GEOMETRIC SCATTERING THEORY FOR LONG-RANGE POTENTIALS AND METRICS

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1. Introduction

Let X be a compact manifold with boundary, $n = \dim X$, and let x be a boundary defining function, i.e. $x \in \mathcal{C}^\infty(X)$, $x \geq 0$, $\partial X = \{p \in X : x(p) = 0\}$, and dx is not zero at ∂X. We consider the following class of asymptotically flat, complete metrics on X which provide the background for a natural generalization of Euclidian scattering theory, first discussed by Melrose in [13]. A Riemannian metric g in the interior of X is called a long-range scattering metric if it can be brought to the form $g = a\frac{dx^2}{x^4} + \frac{h'}{x^2}$ near ∂X for some choice of a boundary defining function x, $a-1 \in x\mathcal{C}^\infty(X)$, and for some smooth symmetric 2-cotensor h' on X which restricts to a metric h on ∂X. Also, following [13], we say that g is a (short-range) scattering metric if we can take $a = 1$ above.

A particular example of this setup is the radial compactification of Euclidian space $\mathbb{R}^n$ to a hemisphere (i.e. a ball) $X = \mathbb{S}^n_+$ by a (non-standard) version of the stereographic projection SP, see [13], and the corresponding lifting of the standard Euclidian metric. More generally, near ∂X we can write X as $[0,\epsilon)_x \times \partial X$. Introducing $r = x^{-1}$ and thereby moving ∂X to 'infinity', this region can be regarded as $(\epsilon^{-1}, \infty)_r \times \partial X$. Then metrics of the form $dr^2 + r^2 h$ for large r, h a metric on ∂X, become (short-range) scattering metrics if we reintroduce $x = r^{-1}$, i.e. we can regard this region, when equipped with such a metric, as the 'large end' of a cone (the tip would have been $r = 0$). Apart from the intrinsic geometric interest in the study of scattering metrics on arbitrary manifolds with boundary, understanding these can clarify Euclidian scattering theory by removing the special symmetries. It can also provide the foundations for a more detailed description of such complex subjects as many-body scattering (see [3, 18] and especially [16, 17]).

Let Δ be the (positive) Laplacian of g, and let $H = \Delta + V$, where V is a second-order differential operator, be such that H is self-adjoint, satisfies non-degeneracy (ellipticity) conditions and that the behavior of H at ∂X is dominated by that of Δ in a certain natural sense. If V is multiplication by a real-valued function, these requirements amount to the statement that $V \in x\mathcal{C}^\infty(X)$, i.e. that V vanishes at the boundary ∂X; the general (and precise) setup is discussed in the following section. We remark that $V \in x\mathcal{C}^\infty(\mathbb{S}^n_+)$ means that V is the pull back of a classical (polyhomogeneous) symbol of order -1 from $\mathbb{R}^n$, so Coulomb-type potentials on $\mathbb{R}^n$ (without the singularity at the origin) fit into this framework. Such a perturbation V is 'long-range' in the sense of [13]; an example of a 'short-range' V is $V \in x^2\mathcal{C}^\infty(X)$.

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Thus, suppose that $V \in x\mathcal{C}^\infty(X)$, write $V = xV'$, so $V' \in \mathcal{C}^\infty(X)$, and given $\lambda \neq 0$ introduce $\alpha_\pm = \alpha_{\pm,\lambda} \in \mathcal{C}^\infty(\partial X)$ which for short-range g is defined by

$$\alpha_\pm = \pm\frac{V'|_{\partial X}}{2\lambda}; \tag{1.1}$$

the general definition (for long-range g and arbitrary V) is given in the next section. We sometimes extend α to a smooth function on X in an arbitrary fashion.

We can now define the scattering matrix, $S(\lambda)$, of H, very easily via asymptotic expansions of generalized eigenfunctions of H, i.e. 'Sommerfeld patterns'. We first note that for any $\lambda \in \mathbb{R} \setminus \{0\}$ and $a \in \mathcal{C}^\infty(\partial X)$ there is a unique $u \in \mathcal{C}^{-\infty}(X)$ which satisfies $(H - \lambda^2)u = 0$ and which is of the form

$$\begin{aligned} u = e^{-i\lambda/x} x^{i\alpha_- + (n-1)/2} v_- &+ e^{i\lambda/x} x^{i\alpha_+ + (n-1)/2} v_+ + u', \\ &u' \in L^2(X, dg),\ v_\pm \in \mathcal{C}^\infty(X),\ v_-|_{\partial X} = a. \end{aligned} \tag{1.2}$$

We define the Poisson operator $P(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^{-\infty}(X)$ as the map $P(\lambda)a = u$. Moreover, the scattering matrix $S(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^\infty(\partial X)$ is the map

$$S(\lambda)a = v_+|_{\partial X}. \tag{1.3}$$

Henceforth we take $\lambda > 0$ for definiteness. The structure of $S(\lambda)$ is far from being evident. Melrose and Zworski showed in [14] that in the short-range case, g a (short-range) scattering metric and V a general second order 'short-range' perturbation, $S(\lambda)$ is a classical Fourier integral operator of order 0 which quantizes the geodesic flow of the boundary metric h at distance π. An analogous result was proved by Joshi [8] when g is still short-range, $V \in x\mathcal{C}^\infty(X)$, and $\alpha_\pm$ (hence $V'|_{\partial X}$) are constant ('strictly Coulomb' case).

In this paper we generalize these results to the full long-range setting by using a different and perhaps simpler method. To do that we define a class of Fourier integral operators (FIO's) of orders whose imaginary part varies along the Lagrangian to which the FIO is associated. These FIO's will be in the classes I_ρ of Hörmander [4] for $\rho < 1$, with a behavior specified by the variable order. This allows us to describe these FIO's rather similarly to the standard treatment of classical (one-step polyhomogeneous) ones. One of the main results of this paper is the following theorem of which a precise version is stated in Section 5.

Theorem. *For $\lambda > 0$ the scattering matrix, $S(\lambda)$, is a Fourier integral operator whose canonical relation is given by the (forward) geodesic flow of h on ∂X at distance π; the imaginary part of the order of $S(\lambda)$ varies and depends on $\alpha_\pm$.*

As mentioned, this result was obtained in many special cases by Melrose, Zworski and Joshi. Indeed, we would like to place at least as much (if not more) emphasis on our methods and how they differ from those of these authors, as on the generalization of their result. We also note that Yafaev gave a description of the S-matrix of more general (very long-range) potential perturbations of the Euclidian Laplacian in [20].

In this paper we also compute the principal symbol of $S(\lambda)$ and describe the structure of the Poisson operator. A combination of the methods of this paper and those of Joshi and Sá Barreto could be used to analyze the inverse problem of determining the asymptotics of V from the singularities of $S(\lambda)$, as was done in the short-range and constant coefficient leading term long-range cases by Joshi and Sá Barreto [11, 10, 8].

Without giving the specifics, we recall Melrose's and Zworski's WKB-type construction for the kernel of the Poisson operator [14]. This construction proceeds much as the usual FIO construction for hyperbolic equations, but the associated Legendre submanifold (replacing the usual Lagrangians) has an 'end' (i.e. a boundary) where it has a conic singularity. The singularities of the scattering matrix can be deduced from the full symbol of $P(\lambda)$ at the end of the Legendrian. The crucial part of the construction is the understanding of these singular Legendre submanifolds and distributions associated to them.

In this paper we approach the subject differently. We show how a pairing formula, given here in the simple but central Proposition 5.1, can lead one to understand $S(\lambda)$ using only the construction of the kernel of $P(\pm\lambda)$ at the smooth part of the associated Legendre submanifold. As mentioned above, this is close to the standard FIO constructions and hence relatively easy. Thus, the above theorem is proved without having to resort to singular Legendre submanifolds. Indeed, this could be expected since the S-matrix is a simpler object than the Poisson operator. However, having understood the S-matrix, we can use the formula $P(\lambda) = P(-\lambda)S(\lambda)$ to analyze the structure of $P(\lambda)$ at the singularity of the Legendre submanifold; to do so we only need to understand $P(-\lambda)$ in the region where the standard FIO-type construction works. Hence, we deduce the structure of $P(\lambda)$ near the conic singularity without the need for its *a priori* understanding. This is particularly useful in the long-range problem described above where the structure is harder to describe. Once the structure of $P(\lambda)$ has been completely understood, one may wish to rewrite one's results using singular Legendre distributions because the arguments are 'cleaner' that way; indeed one *should* understand the singular Legendre geometry underlying the analysis. In our long-range case this would include describing the calculus for a generalization of classical FIO's the order of whose symbols are pure imaginary and vary along the Lagrangian (or Legendrian in our setting) and related objects corresponding to the singular Legendre submanifold. Since this can be avoided by our techniques, we will limit our discussion of such distributions to the Lagrangian case and the description of the S-matrix.

The paper is organized as follows. In the next section we review some of the basic properties of the 'scattering geometry' from [13, 14]. Then, in Section 3, we recall from the papers of Melrose and Zworski [14], Joshi [8] and Vasy [17, Appendix A] (see also [16]) the construction of the kernel of $P(\pm\lambda)$ along the smooth part of the Legendre submanifold. In Section 4 we give a brief description of polyhomogeneous Lagrangian distributions of variable order. In Section 5 we prove that $S(\lambda)$ is an FIO, and in Section 6 we show how its principal symbol can be computed. Finally, in Section 7 we analyze the kernel of $P(\pm\lambda)$ near the singularity of the Legendrian.

We use this opportunity to remark that our normalization of $P(\lambda)$ and $S(\lambda)$ (vs. $P(-\lambda)$ and $S(-\lambda)$) follows [14] instead of [13, 8, 17].

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2. Fundamentals of analysis in the setting of 'scattering geometry'

We now very briefly review some basic properties of the 'scattering geometry' from Melrose's paper [13]. First, Melrose defines the Lie algebra of 'scattering vector fields' as $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$ where $\mathcal{V}_{\mathrm{b}}(X)$ is the set of smooth vector fields on X which are tangent to ∂X. If $(x, y_1, \ldots, y_{n-1})$ are coordinates on X where x is a boundary defining function, then locally a basis of $\mathcal{V}_{\mathrm{sc}}(X)$ is given by $x^2\partial_x$, $x\partial_{y_j}$, $j = 1, \ldots, n-1$. Correspondingly, there is a vector bundle ${}^{\mathrm{sc}}TX$ over X, called the scattering tangent bundle of X, such that $\mathcal{V}_{\mathrm{sc}}(X)$ is the set of all smooth sections of ${}^{\mathrm{sc}}TX$: $\mathcal{V}_{\mathrm{sc}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{sc}}TX)$. The dual bundle of ${}^{\mathrm{sc}}TX$ (called the scattering cotangent bundle) is denoted by ${}^{\mathrm{sc}}T^*X$. Thus, covectors $v \in {}^{\mathrm{sc}}T^*_pX$, p near ∂X, can be written as $v = \tau\,\frac{dx}{x^2} + \mu \cdot \frac{dy}{x}$. Hence, we have local coordinates (x, y, τ, μ) on ${}^{\mathrm{sc}}T^*X$ near ∂X. Finally, $\mathrm{Diff}_{\mathrm{sc}}(X)$ is the algebra of differential operators generated by the vector fields in $\mathcal{V}_{\mathrm{sc}}(X)$; $\mathrm{Diff}^m_{\mathrm{sc}}(X)$ stands for scattering differential operators of order (at most) m.

Before proceeding further, it should be emphasized that most local properties of the scattering structure, and in particular of $\mathrm{Diff}_{\mathrm{sc}}(X)$, are very closely related to traditional microlocal analysis. To establish the relationship, we introduce local coordinates on X near $p \in \partial X$ as above, and use these to identify the coordinate neighborhood U of p with a coordinate patch U' on the closed upper hemisphere $\mathbb{S}^n_+$ (which is just a closed ball) near its boundary. Such an identification preserves the scattering structure since this structure is completely natural. We further identify $\mathbb{S}^n_+$ with $\mathbb{R}^n$ via the radial compactification SP mentioned in the introduction; recall from [13] that $\mathrm{SP} : \mathbb{R}^n \to \mathbb{S}^n_+$ is given by

$$\mathrm{SP}(z) = (1/(1+|z|^2)^{1/2}, z/(1+|z|^2)^{1/2}) \in \mathbb{S}^n_+ \subset \mathbb{R}^{n+1}, \quad z \in \mathbb{R}^n. \tag{2.1}$$

The constant coefficent vector fields ∂_{z_j} on $\mathbb{R}^n$ lift under SP to give a basis of ${}^{\mathrm{sc}}T\mathbb{S}^n_+$. Thus, $V \in \mathcal{V}_{\mathrm{sc}}(\mathbb{S}^n_+)$ can be expressed as (ignoring the lifting in the notation)

$$V = \sum_{j=1}^n a_j \partial_{z_j}, \quad a_j \in \mathcal{C}^\infty(\mathbb{S}^n_+). \tag{2.2}$$

As mentioned in the introduction, $a_j \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ is equivalent to requiring that $\mathrm{SP}^*\,a_j$ is a classical (i.e. one-step polyhomogeneous) symbol of order 0 on $\mathbb{R}^n$. Thus, conjugating V by the Fourier transform, $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, gives an operator $\mathcal{F}V\mathcal{F}^{-1}u = \sum_j \mathcal{F}a_j\mathcal{F}^{-1}i\xi_j u$, $u \in \mathcal{S}'(\mathbb{R}^n)$, (the ξ_j denote the dual variables of the z_j), which is a pseudo-differential operator of order 0. The same conclusion holds for scattering differential operators. In particular, for the Euclidian Laplacian, we have $\mathcal{F}(\Delta - \lambda^2)\mathcal{F}^{-1} = |\xi|^2 - \lambda^2$, which is just a multiplication operator, with characteristic variety in the cotangent bundle over the sphere of radius λ centered at the origin.

Although in the following paragraphs we follow Melrose's geometric approach, all local (or microlocal) statements on ${}^{\mathrm{sc}}T^*\mathbb{S}^n_+$ (and hence on ${}^{\mathrm{sc}}T^*X$), such as principal symbols, bicharacteristics, propagation of singularities, translate directly, via the Fourier transform, to conventional microlocal analysis on $T^*\mathbb{R}^n$ (or indeed on the cotangent bundle over bounded subset of $\mathbb{R}^n$), and the reader may want to provide this translation at times. However, later the necessary passage from these local properties to global ones would mean a significant loss of clarity. We also remark that some properties of this relationship are explored, for example, in [13] or in the introduction of [17].

The joint symbol $j_{\mathrm{sc},m,0}(P)$ of a scattering differential operator P of order m consists of two parts. One (which we write as $\sigma_{\mathrm{sc},m}(P)$) is the extension of the usual principal symbol from the interior of X; this is a smooth homogeneous function on ${}^{\mathrm{sc}}T^*X \setminus 0$ (just a polynomial in this case) which, after rescaling corresponding to the homogeneity, can be thought of as a smooth function on the scattering cosphere bundle ${}^{\mathrm{sc}}S^*X$. The other part is a smooth function on ${}^{\mathrm{sc}}T^*_{\partial X}X$; in fact, a polynomial of degree m which is not necessarily homogeneous. Namely, for $p \in \partial X$, $v \in {}^{\mathrm{sc}}T^*_pX$, $P \in \mathrm{Diff}^m_{\mathrm{sc}}(X)$, $j_{\mathrm{sc},m,0}(P)(v) = \hat{P}(v)$ can be defined using the following oscillatory testing description. If $u \in \mathcal{C}^\infty(X)$ and $f \in \mathcal{C}^\infty(X)$ then $\tilde{u} = e^{-if/x}Pe^{if/x}u \in \mathcal{C}^\infty(X)$ and for $p \in \partial X$, $\tilde{u}(p) = \hat{P}(v)u(p)$ where $v \in {}^{\mathrm{sc}}T^*_pX$ is the covector given by $d(f/x)$.

We now give the definition of the class of metrics we are interested in in this paper.

Definition 2.1. A Riemannian metric g in the interior of X is called a long-range scattering metric if can be brought to the form $g = a\frac{dx^2}{x^4} + \frac{h'}{x^2}$ near ∂X for some choice of a boundary defining function x, $a - 1 \in x\mathcal{C}^\infty(X)$, and for some smooth symmetric 2-cotensor h' on X which restricts to a metric h on ∂X.

We note that this fixes x up to $x^2\mathcal{C}^\infty(X)$. Also, following [13], we say that g is a (short-range) scattering metric if we can take $a = 1$ above. We remark that a long-range scattering metric is simply a smooth fiber metric on ${}^{\mathrm{sc}}TX$ which is of a product form *at* ∂X, i.e. the product condition is just on g modulo $x\mathcal{C}^\infty(X; {}^{\mathrm{sc}}T^*X \otimes {}^{\mathrm{sc}}T^*X)$. In the following we let Δ be the Laplacian of the long-range scattering metric g, and assume that x is chosen as required by the definition. Then $\Delta \in \mathrm{Diff}^2_{\mathrm{sc}}(X)$. In fact, choosing a product decomposition $[0,\epsilon)_x \times \partial X$ of a neighborhood of ∂X in X, we have

$$\Delta = (x^2D_x)^2 + x^2\Delta_h + P, \quad P \in x\,\mathrm{Diff}^2_{\mathrm{sc}}(X). \tag{2.3}$$

Next we state the form of the operators that we analyze in this paper.

Definition 2.2. Let Δ be the Laplacian of the long-range scattering metric g. A self-adjoint smooth long-range perturbation V of Δ is an operator $V \in x\,\mathrm{Diff}^2_{\mathrm{sc}}(X)$ such that the principal symbol $\sigma_{\mathrm{sc},2}(H)$ of $H = \Delta + V$ is invertible (i.e. H is elliptic in the usual sense) and H is self-adjoint.

Thus, under this definition H is an unbounded self-adjoint operator on $L^2_{\mathrm{sc}}(X) = L^2(X, dg)$ with domain the weighted Sobolev space $H^{2,0}_{\mathrm{sc}}(X)$. We recall that $H^{r,s}_{\mathrm{sc}}(X)$ is the same as the standard weighted Sobolev space if $X = \mathbb{S}^n_+$ is the radial compactification of $\mathbb{R}^n$; the standard space being $\langle z\rangle^{-r}H^s(\mathbb{R}^n_z)$; in general it can be defined by locally identifying X with $\mathbb{S}^n_+$. Also, notice that this definition reduces the assumptions on H to conditions on the joint symbol of H and self-adjointness; the short-range conditions of [13] involve sub-leading terms.

Equation (2.3) shows that, in the coordinates discussed above, the joint symbol of $\Delta - \lambda^2$ at ${}^{\mathrm{sc}}T^*_{\partial X}X$ is

$$j_{\mathrm{sc},2,0}(\Delta - \lambda^2) = \tau^2 + |\mu|^2 - \lambda^2, \tag{2.4}$$

so its characteristic manifold (and hence that of $H - \lambda^2$) is

$$\Sigma_{\Delta-\lambda^2} = \{(y,\tau,\mu) : \ \tau^2 + |\mu|^2 = \lambda^2\} \subset {}^{\mathrm{sc}}T^*_{\partial X}X \tag{2.5}$$

where $|.|$ denotes the metric length with respect to the metric h on ∂X. Inside $\Sigma_{\Delta-\lambda^2}$, the (rescaled) Hamilton vector field of Δ vanishes only at the radial surfaces

$$R_\lambda^\pm = \{(y,\tau,\mu) :\ \tau = \pm\lambda,\ \mu = 0\}. \tag{2.6}$$

Correspondingly, we have real principal type propagation of singularities for generalized eigenfunctions of H away from $R_\lambda^\pm$. Of course, these are actually smooth in the interior of X since H is elliptic in the usual sense (the part of $j_{\mathrm{sc},2,0}(H)$ over ${}^{\mathrm{sc}}S^*X$, i.e. $\sigma_{\mathrm{sc},2}(H)$, is invertible). Thus, 'singularities' are understood in the sense of the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$. Similarly to the joint symbol of scattering differential operators, this consists of two parts, one being an extension of the usual wave front set, but for us only the part over the boundary matters (since the other part is empty due to the ellipticity mentioned above). This part is a subset of ${}^{\mathrm{sc}}T^*_{\partial X}X$. It measures oscillations and lack of decay (modulo $\dot{\mathcal{C}}^\infty(X)$) as we approach ∂X. In fact, under a local Fourier transform it is intimately related to the usual wave front set via the Legendre diffeomorphism discussed here in (5.20). We also use a weighted version, $\mathrm{WF}_{\mathrm{sc}}^{r,s}$ which measures decay and oscillations modulo the weighted Sobolev space $H_{\mathrm{sc}}^{r,s}(X)$.

We now mention how the long-range setting, g long-range and $V \in x\,\mathrm{Diff}^2_{\mathrm{sc}}(X)$, differs from the short-range one, g a (short-range) scattering metric and $V \in x^2\,\mathrm{Diff}_{\mathrm{sc}}(X)$. First, the Riemannian density will have the same form as beforehand since sub-leading terms from h' are just as significant as sub-leading terms from a, so there is no difference in the treatment of self-adjointness. Moreover, the only interesting new terms in the operator $H = \Delta + V$ are those of the form

$$xb_2(x^2D_x)^2 + xb_1(x^2D_x) + xb_0, \quad b_j \in \mathcal{C}^\infty(X), \tag{2.7}$$

since terms of the form $x(x^2D_x)(xD_{y_j})$, $x(xD_{y_j})(xD_{y_k})$ arise already from the Laplacian of a short-range metric (corresponding to h'), and the terms $x(xD_{y_j})$ are covered by the same type of estimates as the (partially) tangential second order terms. In particular, no modifications are necessary for those arguments of Melrose and Zworski [13, 14] which only use the principal symbol of H, and the treatment of the second and first order terms in (2.7) parallels that of the zeroth order term in [15], see also [17]. We also remark that if we define $g' = x^{-4}\,dx^2 + x^{-2}h'$ near ∂X with h' as discussed in the introduction, then g' is a (short-range) scattering metric, and

$$\Delta_{g'} + (a-1)x(x^2D_x)^2 - \Delta_g \in x^2\,\mathrm{Diff}^2_{\mathrm{sc}}(X), \tag{2.8}$$

so the Laplacian of a long-range scattering metric can be considered a long-range perturbation of a short-range one, though the L^2 pairing (given by the Riemannian densities) is of course different.

We now define the functions $\alpha_\pm \in \mathcal{C}^\infty(\partial X;\mathbb{R})$ by

$$\alpha_\pm(y) = \pm\frac{j_{\mathrm{sc},2,1}(V + (a-1)x(x^2D_x)^2)(y,\mp\lambda,0)}{2\lambda} \tag{2.9}$$

using the coordinates (y,τ,μ) discussed above; (2.8) is the reason for the appearance of the second term. This definition agrees with (1.1) if $V \in x\mathcal{C}^\infty(X)$. Note that as V is self-adjoint, $\alpha_\pm$ are indeed real-valued. Again, we sometimes extend $\alpha_\pm$ to a smooth real-valued function on X in an arbitrary fashion. We have the following three propositions.

Proposition 2.3. *[essentially in Melrose, [13]] Suppose that $u \in \mathcal{C}^{-\infty}(X)$ and $(H-\lambda^2)u = 0$. If $\mathrm{WF}^{r,s}_{sc}(u) \cap R^-_\lambda = \emptyset$ for some $s > -\frac{1}{2}$ and $r \in \mathbb{R}$ then $u = 0$. The same conclusion holds if we replace R^-_λ by R^+_λ.*

Though this proposition does not appear explicitly in [13], it follows easily from the arguments given there. (We remark that some of the results of [13] are explicitly stated for the Laplacian, but as noted in that paper, the proofs are valid for certain perturbations of Δ such as ours.) Namely, under these assumptions the results of [13] allow one to conclude that $\mathrm{WF}_{\mathrm{sc}}(u) \subset R^+_\lambda$. As proved there (see also [15]), this implies that u has an asymptotic expansion of the form stated in the next proposition (with the $-$ sign). (Actually, this is only proved for $V \in x\mathcal{C}^\infty(X)$ in these papers but the proof given in [15] goes through unchanged.) Hence, the boundary pairing lemma of [13] (see (5.3) of this paper) shows that $u \in L^2_{\mathrm{sc}}(X)$ which implies that $u \in \dot{\mathcal{C}}^\infty(X)$ by [13, Proposition 11].

At this point one has to use a unique continuation theorem at 'infinity' (i.e. at ∂X). For first order long-range perturbations $V \in x\,\mathrm{Diff}^1_{\mathrm{sc}}(X)$ (with g short-range) we can quote Hörmander's result [5, Theorem 17.2.8] directly as in Melrose's proof. Presumably that proof can be modified to accommodate the more general long-range second order setting, but here we refer to the version of Froese's and Herbst's proof in many-body scattering [2] which is described in [17, Appendix B] where this extension certainly does not cause any problems. Indeed, [17, Equation (B.7)] allows operators of the form H; we just need to remove the 'cross terms' of the form $x(xD_y)(x^2D_x)$ which can be done as in that Appendix by regarding the second order terms as part of the Laplacian of (another) long-range metric (but leaving the L^2 pairing unchanged). Moreover, Equation (B.8) and hence Lemma B.1 hold as well. Then we just need to note that the Mourre estimate only depends on the joint symbol of H, so although this is not stated in [13], it is easy to show it, and in any case the joint symbol statement shows that the proof given in [17] can be copied verbatim thus completing the proof of unique continuation. We note that Proposition 2.3 can also be proved more directly without using asymptotic expansions, see [17, Section 17] (cf. Isozaki's paper [7]).

Proposition 2.4. *[essentially in Melrose, [13], see also [15]] For $\lambda > 0$, the resolvent $R(\lambda^2 \pm it)$, $t > 0$, has a limit as $t \to 0$ in $\mathcal{B}(H^{r,s}_{sc}(X), H^{r+2,s'}_{sc}(X))$ whenever $s > \frac{1}{2}$, $s' < -\frac{1}{2}$, $r \in \mathbb{R}$, and*

$$f \in H^{r,s}_{sc}(X) \Rightarrow \mathrm{WF}^{r+2,s-1}_{sc}(R(\lambda^2 \pm it)f) \subset R^{\mp}_\lambda. \tag{2.10}$$

Moreover, for $f \in \dot{\mathcal{C}}^\infty(X)$, $R(\lambda^2 \pm i0)f$ has a full asymptotic expansion

$$R(\lambda^2 \pm i0)f = e^{\pm i\lambda/x} x^{(n-1)/2 + i\alpha_\pm(y)} v_\pm, \quad v_\pm \in \mathcal{A}^{\mathcal{K}}_{phg}(X), \tag{2.11}$$

where

$$\mathcal{K} = \{(m,p) : \ m, p \in \mathbb{N}, \ p \leq 2m\} \tag{2.12}$$

is the index set.

Recall that the above statement about $R(\lambda^2 \pm i0)f$, $f \in \dot{\mathcal{C}}^\infty(X)$, just means that

$$R(\lambda^2 \pm i0)f \sim e^{\pm i\lambda/x} x^{(n-1)/2 + i\alpha_\pm(y)} \sum_{j=0}^{\infty} \sum_{r \leq 2j} x^j (\log x)^r a_{j,r,\pm}(y). \tag{2.13}$$

Proposition 2.5. *[essentially in Melrose, [13]] The boundary value of the resolvent, $R(\lambda^2+i0)$, (resp. $R(\lambda^2-i0)$) extends by continuity (from $\dot{\mathcal{C}}^\infty(X)$) to a map from distributions $u\in\mathcal{C}^{-\infty}(X)$ satisfying $\mathrm{WF}_{sc}(u)\cap R^+_\lambda=\emptyset$ (resp. $\mathrm{WF}_{sc}(u)\cap R^-_\lambda=\emptyset$) to distributions satisfying the same condition. In addition, with $v=R(\lambda^2+i0)u$ (resp. $v=R(\lambda^2-i0)u$), u satisfying the above condition, $\mathrm{WF}_{sc}(v)$ is a subset of the union of R^-_λ (resp. R^+_λ) and the image of $\mathrm{WF}_{sc}(u)$ under the forward (resp. backward) bicharacteristic flow.*

Remark 2.6. We can state this proposition with weighted Sobolev spaces as well: we need to assume that $\mathrm{WF}^{r,s}_{\mathrm{sc}}(u)\cap R^+_\lambda=\emptyset$ for some $s>\frac{1}{2}$, and we conclude that $\mathrm{WF}^{r+2,s-1}_{\mathrm{sc}}(v)\cap R^+_\lambda=\emptyset$ as well as analogues of the propagation results stated above.

This proposition is not stated explicitly in [13] either, but again it follows from the arguments given there. Namely, $\langle R(\lambda^2+it)u,f\rangle=\langle u,R(\lambda^2-it)f\rangle$ together with the previous proposition show that $R(\lambda^2+it)u$ has a limit as $t\downarrow 0$ under the conditions of the proposition. This also shows that $\mathrm{WF}^{r,s}_{\mathrm{sc}}(R(\lambda^2+i0)u)\cap R^+_\lambda=\emptyset$ if $s<-\frac{1}{2}$. The uniform propagation estimates of [13] then imply first that $\mathrm{WF}_{\mathrm{sc}}(R(\lambda^2+i0)u)\cap R^+_\lambda=\emptyset$, and then the real-principal type propagation result of [13] (the uniform version) proves the proposition. Again, the fact that g and V are long-range does not cause any problems. See [17] for a similar argument.

We also recall briefly the definition of Legendre distributions from [14]. We write M in place of X for a manifold with boundary since we are particularly interested in the case $M=X\times\partial X$, X as above. Local coordinates (x,y) on X near ∂X give coordinates $(x,y,y')=(x,\bar{y})$ on M (near ∂M), and correspondingly we have coordinates $(x,y,y';\tau,\mu,\mu')=(x,\bar{y};\tau,\bar{\mu})$ on ${}^{\mathrm{sc}}T^*M$. Then the tautological one-form, ${}^{\mathrm{sc}}\alpha\in\mathcal{C}^\infty({}^{\mathrm{sc}}T^*M;{}^{\mathrm{sc}}T^*({}^{\mathrm{sc}}T^*M))$, has the property that $d\,{}^{\mathrm{sc}}\alpha=d({}^{\mathrm{sc}}\tilde{\chi}/x)$ where ${}^{\mathrm{sc}}\tilde{\chi}$ is a smooth 1-form on ${}^{\mathrm{sc}}T^*M$. Indeed, in the local coordinates discussed above, near ${}^{\mathrm{sc}}T^*_{\partial M}M$ we can take ${}^{\mathrm{sc}}\tilde{\chi}=d\tau+\bar{\mu}\cdot d\bar{y}$. The pull-back, ${}^{\mathrm{sc}}\chi$, of ${}^{\mathrm{sc}}\tilde{\chi}$ to ${}^{\mathrm{sc}}T^*_{\partial M}M$ is a one-form which defines a contact structure on ${}^{\mathrm{sc}}T^*_{\partial M}M$. A Legendre submanifold of ${}^{\mathrm{sc}}T^*_{\partial M}M$ is just a smooth submanifold G of dimension $\dim M-1$ on which ${}^{\mathrm{sc}}\chi$ vanishes identically.

Legendrian distributions are just non-degenerate superpositions of oscillatory functions $v=x^q e^{i\phi/x}a$, where $\phi,a\in\mathcal{C}^\infty(M)$, $q\in\mathbb{R}$. More precisely, one-step polyhomogeneous Legendre distributions of order m associated to a Legendre submanifold G are (modulo $\dot{\mathcal{C}}^\infty(M)$) locally finite sums of oscillatory integrals of the form

$$\int_{\mathbb{R}^k} e^{i\phi(\bar{y},u)/x}a(x,\bar{y},u)x^{m-\frac{k}{2}+\frac{\dim M}{4}}\,du,\quad a\in\mathcal{C}^\infty_c([0,\epsilon)_x\times U_{\bar{y}}\times\mathbb{R}^k),\tag{2.14}$$

where ϕ parameterizes G via the map

$$C_\phi=\{(\bar{y},u):\ d_u\phi(\bar{y},u)=0\}\ni(\bar{y},u)\mapsto(d_{(x,\bar{y})}(\phi/x))(0,\bar{y},u)\in G\tag{2.15}$$

in a non-degenerate way. The set of these distributions is denoted by $I^m_{\mathrm{sc,os}}(M,G)$; unlike Melrose and Zworski we emphasize 'one-step' in the notation as we will have the occasion to deal with different classes of such distributions. The reason for the 'one-step' terminology may be clearer if we expand a in Taylor series in x: $a\sim\sum_{j=0}^\infty x^j a_j(\bar{y},u)$, similarly to the expansion of polyhomogeneous symbols (in fact, these two are the same if we introduce $r=x^{-1}$ and think of $(r,\bar{y})$ as polar coordinates. In fact, if we locally identify M with compactified Euclidian space then under the Fourier transform such distributions become one-step polyhomogeneous (classical) Lagrangian distributions, see [14, Proposition 10]. An important property of distributions $v \in I^m_{\mathrm{sc,os}}(M,G)$ is that $\mathrm{WF}_{\mathrm{sc}}(v) \subset G$. Now, if G is a section of ${}^{\mathrm{sc}}T^*_{\partial M}M$, as is often the case (at least locally), then we do not need any parameters u, and Legendre distributions are just sums of oscillatory functions $x^{m+\dim M/4}\, e^{i\phi(\bar y)/x} a(x,\bar y)$. In particular, such oscillatory functions are sufficient to treat the Euclidian setting. We also remark that typically we shall deal with scattering half-density valued Legendre distributions, i.e. elements of $I^m_{\mathrm{sc,os}}(M,G;{}^{\mathrm{sc}}\Omega^{\frac12}M)$; ${}^{\mathrm{sc}}\Omega M$ is the density bundle induced by ${}^{\mathrm{sc}}T^*M$. Melrose and Zworski also define Legendrian distributions associated to intersecting Legendre submanifolds with conic points, see [14, Section 13], but instead of giving the definition here we postpone it until the last section where it will naturally emerge from our construction of $P(\lambda)$.

3. Poisson operators

First, we note that the Poisson operator $P(\lambda)$ for $H = \Delta + V$ can be constructed away from the 'end' of the smooth Legendre submanifold to which it is related much as it was done by Melrose and Zworski [14]. We often make $P(\pm\lambda)$ a map from half-densities to half-densities to simplify some of the notation. The correspondence between the smooth functions and half-densities is given by the trivialization of the half-density bundles by the Riemannian densities; so for example we have

$$P(\pm\lambda)(a|dh|^{1/2}) = (P(\pm\lambda)a)\,|dg|^{1/2}, \quad a \in \mathcal{C}^\infty(\partial X). \tag{3.1}$$

With this normalization the kernel of $P(\pm\lambda)$ will be a section of the kernel density bundle

$$\mathrm{KD}^{\frac12}_{\mathrm{sc}} = \pi_L^*\,{}^{\mathrm{sc}}\Omega^{\frac12}X \otimes \pi_R^*\,\Omega^{\frac12}\,\partial X. \tag{3.2}$$

where $\pi_L : X \times \partial X \to X$ and $\pi_R : X \times \partial X \to \partial X$ are the projections. We remark that

$$\mathcal{C}^\infty(X \times \partial X; \mathrm{KD}^{\frac12}_{\mathrm{sc}}) = x^{(n-1)/2}\mathcal{C}^\infty(X \times \partial X; {}^{\mathrm{sc}}\Omega^{\frac12}(X \times \partial X)); \tag{3.3}$$

smooth sections of $\mathrm{KD}^{\frac12}_{\mathrm{sc}}$ are of the form $a|dg|^{\frac12}|dh|^{\frac12}$, $a \in \mathcal{C}^\infty(X \times \partial X)$, while smooth sections of ${}^{\mathrm{sc}}\Omega^{\frac12}(X \times \partial X)$ are of the form $a|dg|^{\frac12}\frac{|dh|^{\frac12}}{x^{(n-1)/2}}$. Recall that in [14] the kernel of $P(\lambda)$ is constructed as a (distributional) section of ${}^{\mathrm{sc}}\Omega^{\frac12}(X \times \partial X)$, essentially by identifying $\mathrm{KD}^{\frac12}_{\mathrm{sc}}$ with ${}^{\mathrm{sc}}\Omega^{\frac12}(X \times \partial X)$ (via the mapping $a|dg|^{\frac12}|dh|^{\frac12} \mapsto a|dg|^{\frac12}\frac{|dh|^{\frac12}}{x^{(n-1)/2}}$), given a choice of boundary defining function x. We also let $\alpha_{\pm,R} = \pi_R^*\alpha_\pm$. We introduce the following Legendre submanifolds of ${}^{\mathrm{sc}}T^*_{\partial X \times \partial X}(X \times \partial X)$:

$$G^\sharp(\sigma) = \{(y, y', \tau, \mu, \mu') :\ \mu = 0,\ \mu' = 0,\ \tau = -\sigma\}, \tag{3.4}$$

and for $\lambda > 0$ let

$$\begin{aligned} G(\lambda) =& \{(y, y', \tau, \mu, \mu') :\ (y, \hat\mu) = \exp(sH_{\frac12 h})(y', \hat\mu'),\ \tau = \lambda\cos s, \\ & \mu = \lambda(\sin s)\hat\mu,\ \mu' = -\lambda(\sin s)\hat\mu',\ s \in (0,\pi),\ (y', \hat\mu') \in S^*\partial X\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} G(-\lambda) =& \{(y, y', \tau, \mu, \mu') :\ (y, \hat\mu) = \exp((s-\pi)H_{\frac12 h})(y', \hat\mu'),\ \tau = \lambda\cos s, \\ & \mu = \lambda(\sin s)\hat\mu,\ \mu' = -\lambda(\sin s)\hat\mu',\ s \in (0,\pi),\ (y', \hat\mu') \in S^*\partial X\}. \end{aligned} \tag{3.6}$$

Thus, given any neighborhood U_+ of the outgoing radial set $G^\sharp(\lambda)$ in ${}^{\mathrm{sc}}T^*_{\partial X\times\partial X}X\times\partial X$, we can construct a parametrix $K^\flat_+$ for the kernel of $P(\lambda)$ such that

$$x^{-i\alpha_{-,R}}K^\flat_+\in I^{-(2n-1)/4}_{\mathrm{sc,os}}(X\times\partial X,G(\lambda),\mathrm{KD}^{\frac12}_{\mathrm{sc}}) \tag{3.7}$$

and

$$\mathrm{WF}_{\mathrm{sc}}((\Delta_X+V_X-\lambda^2)K^\flat_+)\subset U_+ \tag{3.8}$$

where the subscript X denotes that these operators act on the left factor.

In fact, the construction of $K^\flat_+$ is very similar to the short-range construction described by Melrose and Zworski [14]. It has been described in Appendix A of [17] for $V\in x\mathcal{C}^\infty(X)$ (and $V\in x\,\mathrm{Diff}^2_{\mathrm{sc}}(X)$ does not require a special treatment) near the incoming radial set $G^\sharp(-\lambda)$ (where the normalization was opposite of the one used here), and it can be continued away from $G^\sharp(\lambda)$ just like in the short-range case discussed in [14] by real principal type propagation arguments (i.e. it amounts to solving the transport equations which are non-degenerate linear ODE's). We just cut off the solution before the smooth Legendrian $G(\lambda)$ reaches $G^\sharp(\lambda)$, hence the wave front set result (3.8).

A little more detailed picture is the following. We construct $K^\flat_+$ as an asymptotic sum

$$K^\flat_+\sim\sum_{j=0}^\infty K_j,\quad x^{-i\alpha_{-,R}}K_j\in I^{-(2n-1)/4+j}_{\mathrm{sc,os}}(X\times\partial X,G(\lambda),\mathrm{KD}^{\frac12}_{\mathrm{sc}}). \tag{3.9}$$

It is convenient to construct $K^\flat_+$ near $G^\sharp(-\lambda)$ rather explicitly, so we write

$$K_j=x^{j+i\alpha_-(y')}e^{-i\lambda\cos\mathrm{dist}(y,y')/x}a_j(x,y,y')\nu \tag{3.10}$$

where $\nu=|dg|^{\frac12}|dh|^{\frac12}\in\mathcal{C}^\infty(X\times\partial X;\mathrm{KD}^{\frac12}_{\mathrm{sc}})$ and dist is the metric distance given by the metric h on ∂X. Note that $G(\lambda)$ is a graph over ∂X near $G^\sharp(-\lambda)$. Regarding y as a parameter and introducing Riemannian normal coordinates in y centered at y' we obtain transport equations for $a'_j=a_j|_{x=0}$:

$$(y\cdot\partial_y+j)a'_j+(i(\alpha_-(0)+(2\lambda)^{-1}V'(y))+b_j)a'_j=c_j\in\mathcal{C}^\infty(X\times\partial X) \tag{3.11}$$

near $y=0$ with b_j vanishing at $y=0$ and $c_0\equiv 0$. Here $V'(y)$ is the principal symbol of $V+(a-1)x(x^2D_x)^2$ at the unique point of $G(\lambda)$ which is near $G^\sharp(-\lambda)$ and lies over y, and some terms arising from the action of V have been moved to b_j. Since

$$2\lambda\alpha_-(0)+V'(y) \tag{3.12}$$

vanishes at $y=0$, the transport equation for a'_0 has a unique smooth solution with $a'_0(y,y)\in\mathcal{C}^\infty(\partial X)$ specified, and the equations for a'_j, $j\geq 1$, have unique smooth solutions, just as in Hadamard's parametrix construction, see e.g. [5, Lemma 17.4.1]. The specification of $a'_0(y,y)$ comes from the requirement that for $v\in\mathcal{C}^\infty(\partial X)$

$$\begin{aligned}\tilde P(\lambda)(v|dh|^{\frac12})=\int_{\partial X}K^\flat_+v|dh|^{\frac12}=e^{-i\lambda/x}x^{(n-1)/2+i\alpha_-(y)}v(y)|dg|^{\frac12}+u',\\ u'\in L^2_{\mathrm{sc}}(X;{}^{\mathrm{sc}}\Omega^{\frac12}).\end{aligned} \tag{3.13}$$

Writing out the oscillatory integral explicitly and using the stationary phase lemma determines $a'_0(y,y)$. In fact, notice that the stationary phase lemma gives the full asymptotic expansion of Proposition 2.4.

Away from $G^\sharp(\lambda)\cup G^\sharp(-\lambda)$ we can use the symbolic filtration and (3.9) to continue the solution just constructed microlocally near $G^\sharp(-\lambda)$, by solving transport equations. The additional factor $x^{i\alpha_-(y')}$ causes no complications since it can be commuted through the operator $\Delta_X+V_X-\lambda^2$ thereby adding a term similar to the ones that already exist. Indeed, since this operator contains no differentiation in y', no terms involving $\log x$ arise as they normally would if one acted on oscillatory sections involving such a factor by a general differential operator. We refer to [14] for further details. Finally, near $G^\sharp(\lambda)$ we cut off the solution $K_+^\flat$ by simply multiplying the amplitudes by $\chi_+(s)$, s the parameter given in (3.5), where $\chi_+\in\mathcal{C}^\infty(\mathbb{R})$ is identically 1 on $[0,\pi-2\delta]$, vanishes on $[\pi-\delta,\pi]$, $\delta>0$ sufficiently small. This of course introduces an error term for $(\Delta_X+V_X-\lambda^2)K_+^\flat$, exactly as in (3.8).

Let $\tilde P(\lambda):\mathcal{C}^\infty(\partial X)\to\mathcal{C}^{-\infty}(X)$ denote the operator given by the kernel $K_+^\flat$. Then the actual Poisson operator is

$$P(\lambda)=\tilde P(\lambda)-R(\lambda^2+i0)(H-\lambda^2)\tilde P(\lambda). \tag{3.14}$$

The composition in this formula makes sense due to Proposition 2.5 and (3.8). Namely, for $a\in\mathcal{C}^{-\infty}(\partial X)$,

$$\mathrm{WF}_{\mathrm{sc}}((H-\lambda^2)\tilde P(\lambda)a)\subset\pi_1(U_+) \tag{3.15}$$

where $\pi_1:{}^{\mathrm{sc}}T^*_{\partial X\times\partial X}X\times\partial X\to{}^{\mathrm{sc}}T^*_{\partial X}X$ is the projection (cf. (5.4)), and $\pi_1(U_+)\cap R_\lambda^+=\emptyset$, so we can apply Proposition 2.5. Here (3.15) is a push-forward result; its simplest proof is to employ a localized version of the Fourier transform $\mathcal{F}$ (use a cut-off function and identify X locally with $\mathbb{S}^n_+$) and show the corresponding statement for $\mathrm{WF}(\mathcal{F}\tilde P(\lambda)a)$ using the standard result on the wave front set of a push forward, see [17, Appendix A] for a more detailed version of this argument (and cf. the discussion after (5.18)).

Thus, $P(\lambda):\mathcal{C}^\infty(\partial X)\to\mathcal{C}^{-\infty}(X)$ extends to a continuous operator $\mathcal{C}^{-\infty}(\partial X)\to\mathcal{C}^{-\infty}(X)$, and if $a\in\mathcal{C}^\infty(\partial X)$ then by Proposition 2.4, $P(\lambda)a$ is indeed of the form (1.2) (in fact, we have a full asymptotic expansion), and by Proposition 2.3 it is the only generalized eigenfunction of H with eigenvalue λ^2 which is of the form (1.2). Hence, the operator $P(\lambda)$ we have constructed is indeed the Poisson operator as defined by asymptotic expansions in the introduction.

We also introduce the Poisson operator $P(-\lambda)$ propagating from $G^\sharp(\lambda)$ and its parametrix $\tilde P(-\lambda)$ with kernel $K_-^\flat$. Thus,

$$x^{-i\alpha_{+,R}}K_-^\flat\in I_{\mathrm{sc,os}}^{-(2n-1)/4}(X\times\partial X,G(-\lambda),\mathrm{KD}_{\mathrm{sc}}^{\frac12}) \tag{3.16}$$

and

$$\mathrm{WF}_{\mathrm{sc}}((\Delta_X+V_X-\lambda^2)K_-^\flat)\subset U_-, \tag{3.17}$$

U_- any prescribed neighborhood of $G^\sharp(-\lambda)$. The construction of $P(-\lambda)$ is completely analogous to that of $P(\lambda)$; we take the cut-off (as in the paragraph preceeding (3.14)) to be a function $\chi_-\in\mathcal{C}^\infty(\mathbb{R})$ which is identically 1 on $[2\delta,\pi]$, 0 on $[0,\delta]$, $\delta>0$ sufficiently small. In fact, for $V\in x\mathcal{C}^\infty(X;\mathbb{R})$ (and more generally when the operator V is real) we can deduce from (1.2) that $\overline{P(\lambda)}=P(-\lambda)$ (here $P(\pm\lambda)$ stands for the kernel of the corresponding operator). We also note that in addition to (3.17) we have

$$x^{-i\alpha_{+,R}}(\Delta_X+V_X-\lambda^2)K_-^\flat\in I_{\mathrm{sc,os}}^{-(2n-1)/4+1}(X\times\partial X,G(-\lambda),\mathrm{KD}_{\mathrm{sc}}^{\frac12}), \tag{3.18}$$

since the principal symbol of $H_X - \lambda^2$ vanishes on $G(-\lambda)$ (see [14, Proposition 13]).

4. Variable order FIOs

In this section we give a brief description of polyhomogeneous Fourier integral operators with an order whose imaginary part varies along the Lagrangian. All such FIOs are in the class I_ρ for $\rho < 1$, so the main issue is that of polyhomogeneity. There are several possibilities to define these polyhomogeneous classes. We first describe a geometrically very natural class that has good composition properties. Then we define a smaller class which is not very natural from the geometric point of view, but the scattering matrix will be in it (as well as in the bigger class), and we can use this additional information to give a more precise description of the Poisson operator near the 'end' of $G(\lambda)$.

In fact, dealing with polyhomogeneity just amounts to checking the proofs given in Hörmander's paper [4] to see that allowing the additional logarithmic factors discussed below takes care of the choices involved in defining these distributions. We first introduce our 'normalized' symbol class; i.e. we take these symbols to be order 0 and move the actual order in front of them in the oscillatory integral representation of the Lagrangian distributions. This allows us to use Melrose's notation for polyhomogeneous symbols [12].

Let $\mathcal{K}$ be the index set

$$\mathcal{K} = \{(m,p) :\ m, p \in \mathbb{N},\ p \leq 2m\}. \tag{4.1}$$

Let $U \subset \mathbb{R}^n$ be an open set. In Hörmander's formulation of oscillatory integral representations of Lagrangian distributions one considers symbols on $U \times \mathbb{R}^N$. Here we compactify $\mathbb{R}^N$ to a ball, $\mathbb{S}^N_+$, and use the notation of [12] for conormal functions. Recall that the space of polyhomogeneous function of index set $\mathcal{K}$ is denoted by $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(U \times \mathbb{S}^N_+)$; if they in addition have compact support (which is a statement about their behavior in the U factor) then by $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg,c}}(U \times \mathbb{S}^N_+)$. Thus, $a \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(U \times \mathbb{S}^N_+)$ means that a has a complete asymptotic expansion at $U \times \mathbb{S}^{N-1}$, $\mathbb{S}^{N-1} = \partial \mathbb{S}^N_+$, i.e. at infinity in the Euclidian picture, of the following kind:

$$a(y,\omega,s) \sim \sum_{j=0}^{\infty} \sum_{r \leq 2j} s^j (\log s)^r\, a_{j,r,\pm}(y,\omega,s). \tag{4.2}$$

Here y denotes the variable on U, s is a boundary defining function of $\mathbb{S}^N_+$ and (s,ω) are the (inverse) polar coordinates on $\mathbb{S}^N_+$, i.e. $\theta = s^{-1}\omega$ is the parameter on $\mathbb{R}^N$ in Hörmander's notation. We can now define the spaces of polyhomogeneous Lagrangian distributions of variable order.

Definition 4.1. Let Y be a smooth manifold, and let Λ be a conic Lagrangian submanifold of $T^*Y \setminus 0$. Suppose that $\alpha \in \mathcal{C}^\infty(\Lambda)$ is real-valued and homogeneous of degree 0. Then for $m \in \mathbb{R}$, $I^{m+i\alpha}_{\mathrm{phg}}(Y,\Lambda,\Omega^{\frac12})$ denotes the subset of $\cap_{\rho \in (\frac12,1)} I^m_\rho(Y,\Lambda,\Omega^{\frac12})$ which consists of distributions given by a locally finite sum of oscillatory integral representations of the form

$$(2\pi)^{-\frac{n}{4}-\frac{N}{2}} \int e^{i\psi(y,\omega)/s} s^{-m-i\tilde\alpha(y,\omega)-(n+2N-2)/4} a(y,\omega,s)\, d\omega\, ds\, |dy|^{\frac12} \tag{4.3}$$

where $\phi(y,\omega,s) = \psi(y,\omega)/s$ is a non-degenerate phase function parameterizing Λ, $a \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg,c}}(U \times \mathbb{S}^N_+)$, and the restriction of $\tilde{\alpha} \in \mathcal{C}^\infty(U \times \mathbb{S}^N_+)$ to the critical set

$$C_\phi = \{(y,\omega,s) :\ d_{(\omega,s)}\phi(y,\omega,s) = 0\} \tag{4.4}$$

is the pull back of α by the diffeomorphism

$$p_\phi : C_\phi \ni (y,\omega,s) \mapsto (y, d_y\phi(y,\omega,s)) \in \Lambda. \tag{4.5}$$

It is straightforward to check (following Hörmander's proof in [4]) that the class of distributions does not depend on the parametrization of Λ and the extension $\tilde{\alpha}$ of α used above in the definition. In fact, the reason for allowing the additional factors of $\log s$ in (4.2) is because such factors arise when changing parametrizations (stationary phase in reducing the number of parameters, cf. [4, Section 3.2]) or the extension of α (an integration by parts argument as in the proof of [4, Proposition 1.2.5]). We remark that it is not an accident that (2.12) and (4.1) are the same; from the point of view of doing the push forward $\tilde{P}(\lambda)(v|dh|^{\frac{1}{2}})$ as discussed in (3.13), the logarithmic terms in (2.13) arise by stationary phase arguments which is one of the reasons for their presence in (4.1). We note that the proofs of other results, such as transversal composition of Lagrangian distributions, go through without significant changes. We thus have the following analogue of [4, Theorem 4.2.2].

Proposition 4.2. *Suppose that C_1 and C_2 are homogeneous canonical relations from T^*Y to T^*X and T^*Z to T^*Y respectively, α_j are homogeneous functions of degree 0 on C_j', $j = 1, 2$, $C_1 \times C_2$ intersects the diagonal Δ in $T^*X \times T^*Y \times T^*Y \times T^*Z$ transversally and that the projection from the intersection to $T^*X \times T^*Z$ is injective and proper, thus giving a homogeneous canonical relation $C_1 \circ C_2$ from T^*Z to T^*Y. If $A_1 \in I^{m_1+i\alpha_1}_{phg}(X \times Y, C_1', \Omega^{\frac{1}{2}})$, $A_2 \in I^{m_2+i\alpha_2}_{phg}(Y \times Z, C_2', \Omega^{\frac{1}{2}})$ are properly supported, then*

$$A_1A_2 \in I^{m_1+m_2+i\alpha}_{phg}(X \times Z, (C_1 \circ C_2)', \Omega^{\frac{1}{2}}) \tag{4.6}$$

where α is the pullback of $\alpha_1 + \alpha_2$, defined on $(C_1 \times C_2) \cap \Delta$, by the diffeomorphism given by the projection to $C_1 \circ C_2$.

In fact, clean composition also works the same way if we assume that $\alpha_1 + \alpha_2$ is constant along the fibers of the projection, i.e. that $\alpha_1 + \alpha_2$ (again, defined on $(C_1 \times C_2) \cap \Delta$) is the pull back of a function α on $C_1 \circ C_2$. The proof proceeds just as in the standard case; see the paper of Duistermaat and Guillemin [1], Weinstein's paper [19], or [5, Chapter XXV].

Proposition 4.3. *Suppose that C_1 and C_2 are as above, but the composition $C_1 \circ C_2$ is clean with excess e (instead of being transversal), and it is proper and connected. Suppose that $\alpha_1 + \alpha_2$, regarded as a function on $(C_1 \times C_2) \cap \Delta$, is the pull back of a function α on $C_1 \circ C_2$. Then*

$$A_1A_2 \in I^{m_1+m_2+\frac{e}{2}+i\alpha}_{phg}(X \times Z, (C_1 \circ C_2)', \Omega^{\frac{1}{2}}). \tag{4.7}$$

We had to allow logarithmic terms in the polyhomogeneous expansions in Definition 4.1 to make sure that our definition is independent of the choice of parametrization of the Lagrangian and of the extension of α used to define the class. However, under certain additional assumptions we can make sure that the logarithmic terms do not arise. This is in fact the case if the variable order is given by a globally defined function α on the base manifold.

Definition 4.4. Let Y be a smooth manifold and let Λ be a conic Lagrangian submanifold of $T^*Y\setminus 0$. Suppose that $\alpha\in\mathcal{C}^\infty(Y)$ is real-valued. Then for $m\in\mathbb{R}$, $I^{m+i\alpha}_{\mathrm{b,phg}}(Y,\Lambda,\Omega^{\frac{1}{2}})$ denotes the subset of $\cap_{\rho\in(\frac{1}{2},1)}I^m_\rho(Y,\Lambda,\Omega^{\frac{1}{2}})$ which consists of distributions given by a locally finite sum of oscillatory integral representations of the form

$$(2\pi)^{-\frac{n}{4}-\frac{N}{2}}\int e^{i\psi(y,\omega)/s}s^{-m-i\alpha(y)-(n+2N-2)/4}a(y,\omega,s)\,d\omega\,ds\,|dy|^{\frac{1}{2}} \tag{4.8}$$

where $\phi(y,\omega,s)=\psi(y,\omega)/s$ is a non-degenerate phase function parameterizing Λ and $a\in\mathcal{C}^\infty_c(U\times\mathbb{S}^N_+)$.

Remark 4.5. Note that $a\in\mathcal{C}^\infty_c(U\times\mathbb{S}^N_+)$ means that it is a classical (one-step polyhomogeneous) symbol of order 0 with compact cone support. Also, directly from the definition, if $\alpha\in\mathcal{C}^\infty(Y)$ then

$$I^{m+i\alpha}_{\mathrm{b,phg}}(Y,\Lambda,\Omega^{\frac{1}{2}})\subset I^{m+i\alpha'}_{\mathrm{phg}}(Y,\Lambda,\Omega^{\frac{1}{2}}) \tag{4.9}$$

where α' is the restriction of $\pi^*\alpha$ to Λ; here $\pi:T^*Y\to Y$ is the projection.

Again, this definition is independent of the parametrizations of Λ. In particular the reduction of the number of parameters means that we have to use the stationary phase lemma in the parameters to be eliminated, but the exponent in $s^{-i\alpha(y)}$ does not depend on the parameter variables, so the differentiations in the stationary phase result (see [4, Equation 3.2.8]) do not give rise to logarithmic terms. Note, however, that this class is geometrically less natural since the Lagrangian and the projection $T^*Y\to Y$ generally relate to each other in a fairly complicated way. Also, in the composition formula for FIOs (see e.g. [4, Equation 4.2.8]) the base variables become parameters, so this class is not, in general, closed under (transversal) composition. However, the S-matrix discussed in the following section will be in this class, and we can use the explicit composition formula in Section 7 to obtain a sharp result for the structure of the Poisson operator near the end of the Legendrian $G(\lambda)$. Namely, there will be no logarithmic terms in the part of the representation along $G(\lambda)$ (we know this *a priori*, from the results of Section 3, away from $G^\sharp(\lambda)$), though they will appear on $G^\sharp(\lambda)$.

5. The scattering matrix

The scattering matrix of H can be calculated by the boundary pairing of [13] as discussed in the following Proposition. This proposition and Equation (5.5) allow us to analyze the structure of $S(\lambda)$ using the results of Section 3, i.e. they eliminate the necessity of knowing the precise structure of $P(\lambda)$ at the 'end' of $G(\lambda)$ (which was crucial in the approach taken by Melrose and Zworski in [14]).

Proposition 5.1. *The scattering matrix is given by*

$$S(\lambda)=\frac{1}{2i\lambda}((H-\lambda^2)\tilde{P}(-\lambda))^*P(\lambda). \tag{5.1}$$

Proof. The following pairing formula was proved by Melrose [13, Proposition 13] for short-range g and V, but the same proof also applies when g is long-range and $V\in x\,\mathrm{Diff}^2_{\mathrm{sc}}(X)$. We use the notation of Proposition 2.4. Suppose that

$$u_j=e^{i\lambda/x}x^{(n-1)/2+i\alpha_+}v_{j,+}+e^{-i\lambda/x}x^{(n-1)/2+i\alpha_-}v_{j,-},\quad v_{j,\pm}\in\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X), \tag{5.2}$$

and $f_j = (H-\lambda^2)u_j \in \dot{\mathcal{C}}^\infty(X)$. Let $a_{j,\pm} = v_{j,\pm}|_{\partial X}$. Then

$$2i\lambda \int_{\partial X} (a_{1,+}\,\overline{a_{2,+}} - a_{1,-}\,\overline{a_{2,-}})\,dh = \int_X (u_1\,\overline{f_2} - f_1\,\overline{u_2})\,dg. \tag{5.3}$$

We apply this result with $u_1 = P(\lambda)a_1$, $u_2 = \tilde{P}(-\lambda)a_2$. By the construction of $\tilde{P}(-\lambda)$ we conclude that $a_{2,+} = a_2$, $a_{2,-} = 0$, while for u_1 we see directly from the definition of $S(\lambda)$ and $P(\lambda)$ that $a_{1,-} = a_1$, $a_{1,+} = S(\lambda)a_1$. Substitution into (5.3) proves the proposition. □

We can even replace $P(\lambda)$ by $\tilde{P}(\lambda)$ in the statement of this proposition if we work modulo smoothing operators. To see this, let $\pi_1 : {}^{\mathrm{sc}}T^*(X\times\partial X) \to {}^{\mathrm{sc}}T^*X$, $\pi_2 : {}^{\mathrm{sc}}T^*(X\times\partial X) \to T^*\partial X$ be the projections arising from the isomorphism

$$ {}^{\mathrm{sc}}T^*X \times T^*\partial X \to {}^{\mathrm{sc}}T^*(X\times\partial X),\ (\mu'\,dy', \tau\,\frac{dx}{x^2} + \mu\,\frac{dy}{x}) \mapsto \tau\,\frac{dx}{x^2} + \mu\,\frac{dy}{x} + \mu'\,\frac{dy'}{x} \tag{5.4}$$

given by a choice of x (restricting π_1, π_2, to the boundary we only need to specify x modulo $x^2\mathcal{C}^\infty(X)$). Now suppose that $U_\pm$ are neighborhoods (in ${}^{\mathrm{sc}}T^*_{\partial X\times\partial X}X\times\partial X$) of $G^\sharp(\pm\lambda)$ respectively such that $\pi_1(U_+)\cap\pi_1(U_-) = \emptyset$, and $K^\flat_\pm$ satisfy (3.8) and (3.17) respectively. Then the assumptions on $U_\pm$ ensure that $S(\lambda)$ is given by

$$\tilde{S}(\lambda) = \frac{1}{2i\lambda}((H-\lambda^2)\tilde{P}(-\lambda))^*\tilde{P}(\lambda) \tag{5.5}$$

modulo smoothing operators (i.e. $S(\lambda) - \tilde{S}(\lambda) \in \Psi^{-\infty}(\partial X)$). In fact, for any $a, a' \in \mathcal{C}^{-\infty}(\partial X)$ we have

$$\mathrm{WF}_{\mathrm{sc}}((P(\lambda)-\tilde{P}(\lambda))a) \subset \pi_1(U_+),\ \mathrm{WF}_{\mathrm{sc}}((H-\lambda^2)\tilde{P}(-\lambda)a') \subset \pi_1(U_-), \tag{5.6}$$

see Proposition 2.5 and the discussion after (3.15), so under the assumptions on $U_\pm$, the complex pairing

$$\langle (H-\lambda^2)\tilde{P}(-\lambda)a', (P(\lambda)-\tilde{P}(\lambda))a\rangle \tag{5.7}$$

is defined by continuity (from $a, a' \in \mathcal{C}^\infty(\partial X)$), so

$$((H-\lambda^2)\tilde{P}(-\lambda))^*(P(\lambda)-\tilde{P}(\lambda)) : \mathcal{C}^{-\infty}(\partial X) \to \mathcal{C}^\infty(\partial X) \tag{5.8}$$

indeed.

Now, the composition of the Legendre distributions in (5.5) is clean in the sense of Duistermaat and Guillemin [1], see also [5, Chapter XXV] and [19], and see Joshi's paper [9] for a geometric proof of clean composition. Namely, the excess is 1, coming from the fact that if a point on a bicharacteristic is in both Legendrians, then so is the whole bicharacteristic through that point. In order to apply their result, we use the localized partial Fourier transform in the 'composition variables', i.e. on X, to translate our composition to that of Lagrangian distribution associated to cleanly intersecting Lagrangians. However, we first discuss the Euclidian setting.

As a simple but (in this respect) very illuminating example we now consider the compactified Euclidean space $X = \mathbb{S}^n_+$ with its standard metric, and let $H = \Delta + V$ as above. We trivialize all density bundles by the standard measures. The kernel $K_\pm$ of the Poisson operator $P(\pm\lambda)$ is a distribution on $\mathbb{S}^n_+\times\mathbb{S}^{n-1}$. Since $G(\pm\lambda)$ is a global section of ${}^{\mathrm{sc}}T^*_{\partial X\times\partial X}(X\times\partial X)$ (for either choice of the sign), $K^\flat_\pm$ is an oscillatory function

$$K^\flat_\pm = e^{\mp i\lambda y'\cdot y/x}x^{i\alpha_\mp(y')}a_\pm(x,y,y'),\quad a \in \mathcal{C}^\infty([0,1)\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}), \tag{5.9}$$

here $\operatorname{supp} a_\pm$ are disjoint from $[0,1) \times \{(y,-y) : y \in \mathbb{S}^{n-1}\}$ (as that would correspond to the end of the Legendrian).

To see how the composition in (5.5) works in this setting, note first that the kernel of $(H-\lambda^2)\tilde{P}(-\lambda)$ is also of the form (5.9) but its amplitude (corresponding to $a_\pm$ above) is in $x\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})$ since $H-\lambda^2$ is characteristic on $G(\lambda)$ (corresponding to the fact that the phase function in (5.9) solves the eikonal equation). Now we write the variables on the composition space $\mathbb{S}^{n-1} \times \mathbb{S}^n_+ \times \mathbb{S}^{n-1}$ as $(y', (x,y), y'')$, so the kernel of $\tilde{S}(\lambda)$ takes the form

(5.10)
$$\tilde{S}(\lambda)(y',y'') = \int e^{-i\lambda(y'+y'')\cdot y/x} b(x,y,y',y'')\,\frac{dx\,dy}{x^{n+1}},$$
$$x^{i(-\alpha_-(y')+\alpha_+(y''))}b \in x\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}).$$

We also note the support properties of b: as y' and y'' are both away from $-y$, and when y' is near y the phase function has no critical points (since y'' is away from $-y$) and a similar statement holds for y'', we can take b to be supported away from $y = \pm y'$, and away from $y = \pm y''$. Under these conditions this is a cleanly parameterized oscillatory integral in the sense of [1, Lemma 7.1], so (uncompactifying the notation and writing $z = y/x$ as the Euclidian variable) $\phi(y',y'',z) = (y'+y'')\cdot z$ satisfies $d\phi \neq 0$ (on $\operatorname{supp} b$), ϕ defines a submanifold C_ϕ by $d_z\phi = 0$, namely

$$C_\phi = \{(y',y'',z) : y' = -y''\}, \tag{5.11}$$

and the tangent space of C_ϕ is the space of vectors annihilated by $d\left(\frac{\partial\phi}{\partial z_j}\right)$, $j = 1,\ldots,n$. In fact, a straightforward computation shows that $n-1 = \operatorname{codim} C_\phi$ of these differentials are linearly independent. Thus, the 'excess' of the parametrization is $e = 1$. As a symbol, b is in the class $S^{-1}_\rho = S^{-1}_{\rho,1-\rho}$ for any $\rho < 1$; due to the $\log x$'s appearing after differentiating $x^{i\alpha_+(y')}$ with respect to y', b is in $S^{-1} = S^{-1}_{1,0}$ only if $\alpha_\pm$ are constant. By Lemma 7.1 of [1] we conclude that $\tilde{S}(\lambda) \in I^0_\rho(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}; N^*\operatorname{graph}(p); \Omega_R)$ for any $\rho < 1$ where $p : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the antipodal map. Strictly speaking, the proof of the above Lemma given in that paper assumes polyhomogeneity, i.e. it only applies in the former case (and shows that the result is polyhomogeneous), but it is easy to see that the lack of the polyhomogeneity (or its variable order version in our setting) causes no additional complications, see [5, Chapter XXV]. Of course, we also immediately see that this integral is of the form (4.8) except that $-\lambda(y'+y'')\cdot y/x$ is a clean phase function, not a non-degenerate one, but the argument of Duistermaat and Guillemin still applies and shows that $\tilde{S}(\lambda) \in I^{i(\pi_L^*\alpha_+ - \pi_R^*\alpha_-)}_{\mathrm{b,phg}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}; N^*\operatorname{graph}(p); \Omega_R)$ where $\pi_L, \pi_R : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are the projections to the left and right factors respectively.

Returning to the general geometric setting, we first remark that the kernel of $((H-\lambda^2)\tilde{P}(-\lambda))^*$ is a Legendrian distribution associated to the Legendre submanifold $G(\lambda)^\dagger$ of ${}^{\mathrm{sc}}T^*_{\partial X \times \partial X}\partial X \times X$; here $\dagger$ simply denotes that the factors X and ∂X are interchanged (compared to the definition of $G(\lambda)$). In the following, as customary for Lagrangians, we use the notation G' for the canonical relation corresponding to a Legendre submanifold $G \subset {}^{\mathrm{sc}}T^*_{\partial X \times \partial X}(X \times \partial X)$ (or of ${}^{\mathrm{sc}}T^*_{\partial X \times \partial X}(\partial X \times X)$), i.e. the sign of the covector component in the second factor is switched. We next show that away from $G^\sharp(\pm\lambda)$ in both factors, the intersection of

$$C = (G(\lambda)^\dagger)' \times G(\lambda)' \tag{5.12}$$

with the (partial) diagonal Δ in the X factors of

$$Z = {}^{\mathrm{sc}}T^*_{\partial X\times\partial X}(\partial X \times X) \times {}^{\mathrm{sc}}T^*_{\partial X\times\partial X}(X \times \partial X) \tag{5.13}$$

is clean. Note that Z is just a manifold without boundary.

Let

$$Y = T^*\partial X \times \Sigma_{\Delta-\lambda^2} \times \Sigma_{\Delta-\lambda^2} \times T^*\partial X. \tag{5.14}$$

Noting that $C \subset Y$, we actually show that C intersects $\Delta\cap Y$ transversally inside Y from which the cleanness of the intersection of Δ and C follows easily. We also remark that the excess of the clean intersection will be $e = 1$ since Y has codimension 2 in Z, and $\Delta \cap Y$ has codimension 1 in Δ. Then all we need to show is that

$$p \in G(\lambda)' \Rightarrow (\pi_1)_* T_p G(\lambda)' = T_{\pi_1(p)}\Sigma_{\Delta-\lambda^2}; \tag{5.15}$$

since for any $(q,p) \in \Delta\cap Y$ we have

$$T_{q,p}(\Delta\cap Y) = T_{\pi_2(q)}T^*\partial X \times T_{(\pi_1(q),\pi_1(p))}\,\mathrm{diag}(\Sigma_{\Delta-\lambda^2}\times\Sigma_{\Delta-\lambda^2}) \times T_{\pi_2(p)}T^*\partial X \tag{5.16}$$

and (5.15)-(5.16) imply that

$$T_{q,p}C + T_{q,p}(\Delta\cap Y) = T_{q,p}Y. \tag{5.17}$$

But (5.15) can be seen directly from the parametrization of $G(\lambda)$: instead of the standard coordinates (y,τ,μ) on ${}^{\mathrm{sc}}T^*_{\partial X}X$ we can also use the coordinates $\rho = \tau^2+|\mu|^2$, $s = \arccos(\tau/(\tau^2+|\mu|^2))$ and $\hat\mu = \frac{\mu}{|\mu|}$ on ${}^{\mathrm{sc}}T^*_{\partial X}X$ away from $R^\pm_\lambda$ and near $\Sigma_{\Delta-\lambda^2}$, so the invertibility of the push-forward by the exponential map in (3.5) proves our claim.

Although we could prove a 'clean composition' theorem in this scattering setting directly similarly to how it was done in [1], we simply reduce the proof to the Lagrangian case by employing a localized version of the Fourier transform. Thus, we rewrite the (5.5) by localizing the operator kernels in regions of the form $U\times\partial X$ where $U \subset \mathbb{S}^n_+$ and using the Fourier transform on $\mathbb{S}^n_+$, i.e. we take

$$(\mathcal{F}u)(\xi,y') = \int e^{-i\xi\cdot y/x}u(x,y,y')\,\frac{dx\,dy}{x^{n+1}}. \tag{5.18}$$

This is a priori defined for, say, $u\in\dot{\mathcal{C}}^\infty_c(U\times\partial X)$, but it extends to a map $\mathcal{C}^{-\infty}_c(U\times\partial X)\to\mathcal{S}'(\mathbb{R}^n\times\partial X)$ as usual. It maps Legendre distributions of order m associated to a Legendre submanifold G to Lagrangian distributions of order $-m-\frac{n-1}{2}$ and of compact singular support on $\mathbb{R}^n\times\partial X$ (cf. the completely analogous result when there are no 'parameters' y', discussed in [14]). The (homogeneous) Lagrangian Λ to which the image is associated is given by

$$\Lambda_+(G) = \{(rL(\pi_1(p)), r\pi_2(p)) :\ p\in G,\ r>0\} \tag{5.19}$$

where L is the Legendre diffeomorphism

$$L : {}^{\mathrm{sc}}T^*_{\mathbb{S}^{n-1}}\mathbb{S}^n_+ \to S^*\mathbb{R}^n,\ L(y,\tau,\mu) = (\mu-\tau y, -y) \tag{5.20}$$

of [14, Lemma 5]. In fact, if $\phi(y, y', u)/x$ is a phase function which locally parameterizes G, then

$$\psi(\xi, y', y, u)/x = (-\xi \cdot y + \phi(y, y', u))/x \tag{5.21}$$

is a (non-degenerate) phase function which locally parameterizes $\Lambda_+(G)$. Note that here ξ and y' are the base variables, and x, y, u are the variables in the parameter space of which x is homogeneous of degree -1 (recall that we have symbolic behavior in x as x goes to 0!) but the others of degree 0, so the true phase function in the sense of Hörmander [4] (see also [5, Chapter XXI]) is

$$\tilde\psi(\xi, y', Y, U, r) = \psi(\xi, y', \frac{Y}{r}, \frac{U}{r})r = -\xi \cdot Y + r\phi(\frac{Y}{r}, y', u). \tag{5.22}$$

The corresponding change in the variables of integration and Hörmander's order convention [4] gives rise to the shift of orders to $-m - (n-1)/2$.

If $j : \mathbb{R}^n \to \mathbb{R}^n$ denotes the map $\xi \mapsto -\xi$, then $\mathcal{F}^{-1} = (2\pi)^{-n} j^* \mathcal{F}$. Correspondingly, $\mathcal{F}^{-1}$ maps Legendre distributions associated to G to Lagrangian distributions associated to

$$\Lambda_-(G) = \{(rj^* L(\pi_1(p)), r\pi_2(p)) :\ p \in G,\ r > 0\} \tag{5.23}$$

Now we insert $\mathcal{F}^{-1}\mathcal{F}$ between (localized versions of) the operators $((H-\lambda^2)\tilde P(-\lambda))^*$ and $\tilde P(\lambda)$ in (5.5). The results of Section 3 show that the kernels of these operators are in $I_{\mathrm{sc},\rho}^{-\frac{2n-1}{4}+1}(\partial X \times X; G(\lambda)^\dagger; \mathrm{KD}_{\mathrm{sc}}^{\frac12})$ and $I_{\mathrm{sc},\rho}^{-\frac{2n-1}{4}}(X \times \partial X; G(\lambda); \mathrm{KD}_{\mathrm{sc}}^{\frac12})$ respectively with $\rho = 1$ if $\alpha_\pm$ are a constant; if $\alpha_\pm$ are not constant, we can take any $\rho \in (\frac12, 1)$. Thus, the corresponding (inverse) Fourier transformed operators are in $I_\rho^{-3/4}(\partial X \times \mathbb{R}^n; \Lambda_-(G(\lambda)^\dagger); \Omega^{\frac12})$ and $I_\rho^{1/4}(\mathbb{R}^n \times \partial X; \Lambda_+(G(\lambda)); \Omega^{\frac12})$ respectively, with $\rho = 1$ or $\rho < 1$ as before. In fact, (5.18), together with the polyhomogeneous description of $\tilde P(\pm\lambda)$, see (3.7) and (3.16), also shows the stronger polyhomogeneous statement that we can replace the spaces mentioned above by $I_{\mathrm{b,phg}}^{-3/4+i\alpha_+}(\partial X \times \mathbb{R}^n; \Lambda_-(G(\lambda)^\dagger); \Omega^{\frac12})$ and $I_{\mathrm{b,phg}}^{1/4-i\alpha_-}(\mathbb{R}^n \times \partial X; \Lambda_+(G(\lambda)); \Omega^{\frac12})$ respectively where $\alpha_\pm$ are regarded as functions on $\mathbb{R}^n \times \partial X$ using the projection to ∂X.

We also remark that $\Lambda_+(G(\lambda))' = \Lambda_+(G(\lambda)')$ and $\Lambda_-(G(\lambda)^\dagger)' = \Lambda_+((G(\lambda)^\dagger)')$ (the difference between the two expressions is due to the fact that in one case the (inverse) Fourier transform is in the primed factor, in the other it is not). Therefore, as $(G(\lambda)^\dagger)' \times G(\lambda)'$ intersects Δ cleanly with excess 1, the same follows for $\Lambda_-(G(\lambda)^\dagger)' \times \Lambda_+(G(\lambda))'$ and Δ (here Δ is the partial diagonal in the new space by an abuse of notation). Thus, the kernel of the composite operator, i.e. that of $\tilde S(\lambda)$, is a Lagrangian distribution. Since the excess of the intersection is 1, the kernel of $\tilde S(\lambda)$ (and hence of $S(\lambda)$) is a Lagrangian distribution of order 0 (of class I_ρ^0 with ρ as above) associated to the Lagrangian

$$\Lambda = (\Lambda_-(G(\lambda)^\dagger)' \circ \Lambda_+(G(\lambda))')'. \tag{5.24}$$

In fact, $S(\lambda)$ is even in the class $I_{\mathrm{b,phg}}^{i(\alpha_+-\alpha_-)}$ since $\alpha_\pm$ do not depend on the composition variable $\frac{y}{x}$ in $\mathbb{R}^n$. Unraveling the Fourier transform we see that the associated Lagrangian is

$$\Lambda = ((G(\lambda)^\dagger)' \circ G(\lambda)')'_{\mathrm{hom}} \tag{5.25}$$

where hom denotes that we have to homogenize the submanifold we obtain by the composition $\circ$, i.e. make it invariant under the $\mathbb{R}^+$ action on $T^*(\partial X \times \partial X) \setminus 0$. This homogenization simply corresponds to the presence of $r > 0$ in Equations (5.19) and (5.23). As $(G(\lambda)^\dagger)'$ is the same as $(G(-\lambda)')^\dagger$ (i.e. the sign of ∂X component of the covectors in $G(-\lambda)$ is switched and the order of the variables is reversed), from (3.5)-(3.6) we conclude that

(5.26)
$$\begin{aligned}\Lambda = \{(y', y'', \mu', \mu'') \in T^*(\partial X \times \partial X) \setminus 0 :\ & \mu' = r'\lambda(\sin s')\hat{\mu}',\ \mu'' = -r\lambda(\sin s)\hat{\mu}'',\\ & r\lambda \sin s \exp(sH_{\frac{1}{2}h})(y'', \hat{\mu}'') = r'\lambda \sin s' \exp((s' - \pi)H_{\frac{1}{2}h})(y', \hat{\mu}'),\\ & r\lambda \cos s = r'\lambda \cos s',\ r, r' > 0,\ s, s' \in (0, \pi),\ (y', \hat{\mu}'), (y'', \hat{\mu}'') \in S^*\partial X\}.\end{aligned}$$

This implies that $r = r'$, $s = s'$, so we deduce that $(y', \hat{\mu}') = \exp(\pi H_{\frac{1}{2}h})(y'', \hat{\mu}'')$ and

$$\begin{aligned}\Lambda = \{(y', y'', \mu', \mu'') \in T^*(\partial X \times \partial X) \setminus 0 : (y', \hat{\mu}') = \exp(\pi H_{\frac{1}{2}h})(y'', \hat{\mu}''),\\ \mu' = \rho\hat{\mu}',\ \mu'' = -\rho\hat{\mu}'',\ \rho > 0,\ (y', \hat{\mu}'), (y'', \hat{\mu}'') \in S^*\partial X\}.\end{aligned} \tag{5.27}$$

This is exactly the Lagrangian associated to the (forward) geodesic flow at distance π. We can know use the Riemannian density on X and ∂X to regard $S(\lambda)$ as an operator on $\mathcal{C}^{-\infty}(\partial X)$. We have thus shown the following theorem.

Theorem 5.2. *Let Λ be the Lagrangian associated to the (forward) geodesic flow of h on ∂X at distance π. Then for $\lambda > 0$, $S(\lambda)$ is a Fourier integral operator with kernel in $I^0_\rho(\partial X \times \partial X; \Lambda; \Omega_R)$ with $\rho = 1$ if $\alpha_\pm$ are constant on ∂X and with $\rho \in (\frac{1}{2}, 1)$ arbitrary in general. More precisely, the kernel is in the polyhomogeneous space $I^{i(\pi_L^*\alpha_+ - \pi_R^*\alpha_-)}_{b,phg}(\partial X \times \partial X; \Lambda; \Omega_R)$ where $\pi_L, \pi_R : \partial X \times \partial X \to \partial X$ are the projections.*

Remark 5.3. For $\lambda < 0$, $S(\lambda)$ is associated to the backward geodesic flow at distance π by a similar argument, and it has order $i(\pi_L^*\alpha_- - \pi_R^*\alpha_+)$. Also note that $S(\lambda) = S(-\lambda)^{-1}$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ directly from the definition.

6. The principal symbol calculation

In this section we briefly outline how the principal symbol of $S(\lambda)$ can be calculated using the clean composition formulae of [1] and the calculations of Joshi and Sá Barreto for the principal symbol of the Poisson operator away from the 'end' of the Legendrian. This and other similar calculations can be used to analyze the inverse problem of reconstructing the asymptotics of V (i.e. its Taylor series at ∂X) as done by Joshi and Sá Barreto [11] for short-range potentials and by Joshi [8] in the constant $\alpha_\pm$ case. We, however, shall not analyze this problem in this paper. For this reason, and also since the Poisson operator construction gives a better way of calculating the principal symbol, exactly as was done by Joshi and Sá Barreto, the following discussion will be kept brief. In particular, we drop all Maslov factors.

First we remark that the principal symbol of $K^\flat_\pm$ can be computed just as it was done by the aforementioned authors; we just need to solve a transport equation, i.e. a first order ODE. Let

$$\gamma_+ : (0, \pi)_s \times S^*\partial X_{(y', \hat{\mu}')} \to {}^{sc}T^*_{\partial X}X \tag{6.1}$$

be the map given by the composition of the parametrization of $G(\lambda)$ in terms of these coordinates with projection to ${}^{sc}T^*_{\partial X}X$ (i.e. π_1); define γ_- similarly with $G(-\lambda)$ in place of $G(\lambda)$. We remark that

$$(y', \hat{\mu}') = \exp(\pi H_{\frac{1}{2}h})(y'', \hat{\mu}'') \Rightarrow \gamma_+(s, y'', \hat{\mu}'') = \gamma_-(s, y', \hat{\mu}') \tag{6.2}$$

directly from (3.5)-(3.6). Using the coordinates $(s, y', \hat{\mu}')$ and $(s, y'', \hat{\mu}'')$ (given in (3.5)-(3.6)) on $G(\lambda)$ and $G(-\lambda)$ respectively, the result of solving the transport equations is that away from $U_\pm$ (respectively) the principal symbol is given by

$$\begin{aligned}(6.3)\qquad &\sigma_{\mathrm{sc},-(2n-1)/4}(x^{-i\alpha_{-,R}}\tilde{P}(\lambda))(s, y'', \hat{\mu}'')\\ &\quad = (\sin s)^{(n-2)/2} e^{i\int_0^s f_+(\gamma_+(s', y'', \hat{\mu}''))\,ds'}\, |ds|^{\frac{1}{2}} |dy''|^{\frac{1}{2}} |d\hat{\mu}''|^{\frac{1}{2}} |dx|^{-n/2},\end{aligned}$$

$$\begin{aligned}(6.4)\qquad &\sigma_{\mathrm{sc},-(2n-1)/4}(x^{-i\alpha_{+,R}}\tilde{P}(-\lambda))(s, y', \hat{\mu}')\\ &\quad = (\sin s)^{(n-2)/2} e^{i\int_\pi^s f_-(\gamma_-(s', y', \hat{\mu}'))\,ds'}\, |ds|^{\frac{1}{2}} |dy'|^{\frac{1}{2}} |d\hat{\mu}'|^{\frac{1}{2}} |dx|^{-n/2};\end{aligned}$$

here $f_+ \in \mathcal{C}^\infty([0,\pi) \times S^*\partial X)$, $f_- \in \mathcal{C}^\infty((0,\pi] \times S^*\partial X)$ are real-valued functions (cf. Joshi's and Sá Barreto's paper [11]). We remark that the factors of x that one might expect, appear as a density (see the power of $|dx|$) since these distributions are associated to Legendre submanifolds, i.e. there is no radial factor as in the conic Lagrangians. The functions $f_\pm$ are of the form

$$(6.5)\qquad f_+(\gamma_+(s, y'', \hat{\mu}'')) = -(\cot s)\alpha_-(y'') + f(\gamma_+(s, y'', \hat{\mu}''))$$

$$(6.6)\qquad f_-(\gamma_-(s, y', \hat{\mu}')) = (\cot s)\alpha_+(y') + f(\gamma_-(s, y', \hat{\mu}'))$$

where f is still real-valued and it is the same in the previous two equations, but it is not smooth down to $s = 0$ or up to $s = \pi$ corresponding to the appearance of the term with $\cot s$. Since the (rescaled) Hamilton vector-field ${}^{\mathrm{sc}}H_g$ is $2\lambda \sin s\,\partial_s$ at $G(\pm\lambda)$, Proposition 13 of [14] shows that we have

$$\begin{aligned}(6.7)\qquad &\sigma_{\mathrm{sc},-(2n-1)/4+1}(x^{-i\alpha_{+,R}}(H-\lambda^2)\tilde{P}(-\lambda))(s, y', \hat{\mu}')\\ &\quad = -2\lambda i(\partial_s \chi_-)(\sin s)^{n/2} e^{i\int_\pi^s f_-(\gamma_-(s', y', \hat{\mu}'))\,ds'}\, |ds|^{\frac{1}{2}} |dy'|^{\frac{1}{2}} |d\hat{\mu}'|^{\frac{1}{2}} |dx|^{1-n/2}.\end{aligned}$$

The principal symbol of $(x^{-i\alpha_{+,R}}(H-\lambda^2)\tilde{P}(-\lambda))^*$ is the complex conjugate of this expression.

The fibers of Λ' over $\Lambda_-(G(\lambda)^\dagger)' \times \Lambda_+(G(\lambda))' \cap \Delta$ are given by (y', y'', μ', μ'') being constant, so from (5.26) this implies that $r \sin s$ is constant. In our notation $x = r^{-1}$, so the fibers of the composition are given by $\rho = x^{-1}\sin s$ is constant, $(y', y'', \hat{\mu}', \hat{\mu}'')$ constant, $(y', \hat{\mu}') = \exp(\pi H_{\frac{1}{2}h})(y'', \hat{\mu}'')$. Thus, we change coordinates, we replace x by ρ. By the clean composition theorem, the principal symbol of the composite operator, $\tilde{S}(\lambda)$, is given by the integral in s (keeping ρ, $(y', y'', \hat{\mu}', \hat{\mu}'')$ constant, $(y', \hat{\mu}') = \exp(\pi H_{\frac{1}{2}h})(y'', \hat{\mu}'')$) of the product of (6.3) with the complex conjugate of (6.7) and with the factors $x^{-i\alpha_+(y')+i\alpha_-(y'')}$ which have been omitted from the principal symbols. The product of the two exponential factors is

$$(6.8)\qquad \exp\Big(i\Big(\int_0^s f_+(\gamma_+(s', y', \hat{\mu}'))\,ds' - \int_\pi^s f_-(\gamma_-(s', y'', \hat{\mu}''))\,ds'\Big)\Big).$$

Taking into account (6.2), (6.5)-(6.6), this becomes

$$(6.9)\qquad \exp\Big(i\int_0^\pi f(\gamma_+(s', y'', \hat{\mu}''))\,ds'\Big)(\sin s)^{i(\alpha_+(y')-\alpha_-(y''))};$$

notice that the first factor is independent of s, so it is constant over the fibers of integration.

Instead of the standard coordinates (x, y, τ, μ) on ${}^{\mathrm{sc}}T^*X$, near $\Sigma_{\Delta-\lambda^2} \setminus (R_\lambda^- \cup R_\lambda^+)$ we can use the following coordinates $(x, y, p, \hat{\mu}, s)$:

$$(6.10)\qquad \tau = \lambda\cos s,\ \ \mu = (p - \lambda^2\cos^2 s)^{1/2}\hat{\mu},$$

so $p = \tau^2 + |\mu|^2$ restricted to ${}^{\mathrm{sc}}T^*_{\partial X}X$ is the principal symbol of Δ; it is equal to λ^2 on $\Sigma_{\Delta-\lambda^2}$. The symplectic form on ${}^{\mathrm{sc}}T^*X$ gives a trivialization (via the induced volume form)

$$\frac{1}{2}(\lambda \sin s)(p - \lambda^2(\cos s)^2)^{\frac{n-3}{2}} x^{-(n+1)}|dx|\,|dy|\,|dp|\,|d\hat{\mu}|\,|ds| = 1; \tag{6.11}$$

on $\Sigma_{\Delta-\lambda^2}$, as $\rho = x^{-1}\sin s$, this takes the form

$$C_1(\lambda)(\sin s)^{-2}\rho^{n-1}|d\rho|\,|dy|\,|dp|\,|d\hat{\mu}|\,|ds| = 1 \tag{6.12}$$

where $C_1(\lambda)$ is a 'constant', depending only on λ. Moreover, dp is identified with the Hamilton vector-field H_p via the symplectic form. As $H_p = x\,{}^{\mathrm{sc}}H_g$ (this being the rescaling factor of ${}^{\mathrm{sc}}H_g$, see [13]), the identification becomes that of $2\lambda\rho^{-1}(\sin s)^2\,\partial_s$ with dp, so we have the identification

$$C_2(\lambda)\rho^{n-2}|d\rho|\,|dy|\,|d\hat{\mu}| = 1. \tag{6.13}$$

The factors of powers of $\sin s$ cancel each other in the product of the two principal symbols when we replace x by ρ, and we can use (6.9) and (6.13). Thus,

(6.14)
$$\sigma_0(\tilde{S}(\lambda)) = C_3(\lambda)\exp(i(\int_0^\pi f(\gamma_+(s',y'',\hat{\mu}''))\,ds')$$
$$(\int_0^\pi \partial_s\chi_-\,|ds|)\rho^{i\alpha_+(y')-i\alpha_-(y'')}\rho^{(n-2)/2}\,|d\hat{\mu}''|^{1/2}|dy''|^{1/2}|d\rho|^{1/2},$$

so introducing $\mu'' = \rho\hat{\mu}''$, and noting that the integral is 1 by the fundamental theorem of calculus and the required properties of χ_- ($\chi_-(s)$ is 1 near $s=\pi$, 0 near $s=0$),

(6.15)
$$\sigma_0(\tilde{S}(\lambda))$$
$$= C_4(\lambda)\exp(i(\int_0^\pi f(\gamma_+(s',y'',\hat{\mu}''))\,ds')|\mu''|^{i\alpha_+(y')-i\alpha_-(y'')}\,|d\mu''|^{1/2}|dy''|^{1/2}.$$

7. Poisson operators: analysis at the end of the Legendrians

We now show how the kernel of the Poisson operator $P(\lambda)$ can be analyzed near $G^\sharp(\lambda)$ using $S(\lambda)$. Directly from the definition of these operators,

$$P(\lambda) = P(-\lambda)S(\lambda), \tag{7.1}$$

so the understanding of $S(\lambda)$ together with that of $P(-\lambda)$ *near* $G^\sharp(\lambda)$ gives the desired description of $P(\lambda)$ there. More precisely, choosing $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ such that $\mathrm{WF}_{\mathrm{sc}}(Q)\cap R^-_\lambda = \emptyset$, $\mathrm{WF}_{\mathrm{sc}}(\mathrm{Id}-Q)\cap R^+_\lambda = \emptyset$, it follows from (3.14) and Proposition 2.5 that (if U_+ is small enough) $Q(P(\lambda)-\tilde{P}(\lambda))$ maps $\mathcal{C}^{-\infty}(\partial X)$ to $\dot{\mathcal{C}}^\infty(X)$, so its kernel is in $\dot{\mathcal{C}}^\infty(X\times\partial X;\mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}})$. Similarly, the kernel of $(\mathrm{Id}-Q)(P(-\lambda)-\tilde{P}(-\lambda))$ is in $\dot{\mathcal{C}}^\infty(X\times\partial X;\mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}})$ (if U_- is sufficiently small). Thus, near $G^\sharp(\lambda)$ the kernel of $P(\lambda)$ is given by $(\mathrm{Id}-Q)\tilde{P}(-\lambda)S(\lambda)$, and the kernel of $(\mathrm{Id}-Q)\tilde{P}(-\lambda)$ has the same form as that of $\tilde{P}(-\lambda)$ in this region. Again, this is a composition result, and the easiest way to see it is to consider $\mathcal{F}^{-1}(\mathcal{F}(\mathrm{Id}-Q)\mathcal{F}^{-1})(\mathcal{F}\tilde{P}(-\lambda))$, use the invariance of $\Psi^{0,0}_{\mathrm{sc}}(X)$ under conjugation by the Fourier transform, and that (the kernel of) $\mathcal{F}\tilde{P}(-\lambda)$ is a Lagrangian distribution. Hence, $(\mathrm{Id} - Q)\tilde{P}(-\lambda)$ is given by an oscillatory function in this region; namely it is of the form

$$K_-^\flat = e^{i\lambda \cos \operatorname{dist}(y,y')/x} x^{i\alpha_+(y')} a(x,y,y')\nu, \qquad a \in \mathcal{C}^\infty(X \times \partial X), \tag{7.2}$$

microlocally *near* $G^\sharp(\lambda)$ ($\nu = |dg|^{\frac{1}{2}}|dh|^{\frac{1}{2}}$).

We also write out the oscillatory integral representation of $S(\lambda)$ in local coordinates. We write the parameters in inverse polar coordinates, so $t^{-1}u$ is the Euclidian variable, so (modulo half-density factors)

$$S(\lambda)(y',y'') = \int e^{i\phi(y',y'',u)/t} b(y',y'',t,u) t^{(k-n+1)/2} \frac{dt\,du}{t^{k+1}}, \tag{7.3}$$

$$t^{i(\alpha_+(y') - \alpha_-(y''))} b \in \mathcal{C}_c^\infty(\partial X_{y'} \times \partial X_{y''} \times [0,1)_t \times \mathbb{R}_u^{k-1}). \tag{7.4}$$

In particular, b is a symbol of order 0 in t (in class S_ρ with $\rho = 1$ if $\alpha_\pm$ are constant, and with any $\rho < 1$ otherwise). We thus obtain the following oscillatory integral representation of the kernel of $P(\lambda)$:

$$\int e^{i(\phi(y',y'',u)/t + \lambda \cos \operatorname{dist}(y,y')/x)} c(x,y,y',y'',t,u)\, t^{-(k+n+1)/2}\, dt\, du\, dy', \tag{7.5}$$

where c is still a symbol of order 0 in t (as well as in x). Now we let $s = \frac{x}{t}$, so this integral becomes

$$\int e^{i(\phi(y',y'',u)s + \lambda \cos \operatorname{dist}(y,y'))/x} c(x,y,y',y'',\frac{x}{s},u) \left(\frac{x}{s}\right)^{-(k+n-1)/2} s^{-1}\, ds\, du\, dy'. \tag{7.6}$$

Here the s integral is from 0 to ∞, but ϕ and $d_{(y',u)}\phi$ never vanish at the same time (due to (5.27), $\phi = 0$ and $d_u\phi = 0$ imply that $d_{y'}\phi \neq 0$), so for large s we have $d_{(y',u,s)}(\phi(y',y'',u)s + \lambda \cos \operatorname{dist}(y,y')) \neq 0$, so integration by parts shows that the region where $s > 2\lambda$ contributes only a function in $\dot{\mathcal{C}}^\infty(X \times \partial X)$, so we can replace c by $c'(x,y,y',y'',s,\frac{x}{s},u) = \chi(s)c$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 on $[-2\lambda, 2\lambda]$:

$$\int e^{i(\phi(y',y'',u)s + \lambda \cos \operatorname{dist}(y,y'))/x} c'(x,y,y',y'',s,\frac{x}{s},u) \left(\frac{x}{s}\right)^{-(k+n-1)/2} s^{-1}\, ds\, du\, dy', \tag{7.7}$$

$$\begin{aligned} &x^{-i\alpha_-(y'')} s^{-i(\alpha_+(y') - \alpha_-(y''))} c' \\ &\qquad \in \mathcal{C}_c^\infty([0,\infty)_x \times \partial X_y \times \partial X_{y'} \times \partial X_{y''} \times [0,\infty)_s \times [0,\infty)_{x/s} \times \mathbb{R}_u^{k-1}). \end{aligned} \tag{7.8}$$

We recall from [14] that $\tilde{G}(\lambda) = (G(\lambda), G^\sharp(\lambda))$ is an intersecting pair of Legendre submanifolds with conic points. Equation (7.7) is nearly of the form of Melrose's and Zworski's parametrization of conic Legendrian $\tilde{G}(\lambda)$; for their formula we need a phase function of the form $\psi(y,y'',s,u')s + \lambda$, i.e. we need to replace $\lambda \cos \operatorname{dist}(y,y')$ in the phase function by λ at the cost of modifying the factor in front of s and changing the parameters. Of course, away from $s = 0$ our phase function is already of the correct form, so in the following we may assume that s is sufficiently small. We proceed by remarking that when $s = 0$ then the y' derivative of the phase function is nonzero except when $y = y'$ where it (the phase function) has a non-degenerate critical point with respect to y' (since $\cos \operatorname{dist}(y,y')$ has this property). Hence, for small s the phase function has non-degenerate critical points with respect to y' which, by the implicit function theorem, are given by $y' = f(y, y'', s, u)$ where f is smooth, also note that $f(y, y'', 0, u) = y$. We can carry out the y' integral by the stationary phase lemma; this introduces an additional factor of $x^{(n-1)/2}$ and factors of $\log s$ due to the exponent $s^{i\alpha_+(y')}$. Therefore, (7.7) becomes

$$\int e^{i(\phi(f(y,y'',s,u),y'',u)s+\lambda)/x} c^\flat(x, y, y'', s, \frac{x}{s}, u) \left(\frac{x}{s}\right)^{-k/2} s^{(n-3)/2}\, ds\, du. \tag{7.9}$$

$$\begin{aligned} &x^{-i\alpha_-(y'')} s^{-i(\alpha_+(y)-\alpha_-(y''))} c^\flat \\ &\qquad \in \mathcal{A}^{\mathcal{K}'}_{\mathrm{phg,c}}([0,\infty)_x \times \partial X_y \times \partial X_{y''} \times [0,\infty)_s \times [0,\infty)_{x/s} \times \mathbb{R}^{k-1}_u); \end{aligned} \tag{7.10}$$

here $\mathcal{A}^{\mathcal{K}'}_{\mathrm{phg,c}}$ stands for polyhomogeneous functions of compact support with index family $\mathcal{K}'$ given by $\mathbb{N}$ for the hypersurfaces $x = 0$ and $x/s = 0$, and by $\mathcal{K}$ for $s = 0$. Thus, we allow logarithmic terms in s just as in (4.2) but we demand smoothness in the variables x and x/s down to 0. This is exactly of the form of Melrose's and Zworski's definition of Legendrian distributions with conic singularities (though of course they did not have the variable order and the logarithmic terms). Note that the phase function is indeed non-degenerate in the sense defined in their paper [14], namely with $\psi(y, y'', s, u) = \phi(f(y, y'', s, u), y'', u)$ we have that $d_{(y,y'',u)}\psi$ and $d_{(y,y'',u)}\partial_{u_j}\psi$ are linearly independent at $(y, y'', 0, u)$ if this point is in the critical set since $f(y, y'', 0, u) = y$ reduces this to the assumption that ϕ is non-degenerate. Matching the orders with their definition we have deduced the following theorem.

Theorem 7.1. *The kernel of $P(\lambda)$ (which we also denote by $P(\lambda)$) satisfies*

$$P(\lambda) \in I^{-(2n-1)/4,-1/4}_{sc,\rho}(X \times \partial X, \tilde{G}(\lambda), \mathrm{KD}^{\frac{1}{2}}_{sc}) \tag{7.11}$$

with $\rho = 1$ if $\alpha_\pm$ are constant and with any $\rho \in (\frac{1}{2}, 1)$ otherwise. More precisely, $P(\lambda)$ is in a polyhomogeneous space defined, modulo $\dot{\mathcal{C}}^\infty(X \times \partial X; \mathrm{KD}^{\frac{1}{2}}_{sc})$, by a locally finite sum of oscillatory integral representations of the form (7.9)-(7.10).

We note that the exponent of x in (7.10) matches that in (3.7) as it must. The logarithmic terms in (7.10) just correspond to the logarithmic terms in (2.13) as discussed after Definition 4.1 for the behavior of $P(\lambda)$ near the incoming radial set. Also, the polyhomogeneous space defined in the theorem is indeed independent of the choices made, in particular of the extension of α to X (used to define $s^{i\alpha_+(y)}$), by the arguments of Section 4.

Our argument also shows that near the singularity, $G(\lambda) \cap G^\sharp(\lambda)$, the space $I^{m,p}_{\mathrm{sc,os}}(X \times \partial X, \tilde{G}(\lambda), \mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}})$ consists of distributions which arise as the composition of $e^{i\lambda \cos \mathrm{dist}(y,y')/x} a(x, y, y')$, $a \in \mathcal{C}^\infty(X \times \partial X; \mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}})$, supported near $x = 0$, $y = y'$, non-zero on a smaller set near $x = 0$, $y = y'$, with classical Fourier integral operators on ∂X associated to Λ, and similar statements are true for the non-polyhomogeneous (or other polyhomogeneous) spaces. In addition, the non-vanishing of the principal symbol of $\tilde{P}(-\lambda)$ in this region also shows that the principal symbol of $S(\lambda)$ can be recovered from the asymptotics of the principal symbol of $P(\lambda)$ on approaching $G^\sharp(\lambda)$; this was the method used by Joshi and Sá Barreto in their computations in [11].

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