Geometric optics and the wave equation on manifolds with corners

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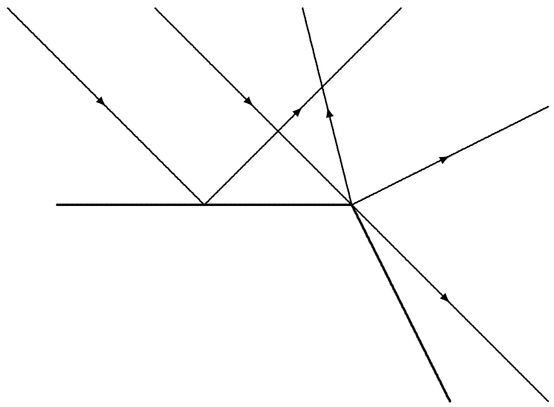
According to geometric optics, light propagates in straight lines (in homogeneous media), reflects/refracts from surfaces according to Snell's law: energy and tangential momentum are conserved.



But light also satisfies the wave equation,

$$Pu = 0, \quad Pu = D_t^2 u - \Delta_g u,$$

Δ_g is the Laplacian, so it is $c^2 \sum_{j=1}^n D_{x_j}^2$ in $\mathbb{R}^n$, where c is the speed of light (this corresponds to a Riemannian metric $g = c^{-2} \sum dx_j^2$), $D_{x_j} = \frac{1}{i}\partial_{x_j}$, with suitable boundary conditions.

One way of discussing the relationship between these is that singularities (lack of smoothness) of solutions of $Pu = 0$ follow geometric optics rays. In order to describe this precisely, I discuss a more general setting.

Let's start with the boundaryless case. So suppose X is a manifold without boundary of dimension n. The standard setting for microlocal analysis is then T^*X. If z_j are local coordinates on X, and we write one-forms as $\sum \zeta_j \, dz_j$, then (z_j, ζ_j), $j = 1, \ldots, n$, are local coordinates on T^*X. (Actually, they are global on the fibres of $T^*X \to X$.)

Now, T^*X, as a vector bundle, is equipped with an $\mathbb{R}^+$-action: $\mathbb{R}^+_s \times T^*X \ni (s, z, \zeta) \mapsto (z, s\zeta)$.

It is also a symplectic manifold, equipped with a canonical symplectic form ω, $\omega = \sum d\zeta_j \wedge dz_j$ in local coordinates.

If P is a differential operator on X, say $P = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha$, one can associate a principal symbol $p = \sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(z) \zeta^\alpha$ to P; this is homogeneous degree m with respect to the $\mathbb{R}^+$-action.

In fact, the same works for pseudodifferential operators, $P \in \Psi^m_{cl}(X)$; now $p = \sigma_m(P)$ is homogeneous degree m on $T^*X \setminus o$, o denoting the zero section.

The symplectic form ω turns p, or dp, into a vector field H_p on T^*X via $\omega(V, H_p) = Vp$ for all vector fields V. Thus,

$$H_p = \sum_j \frac{\partial p}{\partial \zeta_j} \frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j} \frac{\partial}{\partial \zeta_j}.$$

Note that H_p is homogeneous of degree $m-1$.

Bicharacteristics are then integral curves of H_p inside the characteristic set $\Sigma = p^{-1}(\{0\})$.

To do analysis, we also need a notion of singularity of a function or distribution u. The roughest notion is that of the wave front set $\mathrm{WF}(u)$, which locates at which points and in which direction a function u is not smooth, here meaning C^∞. This is defined as a closed conic subset of $T^*X \setminus o$; u is C^∞ if and only if $\mathrm{WF}(u) = \emptyset$. In fact, for any point $z_0 \in X$, z_0 has a neigborhood in X on which u is C^∞ if and only if $\mathrm{WF}(u) \cap (T^*_{z_0}X \setminus o) = \emptyset$.

One way of defining $\mathrm{WF}(u)$ for $u \in \mathcal{D}'(X)$ is to say that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}(u)$ if there exists $A \in \Psi^0(X)$ such that $\sigma_0(A)(q) \neq 0$ (i.e. A is elliptic at q) and $Au \in C^\infty(X)$.

For example: $\mathrm{WF}(\delta_0) = \{(0,\zeta) : \zeta \neq 0\} = N^*\{0\} \setminus o$, and if Ω has a C^∞ boundary, and χ_Ω is the characteristic function of Ω, then $\mathrm{WF}(\chi_\Omega) = N^*\partial\Omega \setminus o$.

One can measure singularities with respect to other spaces: e.g. the Sobolev spaces $H^s_{\mathrm{loc}}(X)$, where we would write $\mathrm{WF}^s(u)$, or with respect to real analytic functions, where we would write $\mathrm{WF}_A(u)$.

The main facts about the analysis of P are:

1. Microlocal elliptic regularity. Let $\Sigma(P) = p^{-1}(\{0\})$ be the characteristic set of P. If $u \in \mathcal{D}'(X)$ then $\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \Sigma(P)$. In particular, if $Pu \in C^\infty(X)$ then $\mathrm{WF}(u) \subset \Sigma(P)$.

2. Propagation of singularities. Suppose that p is real, $Pu \in C^\infty(X)$. Then $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics in $\Sigma(P)$. That is, if $q \in \mathrm{WF}(u)$ (hence in $\Sigma(P)$) then so is the whole bicharacteristic through q.

Note that 2 may be vacuous; indeed, if H_p is radial, i.e. tangent to the orbits of the $\mathbb{R}^+$-action, then it does not give any information on $\mathrm{WF}(u)$, as the latter is already known to be conic. Such points are called radial points, and in recent work with Hassell and Melrose, they have been extensively analyzed under non-degeneracy assumptions. If P is the wave operator, there are no radial points in $\Sigma = \Sigma(P)$, but such points are very important in scattering theory (where the $\mathbb{R}^+$-action, or its remnants, are in the base variables z).

Example: $P = D_t^2 - \Delta_g$ the wave operator, $X = M \times \mathbb{R}$, M a manifold without boundary. Then $p = \sigma_2(P) = \tau^2 - |\xi|_g^2$, where (x, t, ξ, τ) are coordinates on T^*X, and the projection of bicharacteristics to M are geodesics. If $M \subset \mathbb{R}^n$ and g is the Euclidean metric, then $H_p = 2\tau\partial_t - 2\xi \cdot \partial_x$, and bicharacteristics inside $p = 0$, i.e. $|\tau| = |\xi|$, are straight lines

$$s \mapsto (x_0 - 2s\xi_0, t_0 + 2\tau_0 s, \xi_0, \tau_0),$$

which explains geometric optics in the absence of boundaries.

We can now turn to boundaries and corners. So suppose X is a manifold with corners. Locally this means that X is diffeomorphic to an open subset of $[0, \infty)^k \times \mathbb{R}^{n-k}$; we denote the corresponding coordinates by (x, y).

Roughly, the results have the same form as in the boundaryless case, but the definitions of wave front set and the bicharacteristics change significantly. In particular, the relevant wave front set is $\mathrm{WF}_b(u)$, introduced by Melrose. Both $\mathrm{WF}_b(u)$ and the image of the (generalized broken) bicharacteristics are subsets of the b-cotangent bundle ${}^bT^*X$.

Rather than defining this, I describe its main properties. Being a vector bundle, locally in X it is trivial, and in the product decomposition above, it will take the form $[0, \infty)^k_x \times \mathbb{R}^{n-k}_y \times \mathbb{R}^k_\sigma \times \mathbb{R}^{n-k}_\zeta$ where σ is the 'b-dual' variable of x and ζ is the b-dual variable of y.

There is a natural map $\pi : T^*X \to {}^bT^*X$, which in these local coordinates takes the form

$$\pi(x, y, \xi, \zeta) = (x, y, x\xi, \zeta),$$
$$\text{with } x\xi = (x_1\xi_1, \ldots, x_k\xi_k).$$

Thus, π is a C^∞ map, but at ∂X, it is not a diffeomorphism.

Over X°, ${}^bT^*X$ and T^*X are naturally identified via π, and

$$\mathrm{WF}_b(u) \cap {}^bT^*_{X^\circ}X = \pi(\mathrm{WF}(u) \cap T^*_{X^\circ}X).$$

Note that if q is a linear function on each fiber of ${}^bT^*X$, then it has the form

$$q = \sum a_j(x,y)\sigma_j + \sum b_j(x,y)\zeta_j,$$

so

$$\pi^* q = \sum a_j(x,y)x_j\xi_j + \sum b_j(x,y)\zeta_j,$$

which is the principal symbol of

$$Q = \sum a_j(x,y)x_j D_{x_j} + \sum b_j(x,y) D_{y_j}.$$

Vector fields of this form are exactly the vector fields tangent to all boundary faces of X; we denote their space by $\mathcal{V}_b(X)$. There is a pseudodifferential algebra microlocalizing $\mathcal{V}_b(X)$ and the corresponding algebra of differential operators $\mathrm{Diff}_b(X)$, denoted by $\Psi_b(X)$.

If $P \in \mathrm{Diff}^m(X)$, the characteristic set $\Sigma(P) = p^{-1}(\{0\})$ is a subset of T^*X. Let $\dot\Sigma = \pi(\Sigma(P)) \subset {}^bT^*X$. Generalized broken bicharacteristics are curves inside $\dot\Sigma$, namely they are continuous maps $\gamma : I \to \dot\Sigma$, where I is an interval, satisfying a Hamilton vector field condition.

If $X = M \times R$, M a manifold with corners, g a Riemannian metric on M, $P = D_t^2 - \Delta_g$ is the wave operator, then Snell's law is encoded in the statement that γ is continuous. Thus, any (locally defined) smooth functions on ${}^bT^*X$, such as $x, y, t, \sigma, \zeta, \tau$, are continuous along γ. However, $\xi_j = x_j^{-1}\sigma_j$ is *not* continuous, so the normal momentum may jump.

We are now ready to state the main theorem. Recall that $H_0^1(X)$ is the completion of $C_c^\infty(X^\circ)$ in the norm

$$\|u\|^2_{H^1(X)} = \|du\|^2_{L^2(X)} + \|u\|^2_{L^2(X)},$$

and that elements of $H_0^1(X)$ restrict as 0 to ∂X, i.e. $u \in H^1_{0,\mathrm{loc}}(X)$ means that u satisfies the Dirichlet boundary conditions.

Theorem 1 *Suppose $Pu = 0$, $u \in H^1_{0,\mathrm{loc}}(X)$. Then $\mathrm{WF}_b(u) \subset \dot{\Sigma}$, and it is a union of maximally extended generalized broken bicharacteristics of P.*

This theorem can be stated in a completely microlocal manner, and one can also measure the regularity modulo Sobolev spaces. In addition, it also holds for Neumann boundary conditions.

It was proved in the real analytic setting by Lebeau, and in the C^∞ setting with C^∞ boundaries (and no corners) by Melrose, Sjöstrand

and Taylor. This result is thus the C^∞ version of Lebeau's theorem: the geometry is similar in the real analytic vs. C^∞ settings, but the analysis is quite different.

Ideas of the proof: use positive commutator estimates, b-microlocalization, i.e. use elements $A \in \Psi_b^m(X)$ as the commutants, and the Dirichlet form. The proof is thus different from the Melrose-Sjöstrand-Taylor proof even in the C^∞ boundary case at hyperbolic points. As this is too technical, I quickly go through the positive commutator proof in the boundaryless setting.

For this purpose, we also need that operators $A \in \Psi^m(X)$ have an operator wave front set associated to them,

$$\mathrm{WF}'(A) \subset T^*X \setminus o,$$

which is again closed and conic. Roughly, if $A = a(x, y, D_x, D_y)$, then a point $q = (x, y, \xi, \zeta)$ is not in $\mathrm{WF}'(A)$ if a is a symbol of order $-\infty$, i.e. decays rapidly, in an open cone around q. Then the action of A is microlocal:

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

For $P \in \Psi(X)$, we thus want to construct $A \in \Psi(X)$ such that $i[A^*A, P]$ is positive, modulo terms we can control. This is useful since

$$\begin{aligned}
&\langle i[A^*A, P]u, u\rangle \\
&\qquad = \langle iA^*APu, u\rangle - \langle iPA^*Au, u\rangle \\
&\qquad = \langle iA^*APu, u\rangle + \langle u, iA^*APu\rangle,
\end{aligned}$$

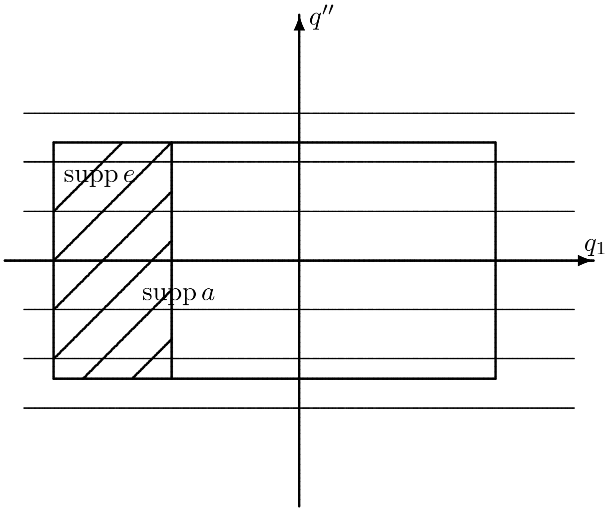
and the PDE gives information about Pu. (This argument needs to be regularized.) Now, if $i[A^*A, P] = B^*B + E + F$ where F is lower order, hence negligible, we deduce that $\|Bu\|^2$ can be controlled provided we have information about $\langle Eu, u\rangle$, i.e. about u microlocally on $\mathrm{WF}'(E)$.

The shape this takes in the positive commutator estimates is that we need to assume that $\mathrm{WF}(u)$ is disjoint from $\mathrm{WF}'(E)$, and then we *propagate* the regularity of u and conclude that $\mathrm{WF}(u)$ is disjoint from the elliptic set of B. (At least after an iterative argument, as before.)

The equation $i[A^*A, P] = B^*B + E + F$ as above is a condition on the principal symbol $\sigma(A)$, namely that $H_p\sigma(A) = -b^2 + e$, with b and e playing the role of B and E. Thus, we want $a = \sigma(A)$ to be decreasing along bicharacteristics except in a region in which we have a priori information about u. It also needs to have small support.

In our simple setting, H_p can locally be made into a constant vector field $\frac{\partial}{\partial q_1}$ by a change of coordinates on $S^*X = (T^*X \setminus o)/\mathbb{R}^+$. Thus, one works with homogeneous degree zero functions $q = (q_1, q'')$ on $T^*X \setminus o$ which give coordinates on S^*X. In such coordinates it is straightforward to construct such an a as a C^∞ function on S^*X, and then regard it as a homogeneous function on $T^*X \setminus o$: one can take a to be the product of a function of q_1, on the derivative of which we have sign conditions, and an arbitrary function of q''.

Below we illustrate the bicharacteristics of H_p, which are now straight lines, and the support of a and e.



In general, at ∂X, the problem is microlocalization: there is no analogue of the functions of q'' – we cannot microlocalize by taking a product of two functions.

Instead, one follows a much more robust construction due to Melrose and Sjöstrand, which they used to prove propagation of singularities at glancing points when ∂X was smooth, which would, however, take us beyond the limits of this talk.