{\hat{s}_\pm(\lambda)+1}\log(1-r^2)_+\mathcal{C}^\infty.$$

Repeating this argument proves this proposition. Note that we obtain arbitrarily large powers of logarithms, but these correspond to increasingly less singular terms in terms of the power s in $(1-r^2)_+^s$. □

7.5. **Obtaining errors rapidly decaying at the front face.** We now have a parametrix in Proposition 7.8, with smooth errors, but these errors do not decay rapidly (faster than any power of x) at the front face. In this subsection we improve the construction to obtain these better errors; this is accomplished in (7.8)-(7.9).

As we would like our operator $E_\pm$ to be localized in the interior of the light cone (for hyperbolic propagation would spread singularities outside otherwise), it is convenient to consider P_σ as an operator on tempered distributions in

$$\mathbb{B}^{n-1}_{1/2} = \{Y : \ |Y| \leq 1\},$$

here equipped with the smooth structure arising from adjoining $\sqrt{1-|Y|^2}$ to the smooth structure induced from the front face (this is what the subscript 1/2 denotes). Let $\nu = (1-r^2)^{1/2}$ be a defining function for $\partial\mathbb{B}^{n-1}_{1/2}$. If only even powers of ν occur as coefficients of products of νD_ν and νV, V a vector field on $\partial\mathbb{B}^{n-1}_{1/2}$ extended to a neighborhood using the polar coordinate decompositions, then one calls the corresponding differential operator even, see [15]. Note that the subspace of even elements of $\mathcal{C}^\infty(\mathbb{B}^{n-1}_{1/2})$ is exactly $\mathcal{C}^\infty(\mathbb{B}^{n-1})$. Then:

Lemma 7.9. *$P_\sigma \in \nu^{-2}\,\mathrm{Diff}_0^2(\mathbb{B}^{n-1}_{1/2})$ is elliptic and even.*

For σ real with $\lambda+\sigma^2-\sigma(n-1)\geq 0$, $-P_\sigma$ is positive with respect to the $L^2(\mathbb{B}^{n-1}_{1/2},(1-\nu^2)^{(n-3)/2}\nu^{1+2\sigma}\,d\nu\,d\omega)$ inner product on

$$\nu H^1_0(\mathbb{B}^{n-1}_{1/2},(1-\nu^2)^{(n-3)/2}\nu^{1+2\sigma}\,d\nu\,d\omega),$$

with $d\omega$ denoting the standard measure on the unit sphere.

Proof. As P_σ is a differential operator with smooth coefficients on all of ff, elliptic for $r<1$, we only need to analyze its behavior near $r=1$. For this purpose it is convenient to use the boundary defining function ν on $\mathbb{B}^{n-1}_{1/2}$. A straightforward calculation using $(1-r^2)^{1/2}D_r=-(1-\nu^2)^{1/2}D_\nu$ gives that in fact

$$\begin{aligned}
-P_\sigma &= (D_\nu+i(2\sigma-1)\nu^{-1}+i(n-3)\nu(1-\nu^2)^{-1})(1-\nu^2)D_\nu\\
&\qquad+\frac{1}{1-\nu^2}\,\Delta_\omega+\lambda+\sigma^2-\sigma(n-1)\\
&=\nu^{-1}\Big((\nu D_\nu+i(2\sigma-1)+\frac{i(n-3)\nu^2}{1-\nu^2})(1-\nu^2)(\nu D_\nu-i)\\
&\qquad+\frac{\nu^2}{1-\nu^2}\,\Delta_\omega+(\lambda+\sigma^2-\sigma(n-1))\nu^2\Big)\,\nu^{-1}
\end{aligned}$$

from which the first claim follows immediately. For the second claim we merely need to notice that the formal adjoint of $D_\nu\nu=\nu D_\nu-i$ with respect to $f\nu^{-1}\,d\nu\,d\omega$ $f=(1-\nu^2)^{(n-3)/2}\nu^{2+2\sigma}$, is $f^{-1}(\nu D_\nu-i)f=\nu D_\nu+i(2\sigma-1)+i(n-3)\nu^2(1-\nu^2)^{-1}$, so

$$\langle u,-P_\sigma u\rangle=\|(1-\nu^2)^{1/2}D_\nu u\|^2+\|(1-\nu^2)^{-1/2}d_\omega u\|^2+(\lambda+\sigma^2-\sigma(n-1))\|u\|^2.$$

□

In fact, it is also convenient to identify the interior of $\mathbb{B}^{n-1}_{1/2}$ with the Poincaré ball model of hyperbolic $(n-1)$-space $\mathbb{H}^{n-1}$ using polar coordinates around the origin, letting $\cosh\rho=\nu^{-1}$, ρ is the distance from the origin. The Laplacian on $\mathbb{H}^{n-1}$ in these coordinates is

$$\Delta_{\mathbb{H}^{n-1}}=D_\rho^2-i(n-2)\coth\rho\,D_\rho+(\sinh\rho)^{-2}\Delta_\omega.$$

Lemma 7.10. *Let s be such that $2s=\sigma-\frac{n}{2}$. Then*

$$\begin{aligned}
&(1-r^2)^{-s}P_\sigma(1-r^2)^s=\nu^{\frac{n}{2}-\sigma}P_\sigma\nu^{\sigma-\frac{n}{2}}\\
&=-\nu^{-1}\left(\Delta_{\mathbb{H}^{n-1}}+\sigma^2-\left(\frac{n-2}{2}\right)^2+\nu^2\left(\lambda-\frac{n(n-2)}{4}\right)\right)\nu^{-1}\\
&=-\cosh\rho\left(\Delta_{\mathbb{H}^{n-1}}+\sigma^2-\left(\frac{n-2}{2}\right)^2+(\cosh\rho)^{-2}\left(\lambda-\frac{n(n-2)}{4}\right)\right)\cosh\rho.
\end{aligned}$$

Thus, this conjugate of P_σ is essentially a compact perturbation of the hyperbolic Laplacian, shifted by the eigenparameter $(n-2)^2/4-\sigma^2$. Note that the spectrum of $\Delta_{\mathbb{H}^{n-1}}$ on $L^2(\mathbb{H}^{n-1})$ is $[(n-2)^2/4,\infty)$. In fact, we have the following result of Mazzeo and Melrose [23]:

Lemma 7.11. *The operator*

$$L_\sigma = \Delta_{\mathbb{H}^{n-1}} + \sigma^2 - \left(\frac{n-2}{2}\right)^2 + \nu^2\left(\lambda - \frac{n(n-2)}{4}\right)$$

is invertible on

$$L^2(\mathbb{H}^{n-1}, \mu_{\mathbb{H}^{n-1}}) = L^2(\mathbb{B}^{n-1}_{1/2}, (1-\nu^2)^{(n-3)/2}\nu^{1-n}\,d\nu) = L^2(\mathbb{B}^{n-1}_{1/2}, (\sinh\rho)^{n-1}\,d\rho)$$

for $\sigma^2 \notin \mathbb{R}$, and it is Fredholm in σ^2 for $\sigma^2 \in \mathbb{C}\setminus[0,\infty)$.

For $\sigma > 0$, any element of the L^2-nullspace of L_σ lies in $\nu^{(n-2)/2+\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}_{1/2})$.

The inverse L_σ^{-1} is meromorphic for $\sigma^2 \in \mathbb{C}\setminus[0,\infty)$ with finite rank residues, maps $\nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^k H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}})$ continuously, provided that $|k| < |\operatorname{Re}\sigma|$. For $k > |\operatorname{Re}\sigma|$, it maps

$$\nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^k H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) + \nu^{(n-2)/2+\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}).$$

In fact, L_σ^{-1}, defined at first in $\operatorname{Re}\sigma > 0$, extends meromorphically to all of $\mathbb{C}$ (i.e. the Riemann surface of σ^2), as shown in [23] with improvements in [15]:

Lemma 7.12. *The operator L_σ^{-1} defined at first for $\operatorname{Re}\sigma > 0$ as the inverse of L_σ, extends to a meromorphic family of operators*

$$R_0(\sigma) : \nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^k H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) + \nu^{(n-2)/2+\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}),$$

$k > |\operatorname{Re}\sigma|$ with no poles for $\sigma \neq 0$ pure imaginary, which satisfies $L_\sigma R_0(\sigma) = \mathrm{Id}$ on $\nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}})$.

Moreover, σ is a pole of R_0, then $L_\sigma u = 0$ has a non-zero solution

$$u \in \nu^{(n-2)/2+\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}).$$

Corollary 7.13. *For $\sigma^2 \in \mathbb{C}\setminus[0,\infty)$, $\operatorname{Re}\sigma > 0$, the operator P_σ is Fredholm, of index 0, as a map*

$$P_\sigma : \nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^{k-2} H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}})$$

for $-\frac{n-2}{2} < k < 2\operatorname{Re}\sigma - \frac{n-2}{2}$, P_σ^{-1} is meromorphic, with finite rank poles, and all poles satisfy $\sigma^2 \in \mathbb{R}$.

Moreover, for $\sigma > 0$, elements of the nullspace of P_σ on $\nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}})$, k as above, lie in $\nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}_{1/2})$.

In addition, for $k > 2\operatorname{Re}\sigma - \frac{n-2}{2}$, whenever P_σ is invertible on L^2,

$$P_\sigma^{-1} : \nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^{k-2} H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) + \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}).$$

Finally, $R(\sigma) = P_\sigma^{-1}$, $\operatorname{Re}\sigma > 0$, extends to a meromorphic family

$$R(\sigma) : \nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) \to \nu^{k-2} H_0^{m+2}(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}}) + \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}),$$

$k > 2|\operatorname{Re}\sigma| - \frac{n-2}{2}$, with no poles for $\sigma \neq 0$ pure imaginary, and $P_\sigma R(\sigma) = \mathrm{Id}$ on $\nu^k H_0^m(\mathbb{B}^{n-1}_{1/2}, \mu_{\mathbb{H}^{n-1}})$, k as above. If σ is a pole of R_0, then $P_\sigma u = 0$ has a non-zero solution

$$u \in \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1}).$$

Proof. $P_\sigma = -\nu^{\sigma-\frac{n}{2}-1} L_\sigma \nu^{-\sigma+\frac{n}{2}-1}$, so

$$P_\sigma^{-1} = -\nu^{\sigma-\frac{n}{2}+1} L_\sigma^{-1} \nu^{-\sigma+\frac{n}{2}+1}.$$

□

Note that 1 just barely fails to be in $\nu^{-(n-2)/2}L^2(\mathbb{B}^{n-1}_{1/2},\mu_{\mathbb{H}^{n-1}})$, while

$$\nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})\subset\nu^k L^2(\mathbb{B}^{n-1}_{1/2},\mu_{\mathbb{H}^{n-1}})$$

for $k<2\operatorname{Re}\sigma-\frac{n-2}{2}$.

If $\lambda<0$, $\hat{s}_+(\lambda)>0$, and $P_{\hat{s}_+(\lambda)}$ fails to be invertible on the spaces listed above as $P_{\hat{s}_+(\lambda)}\nu^{2\hat{s}_+(\lambda)}=0$, and $\nu^{2\hat{s}_+(\lambda)}$ lies in these spaces. However, we claim that P_σ is invertible for $\sigma>\hat{s}_+(\lambda)$. In fact,

$$\begin{aligned}-P_\sigma&=-\nu^{2\sigma}(\nu^{-2\sigma}P_\sigma\nu^{2\sigma})\nu^{-2\sigma}=-\nu^{2\sigma}P_{-\sigma}\nu^{-2\sigma}\\&=\nu^{-1}(\nu D_\nu-i)^*(1-\nu^2)(\nu D_\nu-i)\nu^{-1}+\frac{1}{1-\nu^2}\Delta_\omega+\lambda+\sigma^2+\sigma(n-1),\end{aligned}$$

with adjoint taken relative to $(1-\nu^2)^{(n-3)/2}\nu^{1-2\sigma}\,d\nu\,d\omega$. The first two terms are positive with respect to the corresponding L^2 space, while the roots of $\lambda+\sigma^2+\sigma(n-1)$ are exactly $\hat{s}_\pm(\lambda)$, so $\lambda+\sigma^2+\sigma(n-1)>0$ for $\sigma>\hat{s}_+(\lambda)$. As $\nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})\subset H^1_0(\mathbb{B}^{n-1}_{1/2},\nu^{1-2\sigma}\,d\nu\,d\omega)$, it follows from Corollary 7.13 that P_σ has no nullspace in the listed spaces, so it is invertible. (A different way of arguing would have been to note that $\nu P_{\hat{s}_+(\lambda)}\nu$ has a positive eigenfunction, $\nu^{-1+2\hat{s}_+(\lambda)}$, which thus must correspond to the bottom of the spectrum.)

That for $\lambda\geq 0$ the poles do not occur follows from the following lemma as

$$\nu L^2(\mathbb{B}^{n-1}_{1/2},\nu^{n-2\operatorname{Re}\sigma}\mu_{\mathbb{H}^{n-1}})=\nu^{1-\frac{n}{2}+\operatorname{Re}\sigma}L^2(\mathbb{B}^{n-1}_{1/2},\mu_{\mathbb{H}^{n-1}}),$$

and $1-\frac{n}{2}+\operatorname{Re}\sigma<2\operatorname{Re}\sigma-\frac{n-2}{2}$ if $\operatorname{Re}\sigma>0$.

Lemma 7.14. *P_σ satisfies*

$$\begin{aligned}P_\sigma&=-(D_r-\frac{i\sigma r}{1-r^2})^*(1-r^2)(D_r-\frac{i\sigma r}{1-r^2})-r^{-2}\Delta_\omega-\frac{\sigma^2}{1-r^2}-\lambda\\&=-(D_\nu+i\sigma\nu^{-1})^*(1-\nu^2)(D_\nu+i\sigma\nu^{-1})-\frac{1}{1-\nu^2}\Delta_\omega-\sigma^2\nu^{-2}-\lambda\end{aligned}\tag{7.7}$$

with the (formal) adjoint taken with respect to the measure

$$\mu=r^{n-2}(1-r^2)^{-\operatorname{Re}\sigma}\,dr\,d\omega=(1-\nu^2)^{\frac{n-3}{2}}\nu^{1-2\operatorname{Re}\sigma}\,d\nu\,d\omega=\nu^{n-2\operatorname{Re}\sigma}\mu_{\mathbb{H}^{n-1}}.$$

Corollary 7.15. *Suppose that $\lambda<(n-1)^2/4$. Then P_σ is invertible for $\sigma>\max(0,\hat{s}_+(\lambda))$.*

In fact, we can analyze the poles of the analytic continuation $R(\sigma)$ rather accurately using special algebraic properties of P_σ. Unlike the preceding considerations, which were rather general, i.e. hold for operators of the same form, the following relies on the precise form of P_σ.

Lemma 7.16. *The following identities hold:*

$$P_{\sigma-2}\Delta_Y=\Delta_Y P_\sigma,\ \ P_{\sigma+2}\nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma}=\nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma}P_\sigma.$$

Proof. First, as Δ_Y is homogeneous of degree -2 with respect to dilations on Y, $[y\partial_y,\Delta_Y]=-2\Delta_Y$, so $[YD_Y,\Delta_Y]=2i\Delta_Y$. As

$$P_\sigma=(YD_Y-i(n-1-\sigma))(YD_Y+i\sigma)-\Delta_Y-\lambda,$$

we deduce that

$$\begin{aligned}\Delta_Y P_\sigma &= P_\sigma \Delta_Y + [\Delta_Y, (YD_Y)^2 + i(2\sigma - (n-1))YD_Y] \\ &= P_\sigma\Delta_Y - 2i\Delta_Y(YD_Y) - 2i(YD_Y)\Delta_Y + 2(2\sigma-(n-1))\Delta_Y \\ &= P_\sigma\Delta_Y - 4\Delta_Y - 4i(YD_Y)\Delta_Y + 2(2\sigma-(n-1))\Delta_Y \\ &= \big((YD_Y)^2 + i(2\sigma-(n-1))YD_Y + \sigma(n-1-\sigma) - \Delta_Y - \lambda \\ &\qquad\qquad -4 - 4i(YD_Y) + 4\sigma - 2(n-1)\big)\,\Delta_Y \\ &= \big((YD_Y)^2 + i(2(\sigma-2)-(n-1))YD_Y + (\sigma-2)(n-1-\sigma+2) \\ &\qquad\qquad -\Delta_Y - \lambda\big)\,\Delta_Y \\ &= P_{\sigma-2}\Delta_Y.\end{aligned}$$

Thus, using $P_{-\sigma} = \nu^{-2\sigma}P_\sigma\nu^{2\sigma}$ with σ replaced by $\sigma+2$ first, then with σ replaced by $-\sigma$,

$$\begin{aligned}P_{\sigma+2}\nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma} &= \nu^{2\sigma+4}P_{-\sigma-2}\Delta_Y\nu^{-2\sigma} \\ &= \nu^{2\sigma+4}\Delta_Y P_{-\sigma}\nu^{-2\sigma} = \nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma}P_\sigma\end{aligned}$$

as claimed. □

Lemma 7.17. *Suppose that σ is such that $P_{\sigma+2}w = 0$, $w \in \nu^{2(\sigma+2)}\mathcal{C}^\infty(\mathbb{B}^{n-1})$ implies $w = 0$. If $P_\sigma u = 0$ for some $u \in \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$ then either $\sigma \in \hat{s}_\pm(\lambda) - \mathbb{N}$ or $u = 0$.*

Proof. If $P_\sigma u = 0$, then by the previous lemma, $P_{\sigma+2}\nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma}u = 0$. Moreover, $\nu^{-2\sigma}u \in \mathcal{C}^\infty(\mathbb{B}^{n-1})$, so $w = \nu^{2\sigma+4}\Delta_Y\nu^{-2\sigma}u \in \nu^{2(\sigma+2)}\mathcal{C}^\infty(\mathbb{B}^{n-1})$, hence $w = 0$. Thus, $v = \nu^{-2\sigma}u \in \mathcal{C}^\infty(\mathbb{B}^{n-1})$ satisfies $\Delta_Y v = 0$ and $P_{-\sigma}v = \nu^{-2\sigma}P_{-\sigma}\nu^{2\sigma}v = \nu^{-2\sigma}P_\sigma u = 0$. Thus, $(P_{-\sigma} + \Delta_Y)v = 0$, so

$$\big((YD_Y)^2 - i(n-1+2\sigma)YD_Y - (\lambda + \sigma(n-1+\sigma))\big)\,v = 0.$$

Factoring the operator as $(YD_Y + i\alpha_+)(YD_Y + i\alpha_-)$ with

$$\alpha_\pm = -\frac{n-1}{2} - \sigma \pm \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda} = \hat{s}_\pm(\lambda) - \sigma,$$

we deduce that v satisfies either $(YD_Y + i\alpha_+)v = 0$ or $(YD_Y + i\alpha_-)v = 0$, i.e. v is homogeneous of degree α_+ or degree α_-. But v is $\mathcal{C}^\infty$ at the origin, so, unless $v \equiv 0$, in either case the corresponding α must be a non-negative integer, i.e. $\hat{s}_\pm(\lambda) - \sigma = m \in \mathbb{N}$, so $\sigma \in \hat{s}_\pm(\lambda) - \mathbb{N}$, proving the lemma. □

Corollary 7.18. *The only possible poles of $R(\sigma)$ are $\sigma \in \hat{s}_\pm(\lambda) - \mathbb{N}$. In particular, if m is a positive integer, $R(\sigma)$ is regular at $\sigma = \hat{s}_\pm(\lambda) + m$ unless $s_+(\lambda) - s_-(\lambda) = 2\sqrt{(\frac{n-1}{2})^2 - \lambda} \in \mathbb{N}_+$.*

Proof. As noted in Corollary 7.13, σ is a pole of R if and only if there exists a non-zero $u \in \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$ such that $P_\sigma u = 0$. Moreover, if $\operatorname{Re}\sigma > C$, C sufficiently large (depending on λ), then there exist no such non-trivial u by Corollary 7.15. Correspondingly, if $\operatorname{Re}\sigma \in (C-2, C]$ and σ is a pole of R, then the previous lemma shows that $\sigma \in \hat{s}_\pm(\lambda) - \mathbb{N}$. Proceeding inductively we deduce the corollary. □

Now, if σ is not a pole of R, then given $f \in \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1}_{1/2})$, $P_\sigma v = f$ can be solved with $v \in \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$.

If $\operatorname{Re}\sigma > 0$ and we extend v as 0 to the rest of the fiber of the front face over y', $P_\sigma v$ is thus the extension of f.

In fact, as long as $2\sigma \notin -\mathbb{N}_+$, we can extend v by expanding in Taylor series to finite order, $v = \sum_{j=0}^{N} \nu^{2j} a_j + \nu^{2N+2} v'$, v' $\mathcal{C}^\infty$ near $\partial\mathbb{B}^{n-1}$. If we choose N large enough so that $2\operatorname{Re}\sigma + 2N + 2 > 0$, we can extend $\nu^{2\sigma} v'$ to ff by extending it as 0. On the other hand, we can extend $\nu^{2\sigma+2j} a_j$ as $(1-|Y|^2)_+^{\sigma+j} a_j$. Thus, we obtain a distribution $\tilde{v}$ on ff. Now P_σ is a second order differential operator with $\mathcal{C}^\infty$ coefficients, so $P_\sigma(1-|Y|^2)_+^{\sigma+j} a_j$ has the form $(1-|Y|^2)_+^{\sigma+j-2} b_j'$, with b_j' smooth, and as the principal symbol of P_σ vanishes on the conormal bundle of $\partial\mathbb{B}^{n-1}$, it in fact has the form $(1-|Y|^2)_+^{\sigma+j-1} b_j$, with b_j smooth, as long as $\sigma + j$ is not a non-positive integer. In particular, we deduce that $P_\sigma \tilde{v} = 0$ provided that $P_\sigma v = 0$.

This (namely $P_\sigma v = 0 \Rightarrow P_\sigma \tilde{v} = 0$) is the argument that requires using the analytic extension of R to $\operatorname{Re}\sigma \leq 0$, which gives solutions $v \in \nu^{2\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$ rather than using solutions involving the other indicial root, 0, which would give rise to $v \in \mathcal{C}^\infty(\mathbb{B}^{n-1})$, and hence allow $P_\sigma v$ to have delta distribution terms at $\partial\mathbb{B}^{n-1}$. In particular, for $\operatorname{Re}\sigma < 0$, we cannot simply use the conjugate (in the sense of Lemma 7.10) of $L_{-\sigma}^{-1}$.

If $\sigma \in \hat{s}_-(\lambda) + \mathbb{N}_+$, $2\sigma \in -\mathbb{N}_+$ can hold only if $2\sqrt{(n-1)^2/4 - \lambda} \in \mathbb{N}_+$; it can never hold if $\sigma \in \hat{s}_+(\lambda) + \mathbb{N}_+$. We thus deduce that with $s = \hat{s}_+(\lambda)$, or $s = \hat{s}_-(\lambda)$ under the additional assumption that $\hat{s}_+(\lambda) - \hat{s}_-(\lambda) \notin \mathbb{N}$, we can solve away the error in Taylor series to obtain

$$E_\pm \in x^s I^{m(s)}([X \times Y_+; \operatorname{diag}_{Y_+}], F),\ m(s) = -s - \frac{2n+1}{4},\ s = \hat{s}_\pm(\lambda), \tag{7.8}$$

with $E_\pm - E_{0,\pm} \in x^{s+1} I^{m(s)}([X \times Y_+; \operatorname{diag}_{Y_+}], F)$, and $E_\pm$ supported inside the light cone,

$$PE \in \dot{\mathcal{C}}^\infty(X \times Y_+). \tag{7.9}$$

7.6. **The exact Poisson operator.** The remaining error, (7.9), can be removed using the results of Section 3 to obtain the same conclusion with $PE_\pm = 0$ near Y_+. The standard FIO contruction allows one to obtain $E_\pm$ with the same properties, except $PE_\pm$ supported near Y_-, vanishing in a neighborhood of Y_-. We have thus proved:

Proposition 7.19. *Suppose that $s = \hat{s}_+(\lambda)$, or $s = \hat{s}_-(\lambda)$ under the additional assumption that $\hat{s}_+(\lambda) - \hat{s}_-(\lambda) \notin \mathbb{N}$, i.e. $\lambda \neq \frac{(n-1)^2 - m^2}{4}$, $m \in \mathbb{N}$. Then there exists*

$$E_\pm(\lambda) \in x^s I^{m(s)}([X \times Y_+; \operatorname{diag}_{Y_+}], F),\ m(s) = -s - \frac{2n+1}{4},$$

satisfying $PE_\pm(\lambda) \equiv 0$ near $Y_+ \times Y_+$, $E_\pm(\lambda)$ supported inside the light cone near $Y_+ \times Y_+$, and

$$E_\pm(\lambda)\phi = x^{s_\pm(\lambda)} v,\ v \in \mathcal{C}^\infty(X),\ v|_{Y_+} = \phi$$

for all $\phi \in \mathcal{C}^\infty(Y_+)$. Moreover, $\sigma_{m(s)}(E_\pm)$ never vanishes.

We let $E(\lambda) = E_+(\lambda) \oplus E_-(\lambda)$ be the Poisson operator (near Y_+, where it solves $PE_\pm(\lambda) = 0$). However, much as it is useful in the interior of X to renormalize using powers of the Laplacian, the same holds here. The renormalization depends on the choice of x modulo $x^2\mathcal{C}^\infty(X)$. So let Δ_h denote the Laplacian of the boundary metric h, define Δ_h' analogously to the case of Cauchy surfaces, i.e. is Id on the nullspace of Δ_h, and is Δ_h on its orthocomplement. The renormalized Poisson operator is then

$$\tilde{E}(\lambda) = E_+(\lambda)(\Delta_h')^{(s_+(\lambda)-n/2)/2} \oplus E_-(\lambda)(\Delta_h')^{(s_-(\lambda)-n/2)/2}.$$

The $n/2$ in the exponent of Δ_h' is somewhat arbitrary, it is used to normalize FIO's below to be zeroth order; any quantity differing from $s_\pm(\lambda)$ by a constant (s-independent) amount would work. By Proposition 7.19, the two components of $\tilde{E}(\lambda)$ lie in $x^{1-n/2}I^{-5/4}([X \times Y_+; \mathrm{diag}_{Y_+}], F; (\mathbb{C}^2)^*)$, i.e. they have the same regularity in the interior of $X \times Y_+$ as well as the same behavior at the boundary.

Proposition 7.20. *Suppose $\hat{s}_+(\lambda) - \hat{s}_-(\lambda) \notin \mathbb{N}$, i.e. $\lambda \neq \frac{(n-1)^2-m^2}{4}$, $m \in \mathbb{N}$. For t_0 sufficiently close to 1, the map sending scattering data at Y_+ to Cauchy data at S_{t_0} given by*

$$\mathcal{S}_{+,t_0} : \mathcal{C}^\infty(Y_+)^2 \ni (g_+, g_-) \mapsto (u|_{S_{t_0}}, \partial_x u|_{S_{t_0}}) \in \mathcal{C}^\infty(S_{t_0})^2,$$

where u is the solution of $Pu = 0$ given by Theorem 5.5, is the Fourier integral operator with Schwartz kernel $E(\lambda)|_{\Sigma_+(\epsilon)\times Y_+} \oplus \partial_x E(\lambda)|_{\Sigma_+(\epsilon)\times Y_+}$.

Moreover, the renormalized map

$$\begin{aligned}\tilde{\mathcal{S}}_{+,t_0} = R_{t_0}\tilde{E}(\lambda) =& R_{t_0}E_+(\lambda)(\Delta_h')^{(s_+(\lambda)-n/2)/2} \oplus R_{t_0}E_-(\lambda)(\Delta_h')^{(s_-(\lambda)-n/2)/2} \\ &\in I^0(S_{t_0} \times Y_+, F \cap (S_{t_0} \times Y_+); \mathcal{L}(\mathbb{C}^2, \mathbb{C}^2)),\end{aligned}$$

with R_{t_0} being the Cauchy data map at t_0, $u \mapsto ((\Delta_{t_0}')^{1/2}u|_{S_{t_0}}, Vu|_{S_{t_0}})$, V a vector field transversal to S_{t_0}, is an invertible Fourier integral operator.

Proof. Let $t_1 < t_0$, but still sufficiently close to 1. Let $\chi \in \mathcal{C}^\infty(X)$ be identically 1 in a neighborhood of $T \geq t_0$, supported in $T > t_1$. Let u be the solution of $Pu = 0$ given by Theorem 5.5, and let $v = \int_Y \chi E_+(\lambda)g_+\,dy + \int_Y \chi E_-(\lambda)g_-\,dy$. Then at Y_+, v has the asymptotics required by Theorem 5.5, and $Pv = [P,\chi](\int_Y E_+g_+\,dy + \int_Y E_-g_-\,dy)$ is supported where $T \in (t_1, t_0)$. For $l > \max(\frac{1}{2}, l(\lambda))$, let $v_1 \in H_0^{1,l,-l}(X)$ be the solution of $Pv_1 = -Pv$. As in the proof of Theorem 6.1, v_1 is identically 0 for $T \geq t_0$, is in $\mathcal{C}^\infty(X^\circ)$, and has an asymptotic expansion at Y_- as in Theorem 5.5. Thus, $v + v_1$ has all the properties of u required by the uniqueness part of Theorem 5.5, so

$$u = \int_Y \chi E_+ g_+\,dy + \int_Y \chi E_- g_-\,dy + v_1,$$

and $v_1 \equiv 0$ at S_{t_0}. Thus, $\mathcal{S}_{+,t_0}$ indeed has Schwartz kernel

$$E(\lambda)|_{S_{t_0}\times Y_+} \oplus \partial_x E(\lambda)|_{S_{t_0}\times Y_+},$$

and we have a similar expression for $\tilde{E}(\lambda)$. As F is transversal to S_{t_0} and the restriction map to S_{t_0} is a Fourier integral operator of order 1/4, $\tilde{E}(\lambda)|_{S_{t_0}\times Y_+}$ is an FIO of order -1, while $R_{t_0}\tilde{E}(\lambda)|_{S_{t_0}\times Y_+}$ is an FIO of order 0.

In order to prove the invertibility of $\tilde{\mathcal{S}}_{+,t_0}$, it suffices to show that it is elliptic in the sense that $\tilde{\mathcal{S}}_{+,t_0}^*\tilde{\mathcal{S}}_{+,t_0}$ and $\tilde{\mathcal{S}}_{+,t_0}\tilde{\mathcal{S}}_{+,t_0}^*$ are elliptic pseudo-differential operators, where the adjoint is taken with respect to the Riemannian densities on S_{t_0} and Y_+. Once this is shown, it follows that both the nullspace of $\mathcal{S}_{+,t_0}$ and of its adjoint must lie in smooth matrix-valued functions, and are finite dimensional. Consider for instance $\tilde{\mathcal{S}}_{+,t_0}$. For such smooth Cauchy data (g_+, g_-) at Y_+, the corresponding solution of $\Box u = 0$ is smooth in X°, of the form given by Theorem 5.5, and the vanishing of its Cauchy data at S_{t_0} implies that in fact u vanishes identically, hence $g_\pm = 0$, so $\tilde{\mathcal{S}}_{+,t_0}$ has trivial nullspace. On the other hand, suppose that $\tilde{\mathcal{S}}^*_{+,t_0}$ is not injective, i.e. $\tilde{\mathcal{S}}_{+,t_0}$ is not surjective (e.g. on the L^2-spaces). Any element of the nullspace of $\tilde{\mathcal{S}}^*_{+,t_0}$ is smooth, so in this case there exist smooth non-zero Cauchy data (ψ_0, ψ_1) at S_{t_0} which are L^2-orthogonal to the range of $\tilde{\mathcal{S}}_{+,t_0}$. Let u be the solution of $Pu = 0$ with these Cauchy data. Let (g_+, g_-) be the leading coefficients of the asymptotics at Y_+, as in Theorem 5.5. Then $u = E_+(\lambda)g_+ + E_-(\lambda)g_-$ (since the right hand side has the same asymptotics at Y_+ as the left hand side, so they are equal by the uniqueness part of Theorem 5.5). Therefore (ψ_0, ψ_1) are in the range of $\tilde{\mathcal{S}}_{+,t_0}$, so they vanish, which gives a contradiction. Thus, $\tilde{\mathcal{S}}^*_{+,t_0}$ is also injective. This proves the invertibility of $\tilde{\mathcal{S}}_{+,t_0}$ given its ellipticity.

In order to prove ellipticity, one needs to compute the principal symbol of $\tilde{\mathcal{S}}^*_{+,t_0}\tilde{\mathcal{S}}_{+,t_0}$ and $\tilde{\mathcal{S}}_{+,t_0}\tilde{\mathcal{S}}^*_{+,t_0}$. Consider first the latter. For each $\alpha = (z,\zeta) \in T^*S_{t_0}$ there are two bicharacteristics of $\Box$ which contain a point over $z \in S_{t_0}$ whose image in $T^*S_{t_0} = T^*_{S_{t_0}}X/N^*S_{t_0}$ is (z,ζ). Let the corresponding points in $T^*S_{t_0}$ be $\alpha_j = (t_0, z, \xi_j, \zeta)$, $j = +,-$, where ξ is the dual variable of the first coordinate, T. These bicharacteristics emanate from $S^*_\pm Y_+$ (one from $S^*_+Y_+$, one from $S^*_-Y_+$); let $\beta_j = (y_j, \eta_j)$, $j = +,-$, be the corresponding points. Let $\hat{E}_\pm = E_\pm(\Delta'_h)^{(s_\pm(\lambda)-n/2)/2}$. Then Hörmander's theorem on the composition of FIO's shows that the principal symbol of $\tilde{\mathcal{S}}_{+,t_0}\tilde{\mathcal{S}}^*_{+,t_0}$ at $\alpha = (z,\zeta)$ is a constant times

$$\begin{aligned}
&\sum_j \begin{bmatrix} \sigma(\Delta'_h)^{1/2}(\alpha)\sigma(\hat{E}_+)(\alpha_j,\beta_j) & \sigma(\Delta'_h)^{1/2}(\alpha)\sigma(\hat{E}_-)(\alpha_j,\beta_j) \\ \sigma(V)(\alpha_j)\sigma(\hat{E}_+)(\alpha_j,\beta_j) & \sigma(V)(\alpha_j)\sigma(\hat{E}_-)(\alpha_j,\beta_j) \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \sigma(\Delta'_h)^{1/2}(\alpha)\overline{\sigma(\hat{E}_+)(\alpha_j,\beta_j)} & \overline{\sigma(V)(\alpha_j)}\,\overline{\sigma(\hat{E}_+)(\alpha_j,\beta_j)} \\ \sigma(\Delta'_h)^{1/2}(\alpha)\overline{\sigma(\hat{E}_-)(\alpha_j,\beta_j)} & \overline{\sigma(V)(\alpha_j)}\,\overline{\sigma(\hat{E}_-)(\alpha_j,\beta_j)} \end{bmatrix} \\
&= \sum_j (|\sigma(\hat{E}_+)|^2 + |\sigma(\hat{E}_-)|^2)\sigma(\Delta') \begin{bmatrix} 1 & r_j \\ \bar{r}_j & |r_j|^2 \end{bmatrix},
\end{aligned}$$

where $r_j = \frac{\overline{\sigma(V)}}{\sigma(\Delta')^{1/2}}$, and where on the right hand side the various principal symbols are evaluated at the same points as on the left hand side, but suppressed in notation. Thus, the principal symbol has the form $\sum_j c_j \begin{bmatrix} 1 & r_j \\ \bar{r}_j & |r_j|^2 \end{bmatrix}$ with $c_j > 0$, and a straightforward calculation shows that this matrix is positive definite, hence invertible, provided $r_+ \neq r_-$. But $r_+ = r_-$ would imply that $\alpha_+ = \alpha_-$, which is not the case, so we conclude that $\tilde{\mathcal{S}}_{+,t_0}\tilde{\mathcal{S}}^*_{+,t_0}$ is indeed elliptic.

The calculation for $\tilde{\mathcal{S}}_{+,t_0}\tilde{\mathcal{S}}^*_{+,t_0}$ is similar. In this case, for each $\beta = (y,\eta) \in S^*Y_+$, there are two corresponding bicharacteristics, again one including a point in $S^*_+Y_+$ and one in $S^*_-Y_+$, which then cross $T^*_{S_{t_0}}X$ at $\alpha_j = (x_j, z_j, \xi_j, \eta_j)$, $j = +,-$. Thus, the principal symbol at $\beta = (y,\eta)$ is

$$\sum_j \begin{bmatrix} \sigma(\Delta_h')^{1/2}(\alpha_j)\overline{\sigma(\hat E_+)(\alpha_j,\beta)} & \overline{\sigma(V)(\alpha_j)}\,\overline{\sigma(\hat E_+)(\alpha_j,\beta)} \\ \sigma(\Delta_h')^{1/2}(\alpha_j)\overline{\sigma(\hat E_-)(\alpha_j,\beta)} & \overline{\sigma(V)(\alpha_j)}\,\overline{\sigma(\hat E_-)(\alpha_j,\beta)} \end{bmatrix}$$
$$\times \begin{bmatrix} \sigma(\Delta_h')^{1/2}(\alpha_j)\sigma(\hat E_+)(\alpha_j,\beta) & \sigma(\Delta_h')^{1/2}(\alpha_j)\sigma(\hat E_-)(\alpha_j,\beta) \\ \sigma(V)(\alpha_j)\sigma(\hat E_+)(\alpha_j,\beta) & \sigma(V)(\alpha_j)\sigma(\hat E_-)(\alpha_j,\beta) \end{bmatrix}$$
$$= \sum_j (\sigma(\Delta') + |\sigma(V)|^2)|\sigma(\hat E_+)|^2 \begin{bmatrix} 1 & r_j \\ \bar r_j & |r_j|^2 \end{bmatrix},$$

where now $r_j = \frac{\sigma(\hat E_-)}{\sigma(\hat E_+)}$, and where again on the right hand side the various principal symbols are evaluated at the same points as on the left hand side, but suppressed in notation.

Now $\sigma_{-5/4}(\hat E_+)$ and $\sigma_{-5/4}(\hat E_-)$ satisfy the same first order linear ODE along bicharacteristics, so their ratio along each bicharacteristic is constant, hence are equal to the ratio evaluated at the 'initial point' at the front face of $[X\times Y_+;\mathrm{diag}_{Y_+}]$ (where $\sigma(\hat E_+)$ has to be replaced by $\sigma(x^{n/2-1}\hat E_+)$, etc.). For a given (y',η'), the projection of the two bicharacteristics hit the front face at (y',Y), $Y = \pm\hat\eta'$, and the bicharacteristics themselves hit the cotangent bundle over the front face inside $N^*_{(y',Y)}F$ at $-Y d|Y|$. Thus,

$$r_j = \frac{\sigma(x^{n/2-1}\hat E_-)(y',Y_j,-Y_j d|Y|)}{\sigma(x^{n/2-1}\hat E_+)(y',Y_j,-Y_j d|Y|)}$$

But these can be calculated from the normal operators, which are explicit, hence are easily evaluated as 1, resp. $e^{i\pi(s_+(\lambda)-s_-(\lambda))}$. Thus, if $s_+(\lambda)-s_-(\lambda)$ is not an even integer, which we are assuming, the r_j are unequal, so $\tilde{\mathcal S}^*_{+,t_0}\tilde{\mathcal S}_{+,t_0}$ is indeed elliptic, finishing the proof. □

We are now ready to prove one of our main results, that the scattering operator is a Fourier integral operator.

Theorem 7.21. *Suppose that $\hat s_+(\lambda)-\hat s_-(\lambda)\notin\mathbb N$, i.e. $\lambda \neq \frac{(n-1)^2-m^2}{4}$, $m\in\mathbb N$. Then $\mathcal S(\lambda)$ is a Fourier integral operator with canonical relation given by $\mathcal S_{\mathrm{cl}}$, and $\tilde{\mathcal S} = \tilde{\mathcal S}(\lambda)$ is an invertible elliptic 0th order Fourier integral operator with the same canonical relation.*

Proof. This is immediate from $\tilde{\mathcal S} = \tilde{\mathcal S}^{-1}_{-,-1+\epsilon}\circ C_{1-\epsilon,-1+\epsilon}\circ\tilde{\mathcal S}_{+,1-\epsilon}$ for $\epsilon>0$ small. Indeed, all operators are Fourier integral operators by Proposition 7.20 (applied also at Y_-) and Proposition 6.2, with canonical relation given by the appropriate restriction of the bicharacteristic flow. Thus, the projection of the canonical relation to each factor for each of them has surjective differential, so the composition is transversal, and Hörmander's theorem can be applied. As

$$\tilde{\mathcal S}(\lambda)$$
$$= ((\Delta_h')^{-s_+(\lambda)/2+n/4}\oplus(\Delta_h')^{-s_-(\lambda)+n/4})\mathcal S(\lambda)((\Delta_h')^{s_+(\lambda)/2-n/4}\oplus(\Delta_h')^{s_-(\lambda)/2-n/4}),$$

and the first and last operators are pseudodifferential, the theorem follows. □

Theorem 1.4 follows similarly, as the propagator mapping Cauchy data at different T-slices to each other is an invertible FIO, so it suffices to consider the case t_0 close to 1, in which case the inverse given by Proposition 7.20 proves the claim.

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