THE FEYNMAN PROPAGATOR ON PERTURBATIONS OF MINKOWSKI SPACE

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Abstract. In this paper we analyze the Feynman wave equation on Lorentzian scattering spaces. We prove that the Feynman propagator exists as a map between certain Banach spaces defined by decay and microlocal Sobolev regularity properties. We go on to show that certain nonlinear wave equations arising in QFT are well-posed for small data in the Feynman setting.

1. Introduction

In this paper we use the method introduced in [47], extended in [2] and [27], to analyze the Feynman propagator on spaces (M, g), called spaces with non-trapping Lorentzian scattering metrics (a notion recalled in detail in Section 2), that at infinity resemble Minkowski space in an appropriate manner. As the Feynman propagator is of fundamental importance in quantum field theory, we expect that our result and methods will be useful in a systematic treatment of QFT on curved, non-static, Lorentzian backgrounds.

Here the Feynman propagator is defined as the inverse of the wave operator acting as a map between appropriate function spaces that generalizes the behavior of the standard Feynman propagator on exact Minkowski space. Concretely, the distinguishing feature of propagators from the perspective of function spaces is in terms of the differential order of the Sobolev spaces at the two halves of the (b-)conormal bundle of the 'light cone at infinity' $S_\pm$; components at which the differential order is higher, resp. lower, than a threshold value determine the inverse one obtains. Thus, *we set up function spaces which are weighted microlocal Sobolev spaces of variable order of an appropriate kind such that the wave operator for any non-trapping Lorentzian scattering metric is Fredholm for all but a discrete set of weights – See Theorem 3.3* for a precise statement. Indeed, the same statement holds for more general perturbations of Lorentzian scattering metrics in the sense of smooth sections of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$, defined below in Section 2. Further, we prove Theorem 3.6 below, which we state roughly now.

Theorem (See Theorem 3.6). *For perturbations of Minkowski space, in the sense of smooth sections of* $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$*, the Feynman wave operator, described above, is invertible for a suitable range of weights (rates of decay or growth of functions in the domain). That is to say, its inverse, the Feynman propagator, exists for these space-times.*

In order to give a rough idea for what *the* Feynman propagator is we recall that in their groundbreaking paper [15] Duistermaat and Hörmander constructed distinguished parametrices for wave equations, i.e. distinguished solution operators for

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$\Box u = f$ modulo $\mathcal{C}^\infty(M^\circ)$. Recall that by Hörmander's theorem [29], singularities of solutions of wave equations propagate along bicharacteristics inside the characteristic set in phase space, i.e. T^*M°; the projections of these to the base space are null-geodesics. Here a bicharacteristic is an integral curve of the Hamilton vector field of the principal symbol of the wave operator, which is the dual metric function on T^*M°. For the inhomogeneous wave equation, $\Box u = f$, if, say, f has wave front set (i.e. is singular) at only one point in T^*M°, the different distinguished parametrices produce solutions with different wave front sets, namely either the forward or the backward bicharacteristic through the point in question. Here forward and backward are measured relative to the vector field whose integral curves they are, i.e. the Hamilton vector field. Note, however, that there is a different notion of forward and backward, which one may call future- or past-orientedness, namely whether the underlying time function is increasing or decreasing along the flow. The relative sign between these notions is the opposite in the two halves of the characteristic set of the wave operator over each point. We point out that from the perspective of microlocal analysis the natural direction of propagation *is* given by the Hamilton flow.

As explained by Duistermaat and Hörmander, a distinguished parametrix is obtained by choosing a direction of propagation (of singularities, or estimates) *in each connected component of the characteristic set of the wave operator*. Here the direction of propagation is relative to the Hamilton flow, as above. If the underlying manifold is connected, as one may assume, the characteristic set has two connected components, and there are $2^2 = 4$ choices: propagation forward relative to the Hamilton flow everywhere, propagation backward along the Hamilton flow everywhere (these are the Feynman and anti-Feynman propagators), resp. propagation in the future direction everywhere (the retarded propagator) and in the past direction everywhere (the advanced propagator). A parametrix, however, is only an approximate inverse, modulo smoothing — smoothing operators are not even compact on such a manifold; for actual applications (such as any computations in physics) one would need an actual inverse, and most importantly a *notion of an inverse*. This is exactly what we provide in Theorem 3.6 below.

The historically usual setup for wave equations, and more generally evolution equations, is that of Cauchy problems: one specifies initial data at a time slice, and then one studies local or global solvability. In this sense wave equations are always locally well-posed due to the finite speed of propagation, which in turn is proved by energy estimates. Global well-posedness follows if the local solutions can be pieced together well: global hyperbolicity is a notion that allows one to do so. If one turns this into a setup of inhomogeneous wave equations, $\Box u = f$, by cutting f into two pieces, located in the future, resp. the past, of a Cauchy surface, the choice one is making is that the *support* of u be in the future, resp. the past, of that of f. This necessarily implies, indeed is substantially stronger than, the statement that singularities of solutions are accordingly propagated, so two of the Duistermaat-Hörmander parametrices correspond to these. Thus, due to the energy estimates, even when one considers global solutions, the Cauchy problem, or equivalently the future (or past) oriented problem, for the wave equation is essentially local in character, though, as discussed in [47, 27], in order to understand the global behavior of solutions, it is extremely useful to work directly in a global framework in any case.

What we achieve here is to give an analogous well-posedness framework for the Feynman problems (as opposed to the Cauchy problems). These problems are *necessarily* global in character, very much unlike the Cauchy problems. Thus, they behave similarly, in a certain sense, to elliptic PDE. Indeed, from our perspective, it is an accident (happening for good reasons) that the future/past oriented wave equations are local; one should not normally expect this for any PDE. To be more precise, singularities of solutions behave just as predicted by the Duistermaat-Hörmander construction, but this has no content for $\mathcal{C}^\infty$ solutions — the $\mathcal{C}^\infty$ 'part' of solutions is globally determined.

There has been extensive work in the mathematical physics literature on such QFT problems, often from the perspective of trying to make sense of division by functions with zeros on the characteristic set: for Minkowski space, the Fourier transform gives rise to a multiplier $\xi_n^2 - (\xi_1^2 + \ldots + \xi_{n-1}^2)$; in a $\pm i0$ sense division by this is well-behaved away from the origin, but at the origin delicate questions arise. This is usually thought of as a degree of freedom in defining propagators: precisely how one extends the distribution to 0 even in this constant coefficient setting. (See [6, Section 5] for a discussion of this in the QFT context, and [49] for a recent treatment of renormalization as such extensions.) From our perspective, this is due to translational invariance of the problem being emphasized at the expense of its homogeneity; Mellin transforming in the radial variable gives rise to a much better behaved problem. Indeed, a generalization of this is what Melrose's framework of b-analysis [38] relies on; we further explore it here in the non-elliptic setting following [47, 27]. When the microlocal structure of the function spaces on which the wave operator acts corresponds to the above propagation statements (propagation in the direction of the Hamilton flow in the Feynman case, and in the opposite direction in the anti-Feynman case), the remaining choice is that of a weight: in the case of Minkowski space it turns out that weights l with $|l| < \frac{n-2}{2}$ give rise to invertibility, while outside this range the index of the operator changes, with jumps at weight values corresponding to resonances of the Mellin transformed wave operator family, which in turn correspond to eigenvalues of the Laplacian on the sphere $\Delta_{\mathbb{S}^{n-1}}$ as we show by a complex scaling (Wick rotation) argument in Section 4.

For QFT on curved space-times, the work of Duistermaat and Hörmander was used to introduce a microlocal characterization of Hadamard states, which are considered as physical states of non-interacting QFT, by Radzikowski [43]. (Indeed, part of the paper of Duistermaat and Hörmander was motivated by QFT questions.) This in turn was then extended by Brunetti, Fredenhagen and Köhler [5, 6]. Gérard and Wrochna gave a new pseudodifferential construction of Hadamard states [19, 20]. In a different direction, Finster and Strohmaier extended the general theory to Maxwell fields [18]. However, in all these cases, there is no way of fixing a preferred state: one is always working modulo smoothing operators. Our framework on the other hand gives exactly such a preferred choice. Note also that the Feynman propagator we construct relates to an Hadamard-type condition; see Remark 3.5 below.

In the settings with extra structure, involving time-like Killing vector fields, one can construct Feynman propagators in terms of elliptic operators, e.g. via Cauchy data. Other constructions (such as extensions across null-infinity) in similar settings are investigated by Dappiaggi, Moretti and Pinamonti [11, 41, 12]. In fact, these latter results bear the closest connections to ours in that a canonical state is constructed using the structure on null-infinity. Our results deal directly with the 'bulk', thanks to the Fredholm formulation, with the linear results having considerable perturbation stability in particular. (It is due to the module structure required in Section 5 that the non-linear problem is more restrictive.)

Along with setting up such a Fredholm framework, we also study semilinear wave equations, following the general scheme of [27]; we think of these as a first step towards interacting QFT in this setting. However, being fully microlocal, the necessary framework requires more sophisticated function spaces than those discussed in [27]. We prove small data well-posedness results in the Feynman setting for certain semilinear wave equations in Theorems 5.15 and 5.22 below. In particular, Theorem 5.22 can be summarized as follows

Theorem. *In $\mathbb{R}^{3+1}$, if g is a perturbation of the Minkowski metric for which both the invertibility statements in Theorem 3.6 and Theorem 5.1 hold, the problem*

$$\Box u + \lambda u^3 = f$$

is well-posed for small f, where f lies in the range, and u in the domain, of the Feynman wave operator, in particular $u = \Box^{-1}_{g,fey}(f - \lambda u^3)$ where $\Box^{-1}_{g,fey}$ is the Feynman propagator mapping as in (3.19) *with $l \geq 0$ sufficiently small.*

While as far as we are aware non-linear problems have not been considered in the Feynman context, for the usual Cauchy problem, i.e. the retarded and advanced propagators, non-linear problems on Minkowski space, as well as perturbations of Minkowski space (as opposed to the more general Lorentzian scattering metrics considered in the linear parts of the paper here), have been very well studied. In particular, even quasilinear equations are well understood due to the work of Christodoulou [8] and Klainerman [32, 31], with their book on the global stability of Einstein's equation [9] being one of the main achievements. Lindblad and Rodnianski [34, 35] simplified some of their arguments, and Bieri [3, 4] relaxed some of the decay conditions. We also mention the work of Wang [50] obtaining asymptotic expansions, of Lindblad [33] for results on a class of quasilinear equations, and of Chruściel and Łęski [10] on improvements when there are no derivatives in the non-linearity. Hörmander's book [30] provides further references in the general area, while the work of Hintz and Vasy [27] develops the analogue of the framework we use here in the general Lorentzian scattering metric setting (but still for the Cauchy problem). Works for the linear problem with implications for non-linear ones, e.g. via Strichartz estimate include the recent work of Metcalfe and Tataru [40] where a parametrix construction is presented in a low regularity setting.

The structure of the paper is as follows. In Section 2 we describe the underlying geometry and study the wave operator microlocally in the sense of smoothness (as opposed to decay). Estimates modulo compact errors, and thus Fredholm properties, are established in Section 3. In Section 4 we show that in Minkowski space, the Feynman propagator is the limit of the inverses of elliptic problems, achieved by a 'Wick rotation'; this means that from the perspective of spectral theory the Feynman and anti-Feynman propagators are the natural replacement for resolvents. This in particular establishes the invertibility of the Minkowski wave operator on the appropriately weighted function spaces. Finally, in Section 5 we study semilinear wave equations in the Feynman framework.

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2. Geometry and the d'Alembertian

The basic object of interest is a manifold M with boundary ∂M equipped with a Lorentzian metric g (which we take to be signature $(1, n-1)$) in its interior which has a certain form at the boundary (which is geometrically infinity) modelled on the Minkowski metric. In order to define the precise class of metrics, it is useful to introduce a more general structure. Thus, ${}^{\mathrm{sc}}T^*M$ is the scattering cotangent bundle, which we describe presently, originally defined in [39]. If ρ is a boundary defining function, meaning a function in $C^\infty(M)$ which is non-negative, has $\{\rho = 0\} = \partial M$, and such that $d\rho$ is non-vanishing on ∂M, smooth sections of ${}^{\mathrm{sc}}T^*M$ near the boundary are locally given by $C^\infty(M)$ linear combinations of the differential forms

$$\frac{d\rho}{\rho^2}, \ \frac{dw_i}{\rho},$$

where $w_1, \ldots, w_{n-1}$ form local coordinates on ∂M. A non-degenerate smooth section of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$ of Lorentzian signature (which we take to be $(1, n-1)$) is called a *Lorentzian sc-metric*. The smooth topology on sc-metrics is the $\mathcal{C}^\infty$ topology on sections of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$, i.e. locally in M (which recall is a manifold *with* boundary, i.e. smoothness is *up to* the boundary) is induced by the $\mathcal{C}^\infty$ topology of the coefficients of the basis

$$\frac{d\rho}{\rho^2} \otimes \frac{d\rho}{\rho^2}, \ \frac{d\rho}{\rho^2} \otimes_s \frac{dw_i}{\rho}, \ \frac{dw_i}{\rho} \otimes_s \frac{dw_j}{\rho}, \ i, j = 1, \ldots, n-1, \ i \leq j, \tag{2.1}$$

where $\otimes_s$ is the symmetric tensor product.

When $M^{int} = \mathbb{R}^n$, the objects above can be described along more familiar lines. Indeed, in this case, the radial compactification of $\mathbb{R}^n$ to a ball $\mathbb{B}^n$ gives the manifold with boundary M, see [39], e.g. by using 'reciprocal spherical coordinates' to glue the sphere at infinity $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$. Then if $z_1, \ldots, z_n$ are the standard Euclidean coordinates, setting as usual $r = (z_1^2 + \cdots + z_n^2)^{1/2}$, we can take $\rho = 1/r$ outside the unit ball $B_1(0)$ (recall that ρ is to be a smooth function on all of M), and thus

$$\frac{d\rho}{\rho^2} = -dr \text{ on } \mathbb{R}^n \setminus B_1(0), \tag{2.2}$$

and the w_i can be taken to be some set of $n-1$ angular variables on $\mathbb{S}^{n-1}$ which give local coordinates on the sphere. Thus in this case $dw_i/\rho = r\, d\omega_i$ and one sees that the volume form $dz = r^{n-1}\, dr\, d\,\mathrm{Vol}_{\mathbb{S}^{n-1}}$ is a smooth non-vanishing section of the top degree form bundle, $\wedge^n {}^{\mathrm{sc}}T^*M$, up to and including ∂M. (Put differently, the volume form is equal to a wedge product of $d\rho/\rho^2$ and the dw_i/ρ times a smooth non-vanishing function a which is smooth up to $r = \infty$.) Moreover, $\mathcal{C}^\infty(\mathbb{B}^n)$ consists exactly of the space of classical (one step polyhomogeneous) symbols of order 0, while the standard coordinate differentials dz_j lift to $\mathbb{B}^n$ to give a basis, over $\mathcal{C}^\infty(\mathbb{B}^n)$, of all smooth sections of ${}^{\mathrm{sc}}T^*\mathbb{B}^n$. In particular, any translation invariant Lorentzian metric on $\mathbb{R}^n$ is (after this identification) a sc-metric; and remains so under perturbations of its coefficients by classical symbols of order 0. Moreover, $dz_i \otimes_s dz_j$, $i, j = 1, \ldots, n$, $i \leq j$, forming a basis of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*\mathbb{B}^n$, the $\mathcal{C}^\infty$ topology on sections of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*\mathbb{B}^n$ is simply the $\mathcal{C}^\infty(\mathbb{B}^n)$ topology on the $n(n+1)/2$-tuple of coefficients with respect to this basis.

We next recall the definition of the more refined structure of a *Lorentzian scattering space* from [2] (see also [27, Section 5]), of which the Minkowski metric is an example via the radial compactification of $\mathbb{R}^n$, depicted in Figure 2. For this, we assume that there is a $\mathcal{C}^\infty$ function v defined near ∂M, with $v|_{\partial M}$ having a non-degenerate differential at the zero-set $S = \{v = 0, \rho = 0\}$ of v in ∂M (which we call the *light cone at infinity*); here ρ is a boundary defining function with the property that the scattering normal vector field $V = \rho^2 \partial_\rho$ modulo $\rho\mathcal{V}_{\mathrm{sc}}(M)$ (it is well-defined in this sense) satisfies that $g(V, V)$ has the same sign as v at each point in ∂M, g has the form

$$g = v\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \frac{\tilde{g}}{\rho^2}, \tag{2.3}$$

where $\tilde{g} \in \mathcal{C}^\infty(M; \mathrm{Sym}^2 T^*M)$, $\alpha \in \mathcal{C}^\infty(M; T^*M)$, $\alpha|_S = \frac{1}{2}\, dv$ and $\tilde{g}|_{\mathrm{Ann}(d\rho, dv)}$ at S is positive definite, where, for a set of one forms $\beta_1, \ldots, \beta_k$, $\mathrm{Ann}(\beta_1, \ldots, \beta_k)$ is the set of vectors in the intersection of the kernels of the β_i. Apart from the fact that the Minkowski metric is of this form, the assumption in (2.3) is natural because it guarantees the structure of the Hamiltonian dynamics in the cotangent bundle which is required for our analysis.

This is not quite a statement about $g|_{\partial M}$ as a metric on ${}^{\mathrm{sc}}TM$, i.e. as a section of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$, because of the implied absence of a $O(\rho)\frac{d\rho^2}{\rho^4}$ term. Adding such a term results in a *long-range Lorentzian scattering metric*, the whole theory relevant to the discussion below goes through in this setting, as shown in the work of Baskin, Vasy and Wunsch [1]; e.g. Schwarzschild space-time is of this form near the boundary of the light cone at infinity. (The difference is in the precise form of the asymptotics of the linear waves; they are well-behaved on a logarithmically different blow-up of M at S.)

Note that a perturbation of a Lorentzian scattering metric in the sense of sc-metrics (smooth sections of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$) is a Lorentzian sc-metric, but it need *not* be (even a long-range) Lorentzian scattering metric, since the above form of the metric (2.3) need not be preserved. However, the subspace of sc-metrics of the form (2.3) is a closed subset in the $\mathcal{C}^\infty$ topology of sc-metrics within the open set of Lorentzian sc-metrics (in the space of smooth sections of $\mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M$); by a perturbation in the sense of Lorentzian scattering metrics we mean a perturbation within this closed subset.

We remark here that, as is generally the case, only finite regularity (not being $\mathcal{C}^\infty$) is relevant in any of the discussion below, though the specific regularity needed would be a priori rather high. However, using the low regularity results of Hintz [24] on b-pseudodifferential operators one could easily obtain rather precise low-regularity versions of the linear results presented here.

For statements beyond Fredholm properties, based on the work in Section 4, M will be the ball $\mathbb{B}^n$, i.e. the radial compactification of $\mathbb{R}^n$, equipped with a smooth perturbation of the Minkowski metric,

$$g = dz_n^2 - dz_1^2 - dz_2^2 - \cdots - dz_{n-1}^2, \tag{2.4}$$

with perturbation understood in the set of sc-metrics. (Later, in Section 5, it will be important to have perturbations within scattering metrics to preserve the module structure discussed there.) To see that this takes the form in (2.3), following [2, Sect. 3.1], write g in the coordinates (ρ, v, ω) defined by $z_n = \rho^{-1}\cos\theta, z_j =$

$\rho^{-1}\omega_j \sin\theta$ for $1 \le j \le n-1$, where $\rho = |z|^{-1}$, where $|z| = \sqrt{z_1^2 + \cdots + z_n^2}$ and $\omega_j = z_j/(|z|^2 - z_n^2)^{1/2}$, and take $v = \cos 2\theta$. In this case $\alpha = dv/2$ identically.

The main object of study here is the wave operator, defined in local coordinates by

$$\Box_g := \frac{1}{\sqrt{g}} \partial_i G^{ij} \sqrt{g} \partial_j, \tag{2.5}$$

where G denotes the inverse of g, i.e. the dual metric on $1-$forms defined by g.

We further assume that g is *non-trapping*, which is to say we assume that $S = S_+ \cup S_-$ (each $S_\pm$ being the disjoint union of possibly several connected components),

$$\{\rho = 0,\ v > 0\} = C_+ \cup C_-,$$

$C_\pm$ open, $\partial C_\pm = S_\pm$, and such that the null-geodesics of g tend to S_+ as the parameter goes to $+\infty$, S_- as the parameter goes to $-\infty$, or vice versa. We also let

$$C_0 = \{\rho = 0,\ v < 0\}.$$

We then consider $\Box_g$, on functions (or in the future differential forms or various other squares of Dirac-type operators), and we wish to analyze the invertibility of the Feynman propagator.

For this purpose it is convenient, as we explain further in the next paragraph, to consider

$$L = \rho^{-(n-2)/2} \rho^{-2} \Box_g \rho^{(n-2)/2}; \tag{2.6}$$

then $L \in \mathrm{Diff}_{\mathrm{b}}^2(M)$, the space of b-differential operators, meaning that locally near ∂M, using coordinates $(\rho, w_1, \dots, w_{n-1})$ where ρ is the boundary defining function from (2.3) and w_i are any coordinates on ∂M, there are smooth functions $a_{i,\alpha} \in C^\infty(M)$, such that

$$L = \sum_{j+|\alpha|\le 2} a_{j,\alpha} (\rho\partial_\rho)^j \partial_w^\alpha. \tag{2.7}$$

Its principal symbol is the dual metric $\hat{G}$ of the Lorentzian b-metric

$$\hat{g} = \rho^2 g. \tag{2.8}$$

In general, $\mathrm{Diff}_{\mathrm{b}}^*(M)$ is the algebra of differential operators generated by

$$\mathcal{V}_{\mathrm{b}} := C^\infty(M; {}^{\mathrm{b}}T(M)), \tag{2.9}$$

which more concretely is the $C^\infty(M)$ span of the vector fields

$$\rho\partial_\rho, \quad \partial_{w_i}, \tag{2.10}$$

That L is indeed in $\mathrm{Diff}_{\mathrm{b}}^2(M)$ can be checked directly from (2.3) and (2.5). In the definition of L in (2.6), $\rho^{(n-2)/2}$ is introduced to make L formally self-adjoint with respect to the b-metric $\hat{g}$. The conformal factor ρ merely reparameterizes null-bicharacteristics, so our assumption is equivalent to the statement that null-bicharacteristics of L tend to $S_\pm$.

While we could consider $\Box_g$ or in fact $\Box_g + \lambda$, $\lambda \in \mathbb{C}$, as a scattering differential operator, corresponding to the sc-structure, we instead work with L because $\Box_g$ is rather degenerate as a sc-operator due to the quadratic vanishing of its principal symbol at the zero section at ∂M (i.e. infinity). Note that the option of the b-framework is not available if a non-zero spectral parameter λ is added (it would result in a singular term), but on the other hand in these cases there is no degeneracy in the operator in the sc-framework! The latter phenomenon appears also in analogous work in the elliptic setting, namely in analysis of $\Delta_g + \lambda$ for g a *Riemannian* scattering metric, in particular in Melrose's work on scattering manifolds [39], which incidentally proves and uses radial points estimates. See also [22].

One of the main features of our analysis, parallel to the recent work [26, 27] as well as much other work on analysis on non-compact spaces going back to Melrose [38], is that we use an extension of the vector bundle $T^*(M^{int})$ up to the boundary which is better suited to the analysis than T^*M, and for which in particular the beginnings and ends of null-bicharacteristics become tractable objects. Concretely, we use the b-cotangent bundle, ${}^{\mathrm{b}}T^*M$, the dual bundle of the b-tangent bundle ${}^{\mathrm{b}}TM$, whose local sections near the boundary are $C^\infty(M)$ linear combinations of

$$\frac{d\rho}{\rho}, \qquad dw_i,$$

with coordinates as above. Notation as in the paragraph containing (2.2), in $\mathbb{R}^n$ we can assume that near infinity the differential form $d\rho/\rho = -dr/r$. (The w_i remain coordinates on the sphere.)

For an operator $P \in \mathrm{Diff}^m_{\mathrm{b}}(M)$, the (b-)principal symbol $\sigma_{\mathrm{b},m}(P)$ is a smooth function on ${}^{\mathrm{b}}T^*M$ which is a homogeneous polynomial of degree m on the fibers of ${}^{\mathrm{b}}T^*M$, extending the standard principal symbol from T^*M^{int} to ${}^{\mathrm{b}}T^*M$. Notice that a vector in ${}^{\mathrm{b}}T_qM$, $q \in M$, defines a linear function on ${}^{\mathrm{b}}T^*_qM$; the principal symbol of a vector field is given by i times this fiber-linear function; it is extended to differential operators by making it multiplicative, in the process keeping only the leading (mth order) terms of the operator. Concretely, writing b-covectors as $\sigma\,\frac{d\rho}{\rho} + \sum_j \zeta_j\,dw_j$, (ρ, w, σ, ζ) are local coordinates on ${}^{\mathrm{b}}T^*M$ (global in the fibers), the b-principal symbol of an operator of the form (2.7) is

$$\sum_{j+|\alpha|=2} a_{j,\alpha}(\rho, w)(i\sigma)^j(i\zeta)^\alpha,$$

since the symbol of $\rho\partial_\rho$ is $i\sigma$ in the coordinates (ρ, w, σ, ζ), because $\rho\partial_\rho$ acting on $\sigma\,\frac{d\rho}{\rho} + \sum_j \zeta_j\,dw_j$ covector is σ.

We describe the structure of the null-bicharacteristics at the boundary in detail now. The Hamilton flow on null-bicharacteristics corresponding to L descends from a flow on $T^*(M^{int})$ to a flow on the spherical cotangent bundle $S^*(M^{int})$. On can think of $S^*(M^{int})$ as either the quotient $(T^*(M^{int}) - o)/\mathbb{R}_+$, where o denotes the zero section and the action of $\mathbb{R}_+$ is the standard dilation action on the fibers, or as the bundle obtained by radially compactifying the fibers of $T^*(M^{int})$ to obtain a ball bundle $\overline{T}^*(M^{int})$ and taking the corresponding sphere bundle whose fibers are the boundary of fibers of $\overline{T}^*(M^{int})$. The latter process can just as well be done on on ${}^{\mathrm{b}}T^*M$ to obtain ${}^{\mathrm{b}}\overline{T}^*M$ and taking the boundaries of the fibers gives the spherical b-conormal bundle ${}^{\mathrm{b}}S^*M$.

For the convenience of the reader, we will give a brief summary of the properties of the Hamilton flow used in the propagation estimates, which are discussed in detail in [2, Section 3]. The null-bicharacteristic flow of $\Box_g$ is the flow of the Hamilton vector field $\mathsf{H} = (\partial_\zeta p)\partial_z - (\partial_z p)\partial_\zeta$ restricted to $p = 0$, where z are the coordinates on $\mathbb{R}^n$, ζ is dual to z, and p is the principal symbol of $\Box_g$, which is in fact just the dual metric function $g^{-1} \colon T^*M \longrightarrow \mathbb{R}$, $g^{-1}(z, \zeta) = |\zeta|^2_{g^{-1}(z)}$. (Thus on Minkowski space $p(\zeta) = \zeta_n^2 - \zeta_1^2 - \cdots - \zeta_{n-1}^2$ and the null-bicharacteristic flow is a straight line flow in phase space on the space of null vectors, ζ with $p(\zeta) = 0$, keeping ζ fixed and evolving z affinely.) The b-principal symbol of L, $\sigma_b(L) = \lambda$, can be understood as the extension of the principal symbol of L in the standard sense to the b-cotangent bundle ${}^{\mathrm{b}}T^*M$, thus as discussed λ is equal to the dual metric function of $\hat{g} = \rho^2 g$, which extends smoothly to all ${}^{\mathrm{b}}T^*M$. Concretely, in the variables ρ, v, y, writing forms as

$$\sigma \frac{d\rho}{\rho} + \gamma\, dv + \eta\, dy,$$

λ satisfies

$$\begin{aligned} \lambda = \sigma_{\mathrm{b}}(L) = g^{\rho\rho}\sigma^2 - (4v - \beta v^2 + O(\rho v) + O(\rho^2))\gamma^2 - 2(2 - \alpha v + O(\rho))\sigma\gamma \\ + 2g^{\rho y} \cdot \eta\sigma + \big(2v\Upsilon + O(\rho)\big) \cdot \eta\gamma + g^{y_i y_j}\eta_i\eta_j, \end{aligned} \tag{2.11}$$

where all the $O(.)$ terms are smooth, and β, Υ are smooth as well as are $g^{\rho\rho}$, etc., which are the dual metric components (in the b-basis, for $\hat{g}$, or equivalently in the sc-basis for g); see [2, Equation (3.18)]. (Thus, this formula specifies the structure of certain dual metric components for $\hat{g}$, such as ∂_v^2 and $\partial_v(\rho\partial_\rho)$.)

Consider the b-Hamilton vector field, i.e. the Hamilton vector field of L thought of as a vector field on ${}^{\mathrm{b}}T^*M$. (More precisely, this is the (unique) smooth extension of the Hamilton vector field of L from a vector field on T^*M^{int} to ${}^{\mathrm{b}}T^*M$.) Almost as in [2] (which used ξ in place of σ), this is given in the variables $(\rho, v, y, \sigma, \gamma, \eta)$ by

$$\mathsf{H}_{\mathrm{b}} := (\partial_\sigma\lambda)\rho\partial_\rho + (\partial_\gamma\lambda)\partial_v + (\partial_\eta\lambda)\partial_y - (\rho\partial_\rho\lambda)\partial_\sigma - (\partial_v\lambda)\partial_\gamma - (\partial_y\lambda)\partial_\eta \tag{2.12}$$

where λ is the dual metric function of $\hat{g}$ on ${}^{\mathrm{b}}T^*M$; see [2, Equation (3.20)]. Note that H_{b} is automatically a vector field tangent to ${}^{\mathrm{b}}T^*_{\partial M}M$, i.e. a b-vector field on ${}^{\mathrm{b}}T^*M$ (thus a section of ${}^{\mathrm{b}}T\,{}^{\mathrm{b}}T^*M$). Since taking the Hamilton vector field is a derivation on functions on the cotangent bundle, the flow of H_{b} (this being the Hamilton vector field of $\hat{g} = \rho^2 g$) restricted to the set $\Sigma = \{\lambda = 0\}$ over M^{int} (where $\rho > 0$) is a rescaling of the Hamilton flow of $\Box_g$ restricted to Σ; the rescaling becomes singular at $\rho = 0$. In our case,

$$\begin{aligned} \mathsf{H}_{\mathrm{b}} = &\big(2g^{\rho\rho}\sigma + 2g^{\rho y}\eta - 2\gamma(2 - \alpha v + O(\rho))\big)(\rho\partial_\rho) \\ &- 2\big((4v - \beta v^2 + O(\rho v) + O(\rho^2))\gamma \\ &\qquad + (2 - \alpha v + O(\rho))\sigma + (v\Upsilon + O(\rho))\eta\big)\partial_v \\ &+ 2\big(g^{\rho y}\sigma + (v\Upsilon + O(\rho))\gamma + g^{y_i y_j}\eta_j\big)\partial_y \\ &- (\rho\partial_\rho\lambda)\partial_\sigma - (\partial_v\lambda)\partial_\gamma - (\partial_y\lambda)\partial_\eta; \end{aligned} \tag{2.13}$$

see [2, Equation (3.21)].

To describe how the null-bicharacteristic flow acts at infinity we must define and analyze the b-conormal bundle of the submanifold S. There is a natural map of the b-tangent space ${}^{\mathrm{b}}TM \longrightarrow TM$ defined on sections, i.e. elements of $\mathcal{V}_{\mathrm{b}}$, by considering a b-vector field as a standard vector field. (Thus the map is not surjective over the boundary; $\rho\partial_\rho$ vanishes there.) We can use the dual map $T^*M \longrightarrow {}^{\mathrm{b}}T^*M$ to define the b-conormal bundle of submanifolds; specifically, for our submanifold S, the conormal bundle ${}^{\mathrm{b}}N^*S$ equal to the image in ${}^{\mathrm{b}}T^*M$ of covectors in T^*M annihilating the image of $TS \subset T_SM$ in ${}^{\mathrm{b}}TM$. It turns out that the null-bicharacteristics of L (see Figure 2) terminate both at S_+ and S_- at the spherical b-conormal bundle

$$^{\mathrm{b}}SN^*S_\pm = (^{\mathrm{b}}N^*S_\pm \setminus o)/\mathbb{R}_+.$$

Before we describe this in more detail, we point out that $^{\mathrm{b}}N^*S$ in fact has one dimensional fibers, since in coordinates ρ, v, y with ρ, v (so $S = \{\rho = 0 = v\}$) as above and y local coordinates on S, so vectors in TS are multiples of ∂_y, are annihilated by forms $a\,dv + b\,d\rho$ in T^*M, which map to forms $a\,dv + b\rho(\rho^{-1}\,d\rho)$ in $^{\mathrm{b}}T^*M$ and thus restrict to $a\,dv$ since S lies in the boundary $\rho = 0$. More concretely, the b-conormal bundle of S is generated by dv, i.e. in the coordinates above is given by the vanishing of ρ, v, σ, η, and thus y, γ are (local) coordinates along it. This means that at each point $p \in S$,

$$^{\mathrm{b}}SN^*_pS = (^{\mathrm{b}}N^*_pS \setminus \{0\})/\mathbb{R}_+ = \{\gamma\,dv : \gamma \neq 0\}/\mathbb{R}_+, \tag{2.14}$$

so in fact $^{\mathrm{b}}N^*S$ is a line bundle over S and $^{\mathrm{b}}SN^*S$ is an $\mathbb{S}^0$ (two points) bundle generated by the images of dv and $-dv$ in $^{\mathrm{b}}T^*_SM$.

The flow on null-bicharacteristics, in view of the structure of the operator at $S_\pm$, as shown in [2, Section 3], see also [27, Section 5], makes the two halves of the spherical b-conormal bundle of S, $^{\mathrm{b}}SN^*S = {}^{\mathrm{b}}SN^*_+S \cup {}^{\mathrm{b}}SN^*_-S$, into a family of sources $(-)$ or sinks $(+)$ for the Hamilton flow, meaning that the null-bicharacteristics approach $^{\mathrm{b}}SN^*_+S_+$ as their parameter goes to $+\infty$ and $^{\mathrm{b}}SN^*_-S_-$ as the parameter goes to $-\infty$, or $^{\mathrm{b}}SN^*_+S_-$ as their parameter goes to $+\infty$ and $^{\mathrm{b}}SN^*_-S_+$ as the parameter goes to $-\infty$. Correspondingly, the characteristic set $\Sigma \subset {}^{\mathrm{b}}T^*M \setminus o$, which we also identify as a subset of $^{\mathrm{b}}S^*M$, of L globally splits into the disjoint union $\Sigma_+ \cup \Sigma_-$, with the first class of bicharacteristics contained in Σ_+, the second in Σ_-. One computes, cf. the discussion after [2, Equation (3.22)], that the b-Hamilton vector field in (2.13) at $^{\mathrm{b}}N^*S$, modulo terms vanishing there quadratically, is satisfies

$$-4\gamma(\rho\partial_\rho) - (8v\gamma + 4\sigma)\partial_v + 2\mu_i\partial_{y_i} + 4\gamma^2\partial_\gamma.$$

with μ_i vanishing on $^{\mathrm{b}}N^*S$. Thus, b-Hamilton vector field is indeed radial at $^{\mathrm{b}}N^*S$ (it is a multiple of ∂_γ as σ, ρ, v, μ_i vanish there by (2.14)); furthermore within $^{\mathrm{b}}N^*S$ one sees that for $\gamma > 0$ fiber infinity is a sink (the flow tends towards it), while for $\gamma < 0$ a source. One also sees that in fact $^{\mathrm{b}}S^*NS$ is a source/sink bundle depending on the sign of γ, i.e. in the normal directions to $^{\mathrm{b}}N^*S$ the flow behaves the same way as within $^{\mathrm{b}}N^*S$, by checking the eigenvalues of the linearization, see [2, Equation (3.23)].

Recall that the basic result for elliptic problems on compact manifolds without boundary is elliptic regularity estimates, which in turn imply Fredholm properties. Indeed, if P is an elliptic operator of order k on a compact manifold without boundary X, then for any $m' < m + k$ one has the estimate

$$\|u\|_{H^{m+k}(X)} \leq C(\|Pu\|_{H^m(X)} + \|u\|_{H^{m'}(X)}). \tag{2.15}$$

That P is a Fredholm map from $H^{m+k}(X)$ to $H^m(X)$ is an immediate consequence of this estimate and the fact that $H^{m+k}(X)$ is a compact subspace of $H^{m'}(X)$, together with the fact that P^*, the formal adjoint of P, is then also elliptic, so analogous estimates hold for P^*.

Here we have real principal type points over M° as $\Box_g$ is non-elliptic, as well as radial points at $^{\mathrm{b}}SN^*_\pm S_\pm$. Recall that real principal type estimates simply propagate regularity along null-bicharacteristics, i.e. given that the estimate holds at a point, one gets it elsewhere as well. The basic result at radial points which are

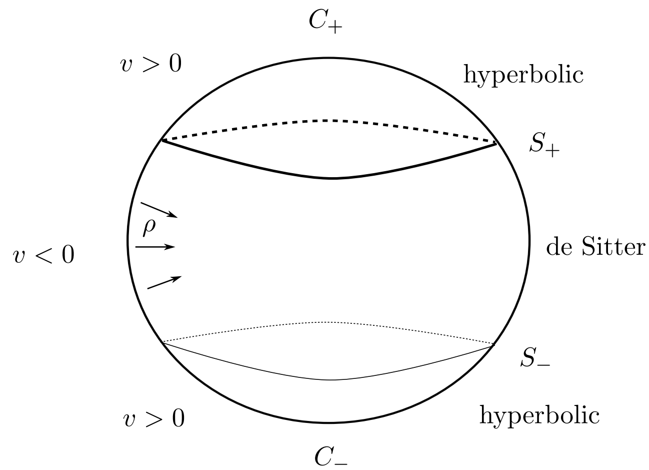

FIGURE 1. ρ equals zero exactly on the boundary and has non-vanishing differential there.

sources or sinks, see [2, Proposition 4.4], [27, Proposition 5.1] and indeed [23] for a precursor in the boundaryless setting (in turn based on [47], which further goes back to [39]), in terms of b-Sobolev spaces, which we proceed to describe in detail, is that subject to restrictions on the decay and regularity orders, in the high regularity regime, one has a real principal type estimate but without an assumption that one has the regularity anywhere, provided one has at least a minimum amount of a priori regularity at the point in question. To clarify, *away* from radial points, real principal type propagation estimates control the norm in H^s (microlocally) near a point on a null bicharacteristic in terms of the H^{s-1} norm of Lu and the H^s norm of u *elsewhere on the bicharacteristic* (together with arbitrarily low regularity norms of u); for high regularity radial points estimates, the "elsewhere..." can be removed, as one gets an estimate for the H^s norm near the radial point in terms of the H^{s-1} norm of Lu (provided one knows the wavefront set does not intersect the radial set at the point in question.) On the other hand, in the low regularity setting, one can propagate estimates into the radial points, much as in the case of real principal type estimates. See Theorem 2.1.

To describe this concretely, we must first say what we mean precisely by regularity and vanishing order. For any manifold with boundary M, fix a non-vanishing b-density μ, i.e. a non-vanishing smooth section of the density bundle of ${}^{\mathrm{b}}TM$, which necessarily takes the form $\rho^{-1}\widetilde{\mu}$ for a non-vanishing density $\widetilde{\mu}$ on the manifold with boundary M (so for $M = \overline{\mathbb{R}^n}$, the radial compactification of $\mathbb{R}^n$, notation as in the paragraph containing (2.2), $\mu = a\rho^{-1}|\,d\rho\, dw|$ where $|dw| = |\prod_{i=1}^{n-1} dw_i|$ and a is a smooth non-vanishing function up to $\rho = 0$.) The density μ is natural here as it can be taken to be the absolute value of the volume form of a b-metric, e.g. on $\mathbb{R}^n$, $|\rho^2\, dz^2|$ is such a b-metric, and we define L^2_b to be the Hilbert space induced by μ, so

$$\langle u, v\rangle_{L^2_b} = \int_M u\,\overline{v}\,\mu. \tag{2.16}$$

We define the weighted b-Sobolev spaces, first for integer orders $k \in \mathbb{N}$ by letting $u \in H_{\mathrm{b}}^k(M)$ if and only if $V^1 \dots V^{k'} u \in L_b^2$ for every $k'-$tuple of b-vector fields $V_i \in \mathcal{V}_b$ with $k' \leq k$. (Recall that $\mathcal{V}_{\mathrm{b}}$ is the space of b-vector fields discussed in (2.9); thus to $u \in H_{\mathrm{b}}^k(M)$ one can apply in particular k'-fold derivatives of the form $\rho\partial_\rho$ and ∂_{w_i} and remain in L_b^2.) For $m \geq 0$ real we have

$$\begin{aligned} H_{\mathrm{b}}^m(M) &= \{u \in C^{-\infty}(M) \mid Au \in L_{\mathrm{b}}^2(M)\ \forall A \in \Psi_{\mathrm{b}}^m(M)\}, \\ H_{\mathrm{b}}^{m,l}(M) &= \rho^l H_{\mathrm{b}}^m(M) \end{aligned} \tag{2.17}$$

where $\Psi_{\mathrm{b}}^m(M) = \Psi_{\mathrm{b}}^{m,0}(M)$ is the space of b-pseudodifferential operators, described in Section 3. For $m < 0$ this can be extended by duality, or instead for all real m, demanding that u lie in $\mathrm{Diff}_{\mathrm{b}}^N(M)L_{\mathrm{b}}^2(M)$ for some N (i.e. be a finite sum of at most Nth b-derivatives of elements of $L_{\mathrm{b}}^2(M)$) and satisfy the regularity under application of b-ps.d.o's:

$$\begin{aligned} H_{\mathrm{b}}^m(M) &= \{u \in \mathrm{Diff}_{\mathrm{b}}^*(M)L_{\mathrm{b}}^2(M) \mid Au \in L_{\mathrm{b}}^2(M)\ \forall A \in \Psi_{\mathrm{b}}^m(M)\}, \\ H_{\mathrm{b}}^{m,l}(M) &= \rho^l H_{\mathrm{b}}^m(M) \end{aligned} \tag{2.18}$$

In general, we will allow a variable $m \in C^\infty({}^{\mathrm{b}}S^*M;\mathbb{R})$, in which case the same definition can be applied, namely one simply takes variable order ps.d.o's (see [2, Appendix]). (Such variable order spaces have a long history, starting with Unterberger [46] and Duistermaat [14]; see the work of Faure and Sjöstrand [17] and Dyatlov and Zworski [16] for other recent applications.) Taking $M = \overline{\mathbb{R}^n}$, note that we may choose the measure μ in (2.16) so that $H_{\mathrm{b}}^{0,n/2} = L^2$ where L^2 here and below denotes the standard Hilbert space on $\mathbb{R}^n$, indeed we can take $\mu = \rho^{-n}|dz|$ there; we remark that the equality as Banach spaces up to equivalence of norms (which is what matters mostly) is automatic. Note that the L_b^2 pairing gives an isomorphism

$$(H_{\mathrm{b}}^{m,l})^* \simeq H_{\mathrm{b}}^{-m,-l}. \tag{2.19}$$

For $s \in \mathbb{R}$, the weighted b-Sobolev wavefront sets of a distribution u, denoted $\mathrm{WF}_{\mathrm{b}}^{s,l}(u)$ are the directions in phase space in which u fails to be in $H_{\mathrm{b}}^{s,l}(M)$. A concrete definition using explicit b-pseudodifferential operators is given in (3.9) below, but for the moment we state that it is defined for $u \in H_{\mathrm{b}}^{-N,l}$ by

$$\mathrm{WF}_{\mathrm{b}}^{s,l}(u) = \bigcap \left\{ \Sigma(A) \subset {}^{\mathrm{b}}TM : Au \in H_{\mathrm{b}}^{s,l}(M) \right\}, \tag{2.20}$$

where the intersection is taken over all $A \in \Psi_{\mathrm{b}}^{0,0}(M)$, i.e. A is a $(0,0)$ order b-pseudodifferential operator (again, see Section 3) and $\Sigma(A)$ is the characteristic set (vanishing set of the principal symbol) of A. Equivalently, a point $(p,\xi) \notin \mathrm{WF}_{\mathrm{b}}^{s,l}(u)$ (where $\xi \in {}^{\mathrm{b}}T_p^*M \setminus o$) if there exists $A \in \Psi_{\mathrm{b}}^{0,0}(M)$ which is elliptic at (p,ξ) such that $Au \in H_{\mathrm{b}}^{m,l}(M)$. We say that u is in $H_{\mathrm{b}}^{s,l}$ microlocally if $(p,\xi) \notin \mathrm{WF}_{\mathrm{b}}^{s,l}(u)$ where $\xi \in {}^{\mathrm{b}}T_p^*M$. There is a completely analogous definition of $\mathrm{WF}_{\mathrm{b}}^{m,l}$ for varying $m \in C^\infty({}^bS^*M)$ and for $l \in \mathbb{R}$.

We have the following result, which is essentially [27, Proposition 5.1]. For the following statement, let $\mathcal{R}$ be any of the above discussed connected components of radial sets ${}^{\mathrm{b}}SN_\pm^*S_\pm$.

Proposition 2.1. *Let (M,g) be a Lorentzian scattering space as in (2.3). Let L be as above and $u \in H_{\mathrm{b}}^{-\infty,l}(M)$.*

If $m+l < \frac{1}{2}$ and m is nonincreasing along the Hamilton flow in the direction that approaches $\mathcal{R}$, then $\mathcal{R}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ provided that $\mathcal{R} \cap \mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu) = \varnothing$ and a punctured neighborhood in $\Sigma \cap {}^{\mathrm{b}}S^*M$ of $\mathcal{R}$ (i.e. a neighborhood of $\mathcal{R}$ with $\mathcal{R}$ removed) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$.

On the other hand, suppose that $m' + l > \frac{1}{2}, m \geq m'$ and m is nonincreasing along the Hamilton flow in the direction that leaves $\mathcal{R}$. Then if $\mathrm{WF}_{\mathrm{b}}^{m',l}(u)$ and $\mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu)$ are both disjoint from $\mathcal{R}$, then $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ is disjoint from $\mathcal{R}$.

For elliptic regularity, the variable order m is completely arbitrary, but for real principal type estimates it has to be non-increasing in the direction along the Hamilton flow in which we wish to propagate the estimates.

So now fixing l and taking m satisfying $m + l > 1/2$ at exactly one of ${}^{\mathrm{b}}SN_+^*S_+$ or ${}^{\mathrm{b}}SN_-^*S_-$ and $m + l < 1/2$ at the other (e.g. if $> 1/2$ at ${}^{\mathrm{b}}SN_+^*S_+$ then $< 1/2$ at ${}^{\mathrm{b}}SN_-^*S_-$), and similarly $m+l > 1/2$ at either ${}^{\mathrm{b}}SN_-^*S_+$ or ${}^{\mathrm{b}}SN_+^*S_-$ and $m+l < 1/2$ at the other, we obtain estimates for u in $H_{\mathrm{b}}^{m,l}$ in terms of the $H_{\mathrm{b}}^{m-1,l}$ norm of Lu plus the $H_{\mathrm{b}}^{m',l}$ norm of u for $m' < m$ (a weaker norm). To make this precise we work with varying order Sobolev spaces $H_{\mathrm{b}}^{m,l}(M)$. These are discussed in detail in [2, Appendix] in the setting of standard Sobolev spaces (i.e. without the "b"), but since the development is nearly identical we discuss them only briefly. Specifically, given a function $m \in C^\infty({}^{\mathrm{b}}S^*M)$ that is monotonic along the Hamilton flow, $u \in H_{\mathrm{b}}^{m,l}(M)$ if and only if $Au \in L^2_{\mathrm{b}}(M)$ for any $A \in \Psi_{\mathrm{b}}^{m,l} = \rho^l \Psi_{\mathrm{b}}^m$, where for $l = 0$ membership of $\Psi_{\mathrm{b}}^{m,l}$ means that A is the quantization of a symbol $a \in C^\infty({}^{\mathrm{b}}T^*M)$ satisfying (among other standard symbol conditions elaborated in [2, Appendix]) that $|a(\rho, w, \sigma, \omega)| \leq C(1 + \sigma^2 + |\omega|^2)^{m/2}$; here ρ is again a boundary defining function, and coordinates on ${}^{\mathrm{b}}T^*M$ are obtained by parametrizing b-covectors as

$$\sigma \frac{d\rho}{\rho} + \omega^i dw_i.$$

For $l \in \mathbb{R}$, we thus have $H_{\mathrm{b}}^{m,l}(M) := \rho^l H_{\mathrm{b}}^{m,0}(M)$; the norm on these spaces is given by any elliptic $A \in \Psi_{\mathrm{b}}^{m,l}$ together with the $H_{\mathrm{b}}^{m',l}$ norm, where $m' < \inf m$. (This is only defined up to equivalence of norms, but that is all we need.) Thus, given any s, r with s monotone along the Hamilton flow and $r \in \mathbb{R}$, consider the spaces

$$\mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(M),\ \mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(M) :\ Lu \in H_{\mathrm{b}}^{s-1,r}(M)\}. \tag{2.21}$$

Then $\mathcal{X}^{s,r}$ is a Hilbert space with norm

$$\|u\|_{\mathcal{X}^{s,r}} = \|u\|_{H_{\mathrm{b}}^{s,r}(X)} + \|Lu\|_{H_{\mathrm{b}}^{s-1,r}(X)}.$$

With m, l and m' as above (in particular m is a function), we have the estimates

$$\|u\|_{H_{\mathrm{b}}^{m,l}(X)} \leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}(X)} + \|u\|_{H_{\mathrm{b}}^{m',l}(X)}). \tag{2.22}$$

(Here $m' < m$ can be taken to be a function, but this is not important. It can, for instance, be taken to be an integer $N < \inf m$.)

Note that the 'end' of the bicharacteristics at which $m + l < 1/2$ is the direction in which the estimates are propagated, thus the choices

$$\begin{aligned} &\pm(m + l - 1/2) < 0 \text{ at } {}^{\mathrm{b}}SN_+^*S_+ \text{ (sinks), and} \\ &\pm(m + l - 1/2) < 0 \text{ at } {}^{\mathrm{b}}SN_-^*S_+ \text{ (sources)} \end{aligned} \tag{2.23}$$

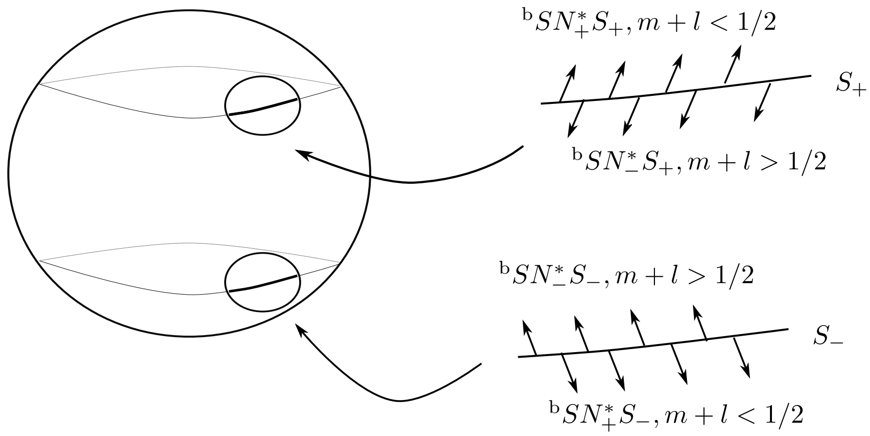

FIGURE 2. For the operator L_{+-}, corresponding to the forward Feynman problem, high regularity is imposed at the 'beginning' (near ${}^{\mathrm{b}}SN^*_-S$) of each null bicharacteristic, whether they begin at S_+ or S_-.

determine what (if any) type of inverse we get for L; we denote L on the corresponding spaces by $L_{\pm\pm}$ with the two $\pm$ corresponding to the two $\pm$ as in (2.23), i.e. the first to the direction of propagation in Σ_+, the second to that in Σ_-, with the signs being positive if the propagation is towards S_+ and negative if the propagation is towards S_-. Notice that by our requirements of $m+l>1/2$ at exactly one end of each bicharacteristic and $m+l<1/2$ at the other, (2.23) is *equivalent* to

$$\begin{aligned} &\mp(m+l-1/2)<0 \text{ at } {}^{\mathrm{b}}SN^*_-S_- \text{ (sources) and} \\ &\mp(m+l-1/2)<0 \text{ at } {}^{\mathrm{b}}SN^*_+S_- \text{ (sinks).} \end{aligned} \tag{2.24}$$

That is to say,

$$L_{\pm\pm} \text{ denotes } any \text{ map } L\colon \mathcal{X}^{m,l} \longrightarrow \mathcal{Y}^{m-1,l}$$

for which the pair (m,l) satisfy (2.23) with the given $\pm,\pm$ combination (the first sign in the first inequality and the second in the second). (To be clear, the fact that we write the choices in (2.23) as taking place at S_+ is arbitrary, as we could just as easily make the choices at S_-. Whichever signs are chosen at S_+, the opposite sign is chosen on the other end of the flow at S_-. For example, for L_{+-}, the condition on m at S_- is that $-(m+l-1/2)<0$ at ${}^{\mathrm{b}}SN^*_-S_-$ and $m+l-1/2<0$ at ${}^{\mathrm{b}}SN^*_+S_-$.) Strictly speaking, $L_{\pm\pm}$ depends on m, but in fact we will see that the choice of m satisfying a particular version of (2.23) is irrelevant. Thus we use the notation

$$\mathcal{X}^{m,l}_{\pm\pm} = \mathcal{X}^{m,l},\ \mathcal{Y}^{m,l}_{\pm\pm} = \mathcal{Y}^{m,l} \text{ for } any\ (m,l) \text{ satisfying (2.23)} \tag{2.25}$$

with the given $\pm$, $\pm$ combination. See Figure 2.

We call L_{++} the forward wave operator (corresponding to the forward solution), L_{--} the backward wave operator, L_{+-} and L_{-+} the Feynman and anti-Feynman wave operators (sometimes we simply call them both Feynman), with L_{+-} propagating forward along the Hamilton flow (i.e. to the sinks), and L_{-+} backward along the Hamilton flow (i.e. to the sources) in *both* Σ_+ and Σ_-. Here we point out that either of the forward and backward wave operators propagate estimates in the

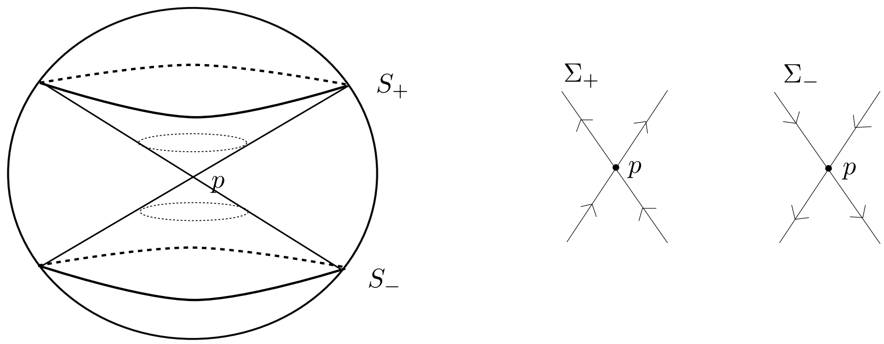

FIGURE 3. The traces (i.e. projections from the cotangent space) of the light rays passing through an arbitrary point p. In the cotangent space these separate into the forward and backward pointing null-bicharacteristics, depicted heuristically at right. The operator L_{+-} corresponds to propagation of singularities along the flow, and corresponds to the choice of $+$ in the first and $-$ in the second inequality in (2.23)

opposite directions relative to the Hamilton flow in Σ_+, resp. Σ_-; the propagation is in the same direction relative to a time function in the underlying space M.

3. Mapping properties of the Feynman propagator

The main result of this section is Theorem 3.3 below, which asserts that $L_{\pm\pm}$ are Fredholm maps between appropriate Hilbert spaces. As mentioned, the estimates in (2.22) are *not* sufficient to conclude that L is Fredholm, since the weaker norm does not possess additional decay. Thus the main technical result of this section is the following. For (m, l) chosen as in (2.23) for any choice of signs $\pm\pm$, and *for certain choices of* l (see the theorem), we have

$$\begin{aligned} \|u\|_{H_{\mathrm{b}}^{m,l}} &\leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + \|u\|_{H_{\mathrm{b}}^{m',l'}}) \\ \|v\|_{H_{\mathrm{b}}^{1-m,-l}} &\leq C(\|Lv\|_{H_{\mathrm{b}}^{-m,-l}} + \|v\|_{H_{\mathrm{b}}^{1-m'',-l''}}) \end{aligned} \tag{3.1}$$

where $m' < m < m''$ and $l' < l < l''$. As explained in the proof of Theorem 3.3, it is then a simple exercise using the fact that $H_{\mathrm{b}}^{m,l} \subset H_{\mathrm{b}}^{m',l'}$ is compact provided $m' < m$ and $l' < l$ to show that $L_{\pm\pm}$ is Fredholm on the spaces in the theorem.

To obtain the improved estimates in (3.1), as in elliptic problems, we also need to consider the Mellin transformed normal operator $\hat{N}(L)(\sigma)$ of L, which is a family of differential operators on ∂M, parameterized by $\sigma \in \mathbb{C}$. Given an arbitrary $P \in \mathrm{Diff}_{\mathrm{b}}^*(M)$ of order k,

$$P = \sum_{i+|\alpha|\leq k} a_{i,\alpha}(\rho, x)(\rho\partial_\rho)^i \partial_x^\alpha, \tag{3.2}$$

the normal operator is locally given by

$$N(P) := \sum_{i+|\alpha|\leq k} a_{i,\alpha}(0, x)(\rho\partial_\rho)^i \partial_x^\alpha \in \mathrm{Diff}_{\mathrm{b}}^k([0,\infty)_\rho \times \partial M). \tag{3.3}$$

The Mellin transform is defined, initially on compactly supported smooth functions $u \in C^\infty(\mathbb{R}^+; \mathbb{C})$, by

$$\mathcal{M}(u)(\sigma) = \hat{u}(\sigma) = \int_0^\infty \rho^{-i\sigma} u(\rho) \frac{d\rho}{\rho}.$$

Note that $\mathcal{M}u(\sigma) = \mathcal{F}v(\sigma)$ where $\mathcal{F}$ is the Fourier transform and $v(x) = u(e^x)$. Writing complex numbers $\sigma = \xi + i\eta$, it extends to a unitary isomorphism

$$\mathcal{M}\colon \rho^l L^2(\mathbb{R}^+, d\rho/\rho) \longrightarrow L^2(\{\operatorname{Im}\sigma = -l\}, d\xi). \tag{3.4}$$

The inverse map of (3.4) is given by

$$\mathcal{M}_l^{-1} f(\rho) = \frac{1}{2\pi} \int_{\{\operatorname{Im}\sigma = -l\}} \rho^{i\sigma} f(\rho) d\sigma. \tag{3.5}$$

Moreover, conjugating $N(P)$ by the Mellin transform in ρ gives

$$\hat{N}(P)(\sigma) = \sum_{i + |\alpha| \le k} a_{i,\alpha}(0, x)\sigma^i \partial_x^\alpha \tag{3.6}$$

We digress briefly to describe following typical example of a b-pseudodifferential operator which is elliptic at a point $p \in {}^{\mathrm{b}}TM$ lying over the boundary, and how it relates to the b-wavefront set discussed above. If $p \in {}^{\mathrm{b}}T^*M$ lies over the boundary, then in coordinates (ρ, y, ξ, η) on ${}^{\mathrm{b}}TM$ where ρ is a boundary defining function, ξ is dual to ρ and η to y, we have $p = (0, y_0, \xi_0, \eta_0)$. We obtain a b-pseudodifferential operator that is elliptic at p by choosing a cutoff function $\chi(\rho, y)$ with $\chi(0, y_0) \neq 0$ and such that χ is supported in $\{\rho < \epsilon\}$ for small ϵ, in particular small enough so that $\{\rho < \epsilon\} \simeq \partial M \times [0, \epsilon)$. Let $\phi(\xi, \eta)$ be a symbol, homogeneous near infinity, non-zero in the cone given by positive multiples of (ξ_0, η_0). With $\mathcal{F}$ the Fourier transform in the y variables, we define

$$Au := \mathcal{M}_0^{-1} \mathcal{F}^{-1} \phi \mathcal{F} \mathcal{M}(\chi u). \tag{3.7}$$

Then $A \in \Psi_{\mathrm{b}}^{0,0}(M)$, and the b-principal symbol of A at order and weight $(m, l) = (0, 0)$ is:

$$\sigma_{0,0}(A)\colon {}^{\mathrm{b}}T^*M \longrightarrow \mathbb{C}, \quad \sigma_{0,0}(A) = \chi\phi \tag{3.8}$$

where we think of $\chi\phi = \chi(\rho, y)\phi(\xi, \eta)$ as a function on ${}^{\mathrm{b}}T^*M$, which near the boundary and with our coordinates is diffeomorphic to $\{\rho < \epsilon, \xi\} \times T^*\partial M$, supported on the neighborhood of $(0, y_0, \xi_0, \eta_0)$ under consideration. In fact, such operators can be used to neatly describe the b-wavefront sets of distributions. Given a distribution $u \in H_{\mathrm{b}}^{-N,l}(M)$, then for $m, l \in \mathbb{R}$,

$$(0, y_0, \xi_0, \eta_0) \notin \mathrm{WF}_{\mathrm{b}}^{m,l}(u) \iff \exists \chi, \phi \text{ with } A\rho^{-l}u \in H_{\mathrm{b}}^{m,0}(M), \tag{3.9}$$

where A is formed from χ and ϕ as in (3.7). (The ρ^{-l} in the front is there so that the inverse Mellin transform $\mathcal{M}_0^{-1}$ of the resulting object is well defined.)

The structure and properties of $\hat{N}(L)(\sigma)$ are discussed at length in [27]. To briefly summarize, for each σ, $\hat{N}(L)(\sigma)$ is a second order differential operator which is elliptic in the interior of the regions $C_\pm$, and hyperbolic on their complement $\partial M \setminus (C_+ \cup C_-)$ whose characteristic set splits into two components $\widetilde{\Sigma}_\pm$, each of which contains a Lagrangian submanifold of radial points lying over $S = \partial C_+ \cup \partial C_-$, and which split the conormal bundle N^*S (in ∂M) into four components $N_\pm^* S_\pm$ which are sources ($N_-^* S$) and sinks ($N_+^* S$) for the Hamilton flow.

The estimates corresponding to those of the previous section allow one to conclude that $\hat{N}(L)(\sigma)$ is Fredholm for each σ on the induced Sobolev spaces, where $\operatorname{Im}\sigma = -l$, i.e. provided

$$\begin{aligned} &\pm(m - \operatorname{Im}\sigma - 1/2) < 0 \text{ at } N_+^* S_+, \text{ and} \\ &\pm(m - \operatorname{Im}\sigma - 1/2) < 0 \text{ at } N_-^* S_+. \end{aligned} \tag{3.10}$$

More precisely here m is replaced by $m|_{S^*\partial M}$, which is a well-defined subbundle of ${}^{\mathrm{b}}S^*_{\partial M}M$. (Indeed, the map ${}^{\mathrm{b}}TM \longrightarrow TM$, restricts to a surjection ${}^{\mathrm{b}}T_{\partial M}M \longrightarrow T\partial M$, and the dual injection $T^*\partial M \longrightarrow {}^{\mathrm{b}}T^*_{\partial M}M$ gives the desired inclusion of $S^*\partial M$ after modding out by the action of $\mathbb{R}_+$). Thinking of σ as the b-dual variable of ρ (which thus depends on the choice of $\frac{d\rho}{\rho}$ in a neighborhood of ∂M but is invariant at ∂M), covectors are of the form $\beta + \sigma\frac{d\rho}{\rho}$, $\beta \in T^*M$, thus (identifying functions on ${}^{\mathrm{b}}S^*M$ with homogeneous degree 0 functions on ${}^{\mathrm{b}}T^*M \setminus o$, where o denotes the zero section) for each $\sigma \neq 0$ one actually has a function on $T^*\partial M$. One thus obtains a family of large parameter norms (as described in the theorem just below), analogous to the usual semiclassical norms: for σ in a compact set, the norms are uniformly equivalent to each other, but as $\sigma \to \infty$ this ceases to be the case. In fact, we have the following applications of [2, Proposition 5.2], [47, Theorem 2.14].

Theorem 3.1. *(Statement of* [47, Theorem 2.14] *in our setting.) In strips in which* $\operatorname{Im}\sigma$ *is bounded,* $\hat{N}(L)(\sigma)^{-1}$ *has finitely many poles.*

Proof. Since this is the statement of [47, Theorem 2.14], with the underlying analysis being carried out in [47, Section 2], we only give a brief sketch.

As we will see momentarily, our family $\hat{N}(L)(\sigma)$ forms an analytic Fredholm family

$$\hat{N}(L)(\sigma)\colon \mathcal{X}^m(\partial M) \longrightarrow \mathcal{Y}^{m-1}(\partial M), \tag{3.11}$$

where $\mathcal{X}^m(\partial M) = \{\phi : \phi \in H^m(\partial M), \hat{N}(L)(\sigma)\phi \in H^{m-1}(\partial M)\}$, and $\mathcal{Y}^m(\partial M) = \{\phi : \phi \in H^m(\partial M)\}$ provided that σ and m are related as in (3.10), whose inverse is thus meromorphic if $\hat{N}(L)(\sigma)$ is invertible for at least one $\sigma = \sigma_0$. For bounded $\operatorname{Im}\sigma$, we can see that $\hat{N}(L)(\sigma)$ is invertible for sufficiently large $\operatorname{Re}\sigma$. This follows exactly as in [2, Proposition 5.2], which in turn follows directly from [47, Theorem 2.14]. The key to this is to consider the semiclassical problem obtained by letting $h = |\sigma|^{-1}$ and $z = \frac{\sigma}{|\sigma|}$, and, letting $P_\sigma = \hat{N}(L)(\sigma)$, studying

$$P_{h,z} := h^2 P_{h^{-1}z} \in \Psi_h^2(\partial M),$$

where $\Psi_h^2(\partial M)$ denotes the space of semiclassical pseudodifferential operators of order 2 on ∂M. This semiclassical family on ∂M has Lagrangian submanifolds of radial points (coming from the b-radial points of L), and, as described in [47, Section 2.8], the standard positive commutator proof of propagation of singularities around Lagrangian submanifolds of radial points carries over to the semiclassical regime without difficulty. This allows us to obtain estimates

$$\begin{aligned} \|u\|_{H_h^m} &\leq C(h^{-1}\|P_{h,z}u\|_{H_h^{m-1}} + h\|u\|_{H_h^{-N}}) \\ \|v\|_{H_h^{1-m}} &\leq C(h^{-1}\|P_{h,z}^*v\|_{H_h^{-m}} + h\|u\|_{H_h^{-N}}) \end{aligned}$$

for arbitrarily large N, within strips of bounded $\operatorname{Im}\sigma$. As described at the beginning of the proof of Theorem 3.3 below, these estimates imply that $N(L)(\sigma)$

mapping in (3.11) is Fredholm. Hence, for sufficiently small h, the $-N$ norm can be absorbed into the left hand side, giving by the first inequality injectivity and by the second surjectivity. (This point is also elaborated in Theorem 3.3.) Note that the statement of [2, Proposition 5.2] is for only the forward and backward propagators, as the results come from microlocal positive commutator estimates which are sufficiently microlocal, the conclusion, with the same proof, also holds for the Feynman operators. □

Remark 3.2. We point out that analogues of the estimates used so far go through if L has sufficiently weak trapping with slight modifications: so-called b-normally hyperbolic trapping, as introduced in [25], gives essentially the same estimates for σ real and large. (However, we do not study this here.)

Following [27] we will prove the following Fredholm mapping result for L, mapping between spaces $\mathcal{X}^{m,l}$ and $\mathcal{Y}^{m-1,l}$ which satisfy not only the threshold properties in (2.23), but furthermore that in the high regularity regime we assume a full order more Sobolev regularity. Specifically, we will assume that (m,l) are chosen as in (2.23) for any choices $\pm,\pm$, with the additional property that when the $-$ sign is valid on the left hand side, i.e. $-(m+l-1/2)<0$, then in fact $-(m+l-3/2)<0$ as well, and thus the complete set of options for m,l shall be

(3.12)

Region	Feynman	Anti-Feynman	Retarded	Advanced
${}^{\mathrm{b}}SN^*_+S_+$	$m+l<1/2$	$m+l>3/2$	$m+l<1/2$	$m+l>3/2$
${}^{\mathrm{b}}SN^*_-S_+$	$m+l>3/2$	$m+l<1/2$	$m+l<1/2$	$m+l>3/2$
${}^{\mathrm{b}}SN^*_+S_-$	$m+l<1/2$	$m+l>3/2$	$m+l>3/2$	$m+l<1/2$
${}^{\mathrm{b}}SN^*_-S_-$	$m+l>3/2$	$m+l<1/2$	$m+l>3/2$	$m+l<1/2$

Theorem 3.3. *Assume that (m,l) satisfy (2.23), and in addition the properties of the previous paragraph, i.e. one of the four columns in (3.12). Moreover, assume that, subject to this choice, there are no poles of $\hat{N}(L)(\sigma)^{-1}$ on the line $\operatorname{Im}\sigma=-l$, (where $\hat{N}(L)$ maps as in (3.11) with $m=m|_{T^*\partial M}$.) Then L is Fredholm as a map*

$$L:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l}. \tag{3.13}$$

Remark 3.4. Note that the Fredholm property is stable under b-perturbations of L, in $\Psi_{\mathrm{b}}^{2,0}$, meaning for operators of the form $L+P$ where $P\in\Psi_{\mathrm{b}}^{2,0}$ is small. For P in $\mathrm{Diff}_{\mathrm{b}}^2$ this means that P also has an expression as in (2.7) but with coefficient functions $a_{j,\alpha}$ which are small in C^∞. In particular, any perturbation of a Lorentzian scattering metric in the sense of sc-metrics gives rise to a similarly Fredholm problem. Indeed, for such a perturbation, the poles of the Feynman resolvent family for the normal operator of the perturbation of L are themselves perturbed in a continuous fashion from the poles of those of the normal operator of L. Since obtaining a Fredholm problem at a weight corresponds to having a line $\{\operatorname{Im}\zeta=c\}$ with no poles, the result follows.

Remark 3.5. Microlocal elliptic regularity states that $\mathrm{WF}_{\mathrm{b}}^{m_0,l}(u)\setminus\Sigma\subset\mathrm{WF}_{\mathrm{b}}^{m_0-2,l}(f)$ if $Lu=f$ and $u\in H_{\mathrm{b}}^{\tilde{m},l}$ for some $\tilde{m}$ (i.e. $u\in H_{\mathrm{b}}^{-\infty,l}$). Propagation of singularities, in the sense of WF_{b}, implies that if $Lu=f$, where $u\in H_{\mathrm{b}}^{m,l}$, $f\in H_{\mathrm{b}}^{m-1,l}$ for some m, l satisfying (2.23) for the $+-$ signs, then a point $\alpha\in\Sigma\setminus({}^{\mathrm{b}}SN^*_+S_+\cup{}^{\mathrm{b}}SN^*_+S_-)$ (i.e. α is not at the radial sink, at which the function spaces have low regularity) is not in $\mathrm{WF}_{\mathrm{b}}^{m_0,l}(u)$ provided that the backward bicharacteristic through α is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(f)$ and provided $\mathrm{WF}_{\mathrm{b}}^{m_0-1,l}(f)$ is disjoint from ${}^{\mathrm{b}}SN_-^*S_+ \cup {}^{\mathrm{b}}SN_-^*S_-$, i.e. the radial sources at which high regularity is imposed. In particular, if f is compactly supported in M°, then $\mathrm{WF}_{\mathrm{b}}^{m_0,l}(u) \setminus ({}^{\mathrm{b}}SN_+^*S_+ \cup {}^{\mathrm{b}}SN_+^*S_-)$ contained in the order m_0-1 wavefront set of f together with the flowout of the intersection of the wavefront set of f with the characteristic set Σ of L. Restricted to the interior, where WF_{b} is just the standard wave front set $\mathrm{WF}(u) \subset S^*M^\circ$, this states that

$$\mathrm{WF}^{m_0}(u) \subset \mathrm{WF}^{m_0-1}(f) \bigcup (\cup_{t\geq 0}\Phi_t(\mathrm{WF}^{m_0-1}(f)\cap\Sigma)) \tag{3.14}$$

where Φ_t is the time t Hamilton flow (on the cosphere bundle). In particular this applies to $u = L_{+-}^{-1}f$ when L_{+-} is actually invertible, so within the characteristic set the wave front set of u is a subset of the forward flowout of that of f.

There are analogous conclusions for the other choices of signs in (2.23) with the wavefront sets of solutions contained in the direction of the Hamilton flowout of the wavefront set of f corresponding to the choice of direction on each component of the characteristic set. In particular, for the $-+$ sign, $\cup_{t\geq 0}\Phi_t(\mathrm{WF}^{m_0-1}(f)\cap\Sigma)$ is replaced by $\cup_{t\leq 0}\Phi_t(\mathrm{WF}^{m_0-1}(f)\cap\Sigma)$.

Further, it is not hard to show that, provided $L_{\pm\pm}^{-1}$ exists, the Schwartz kernel $K_{\pm\pm}$ of $L_{\pm\pm}^{-1}$ satisfies a corresponding wave front set conclusion in $M^\circ \times M^\circ$. For instance, for L_{+-}^{-1}, $\mathrm{WF}(K_{+-})\backslash N^*\mathrm{diag}$ is contained in the forward flowout of $N^*\mathrm{diag}$, the conormal bundle of the diagonal, with respect to the Hamilton vector field in the left factor.

Proof. We wish to obtain the improvements to (2.22) in the estimates in (3.1). These estimates imply that the map in (3.13) is Fredholm. Indeed, using the fact that the containment $H_{\mathrm{b}}^{m,l} \subset H_{\mathrm{b}}^{m',l'}$ is compact provided $m' < m$ and $l' < l$, the first estimate in (3.1) shows that the map has closed range and finite dimensional kernel. Assuming that v lies in $(\mathrm{image}(L : \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l}))^\perp$, where the orthogonal complement is taken with respect to the L^2_b pairing (see (2.16)) between $H_{\mathrm{b}}^{1-m,-l}$ and $\mathcal{Y}^{m-1,l} = H_{\mathrm{b}}^{m-1,l}$, it follows that $Lv = 0$ and thus the second estimate in (3.1) shows that the space of such v is finite dimensional.

Thus we need only obtain the improved estimate in (3.1). The proof is essentially the proof of [27, Proposition 2.3], and we recall it briefly for the convenience of the reader. The condition on $\hat{N}(L)(\sigma)^{-1}$ on the line $\mathrm{Im}(\sigma) = -l$ implies by taking the inverse Mellin transform that the map

$$N(L)\colon \mathcal{X}^{m',l}(\partial M \times \mathbb{R}_+) \longrightarrow \mathcal{Y}^{m'-1,l}(\partial M \times \mathbb{R}_+) \tag{3.15}$$

is bounded and invertible, *where $m'\colon T^*\partial M \longrightarrow \mathbb{R}$ is any function satisfying the constraints in* (2.23) *that m satisfies.* Thus there is a C such that

$$\|u\|_{H_{\mathrm{b}}^{m',l}(\partial M\times\mathbb{R}_+)} \leq C\|N(L)u\|_{H_{\mathrm{b}}^{m'-1,l}(\partial M\times\mathbb{R}_+)},$$

and we may furthermore choose m' so that it satisfies the constraint *and* that $m' < m$. Choosing a cutoff function χ that is supported near ∂M and equal to 1 in a neighborhood thereof, we have (with a constant whose value changes from line to line)

$$\begin{aligned}\|u\|_{H_{\mathrm{b}}^{m,l}(M)} &\leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}(M)} + \|u\|_{H_{\mathrm{b}}^{m',l}(M)}) \\ &\leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}(M)} + \|\chi u\|_{H_{\mathrm{b}}^{m',l}(M)} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{m',l}(M)}) \\ &\leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}(M)} + \|N(L)\chi u\|_{H_{\mathrm{b}}^{m'-1,l}(M)} + \|u\|_{H_{\mathrm{b}}^{m',l'}(M)}).\end{aligned}$$

Now, writing $N(L)\chi = [N(L), \chi] + \chi(N(L) - L) + \chi L$, and using $N(L) - L = \rho P$ where $P \in \mathrm{Diff}_{\mathrm{b}}^2(M)$, and $[N(L), \chi] = \rho P'$ where $P' \in \mathrm{Diff}_{\mathrm{b}}^1(M)$, note that

$$\|u\|_{H_{\mathrm{b}}^{m,l}(M)} \leq C(\|Lu\|_{H_{\mathrm{b}}^{m-1,l}(M)} + \|u\|_{H_{\mathrm{b}}^{m'+1,l'}(M)}),$$

so to obtain the improved estimate in (3.1) we need only make sure that $m'+1 < m$ which can be done due to the $-(m+l-3/2) < 0$ assumption at appropriate radial sets. □

It is important to remark here that $L_{\pm\pm}$ are rather different operators for different choices of $\pm\pm$, on the other hand the choice of m, l *satisfying the constraints* corresponding to a given $\pm$ (i.e. a given one of the two constraints) matter much less. For example, the invertibility of the normal operator $\hat{N}(L)(\sigma)$ is *independent* of these additional choices, so long as the m satisfies that $m - \mathrm{Im}\,\sigma - 1/2$ has the correct sign at the relevant locations and has the correct monotonicity. In the Feynman case see Proposition 4.7 below; the regularity theory shows that the potential kernel of the operator, as well as of the adjoint, is indeed independent of these choices. The choice of l does affect the index of L, however as a Fredholm operator, as we show for the Feynman operator in Theorem 4.3.

We also note that the adjoint of L_{++} is L_{--}, while that of L_{+-} is L_{-+}, so one should *not* think of L as a self-adjoint operator even though it is of course formally self-adjoint.

The standard setting in which $\Box_g$ is considered is that of evolutionary problems, in which the forward or backward propagator L_{++}^{-1} and L_{--}^{-1} are considered. On the other hand, the Feynman propagator arises for instance by Wick-rotating suitable Riemannian problems. Here we are interested in the Feynman propagator, but we first explain the more studied forward and backward problems in order to be able to contrast these.

For the forward or backward problems the usual tools of evolutionary problems, namely standard energy estimates, can be used to compute the index in some cases, as discussed in [27, Theorem 5.2]. Since in this paper we focus on the Feynman propagator, we shall be brief. Thus, recall first from [2, Section 3.2.1] that the Lorentzian scattering metric g in fact induces (even) asymptotically hyperbolic metrics k_+, resp. k_-, on C_+, resp. C_-, with S_+, resp. S_- being conformal infinity for these. Similarly, an asymptotically de Sitter Lorentzian metric is induced on C_0, for which S_+ is future and S_- is past conformal infinity. One thus can consider the spectral family $\Delta_{C_\pm} - \frac{(n-2)^2}{4} - \sigma^2$, as well as its inverse $\mathcal{R}_{C_\pm}(\sigma)$ for $\mathrm{Im}\,\sigma \gg 0$, which continues meromorphically to the complex plane (in σ). Then, as discussed in [2, Section 7] and more systematically in [48], for the forward problem, the poles of $\hat{N}(L)(\sigma)^{-1}$ consist of the poles of the meromorphically continued resolvents $\mathcal{R}_{C_+}(\sigma)$ (i.e. its resonances) and $\mathcal{R}_{C_-}(-\sigma)$ on the asymptotically hyperbolic caps $C_\pm$, as well as possibly a subset of $i\mathbb{Z}\setminus\{0\}$. (The latter correspond to possible differentiated delta distributional resonant states, which exist e.g. in even dimensional Minkowski space and which are responsible for the strong Huygens principle on the one hand and for the absence of poles of the meromorphically continued resolvent on odd dimensional hyperbolic spaces on the other hand.) Further, the resonant states and dual states have a certain support structure (this corresponds to C_0 being a hyperbolic region), namely for ϕ supported in $C_0 \cup \overline{C_+}$, $\hat{N}(L)(\sigma)^{-1}\phi$ can only have poles if σ is either a pole of $\mathcal{R}_{C_+}(\sigma)$ or is in $-i\mathbb{N}^+$, see [2, 48]. Thus, see

[27], suppose that $|l| < 1$ (one could take l larger if one also excludes the possible imaginary integer poles of $\hat{N}(L)(\sigma)^{-1}$), and $\mathcal{R}_{C_\pm}(\sigma)$ have no poles in $\operatorname{Im}\sigma \geq -|l|$, and that there is a boundary-defining function ρ which is globally time-like (in the sense that $\frac{d\rho}{\rho}$ is such with respect to $\hat{g}$) near $\overline{C_+} \cup \overline{C_-}$. (These assumptions hold e.g. on perturbations of Minkowski space.) Then any element of $\operatorname{Ker} L$ would be vanishing to infinite order at $\overline{C_-}$ (and the same for $\operatorname{Ker} L^*$, where L^* is the adjoint of L with respect to the $L^2_{\mathrm{b}} = L^2(\mathbb{R}^n, \mu)$ pairing in (2.16), with C_- replaced by C_+) by the first hypothesis and vanishing in a neighborhood of $\overline{C_-}$ by the second. Finally, a result of Geroch's [21] (relying on a construction of Hawking's) shows that M is globally hyperbolic (there is a Cauchy surface for which every timelike curve intersects it exactly one time) under these assumptions, and in particular L_{++} and L_{--} are invertible since any element of $\operatorname{Ker} L_{++}$ would vanish globally, and similarly for elements of $\operatorname{Ker} L^*_{++}$. One can then use the relative index theorem of Melrose [38, Chapter 6] to compute the index of L on other weighted spaces.

For the Feynman propagator there is no simple direct identification of the poles of $\hat{N}(L)(\sigma)$. However, in Minkowski space, one can compute these exactly by virtue of a Wick rotation (Proposition 4.7), and further even show the invertibility of L on appropriate weighted spaces (Theorem 3.6). Namely, the poles of $\hat{N}(L)(\sigma)$ are exactly those values of σ for which the operator $\Delta_{\mathbb{S}^{n-1}} + (n-2)^2/4 + \sigma^2$ is *not* invertible, i.e. σ is of the form $\pm i\sqrt{\lambda + (n-2)^2/4}$, λ an eigenvalue of $\Delta_{\mathbb{S}^{n-1}}$, i.e. $\lambda = k(k+n-2)$, $k \in \mathbb{N}$, so $\lambda + (n-2)^2/4 = (k+(n-2)/2)^2$, and thus $\sigma = \pm i(\frac{n-2}{2}+k)$. For future reference, we define

$$\Lambda = \left\{ \pm(\frac{n-2}{2} + k) : k \in \mathbb{N}_0 \right\} \tag{3.16}$$

This gives a gap between the two strings of poles with positive and negative imaginary parts, and for $|l| < \frac{n-2}{2}$, L_{+-} and L_{-+} are invertible (and are adjoints of each other on dual spaces). Since the framework we set up is stable under *general b-ps.d.o. perturbations*, we conclude that for general sc-metric perturbations g of the Minkowski metric g_0, $L_{g,+-}$ and $L_{g,-+}$ have the same properties, provided the $|l|$ is taken slightly smaller:

Theorem 3.6. *Let $\delta \in (0, \frac{n-2}{2})$. Then there exists a neighborhood $\mathcal{U}$ of the Minkowski metric g_0 in $\mathcal{C}^\infty(M; \mathrm{Sym}^2\, {}^{\mathrm{sc}}T^*M)$ (i.e. in the sense of sc-metrics, see* (2.1) *and the paragraph following it) such that for $g \in \mathcal{U}$,*

$$L_{g,+-} : \mathcal{X}^{m,l}_{+-} \longrightarrow \mathcal{Y}^{m-1,l}_{+-}, \tag{3.17}$$

with $\mathcal{X}^{m,l}_{+-}, \mathcal{Y}^{m-1,l}_{+-}$ as in (2.25)*, is invertible for $|l| < \frac{n-2}{2} - \delta$ and m satisfying the forward Feynman condition for $+-$ in* (2.23)*, strengthened as in Theorem 3.3, and where $\mathcal{X}^{m,l}_{+-}$ is the domain of the Feynman wave operator defined in* (2.25)*. The same is true for $L_{g,-+}$ with $+-$ replaced by $-+$ in all the spaces.*

Proof. For the actual Minkowski metric g_0, the invertibility is a restatement of Theorem 4.6 below. Since the estimates in (3.1) hold uniformly on a sufficiently small neighborhood $\mathcal{U}'$ of g_0, $L_{g,+-}$ defines a continuous bounded family mapping as in (3.17), and thus is invertible on a possibly smaller neighborhood $\mathcal{U}$. □

Taking into account the construction of L (see (2.6)), for metrics g in the neighborhood $\mathcal{U}$ in the theorem, we deduce that

$$\Box_{g,+-} : \mathcal{X}_{+-}^{m,l+\frac{n-2}{2}} \to \mathcal{Y}_{+-}^{m,l+\frac{n-2}{2}+2} \tag{3.18}$$

is invertible for $|l| < \frac{n-2}{2} - \delta$. Its inverse is indeed the forward Feynman propagator (which is well defined on space $\mathcal{Y}_{+-}^{m,l+\frac{n-2}{2}+2}$ with weight l in the stated range,

$$\Box_{g,fey}^{-1} : \mathcal{Y}_{+-}^{m,l+\frac{n-2}{2}} \to \mathcal{X}_{+-}^{m,l+\frac{n-2}{2}+2}. \tag{3.19}$$

The same for $+-$ replaced by $-+$ and "forward" replaced by "backward".

Remark 3.7. The class of perturbations we consider *does not* preserve the radial point structure at ${}^{\mathrm{b}}SN^*S_\pm$. Nonetheless, the *estimates* the radial point structure implies for L and L^* are preserved, much as discussed for Kerr-de Sitter spaces in [47].

4. Wick rotation (complex scaling)

In this section we work only with the Minkowski metric, which we continue to denote by g. We now explain Wick rotations in Minkowski space, where it amounts to replacing $\Box_g = D_{z_n}^2 - D_{z_1}^2 - \ldots - D_{z_{n-1}}^2$ by

$$\Box_{g,\theta} = e^{-2\theta} D_{z_n}^2 - D_{z_1}^2 - \ldots - D_{z_{n-1}}^2 \tag{4.1}$$

where θ is a complex parameter. While it may seem that we are using rather sophisticated techniques for a simple operator, this is in some ways necessary since we need invertibility on our variable order function spaces, which would not be so easy to show using very simple techniques!

Concretely, consider complex scaling, corresponding to pull-back by the diffeomorphism $\Phi_\theta(z) = (z_1, \ldots, z_{n-1}, e^\theta z_n)$ for $\theta \in \mathbb{R}$, i.e. considering $U_\theta^* \Box (U_\theta^{-1})^*$, where $U_\theta = (\det D\Phi_\theta)^{1/2} \Phi_\theta^* f$, extending the result to an analytic family of operators in $\theta \in \mathbb{C}$ (near the reals). This gives rise to the family $\Box_{g,\theta}$. Letting

$$L_\theta = \rho^{-(n-2)/2} \rho^{-2} \Box_{g,\theta} \rho^{(n-2)/2}, \tag{4.2}$$

as soon as $\operatorname{Im}\theta \in (-\pi, \pi) \setminus \{0\}$, L_θ is a an elliptic b-differential operator; when $\theta = \pm i\pi/2$, one obtains the Euclidean Laplacian $\Box_{g,\pm i\pi/2} = \Delta_{\mathbb{R}^n}$. In the elliptic region the corresponding operator L_θ satisfies the Fredholm estimates uniformly for $L_{\theta,+-}$ (and its adjoint, for which the imaginary part switches sign, but one propagates estimates backwards) when $\operatorname{Im}\theta \geq 0$, and for $L_{\theta,-+}$ when $\operatorname{Im}\theta \leq 0$.

The main analytic property that we will use below for the operators L_θ is that for regularity functions m chosen to satisfy say the forward ($+-$) Feynman condition, the corresponding operators $L_{\theta,+-}$ satisfy estimates

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \leq C(\|L_\theta u\|_{H_{\mathrm{b}}^{m-1,l}} + \|u\|_{H_{\mathrm{b}}^{m',l'}}). \tag{4.3}$$

uniformly in θ for m, l corresponding to $+-$ and $m' < m$, $l' < l$, meaning precisely that there is a constant C such that for $|\theta| < \delta_0, \operatorname{Im}\theta \geq 0$ for $u \in H_{\mathrm{b}}^{m,l}$, (4.3) holds provided m, l satisfy the $+-$ Feynman condition and $-l \notin \Lambda$. For $|\theta| < \delta_0, \operatorname{Im}\theta \leq 0$ they hold provided m, l satisfy the $-+$ Feynman condition and $l \notin \Lambda$. (Note that $\Lambda = -\Lambda$ so actually the conditions on l are the same.) The reason for the uniformity is that all of the ingredients are uniform; this is standard for elliptic estimates. On the other hand, it holds for real principal type estimates where the imaginary part of the principal symbol amounts to complex absorption, provided one propagates estimates in the *forward* direction of the Hamilton flow if the imaginary part of the principal symbol is ≤ 0 (which is the case for $\operatorname{Im}\theta \geq 0$, θ small) and backwards along the Hamilton flow if the imaginary part of the principal symbol is ≥ 0, as shown by Nonnenmacher and Zworski [42] and Datchev and Vasy [13] in the semiclassical microlocal setting and, as is directly relevant here, extended to the general b-setting by Hintz and Vasy [27, Section 2.1.2]. Moreover, at radial points in the standard microlocal setting this was shown by Haber and Vasy [23], and the proof of Proposition 2.1 can be easily modified in the same manner so that non-real principal symbol is also allowed at the b-radial points. Finally, the normal operator constructions are also uniform since they rely on estimates for the Mellin transformed family which are uniform as we stated; the resonances (poles) of the inverse of this family thus a priori vary continuously, so in particular near an invertible weight for $\theta = 0$ one has uniform estimates. (In fact we will show in Proposition 4.7 below that the poles of the complex scaled normal families are constant, i.e. do not vary with θ.)

Note that the estimates in (4.3) are *not* the standard elliptic estimates. Indeed, the term on the left hand side is in a space of differentiability order one lower than ellipticity provides. The point is that the estimates in (4.3) are exactly those which are uniform down to $\operatorname{Im}\theta = 0$.

The family of operators L_θ defines a family of Mellin transformed normal operators on the boundary, $\hat{N}(L_\theta)(\sigma)$ as above, and we have, still for g equal to the Minkowski metric, that

$$\hat{N}(L_{\pm i\pi/2})(\sigma) = \Delta_{\mathbb{S}^{n-1}} + (n-2)^2/4 + \sigma^2. \tag{4.4}$$

We recall the theorem of Melrose describing the behavior of the elliptic operators L_θ for $\operatorname{Im}\theta \neq 0$, which is a special case of our more general framework in that elliptic operators are also Fredholm on variable order Sobolev spaces in view of our results.

Theorem 4.1 (Melrose [38], with Theorem 3.3 here giving the variable order version)**.** *Let P be an elliptic b-differential operator of order k on a manifold with boundary M, and assume that $\hat{N}(P)^{-1}(\sigma)$ has no poles on the line $\operatorname{Im}\sigma = -l$. Then the operator P satisfies*

$$\|u\|_{H_{\mathrm{b}}^{s+k,l}} \leq C(\|Pu\|_{H_{\mathrm{b}}^{s,l}} + \|u\|_{H_{\mathrm{b}}^{-N,l'}}), \tag{4.5}$$

for any $N > 0$ and some $l' < l$. In particular,

$$P\colon H_{\mathrm{b}}^{s+k,l} \longrightarrow H_{\mathrm{b}}^{s,l}$$

is Fredholm.

Thus the set Λ in (3.16) gives the set of weights l for which

$$\Delta_{\mathbb{R}^n}\colon H_{\mathrm{b}}^{m+1,l+(n-2)/2} \longrightarrow H_{\mathrm{b}}^{m-1,l+(n-2)/2+2}$$

is Fredholm; indeed by the definition of L_θ in (4.2), we see that

$$L_{i\pi/2} = \rho^{-(n-2)/2}\rho^{-2}\Delta_{\mathbb{R}^n}\rho^{(n-2)/2}\colon H_{\mathrm{b}}^{m+1,l} \longrightarrow H_{\mathrm{b}}^{m-1,l}$$

is Fredholm exactly when $-l \notin \Lambda$. Consider the elliptic operators L_θ as maps between forward Feynman b-Sobolev spaces

$$L_{\theta,+-}\colon H_{\mathrm{b}}^{m+1,l} \longrightarrow H_{\mathrm{b}}^{m-1,l}. \tag{4.6}$$

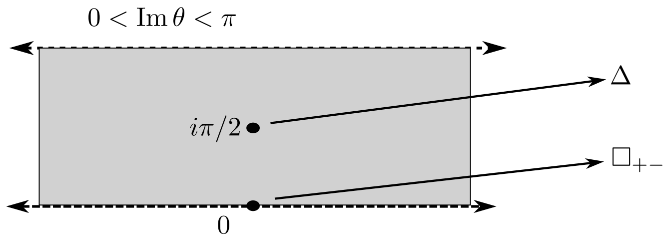

FIGURE 4. The index of L_θ is constant in the grey region by the standard continuity of the index for Fredholm families. In Theorem 4.3 we show the continuity of the index extends up to the dashed line at bottom, i.e. to L_{+-}.

In Section 4.3 below, we will prove in Proposition 4.7 that for the $\theta-$dependent family of Mellin transformed normal operators of the complex scaled Feynman operators, $\hat{N}(L_{\theta,+-})(\sigma)$, *the inverse families have equal poles.* Thus the set Λ in (3.16) is in fact the set of all poles of the inverse families $\hat{N}(L_{\theta,+-})$ in the forward Feynman setting. The same holds for $-+$. As a corollary to Theorem 4.1 and the fact that the index of a continuous family of Fredholm operators is constant (see [44]), we obtain the following:

Lemma 4.2. *For Λ as in* (3.16) *and $-l \notin \Lambda$, the maps in* (4.6) *form a continuous Fredholm family and thus have constant index for $\theta \in (0, \pi/2]$.*

4.1. **Index of $L_{\theta,+-}$.** To prove the invertibility theorem, we will first establish the following

Theorem 4.3. *For fixed m, l satisfying the forward Feynman condition in L_{+-}, and such that $l \notin \Lambda$, for $\operatorname{Im}\,\theta \in (0, \pi/2)$,*

$$\operatorname{Index}(L_\theta \colon H_{\mathrm{b}}^{m+1,l} \longrightarrow H_{\mathrm{b}}^{m-1,l}) = \operatorname{Index}(L_{+-} \colon \mathcal{X}^{m,l} \longrightarrow \mathcal{Y}^{m-1,l}). \tag{4.7}$$

Proof. This follows from the mere fact that the estimates in (4.3) hold uniformly in θ for m, l corresponding to $+-$ and $m' < m$, $l' < l$.

Assume first that L_{+-} on the right hand side of (4.7) is invertible. Then one can drop the compact error terms, and thus then the estimates take the form

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \leq C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}}, \ \|v\|_{H_{\mathrm{b}}^{1-m,-l}} \leq C\|L^* v\|_{H_{\mathrm{b}}^{-m,-l}}, \tag{4.8}$$

where again L^* is the adjoint of L with respect to the L^2_{b} pairing (see (2.16)). To see that the estimate on the right follows, since $H_{\mathrm{b}}^{-m,-l}$ is dual to $H_{\mathrm{b}}^{m,l}$ with respect to the L^2_{b} pairing, using the surjectivity of L to go to the second line we have

$$\begin{aligned}\|L^* v\|_{H_{\mathrm{b}}^{-m,-l}} &= \sup_{\|w\|_{H_{\mathrm{b}}^{m,l}}=1} \langle L^* v, w\rangle_{L^2_{\mathrm{b}}} \geq \sup_{\|w\|_{\mathcal{X}^{m,l}}=1} \langle v, Lw\rangle_{L^2_{\mathrm{b}}} \\ &\geq \frac{1}{C} \sup_{\|g\|_{H_{\mathrm{b}}^{m-1,l}}} \langle v, g\rangle_{L^2_{\mathrm{b}}} = \frac{1}{C}\|v\|_{H_{\mathrm{b}}^{1-m,-l}}.\end{aligned}$$

We claim that the estimates in (4.8) imply the analogous estimates also hold for L_θ, $\operatorname{Im}\theta$ small with $\operatorname{Im}\theta > 0$, namely that

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \leq C\|L_\theta u\|_{H_{\mathrm{b}}^{m-1,l}}, \ \|v\|_{H_{\mathrm{b}}^{1-m,-l}} \leq C\|L_\theta^* v\|_{H_{\mathrm{b}}^{-m,-l}}. \tag{4.9}$$

Otherwise, for example for the first estimate, we would have a sequence $\theta_j \to 0$ with $\operatorname{Im}\theta_j > 0$ and u_j with

$$\|u_j\|_{H_{\mathrm{b}}^{m,l}} = 1 \text{ and } L_{\theta_j} u_j \to 0 \text{ in } H_{\mathrm{b}}^{m-1,l}.$$

Extracting a strongly convergent subsequence of the u_j in $H_{\mathrm{b}}^{m',l'}$ for $m' < m$ and $l' < l$, by the uniform estimates in (4.3) we would obtain a limit $\widetilde{u}$ with $\widetilde{u} \neq 0$ and $L\widetilde{u} = 0$, a contradiction. A similar argument shows that the second estimate also holds for small θ with $\operatorname{Im}\theta > 0$.

Now as soon as $\operatorname{Im}\theta \neq 0$, these give improved estimates by elliptic regularity, namely

$$\|u\|_{H_{\mathrm{b}}^{m+1,l}} \leq C\|L_\theta u\|_{H_{\mathrm{b}}^{m-1,l}},\ \|v\|_{H_{\mathrm{b}}^{2-m,-l}} \leq C\|L_\theta^* v\|_{H_{\mathrm{b}}^{-m,-l}}.$$

Indeed these follow since for $\operatorname{Im}\theta > 0$, L_θ and L_θ^* are Fredholm maps from $H_{\mathrm{b}}^{m'+1,l}$ to $H_{\mathrm{b}}^{m'-1,l}$ for any m' and by (4.9) are injective for the given m and l and thus for any m' by elliptic regularity. Thus, for example taking $m = s$ to be constant in the first inequality and $m = -s + 1$ in the second inequality gives that L_θ is injective and surjective with domain $H_{\mathrm{b}}^{m,l}$ (which again by elliptic regularity means that L_θ is an isomorphism for any m and the given l). This establishes the theorem in the case that L_{+-} is invertible on the spaces under consideration.

If L_{+-} is not invertible but is Fredholm, one can get back to the same setting by adding finite dimensional function spaces to the domain and target as usual, showing that the index is stable under this deformation. Concretely, let

$$\begin{aligned} V &:= \operatorname{Ker}(L_{+-} \colon \mathcal{X}^{m,l} \longrightarrow H_{\mathrm{b}}^{m-1,l}) \\ W &:= \operatorname{Coker}(L_{+-} \colon \mathcal{X}^{m,l} \longrightarrow H_{\mathrm{b}}^{m-1,l}), \end{aligned}$$

where by definition the cokernel in the second line is the orthogonal complement of the range with respect to some (fixed) inner product. The map $\widetilde{L}_\theta$ from $W \oplus \mathcal{X}^{m,l} = W \oplus V \oplus V^\perp$ to $V \oplus H_{\mathrm{b}}^{m-1,l} = V \oplus W \oplus W^\perp$ which takes $w + v + v'$ to $v + w + L_\theta v'$ is an isomorphism for $\theta = 0$, and by the above analysis is also an isomorphism for θ small with $\operatorname{Im}\theta > 0$. Therefore the Fredholm index of the Feynman propagators for Minkowski space is the same as that of $\Delta_{\mathbb{R}^n}$ acting on a weighted b-space with the same weight. □

We can then use the relative index theorem of Melrose [38, Chapter 6], which expresses the difference in the index of a b-differential operator at different weights as the sum of the residues of the normal operator at appropriate indicial roots. This can be extended from the elliptic setting considered there to ours without any difficulties, to compute the index of L on other weighted spaces; here in fact because of the Wick rotation we can use the elliptic result directly.

Corollary 4.4. *Under assumptions as in Lemma 4.2,*

$$\operatorname{Index}(L_{\theta,+-} \colon \mathcal{X}^{m,l} \longrightarrow \mathcal{Y}^{m-1,l}) = -\operatorname{sgn}(l) N(\Delta_{\mathbb{S}^{n-1}} + (n-2)^2/4; l), \tag{4.10}$$

where $N(\Delta_{\mathbb{S}^{n-1}} + (n-2)^2/4; l)$ is the number of eigenvalues λ of $\Delta_{\mathbb{S}^{n-1}} + (n-2)^2/4$ with $\lambda < l^2$. In particular,

$$|l| < (n-2)/2 \implies \operatorname{Index}(L_{g,+-} \colon \mathcal{X}^{m,l} \longrightarrow \mathcal{Y}^{m-1,l}) = 0.$$

Proof. By Theorem 4.3, we have that

$$\begin{aligned}&\mathrm{Index}(L_{\theta,+-}\colon \mathcal{X}^{m,l} \longrightarrow \mathcal{Y}^{m-1,l})\\ &\quad= \mathrm{Index}(L_{i\pi/2}\colon H_{\mathrm{b}}^{m+1,l} \longrightarrow H_{\mathrm{b}}^{m-1,l})\\ &\quad= \mathrm{Index}(\Delta_{\mathbb{R}^n}\colon H_{\mathrm{b}}^{m+1,l+(n-2)/2} \longrightarrow H_{\mathrm{b}}^{m-1,l+(n-2)/2+2}),\end{aligned}$$

and the latter was computed by Melrose, see [38, Section 6.2], or the interpretation in [22, Theorem 2.1] where it is shown to be exactly the right hand side of (4.10). ☐

4.2. **Invertibility of the Feynman problem for $\Box_{g,\theta}$ down to $\theta = 0$.** It follows from Theorem 4.1 and (4.4), together with the spectral theory of the sphere discussed above, that

$$\Delta_{\mathbb{R}^n}\colon H_{\mathrm{b}}^{m+1,l+(n-2)/2} \longrightarrow H_{\mathrm{b}}^{m-1,l+(n-2)/2+2}$$

is Fredholm as long as $-l \notin \Lambda$ where Λ is defined in (3.16). In fact, we have

Theorem 4.5. *The map $\Delta_{\mathbb{R}^n}\colon H_{\mathrm{b}}^{m+1,l+(n-2)/2} \longrightarrow H_{\mathrm{b}}^{m-1,l+(n-2)/2+2}$ is invertible provided $|l| < (n-2)/2$, $m \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*\overline{\mathbb{R}^n})$.*

Proof. This is shown in the proof of [7, Lemma 3.2], for $m \in \mathbb{R}$. Indeed, they show using the maximum principle and elliptic regularity that there can be no nullspace of Δ in $H_{\mathrm{b}}^{m,l}$ for any $l > 0$ (and the same must be true for the formal adjoint), from which the result follows since the operator is Fredholm. Our results give the general Fredholm statement for arbitrary $m \in \mathcal{C}^\infty(S^*\overline{\mathbb{R}^n})$, and elliptic regularity then gives that any element of the kernel is in $H_{\mathrm{b}}^{\infty,l+(n-2)/2}$, with an analogous statement for the cokernel, and these are trivial in turn by the constant m result. ☐

Consider the map

$$\Box_{g,\theta}\colon \mathcal{X}_{+-}^{m,(n-2)/2+l}(\theta) \longrightarrow H_{\mathrm{b}}^{m-1,(n-2)/2+l+2} \tag{4.11}$$

where $\mathcal{X}_{+-}^{m,l}(\theta) = \{u \in H_{\mathrm{b}}^{m,l} : \Box_{g,\theta} u \in H_{\mathrm{b}}^{m-1,l+2}\}$ is a θ-dependent space with the graph norm,

$$\|u\|^2_{\mathcal{X}^{m,l}_{+,-}(\theta)} = \|u\|^2_{H_{\mathrm{b}}^{m,l}} + \|\Box_{g,\theta} u\|^2_{H_{\mathrm{b}}^{m-1,l+2}},$$

so by the elliptic estimates discussed above,

$$\mathcal{X}_{+-}^{m,l}(\theta) = \begin{cases} \mathcal{X}_{+-}^{m,l+(n-2)/2} & \text{if} \quad \theta \in \mathbb{R} \\ H_{\mathrm{b}}^{m+1,l+(n-2)/2} & \text{if} \quad \operatorname{Im}\theta \in (0,\pi) \end{cases},$$

with the equivalence of norms uniform for θ in compact subsets of $\mathbb{R} \times (0,\pi)$. (Here the $+-$ is just to remind us that $m+l$ satisfies the conditions corresponding to $L_{g,+-}$, although this makes no difference in the elliptic region.) We will now study the set

$$\mathcal{D}_l = \{\theta : \operatorname{Im}(\theta) \in [0,\pi/2] \text{ and } \Box_{g,\theta} \text{ mapping as in (4.11) is invertible.}\}$$

We see that for $|l| < (n-2)/2$, $\mathcal{D}_l$ contains $i\pi/2$ and is thus non-empty.

Theorem 4.6. *Let $|l| < (n-2)/2$. The set $\mathcal{D}_l$ contains the entire closed strip $\{\operatorname{Im}\theta \in [0,\pi/2]\}$. In particular $\Box_{g,+-}$ mapping as in* (3.18) *is invertible for g equal to the Minkowski metric and $|l| < (n-2)/2$.*

We will prove Theorem 4.6 by arguing along lines similar to those in [36, 37], which in turn follow the development in [28].

Proof of Theorem 4.6. We will define a subspace $\mathcal{A} \subset L^2 = L^2(\mathbb{R}^n)$ of so-called analytic vectors and a family of maps

$$U_\theta \colon \mathcal{A} \longrightarrow L^2, \tag{4.12}$$

for θ in an open neighborhood $\mathcal{D} \subset \mathbb{C}$ of 0 with the following properties:

(i) For $\theta \in \mathbb{R}$, U_θ is unitary on L^2.

(ii) For $f \in \mathcal{A}$ and $\theta \in \mathcal{D}$,

$$U_\theta \Box_{g,\theta_0} U_\theta^{-1} f = \Box_{g,\theta+\theta_0} f.$$

In particular, U_θ is injective and $\mathcal{A}$ is in the range of U_θ for $\theta \in \mathcal{D}$.

(iii) $U_\theta \mathcal{A}$ is dense in $H_{\mathrm{b}}^{m,l}$ for all $\theta \in \mathcal{D}$ and any $m \colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$, $l \in \mathbb{R}$.

We will then leverage the properties of $\mathcal{A}$ and U_θ to prove Theorem 4.6 as follows. Recall that, by Theorem 4.3, $\Box_{g,\theta}$ as in (4.11) is a Fredholm map of index zero. Since it is invertible for $\theta' = i\pi/2$, it is invertible for θ near θ'. It follows by the analytic Fredholm theorem that

$$\Box_{g,\theta}^{-1} \colon H_{\mathrm{b},+-}^{m-1,l} \longrightarrow H_{\mathrm{b},+-}^{m+1,l} \tag{4.13}$$

extends to a meromorphic family of operators in the strip $\{0 < \operatorname{Im}\theta < \pi\}$ with finite rank poles. In particular, if $\widetilde{\theta}$ is a putative pole, then for θ near 0,

$$\Box_{g,\widetilde{\theta}+\theta}^{-1} = \sum_{j=-N}^{-1} A_j \theta^j + M_\theta, \text{ where } M_\theta \text{ is holomorphic.} \tag{4.14}$$

Thus if $\widetilde{\theta}$ is indeed a pole, by the density of $\mathcal{A}$ we may choose f, h such that, e.g. $\langle f, A_1 h\rangle \neq 0$ and thus $\langle f, \Box_{g,\widetilde{\theta}+\theta}^{-1} h\rangle$ has a pole at $\theta = \widetilde{\theta}$. On the other hand the matrix elements satisfy

$$\langle f, \Box_{g,\widetilde{\theta}+\theta}^{-1} h\rangle_{L^2} = \langle U_{\overline{\theta}} f, \Box_{g,\widetilde{\theta}}^{-1} U_\theta^{-1} h\rangle. \tag{4.15}$$

We will see that for $h \in \mathcal{A}$, both $U_\theta h$ and $U_\theta^{-1} h$ are analytic for $\theta \in \mathcal{D}$, so the matrix elements of $\Box_{g,\widetilde{\theta}+\theta}^{-1}$ are analytic functions for $\theta \in \mathcal{D}$, and thus $\Box_{g,\widetilde{\theta}+\theta}^{-1}$ has no poles in $0 < \operatorname{Im}\theta \leq \pi/2$.

We have proven that $\Box_{g,\theta}$ is invertible only for those θ with $0 < \operatorname{Im}\theta < \pi$. Since $\Box_{g,\theta}$ is not strictly speaking an analytic Fredholm family on an open set containing $\theta = 0$ (the domain changes according to whether $\operatorname{Im}\theta = 0$ or not and $\theta = 0$ lies on the boundary of $0 \leq \operatorname{Im}\theta \leq \pi/2$) we need a different argument there. Pick θ_0 close enough to 0 so that the density statements for $U_\theta \mathcal{A}$ hold on an open set including $\theta = -\theta_0$. Assuming for contradiction that $\Box_{g,+-}$ is not invertible for l in the given range, it will suffice to construct elements $f, h \in \mathcal{A}$ such that

$$\langle f, \Box_{g,\theta_0+\theta}^{-1} h\rangle_{L^2} \text{ diverges as } \theta_0 + \theta \to 0 \text{ in } \operatorname{Im}(\theta + \theta_0) > 0, \tag{4.16}$$

since then by (4.15) with $\widetilde{\theta} = \theta_0$ we will have a contradiction. Note that by this assumption there exists u_0 lying in $H_{\mathrm{b},+-}^{m-1,l+(n-2)/2+2}$ such that

$$u_0 \notin \operatorname{Ran}(\Box_{g,+-} : \mathcal{X}_{+-}^{m,l+(n-2)/2} \longrightarrow H_{\mathrm{b}}^{m-1,l+(n-2)/2+2})$$

since by Theorem 4.3 the map has index zero. Since (4.11) is Fredholm and $\mathcal{A}$ is dense, we may instead choose $h \in \mathcal{A}$ such that also $h \notin \operatorname{Ran}(\Box_{g,0})$. Using the invertibility proved above, for $\mathrm{Im}(\theta_0+\theta)>0$, we consider $\Box_{g,\theta_0+\theta}^{-1}h \in H_{\mathrm{b}}^{m+1,l} \subset \mathcal{X}_b^{m,l+(n-2)/2}$. We claim that

$$\|\Box_{g,\theta_0+\theta}^{-1}h\|_{\mathcal{X}_{+-}^{m,l+(n-2)/2}} \text{ diverges as } \theta+\theta_0 \to 0$$

Indeed, otherwise $\Box_{g,\theta_0+\theta}^{-1}h$ converges subsequentially to some $u \in \mathcal{X}_b^{m,l}$ weakly, and by a standard argument we must have $L_0 u = h$, which is impossible by assumption.

Note that this does not guarantee that (4.16) holds for any $f \in \mathcal{A}$; this requires a further argument. To see this, we use the uniform Fredholm estimates in (4.3), which in terms of $\Box_{g,\theta_0+\theta}$ and applied to $\Box_{g,\theta_0+\theta}^{-1}h$ take the form

$$\begin{aligned}\|\Box_{g,\theta_0+\theta}^{-1}h\|_{H_{\mathrm{b}}^{m,l+(n-2)/2}} \le C(&\|h\|_{H_{\mathrm{b}}^{m-1,l+(n-2)/2+2}} \\ &+ \|\Box_{g,\theta_0+\theta}^{-1}h\|_{H_{\mathrm{b}}^{m',l'+(n-2)/2}}),\end{aligned} \tag{4.17}$$

where $m' < m$ and $l' < l$. Letting θ_j be a sequence with $\theta_j \to -\theta_0$, let $c_j = \|\Box_{g,\theta_0+\theta_j}^{-1}h\|_{H_{\mathrm{b}}^{m,l+(n-2)/2}}$, and let

$$u_j = c_j^{-1}\Box_{g,\theta_0+\theta_j}^{-1}h.$$

By (4.17) and the compact containment of $H_{\mathrm{b}}^{s,\ell} \subset H_{\mathrm{b}}^{s',\ell'}$ when $s' < s$ and $\ell' < \ell$, the u_j converge subsequentially (dropped from the notation) to a non-zero element $u \in H_{\mathrm{b}}^{m,l+(n-2)/2}$. It follows that

$$\langle \Box_{g,\theta_0+\theta_j}^{-1}h, u\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} = c_j(1+o(1)),$$

where $o(1) \to 0$ as $j \to \infty$. We claim that there is a $\delta_0 > 0$ such that for any $\widetilde{f}$ with $\|\widetilde{f}-u\|_{H_{\mathrm{b}}^{m,l+(n-2)/2}} < \delta_0$, that $\langle \Box_{g,\theta_0+\theta_j}^{-1}h, \tilde{f}\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}}$ is also divergent. Indeed,

$$\begin{aligned}\langle \widetilde{f}, \Box_{g,\theta_0+\theta_j}^{-1}h\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} &= \langle u, \Box_{g,\theta_0+\theta_j}^{-1}h\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} \\ &\quad + \langle \widetilde{f}-u, \Box_{g,\theta_0+\theta_j}^{-1}h\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} \\ &\ge c_j(1+o(1)) - Cc_j\delta_0 \ge \frac{1}{2}c_j,\end{aligned} \tag{4.18}$$

for $C\delta_0 < 1/3$ and j large. This is not exactly the desired divergence in (4.16) since the inner product is not L^2. Define

$$\langle z\rangle = (z_1^2 + \cdots + z_n^2 + 1)^{1/2}, \tag{4.19}$$

and let $P \in \Psi_{\mathrm{b}}^m(\mathbb{R}^n)$ be elliptic and self-adjoint. Then (since by the paragraph below (2.16) we have $H_{\mathrm{b}}^{0,n/2} = L^2(\mathbb{R}^n)$)

$$\langle z\rangle^{l-1}(P+i)\colon H_{\mathrm{b}}^{m,l+(n-2)/2} \longrightarrow L^2(\mathbb{R}^n).$$

and we may take the $H_{\mathrm{b}}^{m,l+(n-2)/2}$ inner product to be

$$\langle u, v\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} = \langle \langle z\rangle^{l-1}(P+i)u, \langle z\rangle^{l-1}(P+i)v\rangle_{L^2}.$$

Using the density of $\mathcal{A}$ in all weighted b-Sobolev spaces, we choose

$$\widetilde{f} = (P-i)^{-1}\langle z\rangle^{-2l+2}(P+i)^{-1}f$$

for some $f \in \mathcal{A}$ such that $\widetilde{f}$ within δ_0 of u in $H_{\mathrm{b}}^{m,l+(n-2)/2}$. Thus

$$\langle \widetilde{f}, \Box_{g,\theta_0+\theta_j}^{-1}h\rangle_{H_{\mathrm{b}}^{m,l+(n-2)/2}} = \langle f, \Box_{g,\theta_0+\theta_j}^{-1}h\rangle_{L^2},$$

and (4.16) is established, which means that up to the construction of $\mathcal{A}$ and U_θ and showing that the properties claimed for them hold, the proof is complete.

It remains to define $\mathcal{A}$ and U_θ and prove that they have the properties i)-iii) stated above. Following [36], we define $\mathcal{A}$ to be the space of $f \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R})$ such that, writing $z = (z'', z_n)$ with $z'' \in \mathbb{R}^{n-1}$, we have that $f(z'', z_n)$ is the restriction to $\zeta \in \mathbb{R}$ of an entire function $f(z'', \zeta)$ which satisfies

$$\sup_{|\operatorname{Re}\zeta| < C|\operatorname{Im}\zeta|} |f(z'', \zeta)| \langle \zeta \rangle^N < +\infty, \tag{4.20}$$

for any $C, N > 0$ where $\langle \zeta \rangle = (1 + |\zeta|^2)^{1/2}$, and also assume that

$$\operatorname{supp} f(z'', \zeta) \subset K \times \mathbb{C}, \tag{4.21}$$

where $K \subset \mathbb{R}^{n-1}$ is compact. Finally, for $f \in \mathcal{A}$ let

$$U_\theta(f)(z'', z_n) := e^\theta f(z'', e^\theta z_n). \tag{4.22}$$

By the proof of [36, Proposition 3.6], for $|\operatorname{Im}\theta| < \pi/4$, $U_\theta\mathcal{A}$ is dense in $L^2 = L^2(\mathbb{R}^n, |dz|)$, where $|dz|$ denotes Lebesgue measure. Indeed, given $f \in C_c^\infty(\mathbb{R}^n)$, let

$$f_t(z'', z_n) := \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-(z_n - e^\theta y)^2/4t} e^\theta f(z'', y) dy.$$

Then the reference shows that $f_t \in \mathcal{A}$ and $U_\theta f_t \to f$ in L^2 as $t \to \infty$. Thus $U_\theta\mathcal{A}$ is dense in $L^2 = H_{\mathrm{b}}^{0,n/2}$. To see that $U_\theta\mathcal{A}$ is dense in $H_{\mathrm{b}}^{M,L}$, for any $f \in C_c^\infty$ take a sequence $U_\theta \widetilde{f}_i$ with

$$U_\theta \widetilde{f}_i \to (e^{2\theta} z_n^2 + |z''|^2 + i)^{-L-2M+n/2} (\Box_{i\pi/2+\theta} + i)^M f \in H_{\mathrm{b}}^{0,n/2},$$

and set $F_i := (e^{2\theta} z_n^2 + |z''|^2 + i)^{L+2M-n/2} (\Box_{i\pi/2+\theta} + i)^{-M} \widetilde{f}_i$. Then in fact $F_i = U_\theta f_i$, where $f_i = (z_n^2 + |z''|^2 + i)^{L+2M-n/2} (\Delta + i)^{-M} \widetilde{f}_i$ where again $\Delta = \Box_{i\pi/2}$ is the Laplacian on $\mathbb{R}^n$. Since $F_i = U_\theta f_i \to f$ in $H_b^{M,L}$, and since $H_{\mathrm{b}}^{M,L}$ is dense in $H_{\mathrm{b}}^{m,l}$ provided $M \geq L$ and $L \geq l$, the desired density is established.

□

4.3. **Complex scaling for $N(L_\theta)$.** In this section we will apply another complex scaling to the normal operators corresponding to the L_θ. Namely, let m, l be chosen for the forward Feynman problem L_{+-}, and consider the operators $L_{\theta,+-}$ defined in (4.2). Let $H^m(\partial M)$ denote the variable order Sobolev spaces obtained by restricting m to $T^*\partial M$ as described above. Consider the operators

$$\hat{N}(L_{\theta,+-})(\sigma) \colon \mathcal{X}_\theta^m(\partial M) \longrightarrow H^{m-1}(\partial M), \tag{4.23}$$

where $\mathcal{X}_\theta^m(\partial M) = \{u \in H^m(\partial M) : \hat{N}(L_\theta)(\sigma)u \in H^{m-1}(\partial M)\}$.

Proposition 4.7. *The poles of the inverse family $\hat{N}(L_{\theta,+-})(\sigma)^{-1}$ are independent of θ for $\operatorname{Im}\theta \in [0, \pi/2]$.*

Proof. As in the previous section, we wish to define a set of analytic vectors $\widetilde{\mathcal{A}} \subset L^2(\partial M)$, and a family of maps $\widetilde{U}_\theta \colon \widetilde{\mathcal{A}} \longrightarrow L^2(\partial M)$ defined for θ in an open set which we also call $\mathcal{D} \subset \mathbb{C}$, and such that conditions i), ii), and iii) below (4.12) above hold. The Proposition then follows exactly as in the proof of Theorem 4.6 above.

Consider homogeneous degree zero functions on $\mathbb{R}^n$ of the form

$$F = \frac{p_l(z_1, \dots, z_n)}{(z_1^2 + z_2^2 + \dots + e^{2\omega} z_n^2)^{l/2}}, \tag{4.24}$$

where p_l is a homogeneous polynomial of degree l and $\omega \in \mathbb{C}$ with $|\operatorname{Im}(\omega)| < \pi/4$.

$$\widetilde{\mathcal{A}} = \left\{ f \in C^\infty(\partial M) : f = \sum_{i=1}^k F_i|_{|z|=1} \right\}, \tag{4.25}$$

or in words, $\widetilde{\mathcal{A}}$ consists of all finite sums of restrictions of homogeneous degree zero functions as in (4.24) to the sphere. Note that $\widetilde{\mathcal{A}}$ is dense in every Sobolev space; indeed, $\widetilde{\mathcal{A}}$ contains the spherical harmonics, which are restrictions to the sphere of harmonic polynomials, and which form a basis of every Sobolev space by Fourier series [45]. For $\theta \in \mathbb{R}$, we define

$$V_\theta F = (\det D\Phi_\theta)^{1/2} \Phi_\theta^* F, \tag{4.26}$$

with Φ_θ as above, i.e. $\Phi_\theta(z_1, \dots, z_{n-1}, z_n) = (z_1, \dots, z_{n-1}, e^\theta z_n)$, and thereby define, for $f = \sum_{i=1}^k F_i|_{|z|=1}$,

$$\widetilde{U}_\theta f = \sum_{i=1}^k (V_\theta F_i)|_{|z|=1}. \tag{4.27}$$

To see that condition iii) holds, note that given $0 < \delta < \pi/4$ if we define $\widetilde{\mathcal{A}}_\delta \subset \widetilde{\mathcal{A}}$ to be those elements where the functions F_i in the definition of $\widetilde{\mathcal{A}}$ have parameter ω (in (4.24)) with $|\omega| \le \delta$, then $\widetilde{\mathcal{A}}_\delta$ remains dense in L^2. satisfying $|\omega| < \delta$, then $\widetilde{\mathcal{A}}_\delta$ remains dense since it still contains the spherical harmonics. On the other hand, for $|\operatorname{Im}\theta| < \pi/4$ and δ sufficiently small $\widetilde{\mathcal{A}}_\delta \subset \widetilde{U}_\theta \widetilde{\mathcal{A}}$, so the density result holds also for $\widetilde{U}_\theta \widetilde{\mathcal{A}}$.

Property ii) in the complex scaling process in this context simply says that for $f \in \widetilde{\mathcal{A}}$ and $\theta \in \mathcal{D}$,

$$\widetilde{U}_\theta N(L_{g,\theta_0}) \widetilde{U}_\theta^{-1} f = N(L_{g,\theta+\theta_0}) f,$$

which follows directly from the corresponding statement for $\Box_g$. Thus all three properties hold and the argument proceed exactly as in the previous section.

□

5. Module regularity and semilinear problems

Despite being the *simplest* natural spaces between which the operator $\Box_g$ extends to a Fredholm map, the $\mathcal{X}^{m,l}_{\pm\pm}$ and $\mathcal{Y}^{m,l}_{\pm\pm}$ have the following critical shortcoming when one wants to analyze semilinear equations. If one wishes for example to do a Picard iteration using the inverse of $\Box_g$ on some space (as we do below e.g. in the proof of Proposition 5.15) invertibility restricts the range of weights l. Furthermore, negative weights l are dangerous when the non-linear term involves positive powers, as these will decrease the weight. A lower bound on l produces an upper bound on m in the low regularity region $m + l < 1/2$: if for example one wants to take $l > 0$ then $m < 1/2$. This is exactly the rub: distributions with Sobolev regularity less than 1/2 *cannot* be multiplied together in general, and thus a good weight forces bad regularity from the perspective of semilinear equations. In this section we overcome this problem by adding module derivatives to the $H_{\mathrm{b}}^{m,l}$ spaces, as we describe now, using these in the end to prove Theorems 5.15 and 5.22 below.

We remark that our use of these spaces is related to the use of the infinitesimal generators of the Lorentz group by Klainerman [32] though we work in a purely L^2-based setting while Klainerman proceeds in a $L^\infty L^2$ setting, and more crucially our module regularity is necessarily pseudodifferential. As mentioned earlier, results analogous to ours for the Cauchy problem were obtained by Hintz and Vasy [27]; in that case one can use module generators that are vector fields, and thus one has in a certain sense a geometric generalization of Klainerman's vector fields. For the Feynman propagators we then further generalize this to a microlocal setting.

Elaborating on Proposition 2.1, one can also have a version between spaces with additional module regularity, much as in [27, Section 5]. The module regularity is with respect to pseudodifferential operators characteristic on the halves of the conormal bundles of $S_\pm$ toward which we propagate regularity, e.g. for L_{+-} they are characteristic on ${}^{\mathrm{b}}SN^*_+S_+$ and ${}^{\mathrm{b}}SN^*_+S_-$. Concretely, consider the Ψ^0_{b}-module $\mathcal{M}_{\pm\pm}$ in Ψ^1_{b} consisting of b-pseudodifferential operators A whose b-principal symbols $\sigma_{b,1}(A)$ vanish on the components of ${}^{\mathrm{b}}SN^*_\pm S_\pm$ at which the domain $L_{\pm\pm}$ has *low* regularity. Thus, elements in

$$\mathcal{M}_{+-} \text{ are characteristic at } {}^{\mathrm{b}}SN^*_+S_+ \cup {}^{\mathrm{b}}SN^*_+S_-, \text{ and in}$$
$$\mathcal{M}_{++} \text{ are characteristic at } {}^{\mathrm{b}}SN^*S_+,$$

meaning their principal symbols vanish on the indicated sets. For an integer k we consider spaces

$$H^{m,l,k}_{\mathrm{b},\pm\pm} := \{u \in H^{m,l}_{\mathrm{b}} : \mathcal{M}^k_{\pm\pm} u \subset H^{m,l}_{\mathrm{b}}\}. \tag{5.1}$$

The $H^{m,l,k}_{\mathrm{b},++}$ (and the $--$ whose analysis is essentially identical to the $H^{m,l,k}_{\mathrm{b},++}$) thus have module regularity defined by $\mathcal{M}_{++}$, which consists of first order b-pseudodifferential operators that are characteristic on the b-conormal bundle of S_+ and are allowed to be b-elliptic at S_-. Thus $\mathcal{M}_{++}$ admits differential local generators in the following sense; let $\mathcal{V}_{++}$ denote the $\mathcal{C}^\infty(M)$ module of vector fields V which in the coordinates ρ, v, y satisfy the two conditions: i) in a neighborhood of S_-, V is in the $\mathcal{C}^\infty(M)$ span of $\rho\partial_\rho, \partial_v, \partial_y$, i.e. V is locally a b-vector field there, and ii) near S_+, V is in the $\mathcal{C}^\infty(M)$ span of $\rho\partial_\rho, \rho\partial_v, v\partial_v, \partial_y$, i.e. V is tangent to S_+. Then $\mathcal{M}^j_{++}u \subset H^{m,l}_{\mathrm{b}} \iff (\forall i \le j)\ \mathcal{V}^i_{++}u \subset H^{m,l}_{\mathrm{b}}$, and thus membership of a distribution in $\mathcal{M}_{++}$ can be checked by applying differential and not pseudodifferential operators. The $H^{m,l,k}_{\mathrm{b},++}$ were studied in [27, Section 5]. Note that if we localize near S_-, since elements of $\mathcal{M}_{++}$ are not required to be characteristic on SN^*S_-, we have full b-regularity to order $m+k$ there, which is to say that if χ is a cutoff function supported away from a neighborhood of S_+, then for $u \in H^{m,l,k}_{\mathrm{b},++}$, $\chi u \in H^{m+k,l}_{\mathrm{b}}$.

We have the following regularity result which says that if the right hand side of $L_{\pm\pm}u = f$ has module regularity then the solution u has the appropriate corresponding module regularity. As explained below [27, Theorem 5.3], the following is a consequence of the extension of [2, Proposition 4.4] obtained in [23, Theorem 6.3] in the interior case (i.e. with no "b"). (Recall that $\mathcal{Y}^{m-1,l} = H^{m-1,l}_{\mathrm{b}}$ and that $\mathcal{X}^{m,l} \subset H^{m,l}_{\mathrm{b}}$.)

Theorem 5.1. *(See* [27, Theorem 5.4] *for the retarded/advanced propagators; the proof is identical in the present general case.) Let g be a perturbation of the Minkowski metric in the sense of Lorentzian scattering metrics (see Section 2).*

Let $m\colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$, $l \in \mathbb{R}$ *satisfy the assumptions of Theorem 3.3, corresponding to a particular choice of* $\pm\pm$, *and let* $k \in \mathbb{N}_0$, *and assume that* $L_{\pm\pm}^{-1} : \mathcal{Y}^{m-1,l} \to \mathcal{X}^{m,l}$ *exists (automatic if* $|l| < \frac{n-2}{2}$ *and the perturbation is small by Theorem 3.6). Then* $L_{\pm\pm}^{-1}$ *restricts to a bounded map*

$$L_{\pm\pm}^{-1}\colon H_{\mathrm{b},\pm\pm}^{m-1,l,k} \longrightarrow H_{\mathrm{b},\pm\pm}^{m,l,k}.$$

Thus, $H_{\mathrm{b},+-}^{m,l,k}$ is the subspace of $H_{\mathrm{b}}^{m,l}$ consisting of u such that with $\mathcal{M}_{+-}$ denoting 1st order b-pseudodifferential operators characteristic on ${}^{\mathrm{b}}SN_+^*S_+$ and ${}^{\mathrm{b}}SN_+^*S_-$, $\mathcal{M}_{+-}^k u \in H_{\mathrm{b}}^{m,l}$. Since elements in $\mathcal{M}_{+-}$ can be elliptic wherever they are not required to be characteristic, elements in $H_{\mathrm{b}}^{m,l,k}$ must have $m+k$ b-derivatives at the other halves of the conormal bundles, i.e. at ${}^{\mathrm{b}}SN_-^*S_+$ and ${}^{\mathrm{b}}SN_-^*S_-$. Also notice that $m+l > 1/2$ at the radial sets from which we propagate estimates implies that $m+l+k > 1/2$ for $k \in \mathbb{N}$, so the requirements for the propagation estimates are satisfied there; for the radial sets towards which we propagate the estimates we still need, and have, $m+l < 1/2$ as the module derivatives are 'free' in the sense that one can take k arbitrarily high without sacrificing invertibility. (One could also use a different normalization, so there are no k additional derivatives present at the other halves, but one has to be careful then to make the total weight function behave appropriately; for the present normalization the previous assumptions on m are the appropriate ones.)

These observations also yield the following improvement of the theorem:

Theorem 5.2. *Let* g *be a perturbation of the Minkowski metric in the sense of Lorentzian scattering metrics (see Section 2). Let* $m\colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$, $l \in \mathbb{R}$, $k \in \mathbb{N}_0$ *satisfy the following weakening of the assumptions of Theorem 3.3 corresponding to a particular choice of* $\pm\pm$*: when* $m + l - 3/2 > 0$ *is assumed in Theorem 3.3, assume merely* $m + l + k - 3/2 > 0$. *Assume that* $L_{\pm\pm}^{-1} : H_{\mathrm{b},\pm\pm}^{m-1,l} \to H_{\mathrm{b},\pm\pm}^{m,l}$ *exists (automatic if* $|l| < \frac{n-2}{2}$ *and the perturbation is small by Theorem 3.6). Then* $L_{\pm\pm}^{-1}$ *restricts to a bounded map*

$$L_{\pm\pm}^{-1}\colon H_{\mathrm{b},\pm\pm}^{m-1,l,k} \longrightarrow H_{\mathrm{b},\pm\pm}^{m,l,k}.$$

The assumptions on the regularity function in the theorem thus translate to

(5.2)

Region	Feynman	Retarded
${}^{\mathrm{b}}SN_+^*S_+$	$m+l<1/2$	$m+l<1/2$
${}^{\mathrm{b}}SN_-^*S_+$	$m+l+k>3/2$	$m+l<1/2$
${}^{\mathrm{b}}SN_+^*S_-$	$m+l<1/2$	$m+l+k>3/2$
${}^{\mathrm{b}}SN_-^*S_-$	$m+l+k>3/2$	$m+l+k>3/2$

,

with analogous conditions for the anti-Feynman and advanced propagators.

Remark 5.3. Note that if $\tilde{m}$ is a function on ${}^{\mathrm{b}}S^*M$ such that $m = \tilde{m}$ near the radial sets where the regularity is low and $\tilde{m} = m + k$ where the regularity is high, $\tilde{m}$ monotone along the bicharacteristics, then $H_{\mathrm{b},\pm\pm}^{m,l,k} \subset H_{\mathrm{b},\pm\pm}^{\tilde{m},l}$, explaining the sense of 'restricts' for $L_{\pm\pm}^{-1}$.

Remark 5.4. A very useful consequence of the theorem is that given any l one may take *constant* m with $m+l < 1/2$ and k such that $m+k+l > 3/2$, i.e. one does not need variable order spaces after all. However, note that the variable order spaces are extremely useful in obtaining this conclusion: otherwise at the minimum one would have to do estimates on the dual spaces of spaces with module regularity for the adjoint operator; these dual spaces are more difficult to work with.

Proof. We simply note that with $\tilde{m}$ as remarked after the statement of the proposition, for $f \in H^{m-1,l,k}_{\mathrm{b},\pm\pm}$, $u = L^{-1}_{\pm\pm} f \in H^{\tilde{m},l}_{\mathrm{b}}$ and microlocally away from the low regularity regions in fact u is in $H^{m+k,l}_{\mathrm{b}}$ (as f is in $H^{m+k-1,l}_{\mathrm{b}}$ and $m+k+l > 1/2$ there) by the standard high regularity radial point statement, Proposition 2.1, and the propagation of singularities. On the other hand, at the low regularity radial sets, the extension of [2, Proposition 4.4] as explained below [27, Theorem 5.3] applies, giving that $u$is in $H^{m,l,k}_{\mathrm{b}}$ microlocally since f is in $H^{m-1,l,k}_{\mathrm{b}}$ microlocally, u is in $H^{m+k,l}_{\mathrm{b}}$ in a punctured neighborhood of these radial sets, and $m + l < 1/2$. All these membership statements come with estimates, proving the theorem. □

One reason one may want to develop this is to solve nonlinear equations, as we do in Section 5.3. To this end we will be forced to restrict the class of regularity functions m we consider in the spaces $H^{m,l,k}_{\mathrm{b}}$ so that we can keep track of the wavefront sets of products of distributions therein. Specifically, we will assume that, writing $m_+(x) = \max_{{}^{\mathrm{b}}S^*_x M} m$, that

$$\begin{aligned} &(\forall x)(\forall s < m_+(x)) \quad \{\xi \in {}^{\mathrm{b}}T^*_x M \setminus o :\ m(x,\xi) \le s\} \text{ is a convex cone,} \\ &x \in S_+ \Rightarrow m|_{{}^{\mathrm{b}}S^*_x M} \text{ attains its minimum on } {}^{\mathrm{b}}SN^*_+ S_+, \\ &x \in S_- \Rightarrow m|_{{}^{\mathrm{b}}S^*_x M} \text{ attains its minimum on } {}^{\mathrm{b}}SN^*_+ S_-. \end{aligned} \tag{5.3}$$

The first of these conditions says that all of the non-trivial sublevel sets are convex cones within the fibers of ${}^{\mathrm{b}}T^*_x M$. The last two are only important because of the treatment of module derivatives, where our modules are characteristic on exactly the two above mentioned sets where the minimum is attained.

For this purpose, one then wants to check the following analogue of [27, Lemma 5.4]:

Proposition 5.5. *Assume that* $m\colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$ *satisfies* (5.3)*. If furthermore,* $m > 1/2$ *and* $k \in \mathbb{N}$ *satisfies* $k > (n-1)/2$*. Then*

$$H^{m,l_1,k}_{\mathrm{b},+-} H^{m,l_2,k}_{\mathrm{b},+-} \subset H^{m-0,l_1+l_2,k}_{\mathrm{b},+-}, \tag{5.4}$$

where $H^{m-0,l_1+l_2,k}_{\mathrm{b},+-} = \cap_{\epsilon>0} H^{m-\epsilon,l_1+l_2,k}_{\mathrm{b},+-}$*. Thus for any* $\epsilon > 0$ *and* m, k *satisfying the stated conditions, there is a constant* $C = C(\epsilon, m, k)$ *such that*

$$\|uv\|_{H^{m-\epsilon,l_1+l_2,k}_{\mathrm{b},+-}} \le C \|u\|_{H^{m,l_1,k}_{\mathrm{b},+-}} \|v\|_{H^{m,l_2,k}_{\mathrm{b},+-}}.$$

Furthermore, if $m > 1/2$ *is constant (and thus all assumption on it are satisfied), one can take* $\epsilon = 0$*, and drop* -0 *in* (5.4).

The proof of Proposition 5.5 comes at the end of Section 5.2 below. The condition that $m > 1/2$ can (and will) be relaxed in Section 5.3, but for the moment we use it to simplify arguments below.

To use this proposition for the semilinear Feynman problems, we will need to apply it to the spaces $H_{\mathrm{b},+-}$. For any $l < 0$, one can find a constant $m > 1/2$ such that $m + l < 1/2$ and a positive integer $k > (n-1)/2$ such that $m + k + l > 3/2$ so by Remark 5.4 the Feynman propagator is applicable and Proposition 5.5 is also applicable so that

$$H^{m,l_1,k}_{\mathrm{b},+-} H^{m,l_2,k}_{\mathrm{b},+-} \subset H^{m,l_1+l_2,k}_{\mathrm{b},+-}. \tag{5.5}$$

It is interesting to note that in fact the stronger requirements of Theorem 5.1 can also be arranged:

Corollary 5.6. *For every weight $\ell < 0$, there exists a function $m\colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$ such that: 1) $m > 1/2$, 2) m, ℓ satisfy the same assumptions as the forward Feynman condition in the Theorem 3.3, concretely those in* (3.12)*, and 3) m satisfies the property on the sublevel sets and minima in* (5.3)*. For such m, ℓ and for $k \in \mathbb{N}$ satisfying $k > (n-1)/2$,*

$$H^{m,l_1,k}_{\mathrm{b},+-} H^{m,l_2,k}_{\mathrm{b},+-} \subset H^{m-0,l_1+l_2,k}_{\mathrm{b},+-}. \tag{5.6}$$

In particular, under these assumptions the p-fold products satisfy

$$(H^{m,l,k}_{\mathrm{b},+-})^p \subset H^{m-0,pl,k}_{\mathrm{b},+-}. \tag{5.7}$$

Proof of Corollary 5.6 assuming Proposition 5.5. Fix $\ell < 0$. The corollary follows from the proposition by construction of a regularity function m satisfying the conditions listed in the statement of the corollary. To do so, fix a constant $m_+ > 1/2 - \ell > 1/2$; indeed to satisfy the strengthened form given in Theorem 3.3 take $m_+ > 3/2 - \ell > 3/2$. The function m will be arranged to be equal to m_+ except on a small neighborhood U_+ of ${}^{\mathrm{b}}SN^*_+S_+$ and U_- of ${}^{\mathrm{b}}SN^*_+S_-$, which are the low regularity regions, where it will be arranged to be smaller. We consider U_+; U_- is analogous. Using any (local) defining functions ρ_i, $i = 1, \dots, n+1$, of ${}^{\mathrm{b}}SN^*_+S_+$ in ${}^{\mathrm{b}}S^*M$ and letting $f = \sum \rho_i^2$, the Hamilton derivative of f is monotone in a neighborhood U_+ of ${}^{\mathrm{b}}SN^*_+S_+$ due to the non-degenerate linearization in the normal direction (with the size of the neighborhood of course depending on the choice of the ρ_i), with the monotonicity being strict in the punctured neighborhood. We may assume that U_+ is disjoint from any other component of the radial set; note that if one chooses to, one may always shrink U_+ to lie in any pre-specified neighborhood of ${}^{\mathrm{b}}SN^*_+S_+$. One then takes a cutoff function ϕ, with $\phi \equiv 1$ near 0, $\phi' \leq 0$ on $[0, \infty)$, ϕ supported sufficiently close to 0 so that $\phi \circ f$ is compactly supported in U_+; for $c > 0$, $m = m_+ - c(\phi \circ f)$ satisfies all monotonicity requirements along the Hamilton flow, and if $m_+ - c < 1/2 - \ell$, i.e. $c > m_+ + \ell - 1/2$, then the radial point part of the Feynman condition is also achieved. Note that as $1/2 - \ell > 1/2$, we may arrange in addition that $m_- = \inf m = m_+ - c > 1/2$ by choosing $c < m_+ - 1/2$. This also gives the minimum attaining conditions in (5.3). To arrange the convexity, it is useful to be more definite about the ρ_i: the conormal bundle is $\rho = v = 0$, $\zeta = 0$, $\eta = 0$ where ζ is b-dual to ρ and η is b-dual to the variables y along S_+. Thus, with ξ' the b-dual variable to v (which is thus non-zero on the conormal bundle minus the zero section) a (local) quadratic defining function f is $f = \frac{\zeta^2}{(\xi')^2} + \frac{|\eta|^2}{(\xi')^2} + v^2 + \rho^2$. The convexity requirement for the sublevel sets of m then is implied by one for those of f (only the ones below sufficiently small positive values matter), which is thus a convexity condition for the sets

$$\{(\xi', \zeta, \eta) : \frac{\zeta^2}{(\xi')^2} + \frac{|\eta|^2}{(\xi')^2} \leq \alpha,\ \xi' > 0\},$$

for all sufficiently small $\alpha > 0$, which however certainly holds. This completes the proof of the corollary. □

5.1. **Microlocal multiplicative properties of Sobolev spaces.** Before we turn to multiplicative properties of b-Sobolev spaces as in Proposition 5.5 we first study the analogous properties of standard Sobolev spaces. Here we need to work with variable order Sobolev spaces because of the microlocal nature of the spaces $H_{\mathrm{b},+-}^{m,l_1,k}$, in particular as m is a function in this case, and as k gives additional regularity for one half of a conormal bundle only.

Thus, for a distribution u and for $s \in \mathcal{C}^\infty(S^*\mathbb{R}^n)$ one defines $\mathrm{WF}^s(u)$ as in the case of the b-wave front set (so the definitions agree in the interior): $(p,\xi) \notin \mathrm{WF}^s(u)$ if there exists $A \in \Psi^0(\mathbb{R}^n)$ elliptic at (p,ξ) such that $Au \in H^s(\mathbb{R}^n)$, or equivalently, if there exists $A \in \Psi^s(\mathbb{R}^n)$ elliptic at (p,ξ) such that $Au \in L^2(\mathbb{R}^n)$. We are then interested in questions of the kind: for which functions $r \geq s \geq s_0$ on $S^*\mathbb{R}^n$ does the implication

$$u \in H^r,\ v \in H^{s_0} \Rightarrow \mathrm{WF}^s(uv) \subset \mathrm{WF}^s(v),$$

hold?

By a (Fourier) weight function $w\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, we mean a smooth, measurable, positive function of polynomial growth, meaning $w \leq C\langle\xi\rangle^N$ for some $C, N > 0$. The variable order Sobolev space $H^{(w)}$ corresponding to w is then

$$H^{(w)} = \{u \in S'(\mathbb{R}^n) : w\hat{u} \in L^2(\mathbb{R}^n)\}. \tag{5.8}$$

Thus $w(\xi) = \langle\xi\rangle^s$ for $s \in \mathbb{R}$ defines the standard Sobolev space of order s. The most common weight function we use below is of the form $w(\xi) = \langle\xi\rangle^{s(\hat{\xi})}$ where

$$\hat{\xi} = \xi/|\xi|,$$

so s is a function on the unit sphere, and thus we let

$$H^s := H^{(w)},\ \text{where } w = \langle\xi\rangle^s, \tag{5.9}$$

that is, for s of the form $s = s(\hat{\xi})$ (which is a rather special case of $s \in \mathcal{C}^\infty(S^*\mathbb{R}^n)$!), H^s is the special case of $H^{(w)}$ where w has this form. The reason such special s are sufficient for us is that multiplication is a local operation on the base space $\mathbb{R}^n$, so using the continuity of a general weight in the base variable, up to arbitrarily small losses in the order, one may assume that all weights are in fact dependent only on the Fourier dual variable, cf. the proof of Proposition 5.5 in Section 5.2, explicitly the containment (5.39).

Given an interior point $p \in M$ and local coordinates x near p, we write the induced coordinates on the cotangent space T_p^*M with the variable ξ. The map $\xi \mapsto \hat{\xi}$ then identifies the spherical cotangent bundle at p,

$$S_p^*M := (T_pM \setminus o)/\mathbb{R}^+,$$

with the unit sphere in $\mathbb{R}^n$. Here o denotes the zero section, and the $\mathbb{R}^+$ action is the natural dilation on the fibers. Given $\hat{\xi} \in S_p^*M$ and $s \in \mathbb{R}$, the Sobolev wavefront set of order s at p of a distribution u, $\mathrm{WF}^s(u)$, satisfies $(p,\xi_0) \notin \mathrm{WF}^s(u)$ if and only if there is a cutoff function χ on M supported near p so that $w(\xi)\widehat{\chi u} \in L^2$ for a weight function w satisfying

$$w(\xi) \geq \chi_2(\hat{\xi})\langle\xi\rangle^s \tag{5.10}$$

for χ_2 a cutoff function on the unit sphere $\mathbb{S}^{n-1}$ with $\chi_2(\hat{\xi}_0) \equiv 1$ on a neighborhood of $\hat{\xi}_0$ in $\mathbb{S}^{n-1}$. In particular, $(p,\xi_0) \notin \mathrm{WF}^s(u)$ if and only if, for some cutoff function χ and some s with $\mathsf{s} = \mathsf{s}(\hat{\xi})$, $\mathsf{s}(\hat{\xi}) \equiv s$ on some open set $U \subset S_p^*M$ with $\hat{\xi}_0 \in U$ and $\mathsf{s} \ll 0$ off U, then $\chi u \in H^{\mathsf{s}}$.

We are interested in properties of the Sobolev wavefront sets of products of distributions, and to this end we will exploit [27, Lemma 4.2]:

Lemma 5.7. *Let w_1, w_2, w be weight functions such that one of the quantities*

$$\begin{aligned} M_+ &:= \sup_{\xi\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta \\ M_- &:= \sup_{\xi\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\xi \end{aligned} \tag{5.11}$$

is finite. Then $H^{(w_1)} \cdot H^{(w_2)} \subset H^{(w)}$.

The most well-known algebra property of Sobolev spaces is that H^s is an algebra provided $s > n/2$. We now ask, for example, under what assumption on r, s, s_0 does one have

$$u \in H^r, v \in H^{s_0} \implies \mathrm{WF}^s(uv) \subset \mathrm{WF}^s(v), \tag{5.12}$$

i.e. if u satisfies an a priori high regularity assumption, and v has a priori not too low regularity, can we conclude that if v is H^s microlocally, then uv is also H^s microlocally? We now provide a partial answer using Lemma 5.7. (Note that taking $r = s = s_0 > n/2$ in the following lemma gives the standard statement that H^s is an algebra for $s > n/2$.)

Lemma 5.8. *For distributions u, v (which one may assume to be compactly supported due to the locality of multiplication) and $r, s, s_0 \in \mathbb{R}$,*

$$r \geq s \geq s_0 > 0 \text{ and } r - s + s_0 > n/2 \implies \text{ the containment (5.12) holds.} \tag{5.13}$$

Proof. Indeed, let u, v be compactly supported distributions and let $\xi_0 \neq 0$ have $(x_0, \xi_0) \notin \mathrm{WF}^s(v)$. By definition, there is an open cone $\mathsf{C} \subset \mathbb{R}^n \setminus o$ containing ξ_0 and a function s with

$$\mathsf{s} = \mathsf{s}(\hat{\xi}) \text{ such that } \mathsf{s} \equiv s \text{ on } \mathsf{C}, \mathsf{s} \geq s_0,$$

and a cutoff function χ supported near x_0 such that $\chi v \in H^{\mathsf{s}}$, i.e. $\widehat{\chi v}\langle\xi\rangle^{\mathsf{s}} \in L^2$. To see that $(x_0, \xi_0) \notin \mathsf{WF}^s(uv)$ we choose a conic subset $\mathsf{K} \subset \mathsf{C}$ with compact cross section and $\xi_0 \in \mathsf{K}$, and let $s' = s'(\hat{\xi})$ be such that $s'(\hat{\xi}) \equiv s$ for $\hat{\xi}$ near $\hat{\xi}_0$, $s' \leq s$ everywhere, and such that $s' = s_0$ on a conic neighborhood of K^c. Thus if $\chi uv \in H^{s'}$ for some (possibly different) cutoff χ with $\chi(x_0) \neq 0$, then $(x_0, \xi_0) \notin \mathrm{WF}^s(uv)$. The argument below shows that uv is microlocally $H^{s'}$ outside K, but we concentrate on the statement in K. We will apply Lemma 5.7 with $w = \langle\xi\rangle^{s'}, w_1 = \langle\xi\rangle^r$, and $w_2 = \langle\xi\rangle^{\mathsf{s}}$. That is, we will show that

$$H^r \cdot H^{\mathsf{s}} \subset H^{s'}, \tag{5.14}$$

for r, s, s' chosen as above. Writing

$$I_\xi = \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta, \tag{5.15}$$

we want to show that $\sup_\xi I_\xi \leq C < \infty$.

We first note that for $\xi \in \mathsf{K}$ this is bounded by the analogous expression where $w(\xi)$ is replaced by $\langle\xi\rangle^s$. Thus, we first show

$$\sup_{\xi\in\mathsf{K}} I_\xi = \sup_{\xi\in\mathsf{K}} \int \left(\frac{\langle\xi\rangle^s}{\langle\eta\rangle^{\mathsf{s}(\eta)}\langle\xi-\eta\rangle^r}\right)^2 d\eta \tag{5.16}$$

is finite. Since $s \geq 0$ (recall that s is a constant), $\langle \xi \rangle^{2s} \lesssim \langle \eta \rangle^{2s} + \langle \xi - \eta \rangle^{2s}$, it suffices to prove that

$$\sup_{\xi \in \mathsf{K}} \int \frac{1}{\langle \eta \rangle^{2\mathsf{s}(\eta) - 2s} \langle \xi - \eta \rangle^{2r}} \, d\eta \quad \text{and} \quad \sup_{\xi \in \mathsf{K}} \int \frac{1}{\langle \eta \rangle^{2\mathsf{s}(\eta)} \langle \xi - \eta \rangle^{2r - 2s}} \, d\eta \tag{5.17}$$

are finite. We start by looking at the second of these. We break up the integral into $\langle \eta \rangle \geq \langle \xi - \eta \rangle$ and it complement. In the former region the integral is bounded by

$$\int \frac{1}{\langle \xi - \eta \rangle^{2r - 2s + 2\mathsf{s}(\eta)}} \, d\eta$$

and since $r \geq s$ in the latter region by

$$\int \frac{1}{\langle \eta \rangle^{2r - 2s + 2\mathsf{s}(\eta)}} \, d\eta,$$

both of which are finite under the assumption in (5.13) as $\mathsf{s} \geq s_0$.

Turning to the first integral in (5.17), since $\mathsf{s}(\eta)$ is not necessarily greater than or equal to s, the argument of the previous paragraph does not go through – instead, we break up the integral into one over C and one over C^c. Now, using that $\mathsf{s} \equiv s$ on C,

$$\int_{\mathsf{C}} \frac{1}{\langle \eta \rangle^{2\mathsf{s}(\eta) - 2s} \langle \xi - \eta \rangle^{2r}} \, d\eta = \int_{\mathsf{C}} \frac{1}{\langle \xi - \eta \rangle^{2r}} \, d\eta \leq \int \frac{1}{\langle \xi - \eta \rangle^{2r}} \, d\eta = \int \frac{1}{\langle \eta \rangle^{2r}} \, d\eta$$

is finite, independent of ξ, if $r > n/2$ (which is implied by $r - s + s_0 > n/2$ and $s \geq s_0$). For the integral over C^c, we use that there is a constant $C_0 > 0$ such that $C_0 \langle \xi - \eta \rangle \geq \langle \eta \rangle$ for $\xi \in \mathsf{K}$ and $\eta \in \mathsf{C}^c$. Correspondingly, as $r \geq 0$,

$$\int_{\mathsf{C}^c} \frac{1}{\langle \eta \rangle^{2\mathsf{s}(\eta) - 2s} \langle \xi - \eta \rangle^{2r}} \, d\eta \leq C_0^{2r} \int_{\mathsf{C}^c} \frac{1}{\langle \eta \rangle^{2\mathsf{s}(\eta) - 2s + 2r}} \, d\eta \leq C_0^{2r} \int \frac{1}{\langle \eta \rangle^{2s_0 - 2s + 2r}} \, d\eta$$

which is finite if $r - s + s_0 > n/2$. This proves (5.16).

The bound for I_ξ with $\xi \notin \mathsf{K}$ proceeds along the same lines, where now one can replace $w(\xi)$ by $\langle \xi \rangle^{s_0}$, and $\langle \eta \rangle^{\mathsf{s}(\eta)}$ by $\langle \eta \rangle^{s_0}$, and is left to the reader.

This completes the proof of (5.14) and thus that the conditions on r, s, s_0 in (5.13) imply (5.12). □

For our applications, i.e. to study the module regularity defining the spaces $H^{m,l,k}_{\mathrm{b},\pm\pm}$ we will first study spaces for which one has extra regularity in certain directions. To this end, we write $\mathbb{R}^n$ as $\mathbb{R}^{d+(n-d)}$, i.e. we decompose into $x = (x', x'')$ where $x' \in \mathbb{R}^d$ and $x'' \in \mathbb{R}^{n-d}$, and for functions f, we write the Fourier side variable ξ as (ξ', ξ''). Let

$$\mathcal{Y}^{m,a}_d(\mathbb{R}^{d+(n-d)}) = \{u : \hat{u} \langle \xi \rangle^m \langle \xi'' \rangle^a \in L^2\}, \tag{5.18}$$

so elements have m total derivatives (derivatives in all variables) and in addition a derivatives in x''.

Lemma 5.9. *([27, Lemma 4.4].) Let $m, a \in \mathbb{R}$. If $m > d/2$ and $a > (n-d)/2$ then $\mathcal{Y}^{m,a}_d$ is an algebra. If $a, b \geq 0$, $a + b > n - d$, then $\mathcal{Y}^{m,a}_d \cdot \mathcal{Y}^{m,b}_d \subset H^m$.*

Proof. We begin with the second statement. This is exactly [27, Equation 4.6], namely using $\langle\xi\rangle^p \lesssim \langle\eta\rangle^p + \langle\xi-\eta\rangle^p$ for $p \geq 0$, we have

$$\begin{aligned} &\int \left(\frac{\langle\xi\rangle^m}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta\rangle^m \langle\eta''\rangle^b} \right)^2 d\eta \\ &\leq \int \left(\frac{1}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta''\rangle^b} \right)^2 d\eta + \int \left(\frac{1}{\langle\xi''-\eta''\rangle^a \langle\eta\rangle^m \langle\eta''\rangle^b} \right)^2 d\eta \\ &\leq \int \left(\frac{1}{\langle\xi'-\eta'\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta''\rangle^b} \right)^2 d\eta + \int \left(\frac{1}{\langle\xi''-\eta''\rangle^a \langle\eta'\rangle^m \langle\eta''\rangle^b} \right)^2 d\eta, \end{aligned} \tag{5.19}$$

and integrating first in the primed and then the double primed variable shows this integral is uniformly bounded.

When $a = b > (n-d)/2$, estimating the numerator in (5.19) in the same way, we see that $\mathcal{Y}_d^{m,a}$ is an algebra since

$$\begin{aligned} &\int \left(\frac{\langle\xi\rangle^m \langle\xi''\rangle^a}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta\rangle^m \langle\eta''\rangle^a} \right)^2 d\eta \\ &\leq \sum_{i,j=1}^{2} \int \left(\frac{f_i g_j}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta''\rangle^a} \right)^2 d\eta, \end{aligned} \tag{5.20}$$

where $f_1 = \langle\eta\rangle^m$, $f_2 = \langle\xi-\eta\rangle^m$ and $g_1 = \langle\eta''\rangle^a$, $g_2 = \langle\xi''-\eta''\rangle^a$. We estimate the $f_1 g_1$ term by

$$\begin{aligned} &\int \left(\frac{\langle\eta\rangle^m \langle\eta''\rangle^a}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a \langle\eta\rangle^m \langle\eta''\rangle^a} \right)^2 d\eta \\ &\leq \int \left(\frac{1}{\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a} \right)^2 d\eta \leq \int \left(\frac{1}{\langle\xi'-\eta'\rangle^m \langle\xi''-\eta''\rangle^a} \right)^2 d\eta, \end{aligned}$$

so integrating separately in the primed and double primed variable shows this is uniformly bounded. The other terms are bounded in exactly the same way. ☐

The wavefront set containment in (5.12)–(5.13) can be "improved" when one assumes the distributions lie in the $\mathcal{Y}_d^{m,a}$, in the sense that less total regularity (i.e. smaller m) is required.

Lemma 5.10. *Let $r, a, m \in \mathbb{R}$, and let $u \in H^r, v \in \mathcal{Y}_d^{m,a}$. Then, provided, $m > d/2$, and $a > (n-d)/2$ we have that for any $s \in \mathbb{R}$ with $r \geq s \geq m + a$, that*

$$\mathrm{WF}^s(uv) \subset \mathrm{WF}^s(v). \tag{5.21}$$

Proof. Note first that the conditions on r, s, m, and a imply that $r, s > n/2$ and thus give square integrable weight functions. Let $\xi_0 \neq 0$ and $(x_0, \xi_0) \notin \mathrm{WF}^s(v)$. As above, let $\mathsf{C} \subset \mathbb{R}^n$ be an open, conic set, such that

$$\xi_0 \in \mathsf{C} \text{ and } \{x_0\} \times \mathsf{C} \subset (\mathrm{WF}^s(v))^c,$$

There is a function $\mathsf{s} = \mathsf{s}(\hat{\xi})$ such that $\mathsf{s}|_{\mathsf{C}} = s$, and $\chi v \in H^{(w_1)}$ where

$$w_1 = \langle\xi\rangle^m \langle\xi''\rangle^a + \langle\xi\rangle^{\mathsf{s}}.$$

Furthermore, χ can be chosen such that $\chi u \in H^{(w_2)}$ where $w_2 = \langle\xi\rangle^r$. To show that $(x_0, \xi_0) \notin \mathrm{WF}^s(uv)$, we choose K a conic subset containing ξ_0, and a function $s'(\hat{\xi})$ with

$$s'(\hat{\xi}) = s \text{ for } \hat{\xi}_0 \text{ near } \hat{\xi}, \text{ while } s'(\hat{\xi}) = \delta \text{ on a neighborhood of } \mathsf{K}^c, \tag{5.22}$$

where $\delta > 0$ is fixed and small and also $s' \leq \mathsf{s}$ everywhere. We apply Lemma 5.7 with this w_1, w_2 and $w = \langle\xi\rangle^{s'}$.

Defining I_ξ as in (5.15) with the current w_1, w_2 and w we have

$$I_\xi := \int \Big(\frac{\langle\xi\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})\langle\xi-\eta\rangle^r} \Big)^2 \, d\eta, \tag{5.23}$$

and we want to know that $M_+ = \sup_\xi I_\xi$ is finite. Again we use that $\langle\xi\rangle^p \lesssim \langle\eta\rangle^p + \langle\xi-\eta\rangle^p$ for $p > 0$ to write $I(\xi) \leq I_1(\xi) + I_2(\xi)$ where

$$\begin{aligned} I_1(\xi) &= \int \Big(\frac{\langle\eta\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})\langle\xi-\eta\rangle^r} \Big)^2 \, d\eta \qquad \text{and} \\ I_2(\xi) &= \int \Big(\frac{\langle\xi-\eta\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})\langle\xi-\eta\rangle^r} \Big)^2 \, d\eta. \end{aligned} \tag{5.24}$$

For any ξ, since $\langle\xi-\eta\rangle^s / \langle\xi-\eta\rangle^r \leq 1$,

$$I_2(\xi) \leq \int \Big(\frac{1}{\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}}} \Big)^2 \, d\eta \leq \int \frac{1}{\langle\eta'\rangle^{2m}\langle\eta''\rangle^{2a}} \, d\eta' d\eta'',$$

which is bounded since $m > d/2$ and $a > (n-d)/2$.

Finally consider $I_1(\xi)$ for $\xi \in \mathsf{K}$. We break the integral up into $\eta \in \mathsf{C}$ and $\eta \in \mathsf{C}^c$. Using that, $s'(\xi) \leq s$, and that $\mathsf{s} = s$ on C with $s \geq m + a$,

$$\int_{\mathsf{C}} \Big(\frac{\langle\eta\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})\langle\xi-\eta\rangle^r} \Big)^2 \, d\eta \leq \int_{\mathsf{C}} \frac{1}{\langle\xi-\eta\rangle^{2r}} d\eta,$$

which is bounded uniformly. On the other hand, since K has compact cross section and $\xi \in \mathsf{K}$, we have $\langle\eta\rangle \lesssim \langle\xi-\eta\rangle$ for $\eta \in \mathsf{C}^c$, so as $s \leq r$,

$$\begin{aligned} \int_{\mathsf{C}} \Big(\frac{\langle\eta\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})\langle\xi-\eta\rangle^r} \Big)^2 \, d\eta &\lesssim \int_{\mathsf{C}} \frac{1}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{\mathsf{s}})^2} d\eta \\ &\leq \int_{\mathsf{C}} \frac{1}{\langle\eta'\rangle^{2m}\langle\eta''\rangle^{2a}} d\eta' d\eta'', \end{aligned}$$

and again the last integral is bounded. Again, we leave the estimate for $\xi \notin \mathsf{K}$ to the reader. □

Furthermore, we have

Lemma 5.11. *Let $a, m, s \in \mathbb{R}$. Let $u, v \in \mathcal{Y}^{m,a}_d$. Then, provided $m > d/2$, and $a > (n-d)/2$ with $s \geq m + a$, we have*

$$\mathrm{WF}^s(uv) \subset (\mathrm{WF}^s(u) + \mathrm{WF}^s(v)) \cup \mathrm{WF}^s(u) \cup \mathrm{WF}^s(v). \tag{5.25}$$

Remark 5.12. Lemma 5.11 gives the following elaboration on (5.12)–(5.13). If $u, v \in H^{s_0}$ for $s_0 \in \mathbb{R}$, $s_0 > n/2$, then for any $s \in \mathbb{R}$, (5.25) holds. Indeed, for $s_0 > n/2$, $s > s_0$ (the interesting case) and any d one can find $m, a \in \mathbb{R}$ such that $m > d/2$, $a > (n-d)/2$ and $s \geq m+a$, and for any such $H^{s_0} \subset \mathcal{Y}^{m,a}_d$, so the lemma applies to u, v.

Proof of Lemma 5.11. The idea behind the proof is the following. Working above a fixed point $x_0 \in \mathbb{R}^n$, given

$$\xi \notin (\mathrm{WF}^s(u) + \mathrm{WF}^s(v)) \cup \mathrm{WF}^s(u) \cup \mathrm{WF}^s(v), \tag{5.26}$$

consider the integral

$$\int \Big(\frac{\langle\xi\rangle^s}{(\langle\eta\rangle^m \langle\eta''\rangle^a + \langle\eta\rangle^{s_1})(\langle\xi-\eta\rangle^m \langle\xi''-\eta''\rangle^a + \langle\xi-\eta\rangle^{s_2})} \Big)^2 \, d\eta, \tag{5.27}$$

where s_1 and s_2 are chosen so that $u \in H^{s_1}$ and $v \in H^{s_2}$ (near x_0), and such that $s_i = s > n/2$ on sets that are as large as possible. If one could take $s_1 = s$ on $(\mathrm{WF}^s(u))^c$ and $s_2 = s$ on $(\mathrm{WF}^s(v))^c$ (one cannot) then the integral would be bounded since by (5.26), either $\eta \notin \mathrm{WF}^s(u)$ or $\xi - \eta \notin \mathrm{WF}^s(v)$. In the former case for example, the integral is bounded by

$$\int \frac{1}{\langle \xi - \eta \rangle^{2m} \langle \xi'' - \eta'' \rangle^{2s}} d\eta,$$

which is bounded uniformly in ξ.

For the formal argument, note that for any open conic sets $\widetilde{\mathsf{C}}_1 \supset \mathrm{WF}^s(u)$ and $\widetilde{\mathsf{C}}_2 \supset \mathrm{WF}^s(v)$ there are functions s_1, s_2 so that $s_i \equiv s$ off $\widetilde{\mathsf{C}}_i$ and such that $\chi u \in H^{s_1}, \chi v \in H^{s_2}$ for some cutoff χ with $\chi(x_0) = 1$. Given ξ as in (5.26), assume furthermore that $\xi \notin \widetilde{\mathsf{C}}_1 + \widetilde{\mathsf{C}}_2$, and let C be a conic open set with $C \subset (\widetilde{\mathsf{C}}_1 + \widetilde{\mathsf{C}}_2)^c$ For $\xi \in \mathsf{C}$ where C is an arbitrary open subset with $\mathsf{C} \subset (\widetilde{\mathsf{C}}_1 + \widetilde{\mathsf{C}}_2)^c$. Writing $\langle \xi \rangle^{2s} \lesssim \langle \eta \rangle^{2s} + \langle \xi - \eta \rangle^{2s}$, we can bound the integral above by two terms (one with the η in the numerator and the other with the $\xi - \eta$). The argument to bound each of these is symmetric so we consider only the η term. Then for $\xi \in \mathsf{C}$, the integral in (5.27) can be broken up as an integral over $\widetilde{\mathsf{C}}_1$ and over $\widetilde{\mathsf{C}}_1^c$. Over $\widetilde{\mathsf{C}}_1^c$, we have that $\langle \eta \rangle^m \langle \eta'' \rangle^a + \langle \eta \rangle^{s_1} > (1/2) \langle \eta \rangle^s$, so that part of the integral is bounded by

$$\int \Big(\frac{1}{\langle \xi - \eta \rangle^m \langle \xi'' - \eta'' \rangle^a + \langle \xi - \eta \rangle^{s_2}} \Big)^2 d\eta, \tag{5.28}$$

which is uniformly bounded. On the other hand, for $\eta \in \widetilde{\mathsf{C}}_1$, we have that $s_2(\widehat{\xi - \eta}) = s$, since $\xi \in \mathsf{C}$, that is $\xi = \xi - \eta + \eta \notin \widetilde{\mathsf{C}}_1 + \widetilde{\mathsf{C}}_2$, so $\xi - \eta \notin \widetilde{\mathsf{C}}_2$. Therefore on the $\widetilde{\mathsf{C}}_1$ region, the integral in (5.27) is bounded by

$$\int \Big(\frac{\langle \eta \rangle^s}{(\langle \eta \rangle^m \langle \eta'' \rangle^a + \langle \eta \rangle^{s_1}) \langle \xi - \eta \rangle^s} \Big)^2 d\eta. \tag{5.29}$$

But since $\xi \in \mathsf{C}$ (so in particular $\xi \notin \widetilde{\mathsf{C}}_1$), $\langle \eta \rangle \lesssim \langle \xi - \eta \rangle$, so the integral is uniformly bounded.

Thus we will be able to apply Lemma 5.7 with $w_i = \langle \eta \rangle^m \langle \eta'' \rangle^a + \langle \eta \rangle^{s_i}$ and $w = \langle \xi \rangle^{\mathsf{s}}$ with $\mathsf{s} = s$ on an arbitrary conic subset $\mathsf{K}' \subset \mathsf{C}$ by arguing exactly as in the previous lemmata, namely taking s small but uniformly positive off of K' so that both w/w_1 and w/w_2 bounded off C. In summary we have shown that for such w, w_1, w_2 that

$$H^{(w_1)} \cdot H^{(w_2)} \subset H^{(w)}, \tag{5.30}$$

and thus the lemma follows. □

5.2. **Microlocal multiplicative properties of b-Sobolev spaces and module regularity spaces.** Recall that the b-Sobolev space $H_{\mathrm{b}}^{m,0}$ consists of distributions u which are H^m locally in the interior and whose behavior near infinity is as follows. We consider a tubular neighborhood of ∂M in M (so a tubular neighborhood of infinity) by $\{\rho < \epsilon\}$ for some small $\epsilon > 0$ and boundary defining function ρ. Given a boundary point $p \in \partial M$, we can take coordinates near p in this collar to be of the form (ρ, w) where w form coordinates on ∂M. Defining the following operation on functions v of (ρ, w) by

$$\widetilde{v}(x, w) = v(e^x, w), \tag{5.31}$$

then $u \in H_{\mathrm{b}}^{m,0}$ if in addition to interior regularity, for some cutoff function χ, $\widetilde{\chi u} \in H^m$. Here if $m \colon {}^{\mathrm{b}}T^*M \longrightarrow \mathbb{R}$ is a homogeneous degree zero function (at least outside a compact set) i.e. a function on ${}^{\mathrm{b}}S^*M$, then we mean H^m as defined in (5.9), where $m = m(p, \xi)$ and ξ is a coordinate on ${}^{\mathrm{b}}T_p^*M$. Note that the Fourier transform of $\widetilde{u}$ is equal to the distribution obtained by taking the Fourier transform in the w variables and the Mellin transform (see (3.4)) in ρ. The weighted b-Sobolev spaces are defined by $H_{\mathrm{b}}^{m,l} = \rho^l H_{\mathrm{b}}^{m,0}$. Given $u \in H_{\mathrm{b}}^{-N,0}$ for some N, a covector $(p, \xi) \in {}^{\mathrm{b}}T_p^*(M)$ satisfies $(p, \xi) \notin \mathrm{WF}_{\mathrm{b}}^{m,0}(u)$ if $w(\xi)\widehat{\widetilde{\chi u}} \in L^2$ for some cutoff function, where w satisfies (5.10). Just as stated above (5.10), this is equivalent to having a χ for which $\chi u \in H^{\mathsf{s}}$ where $\mathsf{s} = \mathsf{s}(\hat{\xi})$, $\mathsf{s}(\hat{\xi}_0) \equiv s$ in a neighborhood of $\hat{\xi}_0$ and $\mathsf{s} \ll 0$ away from $\hat{\xi}_0$. Finally, for $u \in H_{\mathrm{b}}^{-N,l}$ with $l \in \mathbb{R}$,

$$\mathrm{WF}_{\mathrm{b}}^{m,l}(u) := \mathrm{WF}_{\mathrm{b}}^{m,0}(\rho^{-l}u). \tag{5.32}$$

Using the previous section we can, for example, easily prove

Lemma 5.13. *Given $r, s_0, s \in \mathbb{R}$, then*

$$u \in H_{\mathrm{b}}^{r,l_1}, v \in H_{\mathrm{b}}^{s_0,l_2} \implies uv \in H_{\mathrm{b}}^{s_0,l_1+l_2} \text{ and } \mathrm{WF}_{\mathrm{b}}^{s}(uv) \subset \mathrm{WF}_{\mathrm{b}}^{s}(v), \tag{5.33}$$

provided (5.13) *above holds, i.e.* $r \geq s \geq s_0 \geq 0$ *and* $r - s + s_0 > n/2$.

Proof. The proof follows from the paragraph following Lemma 5.7, as we explain now.

First let $l_1 = 0 = l_2$. Given such u and v, we want to show first that $uv \in H_{\mathrm{b}}^{s_0,l_1+l_2}$. In the interior of M this follows from (5.12) and (5.13) directly. For $p \in \partial M$, by definition, there is a cutoff function χ so that $\widetilde{\chi u} \in H^r$, $\widetilde{\chi v} \in H^{s_0}$, where the tilded functions are the functions on the cylinder defined in (5.31). Then $\widetilde{\chi u}\widetilde{\chi v} \in H^{s_0}$ by applying (5.12) with $s = s_0$.

Now we show the wavefront set containment, which is almost identical to the paragraph following (5.13). Indeed, for $(p, \xi_0) \notin \mathrm{WF}_{\mathrm{b}}^{s,0}(v)$, let $\mathsf{C} \subset {}^{\mathrm{b}}T_p^*M$ be an open cone with $\xi_0 \in \mathsf{C}$ and $\mathsf{C} \cap \mathrm{WF}_{\mathrm{b}}^{s,0}(v) = \varnothing$. By definition there is a cutoff χ supported near p such that $\widetilde{\chi v} \in H^{\mathsf{s}}$ for some s with $\mathsf{s} \equiv s$ on C, $\mathsf{s} \geq s_0$ and such that $\widetilde{\chi u} \in H^r$. Then let $\mathsf{K} \subset \mathsf{C}$ be a conic set with compact cross section and $\xi_0 \in \mathsf{K}$, and let $s' = s'(\hat{\xi})$ be such that $s'(\hat{\xi}) \equiv s$ for $\hat{\xi}$ near $\hat{\xi}_0$ and such that $s' = s_0$ outside K. It suffices to show that $\widetilde{\chi u}\widetilde{\chi v} \in H^{s'}$, but this is exactly (5.14) above.

The statement for l_1 and l_2 follows by applying the above paragraph to $\rho^{-l_1}u$ and $\rho^{-l_2}v$. ☐

We can now show that the module regularity spaces $H_{\mathrm{b},++}$ have the following algebra property which is closely related to [27, Section 5.2]:

Lemma 5.14. *Let $m \equiv m_0 \in \mathbb{R}$ and let $k \in \mathbb{N}$. Provided $m > 1/2$ and $k > (n-1)/2$, if $u_1 \in H_{\mathrm{b},++}^{m,l_1,k}, u_2 \in H_{\mathrm{b},++}^{m,l_2,k}$, then $u_1 u_2 \in H_{\mathrm{b},++}^{m,l_1+l_2,k}$.*

Proof. Away from the boundary this is just the statement that H^{m+k} is an algebra, and at the boundary but away from S_+ that H_{b}^{m+k,l_j} has the stated multiplicative property. Thus we assume that the u_i are supported in a small neighborhood of a point $x \in S_+$. We begin by showing that

$$\widetilde{u_i} \in \mathcal{Y}_1^{m,k}$$

where $\widetilde{u}_i$ is defined as in (5.31) and $\mathcal{Y}_1^{m,k}$ is the spaces defined in (5.18) with $d = 1$, $a = k$. Indeed, for our coordinates (ρ, v, y) where ρ is a boundary defining function and $\rho = 0 = v$ defines S_+ (see Section 2), recall the Mellin transform (3.4), and consider the Mellin-Fourier transform of test functions $\psi \in H_b^{\infty,\infty}$, $\mathcal{M}\mathcal{F}_{v,y}(\psi)$, where $\mathcal{F}_{v,y}$ denotes the Fourier transform in the v, y variables. Concretely

$$\mathcal{M}\mathcal{F}_{v,y}(\psi) = \int \rho^{-i\zeta} e^{-iv\xi' - iy\cdot\eta} \psi(\rho, v, y) \rho^{-1} d\rho dv dy, \tag{5.34}$$

and we write

$$\xi := (\xi', \zeta, \eta), \qquad \xi'' := (\zeta, \eta), \tag{5.35}$$

so ξ is the total dual variable and ξ' is dual to v. Consider order m elliptic b-pseudodifferential operator A defined by

$$A\psi = \mathcal{F}^{-1}\mathcal{M}_0^{-1}\langle\xi\rangle^m \mathcal{M}\mathcal{F}_{v,y}\psi, \tag{5.36}$$

and the order $\leq k$ b-pseudodifferential operator B_α, $|\alpha| \leq k$, defined by

$$B_\alpha\psi = \mathcal{F}^{-1}\mathcal{M}_0^{-1}(\xi'')^\alpha \mathcal{M}\mathcal{F}_{v,y}\psi. \tag{5.37}$$

By definition of $H_{\mathrm{b},++}^{m,0,k}$ we have $AB_\alpha u_i \in L_b^2$ for all $|\alpha| \leq k$, so since the mellin transform of u is the Fourier transform in $x = \log\rho$ of $\widetilde{u}$ we have $\widetilde{u}_i \in \mathcal{Y}_1^{m,k}$ locally near x, as claimed.

To prove the lemma, we must show that if $a, b, c > 0$ integers, α a multiindex, and $a + b + c + |\alpha| \leq k$, we have that $(\rho\partial_\rho)^a(\rho\partial_v)^b(v\partial_v)^c\partial_y^\alpha(u_1u_2) \in H_{\mathrm{b}}^m$, but this distribution is equal to

$$\sum_{a'\leq a, b'\leq b, c'\leq c, \alpha'\leq\alpha} C_{a',b',c',\alpha'}((\rho\partial_\rho)^{a'}(\rho\partial_v)^{b'}(v\partial_v)^{c'}\partial_y^{\alpha'}u_1) \\ ((\rho\partial_\rho)^{a-a'}(\rho\partial_v)^{b-b'}(v\partial_v)^{c-c'}\partial_y^{\alpha-\alpha'}u_2), \tag{5.38}$$

for some combinatorial constants $C_{a',b',c',\alpha'}$ (which depend on a, b, c, α). In each of these terms we have the product of two elements u_1, u_2, which $u_i \in H_{\mathrm{b},++}^{m,0,k-r_i}$ where $k - r_1 + k - r_2 \geq k$. But by the previous paragraph, locally near S_+, the u_i satisfy that $\widetilde{u}_i$ lies in $\mathcal{Y}_d^{m,k-r_i}$. Thus by the first part of Lemma 5.9, $\widetilde{u_1}\widetilde{u_2} \in H^m$, which is to say that $u_1u_2 \in H_{\mathrm{b}}^{m,0}$, locally near S_+, which is what we wanted in the case $l_1 = l_2 = 0$. For general weights, i.e. $u_i, \in H_{\mathrm{b},++}^{m,l_i,k}$, $i = 1, 2$, apply the above arguements to $\rho^{-l_1-l_2}u_1u_2 = (\rho^{-l_1}u_1)(\rho^{-l_2}u_2)$. □

Finally, we can prove Proposition 5.5 above.

Proof of Proposition 5.5. Using that multiplication is local, we will reduce in the end to considering two compactly supported distributions u_i, $i = 1, 2$, with $u_i \in H_{\mathrm{b}}^{m,l_i,k}(M)$ supported in a neighborhood of a point $x \in M$.

While in fact the boundary case discussed below handles this as well, we first treat interior points. So assume that the u_i are supported in a coordinate chart in M°. Thus the $u_i \in H^{m+k}(\mathbb{R}^n)$, and since $m > 1/2, k > (n-1)/2$, there is a constant $m_0 \in \mathbb{R}$ so that the $u_i \in H^{m_0+k}(\mathbb{R}^n)$ and $m_0 + k > n/2$. Since H^{m_0+k} is an algebra, $u_1u_2 \in H^{m_0+k}$. We claim that $u_1u_2 \in H^{m-\epsilon+k}$. Indeed, for any x and for any $s \in \mathbb{R}$ with $s - k < m_+ = \max_{S_x^*\mathbb{R}^n} m$, the sets $\mathrm{WF}^s(u_i)$ satisfy

$$\mathrm{WF}^s(u_i) \cap T_x^*(\mathbb{R}^n) \subset \{(x, \xi) : m(x, \hat{\xi}) + k \leq s\}. \tag{5.39}$$

By Remark 5.12,

$$\mathrm{WF}^s(u_1u_2) \subset (\mathrm{WF}^s(u_1) + \mathrm{WF}^s(u_2)) \cup \mathrm{WF}^s(u_1) \cup \mathrm{WF}^s(u_2),$$

and thus, *by the assumption that the non-trivial sublevel sets are convex,* we conclude that $\mathrm{WF}^s(u_1u_2) \cap T^*_x\mathbb{R}^n$ is also a subset of $\{(x,\xi) : m(x,\hat{\xi}) + k \le s\}$. To see that $u_1u_2 \in H^{m-\epsilon+k}$ then, for any (x,ξ) let $s = m(x,\xi)-\epsilon/2+k$, and then note that since $(x,\xi) \notin \mathrm{WF}^s(u_i)$ for $i = 1, 2$, by what we just said also $(x,\xi) \notin \mathrm{WF}^s(u_1u_2)$.

For $x \in \partial M$ and the u_i supported near x, we first assume that $l_1 = 0 = l_2$. Since such u_i are also contained in $H^{m_-,0,k}_{\mathrm{b},++}$, by Lemma 5.14 we know that $u_1u_2 \in H^{m_-,0,k}_{\mathrm{b},++}$. Due to the second assumption in (5.3), given $\epsilon > 0$, microlocally near ${}^{\mathrm{b}}SN^*_+S_+$ (with the neighborhood size depending on ϵ), $H^{m_-,0,k}_{\mathrm{b},++}$ is contained in $H^{m-\epsilon,0,k}_{\mathrm{b},+-}$, so microlocally near ${}^{\mathrm{b}}SN^*_+S_+$ we have the conclusion of the proposition (if $l_1 = l_2 = 0$). Thus, it remains to consider points $q \in {}^{\mathrm{b}}SN^*\mathbb{R}^n$ away from ${}^{\mathrm{b}}SN^*_+S_+$, but there the microlocal membership of $H^{m-\epsilon,0,k}_{\mathrm{b},+-}$ is equivalent to not being an element of $\mathrm{WF}^{m+k-\epsilon}_{\mathrm{b}}$. Now $(x,\xi) \in \mathrm{WF}^{s,0}_{\mathrm{b}}(u_i)$ if and only if $\xi \in \mathrm{WF}^s(\chi\widetilde{u_i})$ for each χ with $\chi \equiv 1$ near x. But then by Lemma 5.11 we have (5.25) with $\widetilde{u}_i$ replacing u, v. Thus the same statements hold for this product as for the interior case, and the same argument shows that their product is in $H^{m+k-\epsilon,0}_{\mathrm{b}}$, completing the proof of the proposition in $l_1 = l_2 = 0$. The weights are multiplicative, establishing the proposition apart from the constant m case. In the case of constant m, we only need to observe that in all the arguments we can take $\epsilon = 0$. □

5.3. **A semilinear problem.** Using the above, one has a complete analogue of the semilinear results of [27]. Concretely, one can conclude that the Feynman problem for the equation

$$\Box_g u + \lambda u^p = f \tag{5.40}$$

is well-posed for appropriate $p \in \mathbb{N}$ (and small f), which includes $p \ge 3$ if $n \ge 4$, so in particular the not-yet-second-quantized φ^4 theory is well-behaved on these curved space-times.

Theorem 5.15. *Suppose g is a perturbation of Minkowski space in the sense of Lorentzian scattering metrics (see Section 2) so that in particular Theorem 5.1 holds. Let $p \in \mathbb{N}, \lambda \in \mathbb{R}$ with p and the dimension n satisfying*

$$\frac{2}{p-1} < \frac{n-2}{2}. \tag{5.41}$$

Let $l < 0$ satisfy

$$l \in \Big(\frac{2}{p-1} - \frac{n-2}{2}, 0\Big). \tag{5.42}$$

Note the interval is non-empty. Let $k > \frac{n-1}{2}$ be an integer and m a function on ${}^{\mathrm{b}}S^*M$ given by Corollary 5.6, or instead take $m > 1/2$ constant and $k > \frac{n-1}{2}$ integer with $m+l < 1/2$, $m+l+k > 3/2$ which exist if (5.48) *holds. Then there is a constant $C > 0$ such that the small-data Feynman problem for* (5.40)*, i.e. given*

$$f \in H^{m-1,l+(n-2)/2+2,k}_{\mathrm{b},+-} \text{ with norm } < C$$

finding $u \in H^{m,l+(n-2)/2,k}_{\mathrm{b},+-}$ satisfying the equation, is well-posed, and u can be calculated as the limit of a Picard iteration corresponding to the perturbation series.

In particular the above holds for $p \geq 4$ and $n \geq 4$.

Remark 5.16. As mentioned, the condition on p and n in (5.41) holds in particular if $p \geq 4$ and $n \geq 4$. It holds also if $p = 3$ and $n \geq 5$ and when $n = 3, p \geq 6$, *but fails* for $p = 3, n = 4$. We tackle this case in Theorem 5.22 below.

Remark 5.17. Our argument also works for the retarded and advanced problems considered in [27, Section 5], but it gives a somewhat different result since the multiplicative properties we use are somewhat different, as necessitated by the microlocal nature of the spaces that have to be used for the Feynman problems. For the retarded/advanced problems [27, Section 5.4] considers general first order semilinearities. In the case of no derivatives an analogous result is shown there under the constraint $p > 1 + \frac{3}{n-2}$ which is weaker than (5.41). For $n = 4$ the difference is whether $p = 3$ is admissible; it borderline fails our inequality. However, as noted above, in Section 5.4 we improve our result by showing better multiplicative properties to handle $n = 4$, $p = 3$.

Remark 5.18. By using m constant as in the statement of the theorem we can consider first order derivative nonlinearities as well, provided in addition $m > 3/2$ (so that for the first derivatives the b-order is $m - 1 > 1/2$, note that constant m is crucial since we cannot afford the $\epsilon > 0$ losses in multiplication); the natural assumption is that these are of the form of a finite sum of products of vector fields $V_j \in \mathcal{V}_{\mathrm{sc}}(M)$ applied to u, times u^p: $u^p(V_1 u) \dots (V_q u)$, $p + q \geq 2$. Writing these as $V_j = \rho W_j$, $W_j \in \mathcal{V}_{\mathrm{b}}(M)$, we can proceed as in [27, Section 5.4] and as we proceed below in the proof, provided that we can take some $l < -1$ (necessitated by $m > 3/2$, see Remark 5.4) for the Feynman propagator invertibility considerations. This requires $n \geq 5$ so that $\frac{n-2}{2} > 1$. The numerology paralleling (5.45)-(5.46) is then that nonlinearities satisfying

$$(p-1)\frac{n-2}{2} + q\frac{n}{2} + (p+q)l - 2 \geq l$$

can be handled by the same method. One can satisfy this with l sufficiently close to -1, $l < -1$, if

$$(p-1)(n-4) + q(n-2) > 2,$$

which is weaker than the requirement $(p-1)(n-4) + q(n-2) > 4$ of [27, Equation (5.15)], but recall that here we need $n \geq 5$. In particular, as our results also apply for the advanced/retarded problems, see Remark 5.17, if $n \geq 5$, this slightly improves the result of [27], allowing e.g. $q = 1$ and $p = 1$ for $n = 5$; $q = 1, p = 1$ is not allowed in [27] even after improvements discussed there in Remark 5.16 using analogues of our improvements in Section 5.4.

Proof of Theorem 5.15. As in [27, Section 5], moving the λu^p to the right hand side, we rewrite (5.40) as

$$L\widetilde{u} = \widetilde{f} - \lambda\rho^{-2+(p-1)(n-2)/2}\widetilde{u}^p, \tag{5.43}$$

where $\widetilde{u} := \rho^{-(n-2)/2}u$, $\widetilde{f} := \rho^{-2-(n-2)/2}f$. Assuming that $f \in H^{m-1,l+(n-2)/2+2,k}_{\mathrm{b},+-}$, we have $\widetilde{f} \in H^{m-1,l,k}_{\mathrm{b},+-}$. To apply a Picard iteration to (5.43), we want the right hand side to be in the domain of the forward Feynman inverse of L, $L^{-1}_{+-} : \mathcal{Y}^{m-1,l,k} \longrightarrow \mathcal{X}^{m,l,k}$ (where $\mathcal{Y}^{m,l,k}, \mathcal{X}^{m,l,k}$ are defined as in (2.21) with the $H^{m,l}_{\mathrm{b}}$ replaced by $H^{m,l,k}_{\mathrm{b}}$), so by Theorem 5.1 we want it in $H^{m-1,l,k}_{\mathrm{b},+-}$ where $m + l$ now satisfies the defining properties of the Feynman propagator and such that $|l| < (n-2)/2$. Furthermore, we want to apply the algebra properties in Proposition 5.5; in particular we assume that $m > 1/2$ everywhere. Note that we have both

$$m > 1/2 \text{ and } m + l < 1/2 \text{ near } {}^{\mathrm{b}}SN^*_+S_+ \text{ and } {}^{\mathrm{b}}SN^*_+S_-, \tag{5.44}$$

i.e. in the low regularity regions. Thus l (which is just a real number) *must be negative*, and furthermore for any $l < 0$ there is a function m meeting all of the criteria of Corollary 5.6, in particular both the criteria in (5.44) and the Feynman criteria (since m increases as one approaches the high regularity regions ${}^{\mathrm{b}}SN^*_-S_-$ and ${}^{\mathrm{b}}SN^*_-S_+$), as well as the convexity/minima criteria (5.3). (Indeed, note that by the remarks preceding Corollary 5.6, we could even take m constant if we use Theorem 5.2 in place of Theorem 5.1. In this case we can take $\epsilon = 0$ below.) Under these assumptions, by Proposition 5.5, for $k > (n-1)/2$ we have that $\widetilde{u}^p$ lies in $H^{m-\epsilon,pl,k}_{\mathrm{b},+-}$ for any $\epsilon > 0$, and thus $\rho^{-2+(p-1)(n-2)/2}\widetilde{u}^p$ lies in $H^{m-\epsilon,l',k}_{\mathrm{b},+-}$ where

$$l' = -2 + (p-1)(n-2)/2 + pl, \tag{5.45}$$

and $H^{m-\epsilon,l',k}_{\mathrm{b},+-} \subset H^{m-1,l,k}_{\mathrm{b},+-}$ if and only if

$$l' \geq l \iff l \geq \frac{2}{p-1} - \frac{n-2}{2}, \tag{5.46}$$

where again l is an arbitrary negative number.

For any p, n such that

$$\frac{2}{p-1} < \frac{n-2}{2}, \tag{5.47}$$

taking

$$l' = l \in \Big(\frac{2}{p-1} - \frac{n-2}{2}, 0\Big) \tag{5.48}$$

sufficiently small, and m picked correspondingly as above, we claim that for every $\delta > 0$, there is an $R \geq 0$ such that if both $\|\widetilde{u}\|_{H^{m,l,k}_{\mathrm{b}}}$ and $\|\widetilde{v}\|_{H^{m,l,k}_{\mathrm{b}}}$ are bounded by R then

$$\|\rho^{-2+(p-1)(n-2)/2}\widetilde{u}^p - \rho^{-2+(p-1)(n-2)/2}\widetilde{v}^p\|_{H^{m-1,l,k}_{\mathrm{b}}} \leq \delta\|\widetilde{u} - \widetilde{v}\|_{H^{m,l,k}_{\mathrm{b}}}. \tag{5.49}$$

Assuming the claim for the moment, we see that the map

$$\widetilde{u} \mapsto L^{-1}_{+-}(\widetilde{f} + \lambda\widetilde{u}^p\rho^{-2+(p-1)(n-2)/2})$$

is a contraction mapping on $H^{m,l,k}_{\mathrm{b}}$ and thus the Picard iteration $\widetilde{u}_{n+1} = L^{-1}_{+-}(\widetilde{f} + \lambda\widetilde{u}^p_n\rho^{-2+(p-1)(n-2)/2})$ with $\widetilde{u}_1 = 0$ converges if $\widetilde{f}$ is sufficiently small in $H^{m-1,l,k}_{\mathrm{b}}$ (as assumed in the theorem).

Thus it remains only to prove the claim. For any l and for any $\epsilon > 0$ we have

$$\begin{aligned}\|\widetilde{u}^p - \widetilde{v}^p\|_{H^{m-\epsilon,pl,k}_{\mathrm{b}}} &= \Big\|(\widetilde{u} - \widetilde{v})\sum_{j=0}^{p-1}\widetilde{u}^j\widetilde{v}^{p-1-j}\Big\|_{H^{m-\epsilon,pl,k}_{\mathrm{b}}} \\ &\leq C\|\widetilde{u} - \widetilde{v}\|_{H^{m,l,k}_{\mathrm{b}}}\max(\|\widetilde{u}\|_{H^{m,l,k}_{\mathrm{b}}}, \|\widetilde{u}\|_{H^{m,l,k}_{\mathrm{b}}})^{p-1}\end{aligned} \tag{5.50}$$

provided $m - (p-2)\mu > 1/2$. Since l satisfies (5.46) with l' as in (5.48), by bounding the $H^{m,l,k}_{\mathrm{b}}$ norm with the $H^{m,l',k}_{\mathrm{b}}$ norm,

$$\|\rho^{-2+(p-1)(n-2)/2}\widetilde{u}^p - \rho^{-2+(p-1)(n-2)/2}\widetilde{v}^p\|_{H^{m-\epsilon,l,k}_{\mathrm{b}}} \leq \|\widetilde{u}^p - \widetilde{v}^p\|_{H^{m-\epsilon,pl,k}_{\mathrm{b}}},$$

and combining with (5.50) gives the claim once $\mu > 0$ is taken sufficiently small. □

5.4. **More intricate multiplicative properties and cubic semilinear problems for $n = 4$.** To extend to $p = 3$, $n = 4$, we need improvements of the regularity properties for products which allow us to take the weight l' to be greater than zero. To do so and still have $m + l' < 1/2$ in the low regularity zone, we need $m < 1/2$, which is below the regularity threshold in the work in Section 5.9; thus we need improvements of the results therein. The necessary improvements are based on the ideas in the following.

Lemma 5.19. *Let $s, s', s_0 \in \mathbb{R}$. Let $u, v \in H^{s_0}$, then $\mathrm{WF}^{s'}(uv) \subset \mathrm{WF}^s(v)$, provided $s \geq s_0 \geq s'$ and $s - s' + s_0 > n/2$.*

The point here is that one can take $s_0 < n/2$, and obtain a result for uv which says it is in a worse Sobolev space then H^{s_0} microlocally provided v is in a better one microlocally.

The proof in fact follows the first of the product regularity arguments above, namely that (5.13) implies (5.12). Consider a point ξ_0 with $(x_0, \xi_0) \notin \mathrm{WF}^s(v)$, and a function $\mathsf{s} \geq s_0$ which equals s on an open cone $\mathsf{C} \subset (\mathrm{WF}^s(v))^{comp}$ and take $\mathsf{s}' \equiv s_0$ outside some compact set $\mathsf{K} \subset \mathsf{C}$, $\mathsf{s}' \equiv s'$ near ξ_0 with $\mathsf{s}' \leq \mathsf{s}$ everywhere; as we show this implies that $H^{\mathsf{s}} \cdot H^{s_0} \subset H^{\mathsf{s}'}$. Indeed, this is analogous to (5.14) above, and we break the relevant integral I_ξ up in the same way as in (5.17), so we must bound integrals

$$\sup_{\xi \in K} \int \frac{1}{\langle \eta \rangle^{2\mathsf{s}-2\mathsf{s}'} \langle \xi - \eta \rangle^{2s_0}} \, d\eta \quad \text{and} \quad \sup_{\xi \in \mathsf{K}} \int \frac{1}{\langle \eta \rangle^{2\mathsf{s}} \langle \xi - \eta \rangle^{2s_0 - 2\mathsf{s}'}} \, d\eta.$$

The second integral is bounded by the arguments above, and for the first integral, the only difference is that over the set C, using that $\mathsf{s} - \mathsf{s}' \geq 0$ there, we have

$$\int_{\mathsf{C}} \frac{1}{\langle \eta \rangle^{2\mathsf{s}-2\mathsf{s}'} \langle \xi - \eta \rangle^{2s_0}} \, d\eta \leq \int_{\mathsf{C}} \frac{1}{\langle \xi - \eta \rangle^{2(\mathsf{s}-\mathsf{s}'+s_0)}} \, d\eta \leq \int_{\mathsf{C}} \frac{1}{\langle \xi - \eta \rangle^{2(s-s'+s_0)}} \, d\eta,$$

which is finite since $s - s' + s_0 > n/2$. The rest of the estimates are exactly as in the previous case.

Applying this line of thinking to the model spaces $\mathcal{Y}_d^{m,a}$ defined in (5.18), we can obtain a regularity result for products which allows us to dip under the threshold $d/2$ above.

Lemma 5.20. *For $m, m', m_0, a \in \mathbb{R}$ such that $m - m' + m_0 > d/2$, $a > (n-d)/2$ and $m \geq m_0 \geq m'$, we have $\mathcal{Y}_d^{m_0,a} \cdot \mathcal{Y}_d^{m,a} \subset \mathcal{Y}_d^{m',a}$. Furthermore,*

$$\mathrm{WF}^{m'+a}(uv) \subset (\mathrm{WF}^{m+a}(u) + \mathrm{WF}^{m+a}(v)) \cup \mathrm{WF}^{m+a}(u) \cup \mathrm{WF}^{m+a}(v). \tag{5.51}$$

Proof. To see that the first conclusion holds, we argue as in Lemma 5.9, and thus use the inequality

$$\begin{aligned} &\int \left(\frac{\langle \xi \rangle^{m'} \langle \xi'' \rangle^a}{\langle \xi - \eta \rangle^{m_0} \langle \xi'' - \eta'' \rangle^a \langle \eta \rangle^m \langle \eta'' \rangle^a} \right)^2 d\eta \\ &\leq \sum_{i,j=1}^{2} \int \left(\frac{f_i g_j}{\langle \xi - \eta \rangle^m \langle \xi'' - \eta'' \rangle^a \langle \eta'' \rangle^a} \right)^2 d\eta, \end{aligned} \tag{5.52}$$

where $f_1 = \langle \eta \rangle^{m'}$, $f_2 = \langle \xi - \eta \rangle^{m'}$ and $g_1 = \langle \eta'' \rangle^a$, $g_2 = \langle \xi'' - \eta'' \rangle^a$. Replacing the unprimed variable with primed variables and using that both

$$\int \left(\frac{\langle \eta' \rangle^{m'}}{\langle \xi - \eta' \rangle^{m_0} \langle \eta' \rangle^m} \right)^2 d\eta', \qquad \int \left(\frac{\langle \xi' - \eta' \rangle^{m'}}{\langle \xi - \eta' \rangle^{m_0} \langle \eta' \rangle^m} \right)^2 d\eta'$$

are uniformly bounded under the stated assumptions on m, m', m_0 gives the statement.

The wavefront set containment is obtained by locating similar improvements in the proof of Lemma 5.11. Indeed, as there, we have that $u \in H^{(w_1)}, v \in H^{(w_2)}$ where $w_i(\xi) = \langle \xi \rangle^{m_0} \langle \xi'' \rangle^a + \langle \xi \rangle^{s_i}$ where $s_i = s_i(\xi)$ where the $s_i \equiv m + a$ off open conic sets $\widetilde{\mathsf{C}}_i$ are arbitrary open sets containing, respectively, $\mathrm{WF}^{m+a}(u)$ and $\mathrm{WF}^{m+a}(v)$. We want to show that given

$$\xi \notin (\mathrm{WF}^{m+a}(u) + \mathrm{WF}^{m+a}(v)) \cup \mathrm{WF}^{m+a}(u) \cup \mathrm{WF}^{m+a}(v)$$

and a proper choice of function s with $s = m' + a$ near ξ that $uv \in H^s$, which amounts to applying Lemma 5.7 with $w = s$ and w_1, w_2 exactly as in Lemma 5.11.

We thus want to bound an integral similar to (5.27), namely

$$\int \Big(\frac{\langle \xi \rangle^s}{(\langle \eta \rangle^{m_0} \langle \eta'' \rangle^k + \langle \eta \rangle^{s_1})(\langle \xi - \eta \rangle^{m_0} \langle \xi'' - \eta'' \rangle^k + \langle \xi - \eta \rangle^{s_2})} \Big)^2 d\eta,$$

We choose s so that $s \le m + a$ and bound $\langle \xi \rangle^{2s} \lesssim \langle \eta \rangle^{2(m+a)} + \langle \xi - \eta \rangle^{2(m+a)}$ and as usual break the integral into two parts involving the two terms on the right of this bound. Again we focus on the $\langle \eta \rangle^{2s}$ term. Integrating first over $\widetilde{\mathsf{C}}_1$ and then $(\widetilde{\mathsf{C}}_1)^{comp}$; over $\widetilde{\mathsf{C}}_1^c$, we have that $\langle \eta \rangle^m \langle \eta'' \rangle^k + \langle \eta \rangle^{s_1} > (1/2) \langle \eta \rangle^s$, so that part of the integral with $\langle \eta \rangle^{2s}$ in the numerator over $\widetilde{\mathsf{C}}_1$ is bounded by

$$\begin{aligned} &\int \Big(\frac{1}{\langle \eta \rangle^{s_1 - s} \langle \xi - \eta \rangle^m \langle \xi'' - \eta'' \rangle^k + \langle \xi - \eta \rangle^{s_2}} \Big)^2 d\eta \\ &\le \int \Big(\frac{1}{\langle \eta' \rangle^{m - m'} \langle \xi' - \eta' \rangle^{m_0} \langle \xi'' - \eta'' \rangle^k}, \Big)^2 d\eta. \end{aligned}$$

which, by separating into primed and double primed coordinates and using the assumptions on m, m_0, m' is uniformly bounded. The rest of the bounds proceed analogously and are left to the reader. $\square$

By reducing locally and arguing exactly as in the proof of Proposition 5.5 and Corollary 5.6, we obtain

Proposition 5.21. *For l sufficiently small, there exists $m \colon {}^{\mathrm{b}}S^*M \longrightarrow \mathbb{R}$ satisfying: 1) that $m \ge 1/2 - \delta$ for some $\delta \in (0, 1/2)$, 2) m, l satisfy the forward Feynman condition in the strengthened form given in Theorem 3.3, and 3) m satisfies the condition on the sublevel sets and minima in (5.3). Moreover, for $k \in \mathbb{N}$ satisfying $k > (n-1)/2$,*

$$H^{m,l_1,k}_{\mathrm{b},+-} H^{m,l_2,k}_{\mathrm{b},+-} \subset H^{m-2\delta-0, l_1+l_2,k}_{\mathrm{b},+-}.$$

In particular, for δ sufficiently small $(H^{m,l_1,k}_{\mathrm{b},+-})^3 \subset H^{m-4\delta-0,l_1+l_2,k}_{\mathrm{b},+-}$.

We can now finally prove

Theorem 5.22 ($p = 3, n = 4$)**.** *Suppose g is a perturbation of Minkowski space in the sense of Lorentzian scattering metrics (see Section 2), in particular so that Theorem 5.1 holds. In dimension $n = 4$, given $\lambda \in \mathbb{R}$ and a weight $l \ge 0$ and*

l sufficiently small, and a regularity function m as in Proposition 5.21, there is $C > 0$ such that the small-data Feynman problem, i.e. given

$$f \in H_{\mathrm{b},+-}^{m-1,l+(n-2)/2+2,k} \text{ with norm } < C$$

finding $u \in H_{\mathrm{b},+-}^{m,l+(n-2)/2,k}$ satisfying

$$\Box_{g,+-} u + \lambda u^3 = f, \tag{5.53}$$

is well-posed with $k > (n-1)/2$, and u can again be calculated as the limit of a Picard iteration corresponding to the perturbation series.

Remark 5.23. Again by Remark 5.4, there is a version of this theorem with m constant, $m \geq 1/2 - \delta$, $l \geq 0$, $m + l < 1/2$ and $k > (n-1)/2$ satisfying $m + l + k > 3/2$. We leave the details of this straightforward modification of the following proof to the reader.

Remark 5.24. Although we do not state it here explicitly, Proposition 5.21 also gives improvements to the statement of Theorem 5.15 for other n, p in terms of the spaces in which solvability holds (what l can be), though not for whether there *is* a space of the kind considered there in which solvability holds.

Proof. The proof is identical to that of Theorem 5.15 incorporating the improvements given by Proposition 5.21. We take $l \geq 0$ and find an m such that $m + l$ satisfies the Feynman condition and $m > 1/2 - \delta$ for some small $\delta > 0$. Rewriting the equation as in (5.43) with $\widetilde{f}$ and $\widetilde{u}$ defined in the same way, and assuming that $\widetilde{f} \in H_{\mathrm{b},+-}^{m,l,k}$, the condition that

$$\rho^{2-(p-1)(n-2)/2} \widetilde{u}^p = \rho^{4-n} \widetilde{u}^3 \in H_{\mathrm{b},+-}^{m-5\delta,l,k} \subset H_{\mathrm{b},+-}^{m-1,l,k}$$

is now that δ be less than $1/5$ and that

$$l \leq 4 - n + 3l \iff l \geq n/2 - 2. \tag{5.54}$$

If $n = 4$, we can thus find l and m satisfying the Feynman conditions and (5.54) simultaneously. From now on we assume that $n = 4$. The existence of an m satisfying the conditions in (5.3) is a trivial modification of the proof of Corollary 5.6.

The Picard iteration argument is now identical to that in Theorem 5.15 except incorporating the loss in Proposition 5.21. In this case, the claim in 5.49 is substituted by the following: for every $\delta > 0$ sufficiently small, there is an $R \geq 0$ such that if both $\|\widetilde{u}\|_{H_{\mathrm{b}}^{m-\delta,l,k}}$ and $\|\widetilde{v}\|_{H_{\mathrm{b}}^{m-\delta,l,k}}$ are bounded by R then

$$\|\widetilde{u}^3 - \widetilde{v}^3\|_{H_{\mathrm{b}}^{m-5\delta,l,k}} \leq \delta \|\widetilde{u} - \widetilde{v}\|_{H_{\mathrm{b}}^{m-\delta,l,k}}.$$

This and the rest of the proof follow exactly as in Theorem 5.15 using the improvement in Proposition 5.20. □

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