GLOBAL ANALYSIS OF QUASILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY KERR-DE SITTER SPACES

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Abstract. We consider quasilinear wave equations on manifolds for which infinity has a structure generalizing that of Kerr-de Sitter space; in particular the trapped geodesics form a normally hyperbolic invariant manifold. We prove the global existence and decay, to constants for the actual wave equation, of solutions. The key new ingredient compared to earlier work by the authors in the semilinear case [31] and by the first author in the non-trapping quasilinear case [29] is the use of the Nash-Moser iteration in our framework.

1. Introduction

We consider quasilinear wave equations on manifolds for which infinity has a structure generalizing that of Kerr-de Sitter space. An important feature is that, as in perturbations of Kerr-de Sitter space, the trapped geodesics form a normally hyperbolic invariant manifold. We prove the global existence and decay of solutions; this means decay to constants for the actual wave equation. This result is part of a new framework for solving quasilinear wave equations with normally hyperbolic trapping, which extends the semilinear framework developed by the two authors [31] and the *non-trapping* quasilinear theory developed by the first author [29]. The main new tool introduced here is a Nash-Moser iteration necessitated by the loss of derivatives in the linear estimates at the normally hyperbolic trapping. To our knowledge, this is the first global result for the forward problem for a quasilinear wave equation on either a Kerr or a Kerr-de Sitter background. We remark, however, that Dafermos, Holzegel and Rodnianski [9] have constructed backward solutions for Einstein's equations on the Kerr background; for backward constructions the trapping does not cause difficulties. For concreteness, we state our results first in the special case of Kerr-de Sitter space, but it is important to keep in mind that the setting is more general.

By adding an 'ideal boundary' at infinity in the standard description of Kerr-de Sitter space, the region of Kerr-de Sitter space we are interested in can be considered a (non-compact) 4-dimensional manifold with boundary M. The interior M° is equipped with a Lorentzian metric g_0, recalled below, depending on three parameters $\Lambda > 0$ (the cosmological constant), $M_\bullet > 0$ (the black hole mass) and a (the

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angular momentum), though we usually drop this in the notation.[1] This Lorentzian metric has a specific structure at ∂M, i.e. 'infinity', called a totally characteristic, or b-, structure. Here recall that on any n-dimensional manifold with boundary M, the Lie algebra of smooth vector fields tangent to the boundary is denoted by $\mathcal{V}_b(M)$; in local coordinates $(x, y_1, \ldots, y_{n-1})$, with x a boundary defining function, these are linear combinations of $x\partial_x$ and ∂_{y_j} with $\mathcal{C}^\infty(M)$ coefficients.

Just as a dual metric is a linear combination of symmetric tensor products of coordinate vector fields, a dual metric in this totally characteristic setting, also called a dual b-metric, is a linear combination of

$$x\partial_x \otimes x\partial_x,\ \frac{1}{2}(x\partial_x \otimes \partial_{y_j} + \partial_{y_j} \otimes x\partial_x),\ \frac{1}{2}(\partial_{y_i} \otimes \partial_{y_j} + \partial_{y_j} \otimes \partial_{y_i}).$$

One can think of this as a symmetric bilinear form; then a Lorentzian dual b-metric is a non-degenerate bilinear form of signature $(1, n-1)$. The corresponding wave operator is thus a totally characteristic, or b-, operator, $\Box \in \mathrm{Diff}_b^2(M)$, i.e. is the sum of products of up to two factors of elements of $\mathcal{V}_b(M)$, with $\mathcal{C}^\infty(M)$ coefficients. The actual metric is then a linear combination of

$$\frac{dx}{x} \otimes \frac{dx}{x},\ \frac{1}{2}\Big(\frac{dx}{x} \otimes dy_j +\ dy_j \otimes \frac{dx}{x}\Big),\ \frac{1}{2}(dy_i \otimes dy_j + dy_j \otimes dy_i).$$

We denote linear combinations of these tensors over a point p by $\mathrm{Sym}^2\,{}^bT_p^*M$.

In order to set up our problem, see Figure 1 for an illustration, we consider two functions $\mathfrak{t}_j$, $j = 1, 2$, with forward, resp. backward, time-like differentials near their respective 0-set H_j, which are linearly independent at their joint 0-set, and let $\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty))$, with Ω compact, so Ω is a compact manifold with corners with three boundary hypersurfaces H_1, H_2 and $X = \partial M$, all intersected with Ω. We are interested in solving the forward problem for wave-like equations in Ω, i.e. imposing vanishing Cauchy data at H_1, which we assume is disjoint from X; initial value problems with general Cauchy data can always be converted into an equation of this type.

In order to compress notation for elements of $\mathcal{V}_b(M)$ applied to a function u, it is convenient to introduce the notation

$${}^b du = (x\partial_x u)\frac{dx}{x} + \sum_j (\partial_{y_j} u) dy_j$$

in terms of local coordinates. This is a re-interpretation of the differential du of u in terms of the 1-forms $\frac{dx}{x}$ and dy_j dual to the vector fields $x\partial_x$ and ∂_{y_j}, thus it is in fact invariantly defined. Note that when one writes e.g. $a(u, {}^b du)$, one could instead, at least locally, write

$$a(u, x\partial_x u, \partial_{y_1} u, \ldots, \partial_{y_{n-1}} u);$$

the ${}^b du$ notation is more concise and invariant. One calls linear combinations of $\frac{dx}{x}$ and dy_j over a point p elements of ${}^bT_p^*M$. We note that ${}^b d$ preserves reality.

The wave equations we consider include those of the form

$$\Box_{g(u, {}^b du)} u = f + q(u, {}^b du),$$

[1] We will always assume that Λ, $M_\bullet$ and a are such that the non-degeneracy condition [47, (6.2)] holds, which in particular ensures that the cosmological horizon lies outside the black hole event horizon.

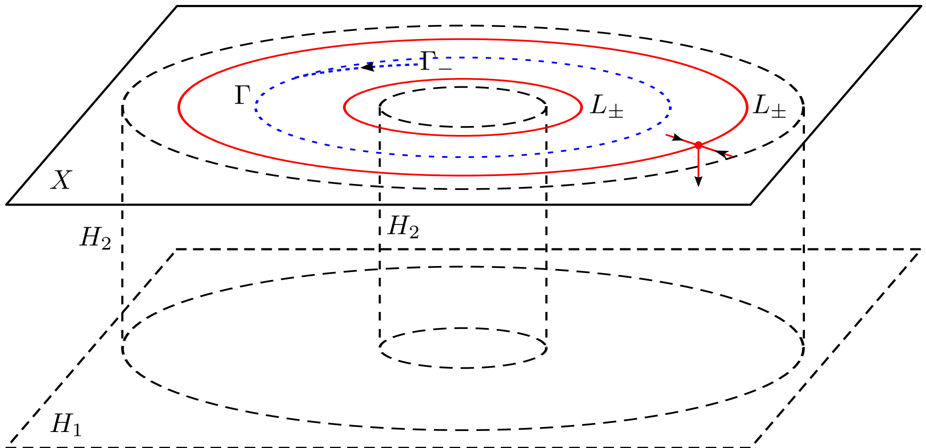

FIGURE 1. Setup for the discussion of the forward problem on Kerr-de Sitter space. Indicated are the ideal boundary X, the Cauchy hypersurface H_1 and the hypersurface H_2, which has two connected components which lie beyond the cosmological horizon and beyond the black hole event horizon, respectively. The horizons at X themselves are the projections to the base of the (generalized) radial sets $L_\pm$, discussed below, each of which has two components, corresponding to the two horizons. The projection to the base of the bicharacteristic flow is indicated near a point on L_+; near L_-, the directions of the flowlines are reversed. Lastly, Γ is the trapped set, and the projection of a trapped trajectory approaching Γ within $\Gamma_- = \Gamma_-^+ \cup \Gamma_-^-$, discussed below, is indicated.

where $g(0,0) = g_0$, and for each $p \in M$, $g_p(v_0, v) : \mathbb{R} \oplus {}^{\mathrm{b}}T_p^*M \to \mathrm{Sym}^2\, {}^{\mathrm{b}}T_p^*M$, depending smoothly on p, and[2]

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} a_j u^{e_j} \prod_{k=1}^{N_j} X_{jk} u,$$
$$e_j, N_j \in \mathbb{N}_0,\ N_j + e_j \geq 2,$$

with

$$a_j \in \mathcal{C}^\infty(M),\ X_{jk} \in \mathcal{V}_{\mathrm{b}}(M). \tag{1.1}$$

Our central result in the form which is easiest to state, without reference to the natural Sobolev spaces, is:

Theorem 1. *On Kerr-de Sitter space with angular momentum $|a| \ll M_\bullet$, for $\alpha > 0$ sufficiently small and $f \in \mathcal{C}_c^\infty(M^\circ)$ with sufficiently small H^{14}-norm, the wave equation $\Box_{g(u, {}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du)$, with q as above with $N_j \geq 1$ for all j, has a unique smooth (in M°) global forward solution of the form $u = u_0 + \tilde{u}$, $x^{-\alpha}\tilde{u}$ bounded, $u_0 = c\chi$, $\chi \in \mathcal{C}^\infty(M)$ identically 1 near ∂M.*

Further, the analogous conclusion holds for the Klein-Gordon operator $\Box - m^2$ with $m > 0$ sufficiently small, without the presence of the u_0 term, i.e. for $\alpha >$

[2]Here a_j is only relevant if $N_j = 0$.

0, $m > 0$ sufficiently small, if $f \in \mathcal{C}_c^\infty(M^\circ)$ has sufficiently small H^{14}-norm, $(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ has a unique smooth global forward solution $u \in x^\alpha L^\infty(\Omega)$. In fact, for Klein-Gordon equations one can also obtain a leading term, analogously to u_0, which now has the form $cx^{i\sigma_1}\chi$, σ_1 the resonance of $\Box_{g(0)} - m^2$ with the largest imaginary part; thus $\operatorname{Im}\sigma_1 < 0$, so this is a decaying solution.

The *only* reason the assumption $|a| \ll M_\bullet$ is made is due to the possible presence (to the extent that we do not disprove it here) of resonances in $\operatorname{Im}\sigma \geq 0$, apart from the 0-resonance with constants as the resonant state, for larger a. Below, in Section 2, we give a general result in a form that makes it clear that this is the only remaining item to check – indeed, this even holds in natural vector bundle settings.

In order to state the natural global regularity assumptions, we now discuss the Sobolev spaces corresponding to our setting: We measure regularity with respect to $\mathcal{V}_{\mathrm{b}}(M)$, and for non-negative integer s, one lets $H_{\mathrm{b}}^s(M)$ be the set of (complex-valued) $u \in L_{\mathrm{b}}^2(M)$ such that $V_1 \dots V_j u \in L_{\mathrm{b}}^2(M)$ for $j \leq s$ and $V_1, \dots, V_j \in \mathcal{V}_{\mathrm{b}}(M)$. Here, L_{b}^2 is the L^2-space with respect to any b-metric, such as the Kerr-de Sitter metric, which is thus in local coordinates given by a density which is a positive smooth multiple of $x^{-1}\,|dx\,dy_1 \dots dy_{n-1}|$. Further, one introduces the weighted Sobolev spaces $H_{\mathrm{b}}^{s,\alpha}(M) = x^\alpha H_{\mathrm{b}}^s(M)$; $H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$ denotes the real-valued elements of these spaces. Sections of vector bundles in $H_{\mathrm{b}}^{s,\alpha}$ are defined by local trivializations; the Sobolev spaces on Ω are defined by restriction.

We then relax (1.1) to

$$a_j \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^\infty(M),\ X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}(M), \tag{1.2}$$

in our assumptions. Generalizing the forcing as well, and making the conclusion more precise, the more natural version of Theorem 1 is, with further generalization given in Theorems 3 and 4:

Theorem 2. *On Kerr-de Sitter space with angular momentum $|a| \ll M_\bullet$, for $\alpha > 0$ sufficiently small and $f \in H_{\mathrm{b}}^{\infty,\alpha}$ with sufficiently small $H_{\mathrm{b}}^{14,\alpha}$-norm, the wave equation $\Box_{g(u,{}^{\mathrm{b}}du)}u = f + q(u, {}^{\mathrm{b}}du)$, with q as above with $N_j \geq 1$ for all j, has a unique, smooth in M°, global forward solution of the form $u = u_0 + \tilde{u}$, $\tilde{u} \in H_{\mathrm{b}}^{\infty,\alpha}$, $u_0 = c\chi$, $\chi \in \mathcal{C}^\infty(M)$ identically 1 near ∂M.*

Further, the analogous conclusion holds for the Klein-Gordon equation $\Box - m^2$ with $m > 0$ sufficiently small, without the presence of the u_0 term, i.e. for $\alpha > 0$, $m > 0$ sufficiently small, if $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$ has sufficiently small $H_{\mathrm{b}}^{14,\alpha}$-norm, $(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ has a unique, smooth in M°, global forward solution $u \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$.

For the proofs, we refer to Corollaries 5.13 and 5.16, which are special cases of Theorems 5.10 and 5.15. For any finite amount of regularity of the solution, our arguments only require a finite number of derivatives: Indeed, for sufficiently large $s_0, C \in \mathbb{R}$ and for $s \geq s_0$, it is sufficient to assume $f \in H_{\mathrm{b}}^{Cs,\alpha}$, with small $H_{\mathrm{b}}^{14,\alpha}$-norm, to ensure the existence of a unique global forward solution u with $H_{\mathrm{b}}^{s,\alpha}$-regularity, i.e. with $\tilde{u} \in H_{\mathrm{b}}^{s,\alpha}$ in the case of wave equations, $u \in H_{\mathrm{b}}^{s,\alpha}$ in the case of Klein-Gordon equations; see Remark 5.12 for details.

We now discuss previous results on Kerr-de Sitter space and its perturbations. There seems to be little work on non-linear equations in Kerr-de Sitter type settings; indeed the only paper the authors are aware of is the earlier paper [31] of the authors in which the semilinear Klein-Gordon equation was studied (with small data well-posedness shown) with non-linearity depending on u only, so that the losses due to the trapping could still be handled by a contraction mapping argument. In addition, the same paper also analyzed non-linearities depending on ${}^{\mathrm{b}}du$ provided these had a special structure at the trapped set. There is more work on the linear equation on perturbations of de Sitter-Schwarzschild and Kerr-de Sitter spaces: a rather complete analysis of the asymptotic behavior of solutions of the linear wave equation was given in [47], upon which the linear analysis of the present paper is ultimately based. Previously in exact Kerr-de Sitter space and for small angular momentum, Dyatlov [21, 20] has shown exponential decay to constants, even across the event horizon; see also the more recent work of Dyatlov [19]. Further, in de Sitter-Schwarzschild space (non-rotating black holes) Bachelot [3] set up the functional analytic scattering theory in the early 1990s, while later Sá Barreto and Zworski [41] and Bony and Häfner [6] studied resonances and decay away from the event horizon, Dafermos and Rodnianski in [14] showed polynomial decay to constants in this setting, and Melrose, Sá Barreto and Vasy [38] improved this result to exponential decay to constants. There is also physics literature on the subject, starting with Carter's discovery of this space-time [8, 7], either using explicit solutions in special cases, or numerical calculations, see in particular [50], and references therein. We also refer to the paper of Dyatlov and Zworski [24] connecting recent mathematical advances with the physics literature.

While it received more attention, the linear, and thus the non-linear, equation on Kerr space (which has vanishing cosmological constant) does not fit directly into our setting; see the introduction of [47] for an explanation and for further references and [15] for more background and additional references. Some of the key works in this area include the polynomial decay on Kerr space which was shown recently by Tataru and Tohaneanu [44, 43] and Dafermos, Rodnianski and Shlapentokh-Rothman [10, 11, 16], after pioneering work of Kay and Wald in [34] and [48] in the Schwarzschild setting. Andersson and Blue [1] proved a decay result for the Maxwell system on slowly rotating Kerr spaces; see also the earlier work of Bachelot [2] in the Schwarzschild setting. The crucial normal hyperbolicity of the trapping, corresponding to null-geodesics that do not escape through the event horizons, in Kerr space was realized and proved by Wunsch and Zworski [49]; later Dyatlov extended and refined the result [22, 23]. Note that a stronger version of normal hyperbolicity is a notion that is stable under perturbations.

On the non-linear side, Luk [35] established global existence for forward problems for semilinear wave equations on Kerr space under a null condition, and Dafermos, Holzegel and Rodnianski [9] constructed *backward* solutions for Einstein's equations on Kerr space. (There was also recent work by Marzuola, Metcalfe, Tataru and Tohaneanu [37] and Tohaneanu [46] on Strichartz estimates, which are applied to the study of semilinear wave equations with power non-linearities, and by Donninger, Schlag and Soffer [18] on L^∞ estimates on Schwarzschild black holes, following L^∞ estimates of Dafermos and Rodnianski [13, 12], of Blue and Soffer [5] on non-rotating charged black holes giving L^6 estimates, and of Finster, Kamran, Smoller and Yau [25, 26] on Dirac waves on Kerr.)

In the next section, Section 2, we explain the ingredients of the proof of Theorem 2, and we also state natural generalizations. At the end of that section we provide a detailed roadmap through this paper.

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2. Overview of the proof and the more general results

Having stated the result, we now explain *why* it holds. Before doing this we recall some notation. The description of $\mathcal{V}_{\mathrm{b}}(M)$ in the introduction in terms of local coordinates shows that it is the space of all $\mathcal{C}^\infty$ sections of a vector bundle, ${}^{\mathrm{b}}TM$, with local basis $x\partial_x$, ∂_{y_j}. The dual bundle of ${}^{\mathrm{b}}TM$ is denoted by ${}^{\mathrm{b}}T^*M$; it has a local basis of $\frac{dx}{x}$, dy_j. A b-metric is a non-degenerate symmetric bilinear form on the fibers of ${}^{\mathrm{b}}TM$ smoothly depending on the base point; a Lorentzian b-metric is one of signature $(1, n-1)$. We point out that the b-differential ${}^{\mathrm{b}}d$, defined locally by

$$
{}^{\mathrm{b}}du = (x\partial_x u)\frac{dx}{x} + \sum_j (\partial_{y_j} u) dy_j
$$

maps $H^{s,\alpha}(M)$ to $H^{s-1,\alpha}(M; {}^{\mathrm{b}}T^*M)$.

In order to start the explanation, it is best to begin with the underlying linear equation; after all, the non-linearity is 'just' a rather serious perturbation! In general, the analysis of b-differential operators (locally finite sums of finite products of elements of $\mathcal{V}_{\mathrm{b}}(M)$), such as $\Box_g \in \mathrm{Diff}^2_{\mathrm{b}}(M)$, has two ingredients, corresponding to the two orders, smoothness and decay, of the Sobolev spaces:

(1) b-regularity analysis. This provides the framework for understanding PDE at high b-frequencies, which in non-degenerate situations involves the b-principal symbol and perhaps a subprincipal term. This is sufficient in order to control solutions u in $H^{s,r}_{\mathrm{b}}$ modulo $H^{s',r}_{\mathrm{b}}$, $s' < s$, i.e. modulo a space with higher regularity, but no additional decay. Since for the inclusion $H^{s,r}_{\mathrm{b}} \to H^{s',r'}_{\mathrm{b}}$ to be compact one needs both $s > s'$ and $r > r'$, this does not control the problem modulo relatively compact errors.

(2) Normal operator analysis. This provides a framework for understanding the decay properties of solutions of the PDE. The normal operator is obtained by freezing coefficients of the differential operator L at ∂M to obtain a dilation-invariant b-operator $N(L)$. One then Mellin transforms the normal operator in the normal variable to obtain a family of operators $\hat{L}(\sigma)$, depending on the Mellin-dual variable σ. The b-regularity analysis, in non-degenerate situations, gives control of this family $\hat{L}(\sigma)$ in a Fredholm sense, uniformly as $|\sigma| \to \infty$ with $\mathrm{Im}\,\sigma$ bounded. However, in any such strip, $\hat{L}(\sigma)^{-1}$ will still typically have finitely many poles σ_j; these poles, called *resonances*, dictate the asymptotic behavior of solutions of the PDE.

In order to have a Fredholm operator L, one needs to work in spaces such as $H^{s,r}_{\mathrm{b}}$, where r is such that there are no resonances σ_j with $\mathrm{Im}\,\sigma_j = -r$. One can also work in slightly more general spaces, such as $\mathbb{C} \oplus H^{s,r}_{\mathrm{b}}$, $r > 0$, identified with a space of distributions via $u = u_0 + \tilde{u}$, $\tilde{u} \in H^{\infty,\alpha}_{\mathrm{b}}$, $u_0 = c\chi$, corresponding to $(c, \tilde{u}) \in \mathbb{C} \oplus H^{s,r}_{\mathrm{b}}$.

Now, the b-regularity analysis for our non-elliptic equation involves the (null)-bicharacteristic flow. In view of the version of Hörmander's theorem on propagation of singularities in this setting, and in view of the a priori control on Cauchy data at

H_1, what one would like is that all bicharacteristics tend to $T^*_{H_1}M$ in one direction. Moreover, for the purposes of the adjoint problem, which effectively imposes Cauchy data at H_2, one would like that the bicharacteristics tend to ${}^{\mathrm{b}}T^*_{H_2}M$ in the other direction.

Unfortunately, bicharacteristics within ${}^{\mathrm{b}}T^*_X M$ can never leave this space, and thus will not tend to $T^*_{H_1}M$. This is mostly resolved, however, by the conormal bundle of the *horizons* at X, which give rise to a bundle of saddle points for the bicharacteristic flow. Since the flow is homogeneous, it is convenient to consider it in ${}^{\mathrm{b}}S^*M = ({}^{\mathrm{b}}T^*M \setminus o)/\mathbb{R}^+$. The characteristic set in ${}^{\mathrm{b}}S^*M$ has two components $\Sigma_\pm$, with Σ_- forward-oriented (i.e. future oriented time functions increase along null-bicharacteristics in Σ_-), Σ_+ backward oriented. Then the images of the conormal bundles of the horizons in the cosphere bundle are submanifolds $L_\pm \subset \Sigma_\pm$ of ${}^{\mathrm{b}}S^*_X M$, with one-dimensional stable $(-)$/unstable $(+)$ manifold $\mathcal{L}_\pm$ transversal to ${}^{\mathrm{b}}S^*_X M$. (The flow within $L_\pm$ need not be trivial; if it is, one has *radial points*, as in the $a = 0$ de Sitter-Schwarzschild space. However, for simplicity we refer to the $L_\pm$ estimates as *radial point estimates* in general.) The realistic ideal situation, called a *non-trapping* one, then is if all (null-)bicharacteristics in ${}^{\mathrm{b}}S^*_\Omega M \cap (\Sigma_+ \setminus L_+)$ tend to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_+$ in the backward direction, and ${}^{\mathrm{b}}S^*_{H_1}M \cup L_+$ in the forward direction, with a similar statement for Σ_-, with backward and forward interchanged.[3] In this non-trapping setting the only subtlety is that the propagation estimates through $L_\pm$ require that the differentiability order s and the decay order r be related by $s > \frac{1}{2} + \beta r$ for a suitable $\beta > 0$ (dictated by the Hamilton dynamics at $L_\pm$), i.e. the more decay one wants, the higher the regularity needs to be.

This is still not the case in Kerr-de Sitter space, though it is true for neighborhoods of the static patch in de Sitter space, and its perturbations. The additional ingredient for Kerr-de Sitter space is normally hyperbolic trapping, introduced in this context by Wunsch and Zworski [49], given by smooth submanifolds $\Gamma^\pm \subset \Sigma_\pm$. Here $\Gamma^\pm$ are invariant submanifolds for the Hamilton flow, given by the transversal intersection of locally defined smooth, Hamilton flow invariant, $\Gamma^\pm = \Gamma^\pm_+ \cap \Gamma^\pm_-$, with $\Gamma^\pm_- \subset \Sigma$ transversal to ${}^{\mathrm{b}}S^*_X M \cap \Sigma$, and $\Gamma^\pm_+ \subset {}^{\mathrm{b}}S^*_X M \cap \Sigma$. Combining results of [22, 23] (which would work directly in a dilation invariant setting) and [30] we show that for $r > 0$ sufficiently small, one can propagate $H^{s,r}_{\mathrm{b}}$ estimates through $\Gamma^\pm$. This suffices to complete the b-regularity setup if the non-trapping requirement is replaced by: All (null-)bicharacteristics in ${}^{\mathrm{b}}S^*_\Omega M \cap (\Sigma_+ \setminus (L_+ \cup \Gamma^+))$ tend to either ${}^{\mathrm{b}}S^*_{H_2}M \cup L_+ \cup \Gamma^+$ in the backward direction, and ${}^{\mathrm{b}}S^*_{H_1}M \cup L_+ \cup \Gamma^+$ in the forward direction, with the tending to Γ^+ allowed in only *one* of the forward and backward directions, with a similar statement for Σ_-, with backward and forward interchanged. Finally, this *is* satisfied in Kerr-de Sitter space, and also in its b-perturbations (the whole setup is perturbation stable).

Next, one needs to know about the resonances of the operator. For the wave operator, the only resonance with non-negative imaginary part is 0, with the kernel of $\hat{L}(\sigma)$ one dimensional, consisting of constants. Since strips can only have finitely many resonances, there is $r > 0$ such that in $\operatorname{Im}\sigma \geq -r$ the only resonance is 0; then $H^{s,r}_{\mathrm{b}} \oplus \mathbb{C}$ works for our Fredholm setup. For the Klein-Gordon equation with

[3] Notice that due to the assumption on the one-dimensional stable/unstable manifold being transversal to ${}^{\mathrm{b}}S^*_X M$, there cannot be non-trivial bicharacteristics in ${}^{\mathrm{b}}S^*M$ tending to L_+ in both the forward and the backward direction, since a bicharacteristic is either completely in ${}^{\mathrm{b}}S^*_X M$, or completely outside it.

$m > 0$ small, the $m = 0$ resonance at 0 moves to $\sigma_1 = \sigma_1(m)$ inside $\operatorname{Im}\sigma < 0$, see [21, 31]. Thus, one can either work with $H_b^{s,r'}$ where r' is sufficiently small (depending on m), or with $H_b^{s,r} \oplus \mathbb{C}$, though with $\mathbb{C}$ now identified with $cx^{i\sigma_1}\chi$.

We now discuss the non-linear terms. Here the basic point is that $H_b^{s,0}$ is an algebra if $s > n/2$, and thus for such s, products of elements of $H_b^{s,r}$ possess even more decay if $r > 0$, but they become more growing if $r < 0$. Thus, one is forced to work with $r \geq 0$.

First, with the simplest semilinear equation, with no derivatives in the non-linearity q (so $N_j \geq 2$ is replaced by $N_j = 0$), the regularity losses due to the normally hyperbolic trapping are in principle sufficiently small to allow for a contraction mapping principle (Picard iteration) based argument. However, for the actual wave equation on Kerr-de Sitter space, the 0-resonance prohibits this, as the iteration maps outside the space $H_b^{s,r} \oplus \mathbb{C}$. Thus, it is the semilinear Klein-Gordon equation that is well-behaved from this perspective, and this was solved by the authors in [31]. On the other hand, if derivatives are allowed, with an at least quadratic behavior in ${}^b du$, then the non-linearity annihilates the 0-resonance. Unfortunately, since the normally hyperbolic estimate loses $1 + \epsilon$ derivatives, as opposed to the usual real principal type/radial point loss of one derivative, the solution operator for $\Box_g$ will not map $q(u, {}^b du)$ back into the desired Sobolev space, preventing a non-linear analysis based on the contraction mapping principle.

Fortunately, the Nash-Moser iteration is designed to deal with just such a situation. In this paper we adapt the iteration to our requirements, and in particular show that semilinear equations of the kind just described are in fact solvable. In particular, we prove that all the estimates used in the linear problems are *tame*. Here we remark that Klainerman's early work on global solvability involved the Nash-Moser scheme [32], though this was later removed by Klainerman and Ponce [33]. In the present situation the loss of derivatives seems much more serious, however, due to the trapping, so it seems unlikely that the solution scheme can be made more 'classical'.

However, we are also interested in quasilinear equations. Quasilinear versions of the above non-trapping scenario were studied by the first author [29], who showed the solvability of quasilinear wave equations on perturbations of de Sitter space. The key ingredient in dealing with quasilinear equations is to allow operators with coefficients with regularity the same kind as what one is proving for the solutions, in this case $H_b^{s,r}$-regularity. All of the smooth linear ingredients (microlocal elliptic regularity, propagation of singularities, radial points) have their analogue for $H_b^{s,r}$ coefficients if s is sufficiently large. Thus, in [29] a Picard-type iteration, $u_{k+1} = \Box_{g(u_k)}^{-1}(f + q(u_k, {}^b du_k))$ was used to solve the quasilinear wave equations on de Sitter space. Notice that $\Box_{g(u_k)}$ has non-smooth coefficients; indeed, these lie in a weighted b-Sobolev space.

In our Kerr-de Sitter situation there is normally hyperbolic trapping. However, notice that as we work in decaying Sobolev spaces modulo constants, $\Box_{g(u)}$ differs from a Kerr-de Sitter operator with smooth coefficients, $\Box_{g(c)}$, by one with *decaying coefficients*. This means that one can combine the smooth coefficient normally hyperbolic theory, as in the work of Dyatlov [22], with a tame estimate in $H_b^{s,r}$ with $r < 0$; the sign of r here is a crucial gain since for $r < 0$ the propagation estimates through normally hyperbolic trapped sets behave in exactly the same way as real principal type estimates. In combination this provides the required tame estimates for Kerr-de Sitter wave equations, and Nash-Moser iteration completes the proof of the main theorem.

We emphasize that our treatment of these quasilinear equations is systematic and general. Thus, quasilinear equations which at $X = \partial M$ are modelled on a finite dimensional family $L = L(v_0)$, $v_0 \in \mathbb{C}^d$ small corresponding to the zero resonances (thus the family is 0-dimensional without 0-resonances!), of smooth b-differential operators on a vector bundle with scalar principal symbol which has the bicharacteristic dynamics described above (radial sets, normally hyperbolic trapping, etc.) fits into it, *provided two conditions hold for the normal operator* (i.e. the dilation invariant model associated to L at ∂M).[4]

1. (1) First, *the resonances for the model* $L(v_0)$ have negative imaginary part, or if they have 0 imaginary part, the non-linearity annihilates them.
2. (2) Second, the *normally hyperbolic trapping estimates* of Dyatlov [22] hold for $\hat{L}(\sigma)$ (as $|\operatorname{Re}\sigma| \to \infty$) in $\operatorname{Im}\sigma > -r_0$ for some $r_0 > 0$. In the semiclassical rescaling, with $\sigma = h^{-1}z$, $h = |\sigma|^{-1}$, this is a statement about $\hat{L}_{h,z} = h^m \hat{L}(h^{-1}z)$, $\operatorname{Im} z > -r_0 h$. By Dyatlov's recent result[5] [23] this indeed is the case if $\hat{L}_{h,z}$ satisfies that *at* Γ its skew-adjoint part, $\frac{1}{2i}(\hat{L}_{h,z} - \hat{L}^*_{h,z}) \in h\mathrm{Diff}^1_\hbar(X)$, for $z \in \mathbb{R}$ has semiclassical principal symbol bounded above by $h\nu_{\min}/2$ for some $\epsilon > 0$, where $\nu_{\min}$ is the minimal expansion rate in the normal directions at Γ; see [23, Theorem 1] and the remark below it (which allows the non-trivial skew-adjoint part, denoted by Q there, microlocally at Γ).

It is important to point out that in view of the decay of the solutions either to 0 if there are no real resonance, or to the space of resonant states corresponding to real resonances, the conditions must be checked for at most a finite dimensional family of elements of the 'smooth' algebra $\Psi_{\mathrm{b}}(M)$, and moreover there is no need to prove tame estimates, deal with rough coefficients, etc., for this point, and one is in a dilation invariant setting, i.e. can simply Mellin transform the problem. Thus, in principle, solving wave-type equations on more complicated bundles is reduced to analyzing these two aspects of the associated linear model operator at infinity. Concretely, we have the following two theorems:

Theorem 3. *Let M be a Kerr-de Sitter space with angular momentum $|a| < \frac{\sqrt{3}}{2}M_\bullet$ that satisfies [47, (6.13)],[6] E a vector bundle over it with a positive definite metric k on E, and let $L_{g(u,{}^{\mathrm{b}}du)} \in \mathrm{Diff}^2_{\mathrm{b}}(M;E)$ have principal symbol $G = g^{-1}(u, {}^{\mathrm{b}}du)$ (times the identity), and suppose that $L_0 = L_{g(0,0)}$ satisfies that*

1. *(1) the large parameter principal symbol of $\frac{1}{2i|\sigma|}(L_0 - L_0^*)$, with the adjoint taken relative to $k\,|dg|$, at the trapped set Γ is $< \nu_{\min}/2$ as an endomorphism of E,*
2. *(2) $\hat{L}_0(\sigma)$ has no resonances in $\operatorname{Im}\sigma \geq 0$.*

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[4]The differential operator needs to be second order, with principal symbol a Lorentzian dual metric near the Cauchy hypersurfaces if the latter are used; otherwise the order m of the operator is irrelevant.

[5]This could presumably also be seen from the work of Nonnenmacher and Zworski [40] by checking that this extension goes through without significant changes in the proof.

[6]This condition on $\Lambda, M_\bullet$ and a ensures non-trapping classical dynamics for the null-geodesic flow.

Then for $\alpha > 0$ sufficiently small, there exists[7] *$d > 0$ such that the following holds: If $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$ has a sufficiently small H_{b}^{2d}-norm, then the equation $L_{g(u,{}^{\mathrm{b}}du)}u = f + q(u, {}^{\mathrm{b}}du)$ has a unique, smooth in M°, global forward solution $u \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)$.*

In particular, the conditions at Γ for the theorem hold if $|a| \ll M_\bullet$, $E = {}^{\mathrm{b}}\Lambda^* M$, $L_{g(u,{}^{\mathrm{b}}du)} = \Box_{g(u,{}^{\mathrm{b}}du)}$ the differential form d'Alembertian, or indeed if $L_{g(u,{}^{\mathrm{b}}du)} - \Box_{g(u,{}^{\mathrm{b}}du)}$ is a 0th order operator, since hyperbolicity is shown in [47] in the full stated range of a, while for $a = 0$, $\frac{1}{2i}(L_0 - L_0^*)$ can be computed explicitly at Γ, with k being the Riemannian metric of the form $\alpha^2\, d\tilde{x}^2 + h$ near the projection of Γ, where g has the form $\alpha^2\, d\tilde{x}^2 - h$, $\tilde{x}$ an appropriate boundary defining function on M strictly away from the horizons. Thus, in this case the only assumption in the theorem remaining to be checked is the second one, concerning resonances.

Theorem 4. *Let M be a Kerr-de Sitter space with angular momentum $|a| < \frac{\sqrt{3}}{2}M_\bullet$ that satisfies [47, (6.13)], E a vector bundle over it with a positive definite metric k on E, and let $L_{g(u,{}^{\mathrm{b}}du)} \in \mathrm{Diff}_{\mathrm{b}}^2(M; E)$ have principal symbol $G = g^{-1}(u, {}^{\mathrm{b}}du)$ (times the identity). Suppose that $L_0 = L_{g(0,0)}$ is such that $\hat{L}_0(\sigma)$ has a simple resonance at 0, with resonant states spanned by $u_{0,1}, \ldots, u_{0,d}$, and no other resonances in $\operatorname{Im}\sigma \geq 0$. Consider the family $\hat{L}_{g(u_0,{}^{\mathrm{b}}du_0)}(\sigma)$, $u_0 \in \mathrm{Span}\{u_{0,1}, \ldots, u_{0,d}\}$ with small enough norm. Suppose that*

1. *this family only has resonances at 0 in $\operatorname{Im}\sigma \geq 0$, and these are given by $\mathrm{Span}\{u_{0,1}, \ldots, u_{0,d}\}$,*
2. *Γ is uniformly normally hyperbolic for $\hat{L}_{g(u_0,{}^{\mathrm{b}}du_0)}(\sigma)$ for u_0 of small norm,*
3. *the large parameter principal symbol of $\frac{1}{2i|\sigma|}(L_0 - L_0^*)$, with the adjoint taken relative to $k\,|dg|$, at the trapped set Γ is $< \nu_{\min}/2$,*
4. *$q(u_0, {}^{\mathrm{b}}du_0) = 0$ for $u_0 \in \mathrm{Span}\{u_{0,1}, \ldots, u_{0,d}\}$.*

Then for $\alpha > 0$ sufficiently small, there exists[8] *$d > 0$ such that the following holds: If $f \in H_{\mathrm{b}}^{\infty,\alpha}$ has a sufficiently small $H_{\mathrm{b}}^{2d,\alpha}$-norm, then the equation $L_{g(u,{}^{\mathrm{b}}du)}u = f + q(u, {}^{\mathrm{b}}du)$ has a unique, smooth in M°, global forward solution of the form $u = u_0 + \tilde{u}$, $\tilde{u} \in H_{\mathrm{b}}^{\infty,\alpha}$, $u_0 = \chi \sum_{j=1}^d c_j u_{0,j}$, $\chi \in \mathcal{C}^\infty(M)$ identically 1 near ∂M.*

Here 'uniformly normally hyperbolic' in the theorem means that one has a smooth family $\Gamma = \Gamma_{u_0}$ of trapped sets, with a smooth family of stable/unstable manifolds, with uniform bounds (within this family) on the normal expansion rates for the flow, which ensures that the normally hyperbolic estimates are uniform within the family (for small u_0); see the discussion around (4.27) for details.

Again, the conditions at Γ for the theorem hold if $|a| \ll M_\bullet$, $E = {}^{\mathrm{b}}\Lambda^* M$, if $L_{g(u,{}^{\mathrm{b}}du)} - \Box_{g(u,{}^{\mathrm{b}}du)}$ is a 0th order operator, $\Box_{g(u,{}^{\mathrm{b}}du)}$ the differential form d'Alembertian, since the structurally stable r-normally hyperbolic statement is shown in [47] (which implies the uniform normal hyperbolicity required in the theorem), while for $a = 0$, $\frac{1}{2i}(L_0 - L_0^*)$ can be computed explicitly at Γ, as mentioned above, and upper bounds on this are stable under perturbations.

The uniform normal hyperbolicity condition at Γ holds if $|a| < \frac{\sqrt{3}}{2}M_\bullet$, $E = {}^{\mathrm{b}}\Lambda^* M$, $L_{g(u,{}^{\mathrm{b}}du)} = \Box_{g(u,{}^{\mathrm{b}}du)}$ the differential form d'Alembertian, with $g(u_0, {}^{\mathrm{b}}du_0)$ being a Kerr-de Sitter metric for $u_0 \in \mathrm{Span}\{u_{0,1}, \ldots, u_{0,d}\}$ with small norm since

[7] See the proof of this theorem in Section 5.4, in particular (5.27), for the value of d.

[8] The value of d is given in (5.27) in the course of the proof of this theorem in Section 5.4.

the hyperbolicity of Γ was shown in this generality in [47]. However, the computation of $\frac{1}{2i}(L_0 - L_0^*)$ is more involved.

The plan of the rest of this paper is the following. In Section 3 we show that the non-smooth pseudodifferential operators of [29] facilitate tame estimates (operator bounds, composition, etc.), with Section 4 establishing tame elliptic estimates in Section 4.1, tame real principal type and radial point estimates in Section 4.2 and tame estimates at normally hyperbolic trapping in Section 4.3 for $r < 0$. In Section 4.4, we adapt Dyatlov's analysis at normally hyperbolic trapping given in [23] to our needs. Finally, in Section 5 we solve our quasilinear equations by first showing that the microlocal results of Section 4 combine with the high energy estimates for the relevant normal operators following from the discussion of Section 4.4 to give tame estimates for the forward propagator in Section 5.1, and then showing in Section 5.2 that the Nash-Moser iteration indeed allows for solving our wave equations. Section 5.3 then explains the changes required for quasilinear Klein-Gordon equations. Finally, in Section 5.4 we show how our methods apply in the general settings of Theorems 3 and 4.

3. Tame estimates in the non-smooth operator calculus

In this section we prove the basic tame estimates for the H_{b}-coefficient, or simply non-smooth, b-pseudodifferential operators defined in [29].

3.1. Mapping properties.

We start with the tame mapping estimate, Proposition 3.1, which essentially states that for non-smooth pseudodifferential operators A, a high regularity norm of Au can be estimated by a high regularity norm of A times a low regularity norm of u, plus a low regularity norm of A times a high regularity norm of u. This is stronger than the a priori continuity estimate one gets from the bilinear map $(A, u) \mapsto Au$, which would require a product of high norms of both. In case A is a multiplication operator, this is essentially a b-version of a (weak) Moser estimate, see Corollary 3.2.

We work on the half space $\overline{\mathbb{R}^n_+}$ with coordinates $z = (x, y) \in [0, \infty) \times \mathbb{R}^{n-1}$; the coordinates in the fiber of the b-cotangent bundle are denoted $\zeta = (\lambda, \eta)$, i.e. we write b-covectors as $\lambda \, \frac{dx}{x} + \eta \, dy$. Recall from [29] the symbol class

$$S^{m;0} H_{\mathrm{b}}^s := \{a(z, \zeta) \colon \|\langle\zeta\rangle^{-m} a(z, \zeta)\|_{H_{\mathrm{b}}^s} \in L^\infty_\zeta\}$$

with the norm

$$\|a\|_{S^{m;0}H_{\mathrm{b}}^s} = \left\| \frac{\langle\xi\rangle^s \hat{a}(\xi, \zeta)}{\langle\zeta\rangle^m} \right\|_{L^\infty_\zeta L^2_\xi},$$

where $\hat{a}$ denotes the Mellin transform in x and Fourier transform in y of a. Left quantizations of such symbols, denoted $\mathrm{Op}(a) \in \Psi^{m;0} H_{\mathrm{b}}^s$, act on $u \in \dot{\mathcal{C}}^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+})$ by

$$\mathrm{Op}(a) u(z) = \int e^{iz\zeta} a(z, \zeta) \hat{u}(\zeta) \, d\zeta.$$

Also recall

$$S^{m;k} H_{\mathrm{b}}^s = \{a \in S^{m;0} H_{\mathrm{b}}^s \colon \partial_\zeta^\alpha a \in S^{m-|\alpha|;0} H_{\mathrm{b}}^s, |\alpha| \leq k\}.$$

and $\Psi^{m;k} H_{\mathrm{b}}^s = \mathrm{Op}\, S^{m;k} H_{\mathrm{b}}^s$. For brevity, we will use the following notation for Sobolev, symbol class and operator class norms, with the distinction between symbolic and b-Sobolev norms being clear from the context:

$$\|u\|_s := \|u\|_{H_{\mathrm{b}}^s}, \qquad \|u\|_{s,r} := \|u\|_{H_{\mathrm{b}}^{s,r}},$$

$$\|a\|_{m,s} := \|a\|_{S^{m;0}H_b^s}, \qquad \|a\|_{(m;k),s} := \|a\|_{S^{m;k}H_b^s},$$
$$\|A\|_{m,s} := \|A\|_{\Psi^{m;0}H_b^s}, \qquad \|A\|_{(m;k),s} := \|A\|_{\Psi^{m;k}H_b^s}.$$

If A is a b-operator acting on an element of a weighted b-Sobolev space with weight r (which will be apparent from the context), then $\|A\|_{m,s}$ is to be understood as $\|x^{-r}Ax^r\|_{m,s}$, similarly for $\|A\|_{(m;k),s}$. Lastly, for $A \in H_b^s\Psi_b^m$, we write $\|A\|_{H_b^s\Psi_b^m}$, by an abuse of notation, for an unspecified $H_b^s\Psi_b^m$-seminorm of A.

Recall the notation $x_+ = \max(x, 0)$ for $x \in \mathbb{R}$.

Proposition 3.1. *(Extension of* [29, Proposition 3.9]*.) Let $s \in \mathbb{R}$, $A = \mathrm{Op}(a) \in \Psi^{m;0}H_b^s$, and suppose $s' \in \mathbb{R}$ is such that $s \geq s' - m$, $s > n/2 + (m - s')_+$. Then A defines a bounded map $H_b^{s'} \to H_b^{s'-m}$, and for all fixed μ, ν with*

$$\mu > n/2 + (m - s')_+, \quad \nu > n/2 + (m - s')_+ + s' - s,$$

there is a constant $C > 0$ such that

$$\|Au\|_{s'-m} \leq C(\|A\|_{m,\mu}\|u\|_{s'} + \|A\|_{m,s}\|u\|_\nu). \tag{3.1}$$

Observe that by the assumptions on s and s', the intervals of allowed μ, ν are always non-empty (since they contain $\mu = s$ and $\nu = s'$). Estimates of the form (3.1), called 'tame estimates' e.g. in [28, 42], are crucial for applications in a Nash-Moser iteration scheme.

Proof of Proposition 3.1. We compute

$$\begin{aligned} \|Au\|_{s'-m}^2 &= \int \langle\zeta\rangle^{2(s'-m)} |\widehat{Au}(\zeta)|^2 \, d\zeta \\ &\leq \int \langle\zeta\rangle^{2(s'-m)} \left(\int |\hat{a}(\zeta - \xi, \xi)\hat{u}(\xi)| \, d\xi \right)^2 d\zeta. \end{aligned}$$

We split the inner integral into two pieces, corresponding to the domains of integration $|\zeta - \xi| \leq |\xi|$ and $|\xi| \leq |\zeta - \xi|$, which can be thought of as splitting up the action of A on u into a low-high and a high-low frequency interaction. We estimate

$$\begin{aligned} &\int \langle\zeta\rangle^{2(s'-m)} \left(\int_{|\zeta-\xi|\leq|\xi|} |\hat{a}(\zeta - \xi, \xi)\hat{u}(\xi)| \, d\xi \right)^2 d\zeta \\ &\quad \leq \int \left(\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}\langle\xi\rangle^{2m}}{\langle\zeta-\xi\rangle^{2\mu}\langle\xi\rangle^{2s'}} \, d\xi \right) \\ &\qquad\qquad \times \left(\int \frac{\langle\zeta-\xi\rangle^{2\mu}|\hat{a}(\zeta-\xi,\xi)|^2}{\langle\xi\rangle^{2m}} \langle\xi\rangle^{2s'} |\hat{u}(\xi)|^2 \, d\xi \right) d\zeta, \end{aligned} \tag{3.2}$$

and we claim that the integral which is the first factor on the right hand side is uniformly bounded in ζ: Indeed, if $s' - m \geq 0$, then we use $|\zeta| \leq 2|\xi|$ on the domain of integration, thus

$$\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}}{\langle\zeta-\xi\rangle^{2\mu}\langle\xi\rangle^{2(s'-m)}} \, d\xi \lesssim \int \frac{1}{\langle\zeta-\xi\rangle^{2\mu}} \, d\xi \in L_\zeta^\infty,$$

since $\mu > n/2$; if, on the other hand, $s' - m \leq 0$, then $|\xi| \leq |\zeta - \xi| + |\zeta|$ gives

$$\int_{|\zeta-\xi|\leq|\xi|} \frac{\langle\xi\rangle^{2(m-s')}}{\langle\zeta-\xi\rangle^{2\mu}\langle\zeta\rangle^{2(m-s')}} \, d\xi \lesssim \int \frac{1}{\langle\zeta-\xi\rangle^{2(\mu-(m-s'))}} + \frac{1}{\langle\zeta-\xi\rangle^{2\mu}} \, d\xi \in L_\zeta^\infty,$$

since $\mu > n/2 + (m - s')$; hence, from (3.2), the $H_{\mathrm{b}}^{s'-m}$ norm of the low-high frequency interaction in Au is bounded by $C_\mu \|a\|_{m,\mu}\|u\|_{s'}$.

We estimate the norm of high-low interaction in a similar way: We have

$$\begin{aligned}\int \langle\zeta\rangle^{2(s'-m)} &\left(\int_{|\xi|\leq|\zeta-\xi|} |\hat{a}(\zeta-\xi,\xi)\hat{u}(\xi)|\,d\xi\right)^2 d\zeta \\ &\leq \int \left(\int_{|\xi|\leq|\zeta-\xi|} \frac{\langle\zeta\rangle^{2(s'-m)}\langle\xi\rangle^{2m}}{\langle\zeta-\xi\rangle^{2s}\langle\xi\rangle^{2\nu}}\,d\xi\right) \\ &\qquad\qquad \times\left(\int \frac{\langle\zeta-\xi\rangle^{2s}|\hat{a}(\zeta-\xi,\xi)|^2}{\langle\xi\rangle^{2m}}\langle\xi\rangle^{2\nu}|\hat{u}(\xi)|^2\,d\xi\right) d\zeta.\end{aligned} \tag{3.3}$$

If $s' - m \geq 0$, the first inner integral on the right hand side is bounded by

$$\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle\zeta-\xi\rangle^{2(s-s'+m)}\langle\xi\rangle^{2(\nu-m)}}\,d\xi \leq \int \frac{1}{\langle\xi\rangle^{2(s-s'+\nu)}}\,d\xi,$$

where we use $s \geq s' - m$, and this integral is finite in view of $\nu > n/2 + s' - s$; if $s' - m \leq 0$, then

$$\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle\zeta\rangle^{2(m-s')}\langle\zeta-\xi\rangle^{2s}\langle\xi\rangle^{2(\nu-m)}}\,d\xi \leq \int \frac{1}{\langle\xi\rangle^{2(\nu-m+s)}}\,d\xi,$$

which is finite in view of $\nu > n/2 + m - s$. In summary, we need $\nu > n/2 + \max(m, s') - s = n/2 + (m - s')_+ + s' - s$ and can then bound the $H_{\mathrm{b}}^{s'-m}$ norm of the high-low interaction by $C_\nu\|a\|_{m,s}\|u\|_\nu$. The proof is complete. □

Using $H_{\mathrm{b}}^s \subset S^{0;0}H_{\mathrm{b}}^s$, we obtain the following weak version of the Moser estimate for the product of two b-Sobolev functions:

Corollary 3.2. *Let $s > n/2$, $|s'| \leq s$. If $u \in H_{\mathrm{b}}^s, v \in H_{\mathrm{b}}^{s'}$, then $uv \in H_{\mathrm{b}}^{s'}$, and one has an estimate*

$$\|uv\|_{s'} \leq C(\|u\|_\mu\|v\|_{s'} + \|u\|_s\|v\|_\nu)$$

for fixed $\mu > n/2 + (-s')_+, \nu > n/2 + s'_+ - s$. In particular, for $u, v \in H_{\mathrm{b}}^s$,

$$\|uv\|_s \leq C(\|u\|_\mu\|v\|_s + \|u\|_s\|v\|_\mu)$$

for fixed $\mu > n/2$.

3.2. **Operator compositions.** We give a tame estimate for the norms of expansion and remainder terms arising in the composition of two non-smooth operators:

Proposition 3.3. *Suppose $s, m, m' \in \mathbb{R}$, $k, k' \in \mathbb{N}_0$ are such that $s > n/2$, $s \leq s' - k$ and $k \geq m + k'$. Suppose $P = p(z, {}^{\mathrm{b}}D) \in \Psi^{m;k}H_{\mathrm{b}}^s$, $Q = q(z, {}^{\mathrm{b}}D) \in \Psi^{m';0}H_{\mathrm{b}}^{s'}$. Put*

$$E_j := \sum_{|\beta|=j} \frac{1}{\beta!}(\partial_\zeta^\beta p\,{}^{\mathrm{b}}D_z^\beta q)(z, {}^{\mathrm{b}}D),$$

$$R := P \circ Q - \sum_{0\leq j<k} E_j.$$

Then $E_j \in \Psi^{m+m'-j;0}H_{\mathrm{b}}^s$ and $R \in \Psi^{m'-k';0}H_{\mathrm{b}}^s$, and for $\mu > n/2$ fixed,

$$\|E_j\|_{\Psi^{m+m'-j;0}H_{\mathrm{b}}^s} \leq C(\|P\|_{\Psi^{m;j}H_{\mathrm{b}}^\mu}\|Q\|_{\Psi^{m';0}H_{\mathrm{b}}^{s+j}} + \|P\|_{\Psi^{m;j}H_{\mathrm{b}}^s}\|Q\|_{\Psi^{m';0}H_{\mathrm{b}}^{\mu+j}}),$$

$$\|R\|_{\Psi^{m'-k';0}H_{\mathrm{b}}^s} \leq C(\|P\|_{\Psi^{m;k}H_{\mathrm{b}}^\mu}\|Q\|_{\Psi^{m';0}H_{\mathrm{b}}^{s+k}} + \|P\|_{\Psi^{m;k}H_{\mathrm{b}}^s}\|Q\|_{\Psi^{m';0}H_{\mathrm{b}}^{\mu+k}}).$$

Proof. The statements about the E_j follow from Corollary 3.2. For the purpose of proving the estimate for R, we define

$$p_0 = \partial_\zeta^k p \in S^{m-k;0} H_{\mathrm{b}}^s, \quad {}^{\mathrm{b}}D_z^k q \in S^{m';0} H_{\mathrm{b}}^{s'-k},$$

where we write $\partial_\zeta^k = (\partial_\zeta^\beta)_{|\beta|=k}$, similarly for ${}^{\mathrm{b}}D_z^k$. Notice that in particular $p_0 \in S^{0;0} H_{\mathrm{b}}^s$. Then $R = r(z, {}^{\mathrm{b}}D)$ with

$$|\hat{r}(\eta;\zeta)| \lesssim \int \left(\int_0^1 p_0(\eta-\xi;\zeta+t\xi)\,dt \right) q_0(\xi;\zeta)\,d\xi$$

by Taylor's formula, hence

$$\begin{aligned}
&\int \frac{\langle\eta\rangle^{2s}|\hat{r}(\eta;\zeta)|^2}{\langle\zeta\rangle^{2m'}}\,d\eta \\
&\quad \lesssim \int \left(\int_{|\eta-\xi|\leq|\xi|} \frac{\langle\eta\rangle^{2s}}{\langle\eta-\xi\rangle^{2\mu}\langle\xi\rangle^{2s}}\,d\xi \right) \\
&\qquad\qquad \times \left(\int \left(\int_0^1 \langle\eta-\xi\rangle^{2\mu} |p_0(\eta-\xi,\zeta+t\xi)|^2\,dt \right) \frac{\langle\xi\rangle^{2s}|q_0(\xi;\zeta)|^2}{\langle\zeta\rangle^{2m'}}\,d\xi \right) d\eta \\
&\qquad + \int \left(\int_{|\xi|\leq|\eta-\xi|} \frac{\langle\eta\rangle^{2s}}{\langle\eta-\xi\rangle^{2s}\langle\xi\rangle^{2\mu}}\,d\xi \right) \\
&\qquad\qquad \times \left(\int \left(\int_0^1 \langle\eta-\xi\rangle^{2s} |p_0(\eta-\xi,\zeta+t\xi)|^2\,dt \right) \frac{\langle\xi\rangle^{2\mu}|q_0(\xi;\zeta)|^2}{\langle\zeta\rangle^{2m'}}\,d\xi \right) d\eta,
\end{aligned}$$

which implies the claimed estimate for $k'=0$. For $k'>0$, we use a trick of Beals and Reed [4] as in the proof of Theorem 3.12 in [29] to reduce the statement to the case $k'=0$: Recall that the idea is to split up $q(z,\zeta)$ into a 'trivial' part q_0 with compact support in ζ and n parts q_i, where q_i has support in $|\zeta_i| \geq 1$, and then writing

$$P \circ Q_i = \sum_{j=0}^{k'} c_{jk'} P\,{}^{\mathrm{b}}D_{z_i}^{k'-j} \circ ({}^{\mathrm{b}}D_{z_i}^j q_i)(z, {}^{\mathrm{b}}D)\,{}^{\mathrm{b}}D_{z_i}^{-k'}$$

for some constants $c_{jk'} \in \mathbb{R}$ using the Leibniz rule; then what we have proved above for $k'=0$ can be applied to the j-th summand on the right hand side, which we expand to order $k-j$, giving the result. □

3.3. **Reciprocals of and compositions with H_{b}^s functions.** We also need sharper bounds for reciprocals and compositions of b-Sobolev functions on a compact n-dimensional manifold with boundary. Localizing using a partition of unity, we can simply work on $\overline{\mathbb{R}^n_+}$.

Proposition 3.4. *(Extension of* [29, Lemma 4.1].*) Let $s > n/2+1$, $u, w \in H_{\mathrm{b}}^s$, $a \in \mathcal{C}^\infty$, and suppose that $|a+u| \geq c_0$ near $\operatorname{supp} w$. Then $w/(a+u) \in H_{\mathrm{b}}^s$, and one has an estimate*

$$\left\| \frac{w}{a+u} \right\|_s \leq C(\|u\|_\mu, \|a\|_{C^N}) c_0^{-1} \max(c_0^{-\lceil s\rceil}, 1)\big(\|w\|_s + \|w\|_\mu(1+\|u\|_s)\big). \tag{3.4}$$

for any fixed $\mu > n/2+1$ and some s-dependent $N \in \mathbb{N}$.

Proof. Choose $\psi_0, \psi \in \mathcal{C}^\infty$ such that $\psi_0 \equiv 1$ on $\operatorname{supp} w$, $\psi \equiv 1$ on $\operatorname{supp}\psi_0$, and such that moreover $|a+u| \geq c_0 > 0$ on $\operatorname{supp}\psi$. Then we have $\|w/(a+u)\|_0 \leq c_0^{-1}\|w\|_0$. We now iteratively prove higher regularity of $w/(a+u)$ and an accompanying 'tame' estimate: Let us assume $w/(a+u) \in H_{\mathrm{b}}^{s'-1}$ for some $1 \leq s' \leq s$. Let $\Lambda_{s'} = \lambda_{s'}({}^{\mathrm{b}}D) \in \Psi_{\mathrm{b}}^{s'}$ be an operator with principal symbol $\langle\zeta\rangle^{s'}$. Then

$$\begin{aligned} \Big\|\Lambda_{s'}\frac{w}{a+u}\Big\|_0 &\leq \Big\|(1-\psi)\Lambda_{s'}\frac{\psi_0 w}{a+u}\Big\|_0 + \Big\|\psi\Lambda_{s'}\frac{\psi_0 w}{a+u}\Big\|_0 \\ &\lesssim \Big\|\frac{w}{a+u}\Big\|_0 + c_0^{-1}\Big\|\psi(a+u)\Lambda_{s'}\frac{w}{a+u}\Big\|_0 \\ &\leq c_0^{-1}\|w\|_0 + c_0^{-1}\left(\|\psi\Lambda_{s'}w\|_0 + \Big\|\psi[\Lambda_{s'}, a+u]\frac{w}{a+u}\Big\|_0\right) \\ &\lesssim c_0^{-1}\left(\|w\|_{s'} + \Big\|\frac{w}{a+u}\Big\|_{s'-1} + \Big\|\psi[\Lambda_{s'}, u]\frac{w}{a+u}\Big\|_0\right), \end{aligned} \tag{3.5}$$

where we used that the support assumptions on ψ_0 and ψ imply $(1-\psi)\Lambda_{s'}\psi_0 \in \Psi_{\mathrm{b}}^{-\infty}$, and $\psi[\Lambda_{s'}, a] \in \Psi_{\mathrm{b}}^{s'-1}$. Hence, in order to prove that $w/(a+u) \in H_{\mathrm{b}}^{s'}$, it suffices to show that $[\Lambda_{s'}, u] \colon H_{\mathrm{b}}^{s'-1} \to H_{\mathrm{b}}^0$. Let $v \in H_{\mathrm{b}}^{s'-1}$. Since

$$\begin{aligned} (\Lambda_{s'}uv)\widehat{\ }(\zeta) &= \int \lambda_{s'}(\zeta)\hat{u}(\zeta-\xi)\hat{v}(\xi)\,d\xi \\ (u\Lambda_{s'}v)\widehat{\ }(\zeta) &= \int \hat{u}(\zeta-\xi)\lambda_{s'}(\xi)\hat{v}(\xi)\,d\xi, \end{aligned}$$

we have, by taking a first order Taylor expansion of $\lambda_{s'}(\zeta) = \lambda_{s'}(\xi + (\zeta-\xi))$ around $\zeta = \xi$,

$$([\Lambda_{s'}, u]v)\widehat{\ }(\zeta) = \sum_{|\beta|=1}\int\left(\int_0^1 \partial_\zeta^\beta\lambda_{s'}(\xi + t(\zeta-\xi))\,dt\right)({}^{\mathrm{b}}D_z^\beta u)\widehat{\ }(\zeta-\xi)\hat{v}(\xi)\,d\xi,$$

thus, writing $u' = {}^{\mathrm{b}}D_z u \in H_{\mathrm{b}}^{s-1}$,

$$|([\Lambda_{s'}, u]v)\widehat{\ }(\zeta)| \lesssim \int\left(\int_0^1 \langle\xi + t(\zeta-\xi)\rangle^{s'-1}\,dt\right)\big|\widehat{u'}(\zeta-\xi)\big||\hat{v}(\xi)|\,d\xi.$$

To obtain a tame estimate for the L^2_ζ norm of this expression, we again use the method of decomposing the integral into low-high and high-low components: The low-high component is bounded by

$$\begin{aligned} \int\left(\int_{|\zeta-\xi|\leq|\xi|}\frac{\sup_{0\leq t\leq 1}\langle\xi + t(\zeta-\xi)\rangle^{2(s'-1)}}{\langle\zeta-\xi\rangle^{2(\mu-1)}\langle\xi\rangle^{2(s'-1)}}\,d\xi\right) & \\ \times\left(\int\langle\zeta-\xi\rangle^{2(\mu-1)}\big|\widehat{u'}(\zeta-\xi)\big|^2\langle\xi\rangle^{2(s'-1)}|\hat{v}(\xi)|^2\,d\xi\right)&\,d\zeta; \end{aligned}$$

the first inner integral, in view of $s' \geq 1$, so the sup is bounded by $\langle\xi\rangle^{2(s'-1)}$, which cancels the corresponding term in the denominator, is finite for $\mu > n/2+1$. For the high-low component, we likewise estimate

$$\begin{aligned} \int\left(\int_{|\xi|\leq|\zeta-\xi|}\frac{\sup_{0\leq t\leq 1}\langle\xi + t(\zeta-\xi)\rangle^{2(s'-1)}}{\langle\zeta-\xi\rangle^{2(s-1)}\langle\xi\rangle^{2\nu}}\,d\xi\right) & \\ \times\left(\int\langle\zeta-\xi\rangle^{2s}\big|\widehat{u'}(\zeta-\xi)\big|^2\langle\xi\rangle^{2\nu}|\hat{v}(\xi)|^2\,d\xi\right)&\,d\zeta, \end{aligned}$$

and the first inner integral on the right hand side is bounded by

$$\int_{|\xi|\leq|\zeta-\xi|} \frac{1}{\langle\zeta-\xi\rangle^{2(s-s')}\langle\xi\rangle^{2\nu}}\,d\xi \leq \int \frac{1}{\langle\xi\rangle^{2(s-s'+\nu)}}\,d\xi$$

because of $s \geq s'$, which is finite for $\nu > n/2 + s' - s$. We conclude that

$$\|[\Lambda_{s'}, u]v\|_0 \leq C_{\mu\nu}(\|u\|_\mu\|v\|_{s'-1} + \|u\|_{s'}\|v\|_\nu),$$

for $\mu > n/2+1, \nu > n/2 + s' - s$. Plugging this into (3.5) yields

$$\left\|\frac{w}{a+u}\right\|_{s'} \lesssim c_0^{-1}\left(\|w\|_{s'} + (1+\|u\|_\mu)\left\|\frac{w}{a+u}\right\|_{s'-1} + \|u\|_{s'}\left\|\frac{w}{a+u}\right\|_\nu\right),$$

where the implicit constant in the inequality is independent of c_0, w and u. Using the abbreviations $q_\sigma := \|w/(a+u)\|_\sigma$, $u_\sigma = \|u\|_\sigma$, $w_\sigma = \|w\|_\sigma$ and fixing $\mu > n/2+1$, this means

$$q_{s'} \lesssim c_0^{-1}(w_{s'} + (1+u_\mu)q_{s'-1} + u_{s'}q_\nu), \quad \nu > n/2 + s' - s,$$

with the implicit constant being independent of c_0, w, a, u, μ. We will use this for $s' \leq \gamma := \lfloor n/2 \rfloor + 1$ with $\nu = s' - 1$, and for $s' > \gamma$, we will take $\nu = \gamma$, thus obtaining a tame estimate for q_s. In more detail, for $1 \leq s' \leq \gamma$, we have

$$q_{s'} \lesssim c_0^{-1}(w_{s'} + (1+u_{s'})q_{s'-1}),$$

which gives, with $C_0 = \max(1, c_0^{-1})$,

$$q_\gamma \lesssim c_0^{-1}w_\gamma\sum_{j=0}^{\gamma-1}(c_0^{-1}(1+u_\gamma))^j + (c_0^{-1}(1+u_\gamma))^\gamma q_0 \lesssim c_0^{-1}C_0^\gamma w_\gamma(1+u_\gamma)^\gamma$$

using the bound $q_0 \leq c_0^{-1}w_0 \leq c_0^{-1}w_\gamma$. For $\gamma < s' \leq s$, we have

$$q_{s'} \lesssim c_0^{-1}(w_s + u_s q_\gamma + (1+u_\mu)q_{s'-1}),$$

thus for integer $k \geq 1$ with $\gamma + k \leq s$,

$$\begin{aligned} q_{\gamma+k} &\leq c_0^{-1}(w_s + u_s q_\gamma)\sum_{j=0}^{k-1}(c_0^{-1}(1+u_\mu))^j + (c_0^{-1}(1+u_\mu))^k q_\gamma \\ &\lesssim c_0^{-1}C_0^{k-1}(1+u_\mu)^k(w_s + (1+u_s)q_\gamma) \\ &\lesssim c_0^{-1}C_0^{\gamma+k}(1+u_\mu)^{\gamma+k}(w_s + (1+u_s)w_\gamma), \end{aligned}$$

where we used $\mu > \gamma$ in the last inequality, thus proving the estimate (3.4) in case s is an integer; in the general case, we just use $q_{\gamma'} \leq q_\gamma$ for $\gamma' < \gamma$, in particular for $\gamma' = s - \lceil s - \gamma \rceil$, and use the above with $q_{\gamma+k}$ replaced by $q_{\gamma'+k}$. □

As in [29], one thus obtains regularity results for compositions, but now with sharper estimates. To illustrate how to obtain these, let us prove an extension of [29, Proposition 4.5]. Let M be a compact n-dimensional manifold with boundary, $s > n/2 + 1$, $\alpha \geq 0$.

Proposition 3.5. *Let $u \in H_{\mathrm{b}}^{s,\alpha}(M)$. If $F\colon \Omega \to \mathbb{C}$, $F(0) = 0$, is holomorphic in a simply connected neighborhood Ω of the range of u, then $F(u) \in H_{\mathrm{b}}^{s,\alpha}(M)$, and*

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_{\mu,\alpha})(1 + \|u\|_{s,\alpha}) \tag{3.6}$$

for fixed $\mu > n/2 + 1$. Moreover, there exists $\epsilon > 0$ such that $F(v) \in H_{\mathrm{b}}^{s,\alpha}(M)$ depends continuously on $v \in H_{\mathrm{b}}^{s,\alpha}(M)$, $\|u - v\|_{s,\alpha} < \epsilon$.

Proof. Observe that $u(M)$ is compact. Let $\gamma \subset \mathbb{C}$ denote a smooth contour which is disjoint from $u(M)$, has winding number 1 around every point in $u(M)$, and lies within the region of holomorphicity of F. Then, writing $F(z) = zF_1(z)$ with F_1 holomorphic in Ω, we have

$$F(u) = \frac{1}{2\pi i}\oint_\gamma F_1(\zeta)\frac{u}{\zeta - u}\,d\zeta,$$

Since $\gamma \ni \zeta \mapsto u/(\zeta - u) \in H_{\mathrm{b}}^{s,\alpha}(M)$ is continuous by Proposition 3.4, we obtain, using the estimate (3.4),

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_\mu)\big(\|u\|_{s,\alpha} + \|u\|_{\mu,\alpha}(1 + \|u\|_s)\big),$$

which implies (3.6) in view of $\alpha \geq 0$. The continuous (in fact, Lipschitz) dependence of $F(v)$ on v is a consequence of Proposition 3.4 and Corollary 3.2. □

We also study compositions $F(u)$ for $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ and real-valued u.

Proposition 3.6. *(Extension of* [29, Proposition 4.7]*.) Let $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$, $F(0) = 0$. Then for $u \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$, we have $F(u) \in H_{\mathrm{b}}^{s,\alpha}(M)$, and one has an estimate*

$$\|F(u)\|_{s,\alpha} \leq C(\|u\|_{\mu,\alpha})(1 + \|u\|_{s,\alpha}) \tag{3.7}$$

for fixed $\mu > n/2 + 1$. In fact, $F(u)$ depends continuously on u.

Proof. The proof is the same as in [29], using almost analytic extensions, only we now use the sharper estimate (3.4) to obtain (3.7). □

Proposition 3.7. *(Extension of* [29, Proposition 4.8]*.) Let $F \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$, and $u' \in \mathcal{C}^\infty(M;\mathbb{R}), u'' \in H_{\mathrm{b}}^{s,\alpha}(M;\mathbb{R})$; put $u = u' + u''$. Then $F(u) \in \mathcal{C}^\infty(M) + H_{\mathrm{b}}^{s,\alpha}(M)$, and one has an estimate*

$$\|F(u) - F(u')\|_{s,\alpha} \leq C(\|u'\|_{C^N}, \|u''\|_{\mu,\alpha})(1 + \|u''\|_{s,\alpha})$$

for fixed $\mu > n/2 + 1$ and some $N \in \mathbb{N}$. In fact, $F(u)$ depends continuously on u.

Proof. The proof is the same as in [29], but now uses the sharper estimate (3.4). □

4. Microlocal regularity: tame estimates

When stating microlocal regularity estimates (like elliptic regularity, real principal type propagation, etc.) for operators with coefficients in $H_{\mathrm{b}}^s(\overline{\mathbb{R}^n_+})$, we will give two quantitative statements, one for 'low' regularities $\sigma \lessapprox n/2$, in which we will not make use of any tame estimates established earlier, and one for 'high' regularities $n/2 \lessapprox \sigma \lessapprox s$, in which the tame estimates will be used.

To concisely write down tame estimates, we use the following notation: The right hand side of a tame estimate will be a real-valued function, denoted by L, of the form

$$\begin{aligned} &L(p_1^\ell, \ldots, p_a^\ell; p_1^h, \ldots, p_b^h; u_1^\ell, \ldots, u_c^\ell; u_1^h, \ldots, u_d^h) \\ &\qquad = \sum_{j=1}^d c_j(p_1^\ell, \ldots, p_a^\ell)u_j^h + \sum_{j=1}^b \sum_{k=1}^c c_{jk}(p_1^\ell, \ldots, p_a^\ell)p_j^h u_k^\ell \end{aligned} \tag{4.1}$$

here, the c_j and c_{jk} are continuous functions. In applications, $p_j^{\ell/h}$ will be a low/high regularity norm of the coefficients of a non-smooth operator, and $u_j^{\ell/h}$ will be a low/high regularity norm of a function that an operator is applied to. The important feature of such functions L is that they are linear in the $u_j^{\ell/h}$, and all p_j^h, u_j^h, corresponding to high regularity norms, only appear in the first power.

4.1. **Elliptic regularity.** Concretely, we have the following quantitative elliptic estimate:

Proposition 4.1. *(Cf. [29, Theorem 5.1].) Let $m, s, r \in \mathbb{R}$ and $\zeta_0 \in {}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+} \setminus o$. Suppose $P' = p'(z, {}^{\mathrm{b}}D) \in H^s_{\mathrm{b}}\Psi^m_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ has a homogeneous principal symbol p'_m. Moreover, let $R \in \Psi^{m-1;0}_{\mathrm{b}} H^{s-1}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$. Let $P = P' + R$, and suppose $p_m \equiv p'_m$ is elliptic at ζ_0. Let $\tilde{s} \in \mathbb{R}$ be such that $\tilde{s} \leq s-1$ and $s > n/2 + 1 + (-\tilde{s})_+$, and suppose that $u \in H^{\tilde{s}+m-1,r}_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ satisfies*

$$Pu = f \in H^{\tilde{s},r}_{\mathrm{b}}(\overline{\mathbb{R}^n_+}).$$

Then there exists $B \in \Psi^0_{\mathrm{b}}(\overline{\mathbb{R}^n_+})$ elliptic at ζ_0 such that $Bu \in H^{\tilde{s}+m}_{\mathrm{b}}$, and for $\tilde{s} \leq n/2 + t$, $t > 0$, the estimate

$$\begin{aligned} \|Bu\|_{\tilde{s}+m,r} \leq C(\|P'\|_{(m;1),n/2+1+(-\tilde{s})_++t}, \|R\|_{m-1,n/2+(-\tilde{s})_++t}) \\ \times (\|u\|_{\tilde{s}+m-1,r} + \|f\|_{\tilde{s},r}) \end{aligned} \tag{4.2}$$

holds. For $\tilde{s} > n/2$, $\epsilon > 0$, there is a tame estimate

$$\begin{aligned} \|Bu\|_{\tilde{s}+m,r} \leq L(&\|P'\|_{(m;1),n/2+1+\epsilon}, \|R\|_{m-1,n/2+\epsilon}; \|P'\|_{(m;1),s}, \|R\|_{m-1,s-1}; \\ &\|u\|_{n/2+m-1+\epsilon,r}, \|f\|_{n/2-1+\epsilon,r}; \|u\|_{\tilde{s}+m-1,r}, \|f\|_{\tilde{s},r}). \end{aligned} \tag{4.3}$$

Let us point out that in our application of such an estimate to the study of nonlinear equations it will be irrelevant what exactly the low regularity norms in (4.3) are; in fact, it will be sufficient to know that there is *some* tame estimate of the general form (4.3), and this in turn is in fact clear without any computation, namely it follows directly from the fact that we have tame estimates for all 'non-smooth' operations involved in the proof of this proposition. The same remark applies to all further tame microlocal regularity results below. The only point where the precise numerology does matter is when one wants to find an explicit bound on the number of required derivatives for the forcing term in Theorems 2, 3 and 4, as we will do.

Proof of Proposition 4.1. We can assume that $r = 0$ by conjugating P by x^{-r}. Choose $a_0 \in S^0$ elliptic at ζ_0 such that p_m is elliptic[9] on $\operatorname{supp} a_0$. Let $\Lambda_m \in \Psi^m_{\mathrm{b}}$ be a b-ps.d.o with full symbol $\lambda_m(\zeta)$ independent of z, whose principal symbol is $\langle\zeta\rangle^m$, and define

$$q(z,\zeta) := a_0(z,\zeta)\lambda_m(\zeta)/p_m(z,\zeta) \in S^{0;\infty}H^s_{\mathrm{b}}, \quad Q = q(z, {}^{\mathrm{b}}D),$$

then by Proposition 3.4 and Corollary 3.2, we have

$$\|Q\|_{(0;k),\sigma} \leq C(\|P'\|_{(m;k),n/2+1+\epsilon})(1 + \|P'\|_{(m;k),\sigma}), \quad \sigma > n/2+1, \epsilon > 0. \tag{4.4}$$

Put $B = a_0(z, {}^{\mathrm{b}}D)\Lambda_m$, then

$$Q \circ P' = B + R'$$

with $R' \in \Psi^{m-1;0}H^{s-1}_{\mathrm{b}}$; by Proposition 3.3, we have for $n/2 < \sigma \leq s-1$

$$\|R'\|_{m-1,\sigma} \lesssim \|Q\|_{(0;1),\mu}\|P'\|_{(m;1),\sigma+1} + \|Q\|_{(0;1),\sigma}\|P'\|_{(m;1),\mu+1}, \quad \mu > n/2. \tag{4.5}$$

[9] And non-vanishing, which only matters near the zero section.

Now, since $Bu = QP'u - R'u = Qf - QRu - R'u$, we need to estimate the $H_{\mathrm{b}}^{\tilde{s}}$-norms of Qf, QRu and $R'u$, which we will do using Proposition 3.1. In the low regularity regime, we have, for $t > 0$ and $\tilde{s} \leq n/2 + t$, using (4.4) and (4.5):

$$\begin{aligned}\|Qf\|_{\tilde{s}} &\lesssim \|Q\|_{0,n/2+(-\tilde{s})_++t}\|f\|_{\tilde{s}} \leq C(\|P'\|_{m,n/2+1+(-\tilde{s})_++t})\|f\|_{\tilde{s}},\\ \|R'u\|_{\tilde{s}} &\lesssim \|R'\|_{m-1,n/2+(-\tilde{s})_++t}\|u\|_{\tilde{s}+m-1}\\ &\qquad \leq C(\|P'\|_{(m;1),n/2+1+(-\tilde{s})_++t})\|u\|_{\tilde{s}+m-1},\\ \|QRu\|_{\tilde{s}} &\leq C(\|P'\|_{m,n/2+1+(-\tilde{s})_++t})\|R\|_{m-1,n/2+(-\tilde{s})_++t}\|u\|_{\tilde{s}+m-1},\end{aligned}$$

giving (4.2). In the high regularity regime, in fact for $0 \leq \tilde{s} \leq s-1$, we have, for $\epsilon > 0$,

$$\begin{aligned}\|Qf\|_{\tilde{s}} &\lesssim \|Q\|_{0,n/2+\epsilon}\|f\|_{\tilde{s}} + \|Q\|_{0,s}\|f\|_{n/2-1+\epsilon}\\ &\qquad \leq C(\|P'\|_{m,n/2+1+\epsilon})(\|f\|_{\tilde{s}} + (1+\|P'\|_{m,s})\|f\|_{n/2-1+\epsilon}),\\ \|R'u\|_{\tilde{s}} &\lesssim \|R'\|_{m-1,n/2+\epsilon}\|u\|_{\tilde{s}+m-1} + \|R'\|_{m-1,s-1}\|u\|_{n/2+m-1+\epsilon}\\ &\qquad \leq C(\|P'\|_{(m;1),n/2+1+\epsilon})(\|u\|_{\tilde{s}+m-1} + (1+\|P'\|_{(m;1),s})\|u\|_{n/2+m-1+\epsilon}),\\ \|QRu\|_{\tilde{s}} &\leq L(\|P'\|_{m,n/2+1+\epsilon}, \|R\|_{m-1,n/2+\epsilon}; \|P'\|_{m,s}, \|R\|_{m-1,s-1};\\ &\qquad\qquad \|u\|_{n/2+m-1+\epsilon}; \|u\|_{\tilde{s}+m-1}),\end{aligned}$$

giving (4.3). The proof is complete. □

There is a similar tame microlocal elliptic estimate for operators of the form $P = P' + P'' + R$ with P', R as above and $P'' \in \Psi_{\mathrm{b}}^m$, as in part (2) of [29, Theorem 5.1], where the tame estimate now also involves the C^N-norm of the 'smooth part' P'' of the operator for some (s-dependent) N.[10]

4.2. **Real principal type propagation; radial points.** Tame estimates for real principal type propagation and propagation near radial points can be deduced from a careful analysis of the proofs of the corresponding results in [29]. The main observation is that the regularity requirements, given in the footnotes to the proofs of these results in [29], indicate what regularity is needed to estimate the corresponding terms: For example, an operator in $A \in \Psi^{m;0}H_{\mathrm{b}}^s$ with $m \geq 0$ maps $H_{\mathrm{b}}^{m/2}$ to $H_{\mathrm{b}}^{-m/2}$ under the condition $s > n/2 + m/2$, which is to say that one has a bound

$$\|A\tilde{u}\|_{-m/2} \lesssim \|A\|_{m-1,n/2+m/2+\epsilon}\|\tilde{u}\|_{m/2}, \quad \epsilon > 0.$$

This means that the only places where one needs to use tame operator bounds for operators with coefficients of regularity s are those where the condition for mapping properties etc. to hold reads $s \gtrsim \sigma$ where σ is the regularity of the target space, i.e. where σ is comparable to the regularity s of the coefficients.

We again only prove the tame real principal type estimate in the interior; the estimate near the boundary is proved in the same way, see also the discussion at the end of Section 4.1.

Proposition 4.2. *(Cf. [29, Theorem 6.6].) Let $m, r, s, \tilde{s} \in \mathbb{R}$. Suppose $P_m \in H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ has a real, scalar, homogeneous principal symbol p_m, and let $P_{m-1} \in$*

[10]Since in our application P'' will only depend on finitely many complex parameters, there is no need to prove an estimate which is also tame with respect to the C^N-norm of P''; however, this could easily be done in principle.

$H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+})$, $R\in\Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1}(\overline{\mathbb{R}^n_+})$. *Let* $P=P_m+P_{m-1}+R$. *Suppose* s *and* $\tilde s$ *are such that*

$$\tilde s\leq s-1,\quad s>n/2+7/2+(2-\tilde s)_+,$$

and suppose $u\in H_{\mathrm{b}}^{\tilde s+m-3/2,r}(\overline{\mathbb{R}^n_+})$ *satisfies* $Pu=f\in H_{\mathrm{b}}^{\tilde s,r}(\overline{\mathbb{R}^n_+})$. *Suppose* $\zeta_0\notin \mathrm{WF}_{\mathrm{b}}^{\tilde s+m-1,r}(u)$, *and let* $\gamma\colon[0,T]\to {}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}\setminus o$ *be a segment of a null-bicharacteristic of* p_m *with* $\gamma(0)=\zeta_0$, *then* $\gamma(t)\notin\mathrm{WF}_{\mathrm{b}}^{\tilde s+m-1,r}(u)$ *for all* $t\in[0,T]$. *Moreover, for all* $A\in\Psi_{\mathrm{b}}^0$ *elliptic at* ζ_0 *there exist* $B\in\Psi_{\mathrm{b}}^0$ *elliptic at* $\gamma(T)$ *and* $G\in\Psi_{\mathrm{b}}^0$ *elliptic on* $\gamma([0,T])$ *such that for* $\tilde s\leq n/2+1$, $\epsilon>0$,

$$\begin{aligned}&\|Bu\|_{\tilde s+m-1,r}\\&\quad\leq C(\|P_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\tilde s)_++\epsilon}\Psi_{\mathrm{b}}^m},\|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\tilde s)_++\epsilon}\Psi_{\mathrm{b}}^{m-1}},\|R\|_{n/2+1+(-\tilde s)_+})\\&\qquad\times(\|u\|_{\tilde s+m-3/2,r}+\|Au\|_{\tilde s+m-1,r}+\|Gf\|_{\tilde s,r}).\end{aligned}\tag{4.6}$$

Moreover, for $\tilde s>n/2+1$, $\epsilon>0$, *there is a tame estimate*

$$\begin{aligned}\|Bu\|_{\tilde s+m-1,r}\leq L(&\|P_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon}\Psi_{\mathrm{b}}^m},\|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon}\Psi_{\mathrm{b}}^{m-1}},\|R\|_{n/2+\epsilon};\\&\|P_m\|_{H_{\mathrm{b}}^s\Psi_{\mathrm{b}}^m},\|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}},\|R\|_{m-2,s-1};\\&\|u\|_{n/2-1/2+m+\epsilon};\|u\|_{\tilde s+m-3/2,r},\|Au\|_{\tilde s+m-1,r},\|Gf\|_{\tilde s,r}).\end{aligned}\tag{4.7}$$

Proof. We follow the proof of the regularity result in [29] and state the estimates needed to establish (4.6) and (4.7) along the way. Using the notation of the proof of [29, Theorem 6.6], but now calling the regularization parameter δ, in particular $\check A_\delta\in\Psi_{\mathrm{b}}^{\tilde s+(m-1)/2}$ is the regularized commutant, which depends on a positive constant M chosen below, and putting $\tilde f=f-Ru$, we have, assuming $m\geq 1$ and $\tilde s\geq(5-m)/2$ for now,

$$\begin{aligned}&\mathrm{Re}\langle i\check A_\delta^*[P_m,\check A_\delta]u,u\rangle\\&\quad=\frac12\langle i(P_m-P_m^*)\check A_\delta u,\check A_\delta u\rangle-\mathrm{Re}\langle i\check A_\delta\tilde f,\check A_\delta u\rangle+\mathrm{Re}\langle i\check A_\delta P_{m-1}u,\check A_\delta u\rangle\\&\quad\equiv I+II+III.\end{aligned}$$

For $\epsilon>0$, we can bound the first term by

$$|I|\lesssim\|P_m\|_{H_{\mathrm{b}}^{n/2+1+(m-1)/2+\epsilon}\Psi_{\mathrm{b}}^m}\|\check A_\delta u\|^2_{(m-1)/2},$$

the second one by

$$|II|\lesssim\|\check A_\delta f\|^2_{-(m-1)/2}+\|Ru\|^2_{\tilde s}+\|\check A_\delta u\|^2_{(m-1)/2},$$

where in turn

$$\|Ru\|_{\tilde s}\lesssim\begin{cases}\|R\|_{m-2;n/2+(-\tilde s)_++t}\|u\|_{\tilde s+m-2}, & \tilde s\leq n/2+t,\\ \|R\|_{m-2;n/2+\epsilon}\|u\|_{\tilde s+m-2}+\|R\|_{m-2;s-1}\|u\|_{n/2+m-2+\epsilon}, & \tilde s\geq 0\end{cases}$$

for $t>0$ by Proposition 3.1. We estimate the third term by

$$|III|\lesssim\|P_{m-1}\|_{H_{\mathrm{b}}^{\max(n/2+\epsilon,(m-1)/2)}\Psi_{\mathrm{b}}^{m-1}}\|\check A_\delta u\|^2_{(m-1)/2}+|\langle[\check A_\delta,P_{m-1}]u,\check A_\delta u\rangle|$$

and further, with $R_2\in\Psi_{\mathrm{b}}^{\tilde s+(m-1)/2-1}\circ\Psi^{m-1;0}H_{\mathrm{b}}^{s-2}$ denoting a part of the expansion of $[\check A_\delta,P_{m-1}]$ as defined after [29, Footnote 28],

$$|\langle[\check A_\delta,P_{m-1}]u,\check A_\delta u\rangle|\leq C(M)\|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(m/2-1)_++\epsilon}\Psi_{\mathrm{b}}^{m-1}}\|u\|^2_{\tilde s+m-3/2}$$

$$+ \|R_2 u\|^2_{-(m-1)/2} + \|\check{A}_\delta u\|^2_{(m-1)/2},$$

where

$$\begin{aligned}&\|R_2 u\|_{-(m-1)/2} \\ &\quad \leq C(M) \begin{cases} \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(1-\tilde{s})_+ +\epsilon}\Psi_{\mathrm{b}}^{m-1}} \|u\|_{\tilde{s}+m-2}, & \tilde{s} \leq n/2+1+\epsilon, \\ \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon}\Psi_{\mathrm{b}}^{m-1}} \|u\|_{\tilde{s}+m-2} & \\ \qquad + \|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}} \|u\|_{n/2+m-1+\epsilon}, & \tilde{s} \geq 1. \end{cases}\end{aligned}$$

Therefore, we obtain, see [29, Equation (6.24)],

$$\begin{aligned}\mathrm{Re} \Big\langle \Big(i\check{A}_\delta^*[P_m, \check{A}_\delta] + B_\delta^* B_\delta + M^2(\Lambda \check{A}_\delta)^*(\Lambda \check{A}_\delta) - E_\delta \Big) u, u \Big\rangle \\ \geq -|\langle E_\delta u, u\rangle| - \|\check{A}_\delta f\|^2_{-(m-1)/2} - L^2 + \|B_\delta u\|^2_{L^2_{\mathrm{b}}},\end{aligned} \tag{4.8}$$

where

$$M = M(\|P_m\|_{H_{\mathrm{b}}^{n/2+1+(m-1)/2+\epsilon}\Psi_{\mathrm{b}}^m}, \|P_{m-1}\|_{H_{\mathrm{b}}^{\max(n/2+\epsilon,(m-1)/2)}\Psi_{\mathrm{b}}^{m-1}}),$$

and L is 'tame'; more precisely, for $\tilde{s} \leq n/2+t$, $t > 0$,

$$L \leq C(M, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\max(m/2-1,1-\tilde{s})_+ +\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2;n/2+(-\tilde{s})_+ +t}) \|u\|_{\tilde{s}+m-3/2},$$

and for $\tilde{s} \geq 1$,

$$\begin{aligned}L = L(M, \|P_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(m/2-1)_+ +\epsilon}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2,n/2+\epsilon}; \\ \|P_{m-1}\|_{H_{\mathrm{b}}^{s-1}\Psi_{\mathrm{b}}^{m-1}}, \|R\|_{m-2;s-1}; \|u\|_{n/2+m-1+\epsilon}; \|u\|_{\tilde{s}+m-3/2}).\end{aligned}$$

Next, in order to exploit the positive commutator of the principal symbols of P_m and $\check{A}_\delta$ in the estimate (4.8), we introduce operators $J^\pm \in \Psi_{\mathrm{b}}^{\pm(\tilde{s}+(m-1)/2-1)}$ with principal symbols $j^\pm$ such that $J^+J^- - I \in \Psi_{\mathrm{b}}^{-\infty}$; then

$$iJ^- \check{A}_\delta^*[P_m, \check{A}_\delta] = \mathrm{Op}(j^- \check{a}_\delta H_{p_m} \check{a}_\delta) + R_1 + R_2 + R_3 + R_4,$$

see [29, Equation (6.27)], where

$$|\langle R_j u, (J^+)^* u\rangle| \leq C(M) \|P_m\|_{H_{\mathrm{b}}^{n/2+2+m/2+\epsilon}\Psi_{\mathrm{b}}^m} \|u\|^2_{\tilde{s}+m-3/2}, \quad j = 1, 3, 4,$$

and $R_2 \in \Psi_{\mathrm{b}}^{\tilde{s}+(m-1)/2-1} \circ \Psi^{m;0} H_{\mathrm{b}}^{s-2}$, hence

$$\begin{aligned}&|\langle R_2 u, (J^+)^* u\rangle| \\ &\quad \leq C(M) \begin{cases} (1 + \|P_m\|^2_{H_{\mathrm{b}}^{n/2+2+(3/2-\tilde{s})_+ +\epsilon}\Psi_{\mathrm{b}}^m}) \|u\|^2_{\tilde{s}+m-3/2} & \forall\ \tilde{s}, \\ (1 + \|P_m\|^2_{H_{\mathrm{b}}^{n/2+2+\epsilon}\Psi_{\mathrm{b}}^m}) \|u\|^2_{\tilde{s}+m-3/2} & \\ \qquad + \|P_m\|^2_{H_{\mathrm{b}}^s \Psi_{\mathrm{b}}^m} \|u\|^2_{n/2-1/2+m+\epsilon} & \tilde{s} \geq 3/2. \end{cases}\end{aligned}$$

Thus, further following the proof in [29] to equation (6.28) and beyond, it remains to bound

$$\mathrm{Re}\langle \mathrm{Op}(j^- f_\delta / j^+)(J^+)^* u, (J^+)^* u\rangle + \mathrm{Re}\langle R' u, (J^+)^* u\rangle, \quad R' \in \Psi^{\tilde{s}+3(m-1)/2;0} H_{\mathrm{b}}^{s-1},$$

from below, which is accomplished by

$$|\langle R' u, (J^+)^* u\rangle| \leq C(M) \|P_m\|_{H_{\mathrm{b}}^{n/2+1+m/2+\epsilon}\Psi_{\mathrm{b}}^m} \|u\|^2_{\tilde{s}+m-3/2},$$

$$\mathrm{Re}\langle \mathrm{Op}(j^- f_\delta / j^+)(J^+)^* u, (J^+)^* u\rangle \geq -C(M) \|P_m\|_{H_{\mathrm{b}}^{n/2+3+m/2+\epsilon}\Psi_{\mathrm{b}}^m} \|u\|^2_{\tilde{s}+m-3/2}.$$

Lastly, for general $m \in \mathbb{R}$, we rewrite the equation $Pu = f$ as $P\Lambda^+(\Lambda^- u) = f + PRu$ with $\Lambda^\pm \in \Psi_{\mathrm{b}}^{\mp(m-m_0)}$, $R \in \Psi_{\mathrm{b}}^{-\infty}$, where $m_0 \geq 1$; hence, replacing P by $P\Lambda^+$, u by $\Lambda^- u$ and m by m_0 in the above estimates is equivalent to just replacing m by m_0 in the b-Sobolev norms of the coefficients of P. Choosing $m_0 = 1 + 2(2-\tilde{s})_+$ as in [29] then implies the estimates (4.6) and (4.7) with $B = B_0$, G an elliptic multiple of $\check{A}_0$, and A elliptic on the microsupport of E_0. □

In a similar manner, we can analyze the proof of the radial point estimate, obtaining, in the notation of [29, §6.4]:

Proposition 4.3. *Let $m, r, s, \tilde{s} \in \mathbb{R}$, $\alpha > 0$. Let $P = P_m + P_{m-1} + R$, where $P_j = P'_j + P''_j$, $j = m, m-1$, with $P'_m \in H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ and $P''_m \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ having real, scalar, homogeneous principal symbols p'_m and p''_m, respectively; moreover $P'_{m-1} \in H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+})$, $P''_{m-1} \in \Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+})$ and $R = R' + R''$ with $R' \in \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha}(\overline{\mathbb{R}^n_+})$ and $R'' \in \Psi_{\mathrm{b}}^{m-2}(\overline{\mathbb{R}^n_+})$. Suppose that the conditions (1)-(4) in [29, §6.4] hold for $p = p''_m$, and*

$$\sigma_{\mathrm{b},m-1}\left(\frac{1}{2i}\Big((P''_m + P''_{m-1}) - (P''_m + P''_{m-1})^*\Big)\right) = \pm\hat{\beta}\beta_0\rho^{m-1} \text{ at } L_\pm, \tag{4.9}$$

where $\hat{\beta} \in \mathcal{C}^\infty(L_\pm)$ is self-adjoint at every point. Finally, assume that s and $\tilde{s}$ satisfy

$$\tilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2-\tilde{s})_+. \tag{4.10}$$

Suppose $u \in H_{\mathrm{b}}^{\tilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+})$ is such that $Pu = f \in H_{\mathrm{b}}^{\tilde{s},r}(\overline{\mathbb{R}^n_+})$.

1. *If $\tilde{s} + (m-1)/2 - 1 + \inf_{L_\pm}(\hat{\beta} - r\tilde{\beta}) > 0$, let us assume that in a neighborhood of $L_\pm$, $\mathcal{L}_\pm \cap \{x > 0\}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\tilde{s}+m-1,r}(u)$.*
2. *If $\tilde{s} + (m-1)/2 + \sup_{L_\pm}(\hat{\beta} - r\tilde{\beta}) < 0$, let us assume that a punctured neighborhood of $L_\pm$, with $L_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*_{\partial\overline{\mathbb{R}^n_+}}\overline{\mathbb{R}^n_+}$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\tilde{s}+m-1,r}(u)$.*

Then in both cases, $L_\pm$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{\tilde{s}+m-1,r}(u)$.

Quantitatively, for every neighborhood U of $L_\pm$, there exist $B_0, B_1 \in \Psi_{\mathrm{b}}^0$ elliptic at $L_\pm$, $A \in \Psi_{\mathrm{b}}^0$ with microsupport in the respective a priori control region in the two cases above, with $\mathrm{WF}'_{\mathrm{b}}(A), \mathrm{WF}'_{\mathrm{b}}(B_j) \subset U$, $j = 1, 2$, and $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(U)$, such for $\tilde{s} \leq n/2 + 1$, $\epsilon > 0$, we have, with implicit dependence of the appearing constants on seminorms of the smooth operators P''_m, P''_{m-1} and R'':

$$\begin{aligned} \|B_0 u\|_{\tilde{s}+m-1,r} \leq C(&\|P'_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\tilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \\ &\|P'_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\tilde{s})_++\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|R'\|_{m-2,n/2+1+(-\tilde{s})_+}) \\ &\times(\|u\|_{\tilde{s}+m-3/2,r} + \|Au\|_{\tilde{s}+m-1,r} + \|B_1 f\|_{\tilde{s},r} + \|\chi f\|_{\tilde{s}-1,r}). \end{aligned} \tag{4.11}$$

Moreover, for $\tilde{s} > n/2 + 1$, $\epsilon > 0$, there is a tame estimate

$$\begin{aligned} \|B_0 u\|_{\tilde{s}+m-1,r} \leq L(&\|P'_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \|P'_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|R'\|_{m-2,n/2+\epsilon}; \\ &\|P'_m\|_{H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m}, \|P'_{m-1}\|_{H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|R'\|_{m-2,s-1}; \|u\|_{n/2-1/2+m+\epsilon}, \|f\|_{n/2-1+\epsilon}; \\ &\|u\|_{\tilde{s}+m-3/2,r}, \|Au\|_{\tilde{s}+m-1,r}, \|B_1 f\|_{\tilde{s},r}, \|\chi f\|_{\tilde{s}-1,r}). \end{aligned} \tag{4.12}$$

Proof. One detail changes as compared to the previous proof: While it still suffices to only assume microlocal regularity $B_2 f \in H_{\mathrm{b}}^{\tilde{s},r}$ at $L_\pm$, we now in addition need to assume local regularity $\chi f \in H_{\mathrm{b}}^{\tilde{s}-1,r}$, which is due to the use of elliptic regularity in the proof given in [29]. □

4.3. **Non-trapping estimates at normally hyperbolic trapping.** We now extend the proof of non-trapping estimates on weighted b-Sobolev spaces at normally hyperbolically trapped sets given in [30, Theorem 3.2] to the non-smooth setting.

To set this up, let $P_0 \in \Psi_{\mathrm{b}}^m(\overline{\mathbb{R}^n_+})$ with

$$\frac{1}{2i}(P_0 - P_0^*) = E_1 \in \Psi_{\mathrm{b}}^{m-1}(\overline{\mathbb{R}^n_+}), \tag{4.13}$$

where the adjoint is taken with respect to a fixed smooth b-density; an example to keep in mind here and in what follows is $P_0 = \Box_g$ for a smooth Lorentzian b-metric g on $\overline{\mathbb{R}^n_+}$, considered a coordinate patch of Kerr-de Sitter space, in which case $E_1 = 0$, and the threshold weight in Theorem 4.4 below is $r = 0$. Let p_0 be the principal symbol of P_0. Let us use the coordinates $(z;\zeta) = (x, y; \lambda, \eta)$ on ${}^{\mathrm{b}}T^*\overline{\mathbb{R}^n_+}$ and write $M = \overline{\mathbb{R}^n_+}, X = \partial\overline{\mathbb{R}^n_+}$. With $\Sigma \subset {}^{\mathrm{b}}S^*M$ denoting the characteristic set of P_0, we make the following assumptions:

(1) $\Gamma \subset \Sigma \cap {}^{\mathrm{b}}S^*_X M$ is a smooth submanifold disjoint from the image of $T^*X \setminus o$, so xD_x is elliptic near Γ,
(2) Γ_+ is a smooth submanifold of $\Sigma \cap {}^{\mathrm{b}}S^*_X M$ in a neighborhood U_1 of Γ,
(3) Γ_- is a smooth submanifold of Σ transversal to $\Sigma \cap {}^{\mathrm{b}}S^*_X M$ in U_1,
(4) Γ_+ has codimension 2 in Σ, Γ_- has codimension 1,
(5) Γ_+ and Γ_- intersect transversally in Σ with $\Gamma_+ \cap \Gamma_- = \Gamma$,
(6) the vector field V is tangent to both Γ_+ and Γ_-, and thus to Γ,
(7) Γ_+ is backward trapped for the Hamilton flow, Γ_- is forward trapped; in particular, Γ is a trapped set.

In view of condition (1), we can take

$$\rho = \langle\lambda\rangle \text{ near } \Gamma,$$

appropriately extended to ${}^{\mathrm{b}}\overline{T}^*M$, as the inverse of a boundary defining function of fiber infinity ${}^{\mathrm{b}}S^*M$ in ${}^{\mathrm{b}}\overline{T}^*M$. Then, let

$$V = \rho^{-m+1} H_{p_0},$$

be the rescaled Hamilton vector field of p_0. We make quantitative assumptions related to condition (7): Let $\phi_+ \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ be a defining function of Γ_+ in ${}^{\mathrm{b}}S^*_X M$, and let $\phi_- \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ be a defining function of Γ_-. Thus, Γ_+ is defined within ${}^{\mathrm{b}}S^*M$ by $x = 0, \phi_+ = 0$. Let

$$\hat{p}_0 = \rho^{-m} p_0.$$

We then assume that

(8) ϕ_+ and ϕ_- satisfy

$$V\phi_+ = -c_+^2\phi_+ + \mu_+ x + \nu_+\hat{p}_0, \quad V\phi_- = c_-^2\phi_- + \nu_-\hat{p}_0, \tag{4.14}$$

with $c_\pm > 0$ smooth near Γ and $\mu_+, \nu_\pm$ smooth near Γ. This is consistent with the (in)stability of Γ_- (Γ_+),

(9) x satisfies
$$Vx = -c_\partial x, \quad c_\partial > 0, \tag{4.15}$$
which is consistent with the stability of Γ_-,

(10) near Γ,
$$\rho^{-1} V \rho = c_f x \tag{4.16}$$
for some smooth c_f, which holds in view of our choice of ρ.

Here we recall from [22, Lemma 5.1], see also [23, Lemma 2.4], that in the closely related semiclassical setting[11] one can arrange for any (small) $\epsilon > 0$ that
$$0 < \nu_{\min} - \epsilon < c_\pm^2 < \nu_{\max} + \epsilon, \tag{4.17}$$
where $\nu_{\min}$ and $\nu_{\max}$ are the minimal and maximal normal expansion rates; see [22, Equations (5.1) and (5.2)] for the definition of the latter, with $\nu_{\min}$ also given in (4.27) below. Note that in these works of Dyatlov our $c_\pm^2$ is denoted by $c_\pm$. In particular, if M is replaced by $[0,\infty) \times X$, and if P_0 is dilation invariant, then the semiclassical and the b-settings are equivalent via the Mellin transform and a rescaling, see e.g. [47, Section 3.1]; since in our general case $c_\pm|_{{}^{\mathrm{b}}S^*_X M}$ is what matters, we can replace P_0 by $N(P_0)$, and in particular (4.17) applies, with the expansion rate calculated using $p_0|_{{}^{\mathrm{b}}T^*_X M}$.

We now perturb P_0 by a non-smooth operator $\tilde{P}$, that is, we consider the operator
$$P = P_0 + \tilde{P}, \quad \tilde{P} = \tilde{P}_m + \tilde{P}_{m-1} + \tilde{R}, \tag{4.18}$$
where for some fixed $\alpha > 0$, we have $\tilde{P}_{m-j} \in H_{\mathrm{b}}^{s-j,\alpha}\Psi_{\mathrm{b}}^{m-j}$, $j = 0, 1$, and $\tilde{R} \in \Psi_{\mathrm{b}}^{m-2;0} H_{\mathrm{b}}^{s-1,\alpha}$.

We then have the following tame non-trapping estimate at Γ:

Theorem 4.4. *Using the above notation and making the above assumptions, let $s, \tilde{s} \in \mathbb{R}$ be such that*
$$\tilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2 - \tilde{s})_+. \tag{4.19}$$
Suppose $u \in H_{\mathrm{b}}^{\tilde{s}+m-3/2,r}(\overline{\mathbb{R}^n_+})$ is such that $Pu = f \in H_{\mathrm{b}}^{\tilde{s},r}(\overline{\mathbb{R}^n_+})$.

Then for $r < -\sup_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ and for any neighborhood U of Γ, there exist $B_0 \in \Psi_{\mathrm{b}}^0(M)$ elliptic at Γ and $B_1, B_2 \in \Psi_{\mathrm{b}}^0(M)$ with $\mathrm{WF}_{\mathrm{b}}'(B_j) \subset U$, $j = 0, 1, 2$, $\mathrm{WF}_{\mathrm{b}}'(B_2) \cap \Gamma_+ = \emptyset$, and $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(U)$, such that the following estimate holds for $\tilde{s} \leq n/2 + 1$, $\epsilon > 0$:
$$\begin{aligned} \|B_0 u\|_{\tilde{s}+m-1,r} \leq C(&\|\tilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+(2-\tilde{s})_+ + \epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \\ &\|\tilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+(3/2-\tilde{s})_+ + \epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\tilde{R}\|_{m-2,n/2+1+(-\tilde{s})_+}) \\ &\times (\|u\|_{\tilde{s}+m-3/2,r} + \|B_2 u\|_{\tilde{s}+m-1,r} + \|B_1 f\|_{\tilde{s},r} + \|\chi f\|_{\tilde{s}-1,r}). \end{aligned} \tag{4.20}$$
Moreover, for $\tilde{s} > n/2 + 1$, $\epsilon > 0$, there is a tame estimate
$$\begin{aligned} \|B_0 u\|_{\tilde{s}+m-1,r} \leq L(&\|\tilde{P}_m\|_{H_{\mathrm{b}}^{n/2+7/2+\epsilon,\alpha}\Psi_{\mathrm{b}}^m}, \|\tilde{P}_{m-1}\|_{H_{\mathrm{b}}^{n/2+1+\epsilon,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\tilde{R}\|_{m-2,n/2+\epsilon}; \\ &\|\tilde{P}_m\|_{H_{\mathrm{b}}^{s,\alpha}\Psi_{\mathrm{b}}^m}, \|\tilde{P}_{m-1}\|_{H_{\mathrm{b}}^{s-1,\alpha}\Psi_{\mathrm{b}}^{m-1}}, \|\tilde{R}\|_{m-2,s-1}; \|u\|_{n/2-1/2+m+\epsilon}, \|f\|_{n/2-1+\epsilon}; \\ &\|u\|_{\tilde{s}+m-3/2,r}, \|B_2 u\|_{\tilde{s}+m-1,r}, \|B_1 f\|_{\tilde{s},r}, \|\chi f\|_{\tilde{s}-1,r}). \end{aligned} \tag{4.21}$$

[11] See the discussion prior to Theorem 5.5.

On the other hand, for $r > -\inf_\Gamma \rho^{-m+1}\sigma_{\mathrm{b},m-1}(E_1)/c_\partial$ *and for appropriate* B_2 *with* $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_- = \emptyset$*, the estimates* (4.20) *and* (4.21) *hold as well. These estimates are understood in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds.*

Proof. The main part of the argument, in particular the choice of the commutant, is a slight modification of the positive commutator argument of [30, Theorem 3.2]; the handling of the non-smooth terms is a modification of the proof of the radial point estimate, [29, Theorem 6.10]. In particular, the positivity comes from differentiating the weight x^{-r} in the commutant. To avoid working in weighted b-Sobolev spaces for the non-smooth problem, we will conjugate P by x^{-r}, giving an advantageous (here meaning negative) contribution to the imaginary part of the subprincipal symbol of the conjugated operator near Γ.

Throughout this proof, we denote operators and their symbols by the corresponding capital and lower case letters, respectively.

Concretely, put $\sigma = \tilde{s} + m - 1$, and define

$$u_r := x^{-r}u \in H_{\mathrm{b}}^{\sigma-1/2}, \quad f_r := x^{-r}f \in H_{\mathrm{b}}^{\sigma-m+1},$$
$$P_r := x^{-r}Px^r = P_{0,r} + \tilde{P}_r, \quad P_{0,r} = x^{-r}P_0x^r, \tilde{P}_r = x^{-r}\tilde{P}x^r,$$

where

$$\tilde{P}_r = \tilde{P}_{m,r} + \tilde{P}_{m-1,r} + \tilde{R}_r, \quad \tilde{P}_{m-j,r} \in H_{\mathrm{b}}^{s-j,\alpha}\Psi_{\mathrm{b}}^{m-j}, \tilde{R}_r \in \Psi_{\mathrm{b}}^{m-2;0}H_{\mathrm{b}}^{s-1,\alpha};$$

then $P_ru_r = f_r$, and we must show a non-trapping estimate for u_r on *unweighted* b-Sobolev spaces. A simple computation shows that

$$\frac{1}{2i}(P_{0,r} - P_{0,r}^*) - \left(\frac{1}{2i}(P_0 - P_0^*) - \mathrm{Op}(rx^{-1}H_{p_0}x)\right) \in \Psi_{\mathrm{b}}^{m-2};$$

but $x^{-1}H_{p_0}x = -\rho^{m-1}c_\partial$ with $c_\partial > 0$ near Γ by (4.15), hence, using (4.13),

$$\frac{1}{2i}(P_{0,r} - P_{0,r}^*) = E_1 + E_1' + B \tag{4.22}$$

with $B, E_1' \in \Psi_{\mathrm{b}}^{m-1}$, where B has principal symbol $b = rc_\partial\rho^{m-1}$ near Γ, and $\mathrm{WF}'_{\mathrm{b}}(E_1') \cap \Gamma = \emptyset$. Notice that by assumption on r, $B + E_1$ is elliptic on Γ.

We now turn to the positive commutator argument: Fix $0 < \beta < \min(1, \alpha)$ and define

$$\rho_+ = \phi_+^2 + x^\beta.$$

Let $\chi_0(t) = e^{-1/t}$ for $t > 0$ and $\chi_0(t) = 0$ for $t < 0$, further $\chi \in \mathcal{C}_c^\infty([0, R))$ for $R > 0$ to be chosen below, $\chi \equiv 1$ near 0, $\chi' \leq 0$, and finally $\psi \in \mathcal{C}_c^\infty((-R, R))$, $\psi \equiv 1$ near 0. Define for $\kappa > 0$, specified later,

$$a = \rho^{\sigma-(m-1)/2}\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(\hat{p}_0).$$

On $\operatorname{supp} a$, we have $\rho_+ \leq R$, thus the argument of χ_0 is bounded above by $R + \kappa$. Moreover, $\phi_-^2 \leq R + \kappa$ and $x \leq R^{1/\beta}$, therefore a is supported in any given neighborhood of Γ if one chooses R and κ small. Notice that a is merely a *conormal* symbol which does not grow at the boundary. However, b-analysis for operators with conormal coefficients can easily be seen to work without much additional work, in fact, a logarithmic change of variables essentially reduces such a b-analysis on $\overline{\mathbb{R}^n_+}$ to the analysis of operators corresponding to uniform symbols on $\mathbb{R}^n$. Moreover, the proofs of composition results of smooth and non-smooth b-ps.d.o's presented in

[29] go through without changes if one uses b-ps.d.o's with non-growing conormal, instead of smooth, symbols.[12]

Define the regularizer $\varphi_\delta(\zeta) = (1+\delta\rho)^{-1}$ near Γ, and put $a_\delta = \varphi_\delta a$. Put $\tilde V = \rho^{-m+1} H_{\tilde p_{m,r}}$ and define $\tilde c_\partial, \tilde c_f \in H_{\mathrm{b}}^{s-1,\alpha}$ near Γ by $\tilde V x = -\tilde c_\partial x$, $\rho^{-1}\tilde V\rho = \tilde c_f x$. Then, with $p_{m,r} = p_{0,r} + \tilde p_{m,r}$, we obtain, using (4.14)-(4.16):

$$\begin{aligned} a_\delta H_{p_{m,r}} a_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi^2\psi^2(\sigma-(m-1)/2-\delta\rho\varphi_\delta)(c_f+\tilde c_f)x \\ &\quad - \varphi_\delta^2\rho^{2\sigma}\chi_0\chi_0'\chi^2\psi^2(2c_+^2\phi_+^2 + \beta c_\partial x^\beta - 2\mu_+\phi_+ x - 2\nu_+\phi_+\hat p_0 \\ &\qquad\qquad + 2c_-^2\phi_-^2 + 2\nu_-\phi_-\hat p_0 - \tilde V\phi_+^2 + \beta\tilde c_\partial x^\beta + \tilde V\phi_-^2) \\ &\quad + \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi\chi'\psi^2(V\rho_+ + \tilde V\rho_+) + \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi^2\psi\psi'(V\hat p_0 + \tilde V\hat p_0) \\ &= -c_+^2 a_{+,\delta}^2 - c_-^2 a_{-,\delta}^2 + a_{+,\delta}h_{+,\delta}p_{m,r} + a_{-,\delta}h_{-,\delta}p_{m,r} + e_\delta + g_\delta - f_\delta, \end{aligned} \tag{4.23}$$

where, writing $\hat p_0 = \rho^{-m}p_{m,r} - \rho^{-m}\tilde p_{m,r}$ in the second and third line,

$$\begin{aligned} a_{\pm,\delta} &= \varphi_\delta\rho^\sigma\sqrt{2\chi_0\chi_0'}\chi\psi\phi_\pm, \\ h_{\pm,\delta} &= \pm\varphi_\delta\rho^{\sigma-m}\sqrt{2\chi_0\chi_0'}\chi\psi\nu_\pm, \\ e_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi\chi'\psi^2(V\rho_+ + \tilde V\rho_+), \\ g_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0^2\chi^2\psi\psi'(V\hat p_0 + \tilde V\hat p_0), \\ f_\delta &= \varphi_\delta^2\rho^{2\sigma}\chi_0\chi^2\psi^2\Big[\big(\beta(c_\partial+\tilde c_\partial)x^\beta - 2\mu_+\phi_+x - \tilde V\phi_+^2 + \tilde V\phi_-^2 \\ &\qquad + 2(\nu_+\phi_+ - \nu_-\phi_-)\rho^{-m}\tilde p_{m,r}\big)\chi_0' \\ &\qquad\qquad - (\sigma-(m-1)/2-\delta\rho\varphi_\delta)(c_f+\tilde c_f)x\chi_0\Big] \end{aligned}$$

Note that in the definition of f_δ, by the choice of β and using the fact that χ_0 is bounded by a constant multiple of χ_0' on its support, the constant being uniform for $R+\kappa<1$, the term $c_\partial x^\beta$ dominates all other terms on the support of $f_\delta \in S^{2\sigma;\infty}H_{\mathrm{b}}^{s-1}$ for R and κ small enough, hence $f_\delta \geq 0$, and its contribution will be controlled by virtue of the sharp Gårding inequality. The term arising from e_δ will be controlled using the a priori regularity assumption of u_r on Γ_-, and g_δ, which is supported away from the characteristic set, will be controlled using elliptic regularity.

Proceeding with the argument, we first make the simplification $\tilde R_r = 0$ by replacing f by $f - \tilde R_r u_r$, and we assume $m \geq 1$ and $\tilde s \geq (5-m)/2$ for now. Then we have, as in the proof of [29, Theorem 6.10],

$$\mathrm{Re}\langle iA_\delta^*[P_{0,r}+\tilde P_{m,r}, A_\delta]u_r, u_r\rangle + \left\langle \frac{1}{2i}(P_{0,r}-P_{0,r}^*)A_\delta u_r, A_\delta u_r\right\rangle$$

[12] A somewhat more direct way of dealing with this issue goes as follows: Assume, as one may, that $\ell := \beta^{-1} \in \mathbb{N}$. Then even though a is not a smooth symbol of $\overline{\mathbb{R}^n_+}$ with the standard smooth structure, it becomes smooth if one changes the smooth structure of $\overline{\mathbb{R}^n_+}$ by blowing up the boundary to the ℓ-th order, i.e. by taking $x' = x^\beta$ as a boundary defining function, thus obtaining a manifold M_ℓ, which is $\overline{\mathbb{R}^n_+}$ as a topological manifold, but with a different smooth structure; in particular, the function $x = (x')^\ell$ is smooth on M_ℓ in view of $\ell \in \mathbb{N}$. Moreover, the blow-down map $M_\ell \to \overline{\mathbb{R}^n_+}$ induces isomorphisms (see e.g. [36, §4.18])

$$H_{\mathrm{b}}^{s',\gamma}(\overline{\mathbb{R}^n_+}) \cong H_{\mathrm{b}}^{s',\ell\gamma}(M_\ell), \quad s',\gamma \in \mathbb{R}.$$

Therefore, one can continue to work on $\overline{\mathbb{R}^n_+}$, tacitly assuming that all functions and operators live on, and all computations are carried out on, M_ℓ.

$$= -\left\langle \frac{1}{2i}(\tilde{P}_{m,r} - \tilde{P}_{m,r}^*)A_\delta u_r, A_\delta u_r \right\rangle \\ - \mathrm{Re}\langle iA_\delta f, A_\delta u_r\rangle + \mathrm{Re}\langle iA_\delta \tilde{P}_{m-1,r}u_r, A_\delta u_r\rangle.$$

Estimating each term on the right hand side as in the proof of [29, Theorem 6.10] and using (4.22), we obtain for any $\mu > 0$:

$$\mathrm{Re}\left\langle \left(A_\delta^*(i[P_{0,r} + \tilde{P}_{m,r}, A_\delta] + E_1 + E_1' + B)A_\delta\right)u_r, u_r \right\rangle \geq -C_\mu - \mu\|A_\delta u_r\|^2_{(m-1)/2}. \tag{4.24}$$

Here and in what follows, we in particular absorb all terms involving $\|u_r\|_{\sigma-1/2}$ into the constant C_μ. On the left hand side, the E_1'-term can be dropped because of $\mathrm{WF}_\mathrm{b}'(E_1') \cap \mathrm{WF}_\mathrm{b}'(A) = \emptyset$ for sufficiently localized a. Moreover, the principal symbol of $E_1 + B$ near Γ is $e_1 + b = -q^2$ with q an elliptic symbol of order $(m-1)/2$, since, by assumption on r, we have $e_1 + rc_\partial\rho^{m-1} < 0$ near Γ. Therefore, we can write $E_1 + B = -Q^*Q + E_1'' + E_2$, where $E_1'' \in \Psi_\mathrm{b}^{m-1}$, $E_2 \in \Psi_\mathrm{b}^{m-2}$, $\mathrm{WF}_\mathrm{b}'(E_1'') \cap \Gamma = \emptyset$. Again, the resulting term in the pairing (4.24) involving E_1'' can be dropped; also, the term involving E_2 can be dropped at the cost of changing the constant C_μ, since $u_r \in H_\mathrm{b}^{\sigma-1/2}$.

Hence, introducing $J^\pm \in \Psi_\mathrm{b}^{\pm(\sigma-(m-1)/2-1)}$, with real principal symbols, satisfying $I - J^+J^- \in \Psi_\mathrm{b}^{-\infty}$, we get

$$\mathrm{Re}\left\langle \mathrm{Op}(j^- a_\delta H_{p_{m,r}} a_\delta)u_r, (J^+)^* u_r\right\rangle - \|QA_\delta u_r\|_0^2 \geq -C_\mu - \mu\|A_\delta u_r\|^2_{(m-1)/2}. \tag{4.25}$$

We now plug the commutator relation (4.23) into this estimate. We obtain several terms, which we bound as follows: First, since $j^- e_\delta \in (\mathcal{C}^\infty + H_\mathrm{b}^{s-1,\alpha})S^{\sigma+(m-1)/2+1}$ uniformly, $\mathrm{Op}(j^- e_\delta)$ is a uniformly bounded family of maps $H_\mathrm{b}^\sigma \to H_\mathrm{b}^{-(m+1)/2}$; thus, choosing $\tilde{E} \in \Psi_\mathrm{b}^0$ with $\mathrm{WF}_\mathrm{b}'(\tilde{E}) \subset U$ and with $\mathrm{WF}_\mathrm{b}'(I - \tilde{E})$ disjoint from $\operatorname{supp} e_\delta$, we conclude

$$|\langle \mathrm{Op}(j^- e_\delta)u_r, (J^+)^* u_r\rangle| \leq C + |\langle \mathrm{Op}(j^- e_\delta)u_r, (J^+)^*\tilde{E}u_r\rangle| \leq C + \|B_2 u_r\|_\sigma^2$$

for some $B_2 \in \Psi_\mathrm{b}^0$ with $\mathrm{WF}_\mathrm{b}'(B_2) \cap \Gamma_+ = \emptyset$.

Next, the term $\langle \mathrm{Op}(j^- g_\delta)u_r, (J^+)^* u_r\rangle$ is uniformly bounded, as detailed in the proof of [29, Theorem 6.10]. Moreover, by the sharp Gårding inequality, see the argument in the proof of [29, Theorem 6.6],

$$\mathrm{Re}\langle \mathrm{Op}(-j^- f_\delta)u_r, (J^+)^* u_r\rangle \leq C.$$

Further, we obtain two terms involving $h_{\pm,\delta}$; introducing $B_3 \in \Psi_\mathrm{b}^0$ elliptic on $\mathrm{WF}_\mathrm{b}'(A)$, these can be bounded for $\mu > 0$ by

$$\begin{aligned} &|\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta} p_{m,r})u_r, (J^+)^* u_r\rangle| \\ &\quad\leq C + |\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})(P_{0,r} + \tilde{P}_{m,r})u_r, (J^+)^* u_r\rangle| \\ &\quad\leq C + |\langle H_{\pm,\delta} f_r, A_{\pm,\delta} u_r\rangle| + |\langle \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})\tilde{P}_{m-1,r}u_r, (J^+)^* u_r\rangle| \\ &\quad\leq C + \mu\|A_{\pm,\delta} u_r\|_0^2 + C_\mu\|B_3 f_r\|^2_{\sigma-m}. \end{aligned}$$

Here, for the first estimate, we employ [29, Theorem 3.12 (3)] to obtain

$$\mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta})\tilde{P}_{m,r} - \mathrm{Op}(j^- a_{\pm,\delta} h_{\pm,\delta}\tilde{p}_{m,r}) \\ =: \Upsilon_\delta \in \Psi_\mathrm{b}^{\sigma+(m-1)/2;0} H_\mathrm{b}^{s-1} + \Psi_\mathrm{b}^{\sigma-(m-1)/2-1} \circ \Psi_\mathrm{b}^{m;0} H_\mathrm{b}^{s-1},$$

and Υ_δ is easily seen to be uniformly bounded from $H_{\mathrm{b}}^{\sigma-1/2}$ to $H_{\mathrm{b}}^{-m/2}$, whereas $(J^+)^*u_r \in H_{\mathrm{b}}^{m/2}$, thus $|\langle \Upsilon_\delta u_r, (J^+)^* u_r\rangle| \leq C$. For the second estimate, we simply use $(P_{0,r} + \tilde{P}_{m,r})u_r = f_r - \tilde{P}_{m-1,r}u_r$, and for the third estimate, we apply the Peter–Paul inequality to the first pairing; to bound the second pairing, we use the boundedness of $\tilde{P}_{m-1,r}\colon H_{\mathrm{b}}^{\sigma-1/2} \to H_{\mathrm{b}}^{\sigma-m+1/2}$.

Finally, including the terms $c_\pm^2 a_{\pm,\delta}^2$ into the estimate obtained from (4.25) by making use of the above estimates, we obtain

$$\begin{aligned}\|C_+A_{+,\delta}u_r\|_0^2 + \|C_-A_{-,\delta}u_r\|_0^2 + \|QA_\delta u_r\|_0^2 \\ \leq C_\mu + \mu\|A_{+,\delta}u_r\|_0^2 + \mu\|A_{-,\delta}u_r\|_0^2 + \mu\|A_\delta u_r\|_{(m-1)/2}^2 \\ + \|B_2u_r\|_\sigma^2 + \|B_1f_r\|_{\sigma-m+1}^2 + C_\mu\|\chi f_r\|_{\sigma-m}^2,\end{aligned}$$

where $B_1 \in \Psi_{\mathrm{b}}^0$ is elliptic on $\mathrm{WF}_{\mathrm{b}}'(A)$ with $\mathrm{WF}_{\mathrm{b}}'(B_1) \subset U$, and $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(M)$ is identically 1 near the projection of $\Gamma \subset {}^{\mathrm{b}}S^*M$ to the base M. Since c_+ and c_- have positive lower bounds near Γ, we can absorb the terms on the right involving $A_{\pm,\delta}$ into the left hand side by choosing μ sufficiently small, at the cost of changing the constant C_μ; likewise, $\rho^{-(m-1)/2}q$ has a positive lower bound near $\operatorname{supp} a$, hence the term on the right involving A_δ can be absorbed into the left hand side for small μ. Dropping the first two terms on the left hand side, we obtain the H_{b}^σ-regularity of u_r at Γ, hence $\mathrm{WF}_{\mathrm{b}}^{\sigma,r}(u) \cap \Gamma = \emptyset$, and a corresponding tame estimate, which follows from a careful analysis of the above argument as in the proof of Proposition 4.2.

Next, we remove the restriction $m \geq 1$: Let $m_0 \geq 1$. The idea, as before, is to rewrite $Pu = f$ as $P\Lambda^+(\Lambda^- u) = f + PRu$, where $\Lambda^\pm \in \Psi_{\mathrm{b}}^{\pm(m_0-m)}$, with real principal symbols, satisfy $\Lambda^+\Lambda^- = I + R$. We now have to be a bit careful though to not change the imaginary part of the subprincipal symbol of $P\Lambda^+$ at Γ. Concretely, we choose Λ^+ self-adjoint with principal symbol $\lambda^+ = \rho^{m_0-m}$ near Γ; then

$$P_0\Lambda^+ - (P_0\Lambda^+)^* = \Lambda^+(P_0 - P_0^*) + [P_0, \Lambda^+].$$

Clearly, $\Lambda^+(P_0 - P_0^*) \in x\Psi_{\mathrm{b}}^{m_0-1} + \Psi_{\mathrm{b}}^{m_0-2}$, and the principal symbol of the second term is

$$\sigma_{\mathrm{b},m_0-1}([P_0, \Lambda^+]) = -iH_{p_0}\lambda^+ = -ix(m_0 - m)\rho^{m_0-1}c_f$$

near Γ by (4.16), hence, using (4.13),

$$P_0\Lambda^+ - (P_0\Lambda^+)^* = \Lambda^+E_1 + xE_1' + E_1'' + E_2$$

with $E_1', E_1'' \in \Psi_{\mathrm{b}}^{m_0-1}, E_2 \in \Psi_{\mathrm{b}}^{m_0-2}$ and $\mathrm{WF}_{\mathrm{b}}'(E_1'') \cap \Gamma = \emptyset$; therefore, the first part of the proof with P and u replaced by $P\Lambda^+$ and $\Lambda^- u$, respectively, applies. The proof of the theorem in the case $r < -\sup_\Gamma \rho^{-m+1}e_1/c_\partial$ is complete.

When the role of Γ_+ and Γ_- is reversed, there is an overall sign change, and we thus get a advantageous (now meaning positive) contribution to the subprincipal part of the conjugated operator P_r for $r > -\inf_\Gamma \rho^{-m+1}e_1/c_\partial$; the rest of the argument is unchanged. □

4.4. **Trapping estimates at normally hyperbolic trapping.** Complementing the results above on negatively weighted spaces, we recall results of Dyatlov from [22, 23] on semiclassical estimates for smooth operators at normally hyperbolic trapping, which via the Mellin transform correspond to estimates on non-negatively weighted spaces. Here we present the results in the semiclassical setting, then in Section 5.1 we relate this to the solvability of linear equations with Sobolev coefficients in Theorem 5.5 and Theorem 5.6. We recall that prior to Dyatlov's work, Wunsch and Zworski [49] and Nonnenmacher and Zworski [39] studied semiclassical estimates at normally hyperbolic trapping; this was in turn much preceded by the work of Gérard and Sjöstrand [27] in the analytic category. The advantage of Dyatlov's framework for us, especially as espoused in [23], is the explicit size of the 'spectral gap' (discussed below), which was also shown by Nonnenmacher and Zworski [39], the explicit inclusion of a subprincipal term of the correct sign, and the relative ease with which the parameter dependence can be analyzed.

We first recall Dyatlov's semiclassical setting for

$$\tilde{P}_0 = \tilde{P}_0(h), \tilde{Q}_0 = \tilde{Q}_0(h) \in \Psi_\hbar^m(X),$$

both formally self-adjoint, with $\tilde{Q}_0$ having non-negative principal symbol, $\tilde{P}_0 - i\tilde{Q}_0$ elliptic in the standard sense. In fact, Dyatlov states the results in the special case $m = 0$, but by ellipticity of $\tilde{P}_0 - i\tilde{Q}_0$ in the standard sense, it is straightforward to allow general m; see also the remark [23, Bottom of p. 2]. The main assumption, see [23, p. 3], then is that $\tilde{P}_0$ has normally hyperbolic trapping semiclassically at $\tilde{\Gamma} \subset T^*X$ compact,[13] with all bicharacteristics of $\tilde{P}_0$, except those in the stable $(-)$ and unstable $(+)$ submanifolds $\tilde{\Gamma}_\pm$, entering the elliptic set of $\tilde{Q}_0$ in the forward (the exception being for only the $-$ sign), resp. backward $(+)$ direction, and $\gamma < \nu_{\min}/2$, where $\nu_{\min} > 0$ is the minimal normal expansion rate of the flow at $\tilde{\Gamma}$, discussed above and in (4.27). If $\tilde{Q}_0$ is microlocally in $h\Psi_\hbar(X)$ near $\tilde{\Gamma}$, with $h^{-1}\tilde{Q}_0$ having a non-negative principal symbol there, Dyatlov shows that there is $h_0 > 0$ such that for $\operatorname{Im} z > -\gamma$,

$$\|v\|_{H_\hbar^s} \lesssim h^{-2} \|(\tilde{P}_0 - i\tilde{Q}_0 - hz)v\|_{H_\hbar^{s-m}}, \ h < h_0. \tag{4.26}$$

In view of $\tilde{\Gamma}$ lying in a compact subset of T^*X, the order s is irrelevant in the sense that the estimate for one value of s implies that for all other via elliptic estimates; thus, one may just take $s = 0$, and even replace $s - m$ by 0, in which case this is an L^2-estimate, as stated by Dyatlov.

Suppose now that one has a family of operators $\tilde{P}_0(\omega)$ depending on another parameter, ω, in a compact space S, with $\tilde{P}_0, \tilde{Q}_0$ depending continuously on ω, with values in $\Psi_\hbar^m(X)$, satisfying all of the assumptions listed above. Suppose moreover that this family satisfies the normally hyperbolic assumptions with $\tilde{\Gamma}, \tilde{\Gamma}_\pm$ continuously depending on ω in the $\mathcal{C}^\infty$ topology, and uniform bounds for the normal expansion rates in the sense that both ν and the constant C' in

$$\sup_{\rho \in \Gamma} \|de^{\mp tH_p}(\rho)|_{\mathcal{V}_\pm}\| \leq C' e^{-\nu t}, \ t \geq 0, \tag{4.27}$$

with $\mathcal{V}_\pm$ the unstable and stable normal tangent bundles at Γ, can be chosen uniformly (cf. [22, Equation (5.1)]); $\nu_{\min}$ is then the sup of these possible choices of ν. (Note that since the trapped set dynamics involves arbitrarily large times, it is *not* automatically stable, unlike the dynamics away from the trapped set.) In this case the implied constant C in (4.28), as well as h_0, is uniform in ω. Note that r-normal hyperbolicity for every r implies the local (hence global, in view of compactness) uniformity of the normal dynamics by structural stability; see [49, Section 1] and [22, Section 5.2].

To see this uniformity in C, we first point out that in [22, Lemma 5.1] the construction of $\phi_\pm$ can be done continuously with values in $\mathcal{C}^\infty$ in this case. Then in

[13] Our $\tilde{\Gamma}$ is the intersection of Dyatlov's K with the semiclassical characteristic set of P, and similarly our $\tilde{\Gamma}_\pm$ are the intersection of Dyatlov's $\Gamma_\pm$ with the characteristic set of P.

the proof of (4.28) given in [23], we only need to observe that the direct estimates provided are certainly uniform in this case for families $\tilde{P}_0, \tilde{Q}_0$, and furthermore for the main argument, using semiclassical defect measures, one can pass to an L^2-bounded subsequence u_j such that $(\tilde{P}_0(\omega_j) - i\tilde{Q}_0(\omega_j) - \lambda_j)u_j = O(h^2)$, with $\omega_j \to \omega$ for some $\omega \in S$ in addition to $h^{-1}\lambda_j$ converging to some $\tilde{\lambda}$. Concretely, all of Dyatlov's results in [23, Section 2] are based on elliptic or (positive) commutator identities or estimates which are uniform in this setting. In particular, [23, Lemma 2.3] is valid with $P_j = P(\omega_j) \to P$, $W_j = W(\omega_j) \to W$ with convergence in $\Psi_\hbar(X)$. (This uses that one can take $A_j(h_j)$ in Definition 2.1, with $A_j \to A$, since the difference between $A_j(h_j)$ and $A(h_j)$ is bounded by a constant times the squared L^2-norm of u_j times the operator norm bound of $A_j(h_j) - A(h_j)$, with the latter going to 0.) Then with $\Theta_{+,j}$ in place of Θ_+, one still gets Lemma 3.1, which means that Lemma 3.2 still holds with ϕ_+ (the limiting $\phi_{+,j}$) using Lemma 2.3. Then the displayed equation above [23, Equation (3.9)] still holds with the limiting $\tilde{P}_0 = \tilde{P}_0(\omega)$, again by Lemma 2.3, and then one can finish the argument as Dyatlov did. With this modification, one obtains the desired uniformity. This in particular allows one to apply (4.28) even if $\tilde{P}_0$ and $\tilde{Q}_0$ depend on z (in a manner consistent with the other requirements), which can also be dealt with more directly using Dyatlov's model form [22, Lemma 4.3]. It also allows for uniform estimates for families depending on a small parameter in $\mathbb{C}$, denoted by v_0 below, needed in Section 5.

Allowing $\tilde{P}_0$ and $\tilde{Q}_0$ depending on z means, in particular, that we can replace the requirement on $h^{-1}\tilde{Q}_0$ by the principal symbol of $h^{-1}\tilde{Q}_0$ being $> -\beta$, $\beta < \nu_{\min}/2$, and drop z, so one has

$$\|v\|_{H^s_\hbar} \lesssim h^{-2}\|(\tilde{P}_0 - i\tilde{Q}_0)v\|_{H^{s-m}_\hbar},\ h < h_0. \tag{4.28}$$

At this point it is convenient to rewrite this estimate, removing $\tilde{Q}_0$ from the right hand side at the cost (or benefit!) of making it microlocal.[14] *From here on it is convenient to change the conventions and not require that $\tilde{P}_0$ is formally self-adjoint (though it is at the principal symbol level, namely it has a real principal symbol); translating back into the previous notation, one would replace $\tilde{P}_0$ by its (formally) self-adjoint part, and absorb its skew-adjoint part into $\tilde{Q}_0$.* Namely, we have

Theorem 4.5. *Suppose $\tilde{P}_0$ satisfies the above assumptions, in particular the semiclassical principal symbol of $\frac{1}{2ih}(\tilde{P}_0 - \tilde{P}_0^*)$ being $< \beta < \nu_{\min}/2$ at $\tilde{\Gamma}$.*[15] *With $\tilde{B}_j$ analogous to Theorem 4.4, with wave front set sufficiently close to $\tilde{\Gamma}$, we have, for sufficiently small $h > 0$ and for all N and s_0,*

$$\|\tilde{B}_0 u\|_{H^s_\hbar} \lesssim h^{-2}\|\tilde{B}_1\tilde{P}_0 u\|_{H^{s-m+1}_\hbar} + h^{-1}\|\tilde{B}_2 u\|_{H^s_\hbar} + h^N\|u\|_{H^{s_0}_\hbar}. \tag{4.29}$$

Note that the differential orders are actually irrelevant here due to wave front set conditions.

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[14] An alternative would be using the gluing result of Datchev and Vasy [17], which is closely related in approach.

[15] The apparent sign change here as compared to before comes from the fact that for formally self-adjoint $\tilde{P}_0, \tilde{Q}_0$, one has $\frac{1}{2ih}\big((\tilde{P}_0 - i\tilde{Q}_0) - (\tilde{P}_0 - i\tilde{Q}_0)^*\big) = -h^{-1}\tilde{Q}_0$; notice the minus sign on the right hand side.

Proof. Take $\tilde{Q}_0 \in \Psi_\hbar^0(X)$ with non-negative principal symbol such that $\mathrm{WF}'_\hbar(\tilde{Q}_0)$ is disjoint from $\mathrm{WF}'_\hbar(\tilde{B}_0)$, and so that all backward bicharacteristics from points not in $\tilde{\Gamma}_+$, as well as forward bicharacteristics from points not in $\tilde{\Gamma}_-$, reach the elliptic set of $\tilde{Q}_0$, and with $\tilde{B}_1$ elliptic on the complement of the elliptic set of $\tilde{Q}_0$. Let $\tilde{B}_3 \in \Psi_\hbar^0(X)$ to be such that $\mathrm{WF}'_\hbar(I - \tilde{B}_3)$ is disjoint from $\mathrm{WF}'_\hbar(\tilde{B}_0)$ but $\mathrm{WF}'_\hbar(\tilde{Q}_0) \cap \mathrm{WF}'_\hbar(\tilde{B}_3) = \emptyset$. Let $\tilde{A}_+ \in \Psi_\hbar^0(X)$ have wave front set near $\tilde{\Gamma}_+$, with

$$\mathrm{WF}'_\hbar(I - \tilde{A}_+) \cap \mathrm{WF}'_\hbar(\tilde{B}_3) \cap \tilde{\Gamma}_+ = \emptyset$$

and with

$$\mathrm{WF}'_\hbar(\tilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \tilde{B}_3) \cap \tilde{\Gamma}_- = \emptyset,$$

and with no backward bicharacteristic from $\mathrm{WF}'_\hbar(\tilde{B}_0)$ reaching

$$\mathrm{WF}'_\hbar(\tilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \tilde{B}_3) \cap \tilde{\Gamma}_+.$$

Take $\tilde{Q}_1$ elliptic on $\tilde{\Gamma}$, with $\mathrm{WF}'_\hbar(\tilde{Q}_1) \cap \mathrm{WF}'_\hbar(I - \tilde{B}_3) = \emptyset$, again with non-negative principal symbol, with no backward bicharacteristic from $\mathrm{WF}'_\hbar(\tilde{Q}_1)$ reaching

$$\mathrm{WF}'_\hbar(\tilde{A}_+) \cap \mathrm{WF}'_\hbar(I - \tilde{B}_3).$$

Thus, all backward and forward bicharacteristics of $\tilde{P}_0$ reach the elliptic set of $\tilde{Q}_1$ or $\tilde{Q}_0$. See Figure 2 for the setup.

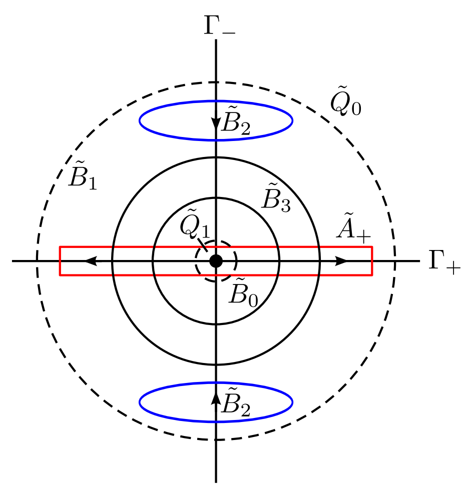

FIGURE 2. Setup for the proof of the microlocalized normally hyperbolic trapping estimate (4.29): Indicated are the backward and forward trapped sets Γ_+ and Γ_-, respectively, which intersect at Γ (large dot). We use complex absorbing potentials $\tilde{Q}_0$ (with $\mathrm{WF}'_\hbar(\tilde{Q}_0)$ outside the large dashed circle) and $\tilde{Q}_1$ (with $\mathrm{WF}'_\hbar(\tilde{Q}_1)$ inside the small dashed circle). We obtain an estimate for $\tilde{B}_0 u$ by combining (4.28) with microlocal propagation from the elliptic set of $\tilde{B}_2$.

Then

$$(\tilde{P}_0 - i\tilde{Q}_0)\tilde{B}_3 u = \tilde{B}_3 \tilde{P}_0 u + \tilde{A}_+[\tilde{P}_0, \tilde{B}_3]u + (I - \tilde{A}_+)[\tilde{P}_0, \tilde{B}_3]u - i\tilde{Q}_0 \tilde{B}_3 u,$$

so

$$\begin{aligned}\tilde{B}_0 u &= \tilde{B}_0\tilde{B}_3 u + \tilde{B}_0(I-\tilde{B}_3)u \\ &= \tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{B}_3\tilde{P}_0 u + \tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u \\ &\qquad + \tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}(I-\tilde{A}_+)[\tilde{P}_0,\tilde{B}_3]u \\ &\qquad - i\tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{Q}_0\tilde{B}_3 u + \tilde{B}_0(I-\tilde{B}_3)u,\end{aligned} \tag{4.30}$$

and by (4.28), for $h < h_0$,

$$\|(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{B}_3\tilde{P}_0 u\|_{H^s_\hbar} \lesssim h^{-2}\|\tilde{B}_3\tilde{P}_0 u\|_{H^{s-m}_\hbar}.$$

Now, $\tilde{Q}_0\tilde{B}_3, \tilde{B}_0(I-\tilde{B}_3) \in h^\infty\Psi_\hbar^{-\infty}(X)$, so the corresponding terms in (4.30) can be absorbed into $h^N\|u\|_{H^{s_0}_\hbar}$. On the other hand, since $\mathrm{WF}'_\hbar((I-\tilde{A}_+)[\tilde{P}_0,\tilde{B}_3])$ is disjoint from $\tilde{\Gamma}_+$, the backward bicharacteristics from it reach the elliptic set of $\tilde{B}_2$, and so we have the microlocal real principal type estimate for u:

$$\|(I-\tilde{A}_+)[\tilde{P}_0,\tilde{B}_3]u\|_{H^{s-m}_\hbar} \lesssim h\|\tilde{B}_2 u\|_{H^{s-1}_\hbar} + \|\tilde{B}_1\tilde{P}_0 u\|_{H^{s-m}_\hbar}$$

as $(I-\tilde{A}_+)[\tilde{P}_0,\tilde{B}_3] \in h\Psi_\hbar^{m-1}(X)$, so by (4.28),

$$\|(\tilde{P}_0 - i\tilde{Q}_0)^{-1}(I-\tilde{A}_+)[\tilde{P}_0,\tilde{B}_3]u\|_{H^s_\hbar} \lesssim h^{-1}\|\tilde{B}_2 u\|_{H^{s-1}_\hbar} + h^{-2}\|\tilde{B}_1\tilde{P}_0 u\|_{H^{s-m}_\hbar}.$$

Thus, (4.29) follows if we can estimate $\|\tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u\|_{H^s_\hbar}$. Now, $\mathrm{WF}'_\hbar(\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]) \cap \tilde{\Gamma}_- = \emptyset$ by arrangement. In order to microlocalize, we now introduce a nontrapping model, $\tilde{P}_0 - i(\tilde{Q}_0 + \tilde{Q}_1)$. We claim that

$$v = \big(\tilde{P}_0 - i(\tilde{Q}_0+\tilde{Q}_1)\big)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u - (\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u$$

satisfies

$$\|v\|_{H^{s'}_\hbar} \lesssim h^N\|u\|_{H^{s_0}_\hbar} \tag{4.31}$$

for all s', N. Notice that for any s'' one certainly has

$$\|v\|_{H^{s''}_\hbar} \lesssim h^{-1}\|u\|_{H^{s''-1}_\hbar}$$

by (4.28) plus its non-trapping analogue. To see (4.31), notice that

$$(\tilde{P}_0 - i\tilde{Q}_0)v = i\tilde{Q}_1\big(\tilde{P}_0 - i(\tilde{Q}_0+\tilde{Q}_1)\big)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u,$$

so by (4.28), with s_0 replaced by any s_0' (since s_0 was arbitrary), and for any N,

$$\|v\|_{H^{s'}_\hbar} \lesssim h^{-2}\|\tilde{Q}_1\big(\tilde{P}_0 - i(\tilde{Q}_0+\tilde{Q}_1)\big)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u\|_{H^{s'-m}_\hbar} \lesssim h^N\|u\|_{H^{s_0}_\hbar},$$

since $\tilde{P}_0 - i(\tilde{Q}_0+\tilde{Q}_1)$ is non-trapping, hence $\big(\tilde{P}_0 - i(\tilde{Q}_0+\tilde{Q}_1)\big)^{-1}$ propagates semiclassical wave front sets along forward bicharacteristics, and no backward bicharacteristic from $\mathrm{WF}'_\hbar(\tilde{Q}_1)$ can reach $\mathrm{WF}'_\hbar(\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]) \subset \mathrm{WF}'_\hbar(\tilde{A}_+) \cap \mathrm{WF}'_\hbar(I-\tilde{B}_3)$, proving the claim. Then, since backward bicharacteristics from $\mathrm{WF}'_\hbar(\tilde{B}_0)$ do not encounter $\mathrm{WF}'_\hbar(\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]) \cap \tilde{\Gamma}_+$ before reaching the elliptic set of $\tilde{Q}_0$ or $\tilde{Q}_1$, we conclude that

$$\begin{aligned}&\|\tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u\|_{H^s_\hbar} \\ &\qquad \le \|\tilde{B}_0(\tilde{P}_0 - i\tilde{Q}_0 - i\tilde{Q}_1)^{-1}\tilde{A}_+[\tilde{P}_0,\tilde{B}_3]u\|_{H^s_\hbar} + \|\tilde{B}_0 v\|_{H^s_\hbar} \\ &\qquad \lesssim h\|\tilde{B}_2 u\|_{H^s_\hbar} + \|\tilde{B}_1\tilde{P}_0 u\|_{H^{s-m+1}_\hbar} + h^N\|u\|_{H^{s_0}_\hbar}.\end{aligned}$$

This proves (4.29), and thus the theorem. □

5. Quasilinear wave and Klein-Gordon equations

5.1. **Forward solution operators.** We now generalize the setting considered in [29, §7.2] for the study of quasilinear equations on static asymptotically de Sitter spaces to allow for normally hyperbolic trapping, as discussed in the previous section.

Thus, working on a compact manifold M with boundary X, we assume that the operator P is of the form $P = P_0 + \tilde{P}$, continuously depending on a small parameter $v = v_0 + \tilde{v} \in \mathcal{X}^{\tilde{s},\alpha}$, where

$$\begin{aligned} P_0 = P_0(v_0) &= \Box_{g(v_0)} + L(v_0) \in \mathrm{Diff}_{\mathrm{b}}^2(M), \\ L(v_0) &\in \mathrm{Diff}_{\mathrm{b}}^1(M), \qquad L(0) - L(0)^* \in \mathrm{Diff}_{\mathrm{b}}^0(M), \\ \tilde{P} = \tilde{P}(v) &\in H_{\mathrm{b}}^{\tilde{s},\alpha}\mathrm{Diff}_{\mathrm{b}}^2(M) \end{aligned} \tag{5.1}$$

for a smooth b-metric g on M that continuously depends on one real parameter; here, $\alpha > 0$.[16] We assume:

(1) The characteristic set $\Sigma \subset {}^{\mathrm{b}}S^*_X M$ of P_0 has the form $\Sigma = \Sigma_+ \cup \Sigma_-$ with $\Sigma_\pm$ a union of connected components of Σ,
(2) P_0 has normally hyperbolic trapping at $\Gamma^\pm \subset \Sigma_\pm$ for small v_0, as detailed in assumptions (1)-(10) in Section 4.2,
(3) P_0 has radial sets $L_\pm \subset {}^{\mathrm{b}}S^*_X M$, which, in appropriate directions transverse to $L_\pm$, are sources $(-)$/sinks $(+)$ for the null-bicharacteristic flow within ${}^{\mathrm{b}}S^*_X M$, with a one-dimensional stable/unstable manifold intersecting ${}^{\mathrm{b}}S^*_X M$ transversally; for details, see [29, §6.4]. In particular, there are $\beta_0, \tilde{\beta} \in \mathcal{C}^\infty(L_\pm)$, $\beta_0, \tilde{\beta} > 0$, such that for a homogeneous degree -1 boundary defining function ρ of fiber infinity in ${}^{\mathrm{b}}\overline{T}^*M$ and with $V = \rho H_{p_0}$,

$$\rho^{-1} V\rho|_{L_\pm} = \mp\beta_0, \qquad -x^{-1}Vx|_{L_\pm} = \mp\tilde{\beta}\beta_0. \tag{5.2}$$

We will set up initial value problems by introducing artificial boundaries as in [29, 31]: We denote by $\mathfrak{t}_1$ and $\mathfrak{t}_2$ two smooth functions on M and put

$$\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty)),$$

where we assume that:

(4) Ω is compact,
(5) putting $H_j := \mathfrak{t}_j^{-1}(0)$, the H_j intersect the boundary ∂M transversally, and H_1 and H_2 intersect only in the interior of M, and they do so transversally,
(6) the differentials of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have opposite timelike characters near their respective zero sets within Ω; more specifically, $\mathfrak{t}_1$ is future timelike, $\mathfrak{t}_2$ past timelike,
(7) there is a boundary defining function x of M such that dx/x is timelike on $\Omega \cap \partial M$ with timelike character opposite to the one of $\mathfrak{t}_1$, i.e. dx/x is past oriented,
(8) the metric g is *non-trapping* in the following sense: All bicharacteristics in $\Sigma_\Omega := \Sigma \cap {}^{\mathrm{b}}S^*_\Omega M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus (L_+ \cup \Gamma^+))$ flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_+ \cup \Gamma^+$ in the forward direction (i.e. either enter ${}^{\mathrm{b}}S^*_{H_1}M$ in finite time or tend to the radial set L_+ or the trapped set Γ^+) and

[16] An example to keep in mind for the remainder of the section is the wave operator on a perturbed (asymptotically) Kerr-de Sitter space, where the metric of the (asymptotically) Kerr-de Sitter space is perturbed in $H_{\mathrm{b}}^{\tilde{s}+1,\alpha}$.

to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_+ \cup \Gamma^+$ in the backward direction, and from any point in $\Sigma_\Omega \cap (\Sigma_- \setminus (L_- \cup \Gamma^-))$ to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_- \cup \Gamma^-$ in the forward direction and to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_- \cup \Gamma^-$ in the backward direction, with tending to $\Gamma^\pm$ allowed in only one of the two directions.

Recall the space $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ of distributions which are supported ($\bullet$) at the 'artificial' boundary hypersurface H_1 and extendible ($-$) at H_2, and the other way around for $H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet}$. Then we have the following global energy estimates:

Lemma 5.1. *(Cf.* [29, Lemma 7.3]*.) Suppose $\tilde{s} > n/2+2$. There exists $r_0 < 0$ such that for $r \le r_0$, $-\tilde{r} \le r_0$, there is $C > 0$ such that for $u \in H_{\mathrm{b}}^{2,r}(\Omega)^{\bullet,-}$, $v \in H_{\mathrm{b}}^{2,\tilde{r}}(\Omega)^{-,\bullet}$, one has*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega)^{\bullet,-}} \le C\|Pu\|_{H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}},$$
$$\|v\|_{H_{\mathrm{b}}^{1,\tilde{r}}(\Omega)^{-,\bullet}} \le C\|P^*v\|_{H_{\mathrm{b}}^{0,\tilde{r}}(\Omega)^{-,\bullet}}.$$

Proof. This result does not rely on the dynamical structure of P at the boundary, but only on the timelike nature of dx/x and of $d\mathfrak{t}_1$ and $d\mathfrak{t}_2$ near H_1 and H_2, respectively, see also [29, Remark 7.4]. ☐

Let us stress that we assume the parameter v to be *small* so that in particular the skew-adjoint part of $P_0(v_0)$ is small and does not affect the radial point and normally hyperbolic trapping estimates which are used in what follows; the general case without symmetry assumptions on $P_0(0)$ will be discussed in Section 5.4. Using a duality argument and the tame estimates for elliptic regularity and the propagation of singularities (real principal type, radial points, normally hyperbolic trapping) given in Propositions 4.1, 4.2 and 4.3 and Theorem 4.4, we thus obtain solvability and higher regularity:

Lemma 5.2. *(Cf.* [29, Lemma 7.5]*.) Let $0 \le s \le \tilde{s}$ and assume $\tilde{s} > n/2+6$, $s_0 > n/2+1/2$. There exists $r_0 < 0$ such that for $r \le r_0$, there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ such that $Pu = f$, and u moreover satisfies*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \lesssim \|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}^{\tilde{s},\alpha}}. \tag{5.3}$$

Here, the implicit constant depends only on s and $\|v\|_{\mathcal{X}^{n/2+6+\epsilon,\alpha}}$ for $\epsilon > 0$.

Proof. The proof proceeds as the proof given in the reference. The tame estimate (5.3) in particular is obtained by iterative use of the aforementioned microlocal regularity estimates; the given bound for s_0 comes from an inspection of the norms in these estimates which correspond to the terms called u_*^ℓ in (4.1). ☐

We deduce analogues of [29, Corollaries 7.6-7.7]:

Corollary 5.3. *Let $0 \le s \le \tilde{s}$ and assume $\tilde{s} > n/2+6$, $s_0 > n/2+1/2$. There exists $r_0 < 0$ such that for $r \le r_0$, there is $C > 0$ with the following property: If $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ is such that $Pu \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then the estimate (5.3) holds.*

Corollary 5.4. *Let $s_0 > n/2+1/2$, $s_0 \le s' \le s \le \tilde{s}$, and assume $\tilde{s} > n/2+6$; moreover, let $r < 0$. Then there is $C > 0$ such that the following holds: Any $u \in H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}$ with $Pu \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$ in fact satisfies $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$, and obeys the estimate*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \lesssim \|Pu\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s',r}(\Omega)^{\bullet,-}}$$

$$+ (\|Pu\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s_0+1,r}(\Omega)^{\bullet,-}})\|v\|_{\mathcal{X}^{\tilde{s},\alpha}}.$$

Proof. The proof of the two corollaries is as in the cited reference. For the radial point estimate involved in the proof of Corollary 5.4, we need the additional assumption $s' - 1 + \sup_{L_\pm}(r\tilde{\beta}) > 0$, which however is automatically satisfied since $s' \geq 1$ and the sup is negative for $r < 0$. □

We now note that the Mellin transformed normal operator $\hat{N}(P)(\sigma)$ satisfies global large parameter estimates corresponding to the semiclassical microlocal estimates of Theorem 4.5. In order to state this precisely we recall the connection between the b-structure, the normal operator (and the large parameter algebra) and the Mellin transform of the latter.

The weighted b-Sobolev spaces $H_{\mathrm{b}}^{s,\gamma}([0,\infty) \times X)$ are isometric to the large parameter Sobolev spaces on X on the line $\operatorname{Im}\sigma = -\gamma$ in $\mathbb{C}$ via the Mellin transform $\mathcal{M}$; see [47, Equation (3.8)]. Further, the latter can be described in terms of semiclassical Sobolev spaces, namely the restriction $r_{-\gamma} \circ \mathcal{M}$ to $\operatorname{Im}\sigma = -\gamma$ of the Mellin transform identifies $H_{\mathrm{b}}^{s,\gamma}([0,\infty) \times X)$ with

$$\langle|\sigma|\rangle^{-s} L^2(\mathbb{R}; H^s_{\langle|\sigma|\rangle^{-1}}(X));$$

see [47, Equation (3.9)]. Now, in order to relate b-microlocal analysis with semiclassical analysis, we first identify $\varpi + \sigma\frac{dx}{x} \in {}^{\mathrm{b}}T^*([0,\infty) \times X)$, $\varpi \in T^*X$, with $(\sigma, \varpi) \in \mathbb{R} \times T^*X$. Under the semiclassical rescaling, say by $|\sigma|^{-1}$, one identifies the latter with $h = |\sigma|^{-1}$, $\tilde{\varpi} = |\sigma|^{-1}\varpi$. In particular, if a conic set is disjoint from T^*X in ${}^{\mathrm{b}}T^*([0,\infty) \times X)$, then its image under the semiclassical identification lies in a compact subset of T^*X. Thus, for $B \in \Psi_{\mathrm{b}}^0([0,\infty) \times X)$ dilation invariant, the large parameter principal symbol and wave front set of the Mellin conjugate $\mathcal{M}B\mathcal{M}^{-1} = \hat{B}$ of B are exactly those of B under the above identification of $\varpi + \sigma\frac{dx}{x} \in {}^{\mathrm{b}}T^*([0,\infty) \times X)$, $\varpi \in T^*X$, with $(\sigma, \varpi) \in \mathbb{R} \times T^*X$, and then the analogous statement also holds for $\hat{B}$ considered as an element $\tilde{B}$ of $\Psi_\hbar(X)$ under the semiclassical identification. In particular, one has, for $B \in \Psi_{\mathrm{b}}^0([0,\infty) \times X)$ dilation invariant, with $\mathrm{WF}'_{\mathrm{b}}(B) \cap T^*X = \emptyset$, that $\tilde{B} \in \Psi^{-\infty}_{|\sigma|^{-1}}(X)$, with semiclassical wave front set in a compact subset of T^*X. Correspondingly, for any s_0,

$$\|Bu\|^2_{H_{\mathrm{b}}^{s,\gamma}} \lesssim \int_{\sigma\in\mathbb{R},\ |\sigma|>h_0^{-1}} |\sigma|^{2s}\|\hat{B}\mathcal{M}u(.-i\gamma)\|^2_{L^2}\, d\sigma + \|u\|^2_{H_{\mathrm{b}}^{s_0,\gamma}}.$$

Now if $P_0 = P_0(v_0) \in \Psi_{\mathrm{b}}^m(M)$, then $N(P_0)$ is dilation invariant on $[0,\infty)\times X$, and its conjugate by the Mellin transform is $\hat{P}_0 = \hat{N}(P_0)$, whose rescaling $\tilde{P}_0 = |\sigma|^{-m}\hat{P}_0$ is an element of $\Psi_\hbar^m(X)$. Further, with P_0 b-normally hyperbolic in the sense discussed above (with the convention changed regarding formal self-adjointness, as stated before Theorem 4.5), $\tilde{P}_0$ is normally hyperbolic in the sense of Dyatlov. Fix a smooth b-density on M near X, identified with $[0,\epsilon_0) \times X$ as above; we require this to be of the product form $\frac{|dx|}{x}\nu$, ν a smooth density on X; we compute adjoints with respect to this density. Then for any $B \in \Psi_{\mathrm{b}}^m(M)$, $\widehat{B^*}(\sigma) = (\hat{B}(\overline{\sigma}))^*$, see [47, Section 3.3] for differential operators, and by a straightforward calculation using the Mellin transform in general. In particular, if $B = B^*$, then $\hat{B}(\sigma) = (\hat{B}(\sigma))^*$ for $\sigma \in \mathbb{R}$. Relaxing (5.1) momentarily, we then assume that

$$\frac{1}{2i}(P_0 - P_0^*) \in \Psi_{\mathrm{b}}^{m-1}(M), \qquad \sigma_{\mathrm{b},m-1}\Big(\frac{1}{2i}(P_0 - P_0^*)\Big)\Big|_\Gamma < |\sigma|^{m-1}\nu_{\min}/2, \qquad (5.4)$$

with $\nu_{\min}$ the minimal normal expansion rate for the Hamilton flow of the principal symbol of P_0 at $\Gamma \subset {}^{\mathrm{b}}T^*_X M$, as above; note that σ is elliptic on Γ. This gives that for $\sigma \in \mathbb{R}$, $\hat{P}_0(\sigma) - \hat{P}_0(\sigma)^*$ is order $m-1$ in the large parameter pseudodifferential algebra, so, defining $z = \sigma/|\sigma|$, the semiclassical version gives

$$\tilde{P}_0 - \tilde{P}_0^* \in h\Psi_\hbar^{m-1}(X),\ z \in \mathbb{R},$$

with

$$\sigma_{\hbar,m-1}\Big(\frac{1}{2ih}(\tilde{P}_0 - \tilde{P}_0^*)\Big)\Big|_{\tilde{\Gamma}} < \nu_{\min}/2,\ z \in \mathbb{R},$$

where $\tilde{\Gamma}$ is the image of Γ under the semiclassical identification. In particular, there is $\gamma_\Gamma > 0$ and $\beta_\Gamma < \nu_{\min}/2$ such that if $|\operatorname{Im} z| < h\gamma_\Gamma$ then

$$\sigma_{\hbar,m-1}\Big(\frac{1}{2ih}(\tilde{P}_0 - \tilde{P}_0^*)\Big)\Big|_{\tilde{\Gamma}} < \beta_\Gamma. \tag{5.5}$$

With this background, under our assumptions on the dynamics, propagating estimates from the radial points towards H_2, in particular through $\tilde{\Gamma}$, and using the uniformity in parameters described above Theorem 4.5, we have:

Theorem 5.5. *Let $C_0 > 0$. Suppose $P_0 = P_0(v_0)$ satisfies (5.4) at Γ, $\tilde{P}_0$ is the semiclassical rescaling of $\hat{P}_0 = \hat{N}(P_0)$, $s > 1/2 + \sup(\tilde{\beta})\gamma$, $s > 1$, $\gamma < \gamma_\Gamma$, $\gamma_\Gamma > 0$ as in (5.5). Then there is $h_0 > 0$ such that for $h < h_0$, $|\operatorname{Im} z| < h\gamma$,*

$$\|u\|_{H_\hbar^s} \lesssim h^{-2}\|\tilde{P}_0 u\|_{H_\hbar^{s-m+1}}, \tag{5.6}$$

with the implied constant and h_0 uniform in v_0 with $|v_0| \leq C_0$.

Proof. This is immediate from piecing together the semiclassical propagation estimates from radial points (which is where $s > 1/2 + \sup(\tilde{\beta})\gamma$ is used, see the corresponding statement in the b-setting given in [31, Proposition 2.1, Footnote 20]) through $\tilde{\Gamma}$, using Theorem 4.5, which is where $\gamma < \gamma_\Gamma$ is used and where h^{-2}, rather than h^{-1}, is obtained for the right hand side, to $H_2 \cap X$, which is where $s > 1$ is used.

An alternative proof would be using Dyatlov's setting [23] directly, together with the gluing of Datchev and Vasy [17], exactly as described in [47, Theorem 2.17]. □

Going back to the operator $P_0(v_0)$ satisfying the conditions stated at the beginning of this section, and under the additional assumption of uniform normal hyperbolicity as explained above, we can now obtain partial expansions of solutions to $Pu = f$ at infinity, i.e. at X:

Theorem 5.6. *(Cf. [29, Theorem 7.9].) Let $0 < \alpha < \min(1, \gamma_\Gamma)$. Suppose P has a simple rank 1 resonance at 0 with resonant state 1, and that all other resonances have imaginary part less than $-\alpha$. Let $\tilde{s} > n/2 + 6$, $s_0 > \max(n/2 + 1/2, 1 + \sup(\tilde{\beta})\alpha)$,*[17] *and assume $s_0 \leq s \leq \tilde{s} - 4$. Let $0 \neq r \leq \alpha$. Then any solution $u \in H_{\mathrm{b}}^{s+4,r_0}(\Omega)^{\bullet,-}$ of $Pu = f$ with $f \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ satisfies $u \in \mathcal{X}^{s',r}$ with $s' = s+4$ for $r < 0$ and $s' = s$ for $r > 0$, and the following tame estimate holds:*

$$\begin{aligned}\|u\|_{\mathcal{X}^{s',r}} \lesssim \|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} &+ \|u\|_{H_{\mathrm{b}}^{s+4,r_0}(\Omega)^{\bullet,-}} \\ &+ (\|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}} + \|u\|_{H_{\mathrm{b}}^{s_0+1,r_0}(\Omega)^{\bullet,-}})\|v\|_{\mathcal{X}^{\tilde{s},\alpha}}.\end{aligned}$$

[17] In particular, if we merely assume $s_0 > n/2 + 1/2$, then the full condition on s_0 holds if we choose $\alpha > 0$ sufficiently small.

Proof. The proof works in the same way as in the reference by an iterative argument that consists of rewriting $Pu = f$ as $N(P)u = f - (P - N(P))u$ and employing a contour deformation argument, see [47, Lemma 3.1] (which uses high-energy estimates for the inverse normal operator family $\widehat{P}(\sigma)^{-1}$ and the location of resonances, i.e. of the poles of this family), to improve on the decay of u by α in each step, but losing an order of differentiability as we are treating $P - N(P)$ as an error term; using tame microlocal regularity for the equation $Pu = f$, Corollary 5.4, one can regain this loss. We obtain $u \in H_{\mathrm{b}}^{s+1,r}$ after a finite number of iterations in case $r < 0$,[18] and $u \in H_{\mathrm{b}}^{s+4,r_0}$ for all $r_0 < 0$ in case $r > 0$.

Assuming we are in the latter case, the next step of the iteration gives a partial expansion $u = c + u'$ with $c \in \mathbb{C}$ (identified, as before, with $c\chi$, where χ is a smooth cutoff near the boundary) and $u' \in H_{\mathrm{b}}^{s+2,r'}$ for any r' satisfying $r' \leq r$ and $r' < \alpha$; here, we need $0 < \alpha < \gamma_\Gamma$ so that the normally hyperbolic trapping estimate (5.6) holds with $\gamma > \alpha$, with loss of two derivatives. If $r = \alpha$, we can use this information to deduce

$$N(P)u = f - (P - N(P))u = f - \tilde{f}, \quad \tilde{f} \in H_{\mathrm{b}}^{\tilde{s},\alpha} + H_{\mathrm{b}}^{s,r'+\alpha} \subset H_{\mathrm{b}}^{s,r},$$

which implies that the expansion $u = c + u'$ in fact holds with the membership $u' \in H_{\mathrm{b}}^{s,r}$; notice the improvement in the weight. Therefore, $u \in \mathcal{X}^{s,r}$, finishing the proof. □

Pipelining this result with the existence of solutions, Lemma 5.2, we therefore obtain:

Theorem 5.7. *Under the assumptions of Theorem 5.6 with $r > 0$ and $s > n/2+2$, define the space*

$$\mathcal{Y}^{s,r} = \{u \in \mathcal{X}^{s,r} \colon Pu \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}\}.$$

Then the operator $P \colon \mathcal{Y}^{s,r} \to H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ has a continuous inverse S that satisfies the tame estimate

$$\|Sf\|_{\mathcal{X}^{s,r}} \leq C(s, \|v\|_{\mathcal{X}^{s_0,\alpha}})(\|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}^{s+4,\alpha}}). \tag{5.7}$$

5.2. **Solving quasilinear wave equations.** We continue to work in the setting of the previous section. With the tame forward solution operator constructed in Theorem 5.7 in our hands, we are now in a position to use a Nash-Moser implicit function theorem to solve quasilinear wave equations. We use the following simple form of Nash-Moser, given in [42]:

Theorem 5.8. *Let $(B^s, |\cdot|_s)$ and $(\mathbf{B}^s, \|\cdot\|_s)$ be Banach spaces for $s \geq 0$ with $B^s \subset B^t$ and indeed $|v|_t \leq |v|_s$ for $s \geq t$, likewise for $\mathbf{B}^*$ and $\|\cdot\|_*$; put $B^\infty = \bigcap_s B^s$ and similarly $\mathbf{B}^\infty = \bigcap_s \mathbf{B}^s$. Assume there are smoothing operators $(S_\theta)_{\theta>1} \colon B^\infty \to B^\infty$ satisfying for every $v \in B^\infty$, $\theta > 1$ and $s, t \geq 0$:*

$$\begin{aligned} |S_\theta v|_s &\leq C_{s,t}\theta^{s-t}|v|_t \text{ if } s \geq t, \\ |v - S_\theta v|_s &\leq C_{s,t}\theta^{s-t}|v|_t \text{ if } s \leq t. \end{aligned} \tag{5.8}$$

[18] In particular, this holds under the weaker conditions $s + 1 \leq \tilde{s}$, $\alpha \leq 1$.

Let $\phi\colon B^\infty \to \mathbf{B}^\infty$ be a C^2 map, and assume that there exist $u_0 \in B^\infty$, $d \in \mathbb{N}$, $\delta > 0$ and constants C_1, C_2 and $(C_s)_{s\geq d}$ such that for any $u, v, w \in B^\infty$,

$$|u-u_0|_{3d} < \delta \Rightarrow \begin{cases} \forall s \geq d, \quad \|\phi(u)\|_s \leq C_s(1+|u|_{s+d}), \\ \|\phi'(u)v\|_{2d} \leq C_1|v|_{3d}, \\ \|\phi''(u)(v,w)\|_{2d} \leq C_2|v|_{3d}|w|_{3d}. \end{cases} \tag{5.9}$$

Moreover, assume that for every $u \in B^\infty$ with $|u-u_0|_{3d} < \delta$ there exists an operator $\psi(u)\colon \mathbf{B}^\infty \to B^\infty$ satisfying

$$\phi'(u)\psi(u)h = h$$

and the tame estimate

$$|\psi(u)h|_s \leq C_s(\|h\|_{s+d} + |u|_{s+d}\|h\|_{2d}), \quad s \geq d, \tag{5.10}$$

for all $h \in \mathbf{B}^\infty$. Then if $\|\phi(u_0)\|_{2d}$ is sufficiently small depending on $\delta, |u_0|_D$ and $(C_s)_{s\leq D}$, where $D = 16d^2 + 43d + 24$, there exists $u \in B^\infty$ such that $\phi(u) = 0$.

To apply this in our setting, we let $B^s = \mathcal{X}^{s,\alpha}(\Omega) = \mathbb{C} \oplus H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ and $\mathbf{B}^s = H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ with the corresponding norms; $\phi(u)$ will be the quasilinear equation, with implicit dependence on the forcing term. We now construct the smoothing operators S_θ; we may assume, using a partition of unity, that Ω is the closure of an open subset of $\overline{\mathbb{R}^n_+}$, say $\Omega = \Omega(1)$, where we let $\Omega(x_0) = \{x \leq x_0, |y| \leq 1\}$. Then there are bounded extension and restriction operators

$$E\colon H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+}), \quad R\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+}) \to H_{\mathrm{b}}^{s,\alpha}(\Omega)^{-,-},$$

for $s \geq 0$; the operator E can be constructed such that $\operatorname{supp} Ev \subset \{x \leq 1\}$ for $v \in H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$. If we then define for $\theta > 1$ and $v = (c,u) \in \mathcal{X}^{s,\alpha}$:

$$S_\theta^1 v = (c, RS'_\theta Ev),$$

where S'_θ is a smoothing operator on $\overline{\mathbb{R}^n_+}$ with properties as in (5.8), then S_θ^1 satisfies (5.8) in view of RE being the identity on $H_{\mathrm{b}}^{s,\alpha}(\Omega)^{\bullet,-}$ if the norms on the left hand side are understood to be $H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n_+})$-norms. However, note that S_θ^1 does not map $\mathcal{X}^{\infty,\alpha}$ into itself, since smoothing operators such as S'_θ enlarge supports; we will thus need to modify S_θ^1 below to obtain the operators S_θ. In order to construct S'_θ on weighted b-Sobolev spaces $H_{\mathrm{b}}^{s,\alpha}$, it suffices by conjugation by the weight to construct it on the unweighted spaces H_{b}^s; then, by a logarithmic change of coordinates, we only need to construct the smoothing operator $\tilde{S}_\theta$ on the standard Sobolev spaces $H^s(\mathbb{R}^n)$, which we will do in Lemma 5.9 below. In order to deal with the issue of S_θ^1 enlarging supports, we will define $\tilde{S}_\theta$ such that

$$v \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n_{x',y'}), \ \operatorname{supp} v \subset \{x' \leq 0\} \ \Rightarrow \ \operatorname{supp} \tilde{S}_\theta v \subset \{x' \leq \theta^{-1/2}\}.$$

In particular, when one undoes the logarithmic change of coordinates, this implies

$$S_\theta^1\colon \mathcal{X}^{s,\alpha}(\Omega(1)) \to \mathcal{X}^{s,\alpha}\left(\Omega\big(\exp(\theta^{-1/2})\big)\right);$$

more generally, with D_λ denoting dilations $D_\lambda(x,y) = (\lambda x, y)$ on $\overline{\mathbb{R}^n_+}$, we have

$$S_\theta^\lambda := (D_\lambda^{-1})^* S_\theta^1 (D_\lambda)^*\colon \mathcal{X}^{s,\alpha}(\Omega(\lambda)) \to \mathcal{X}^{s,\alpha}\left(\Omega\big(\lambda\exp(\theta^{-1/2})\big)\right), \quad \lambda > 0, \tag{5.11}$$

with the operator norm independent of λ near 1. Now, in our application of Theorem 5.8, we will have

$$\phi\colon \mathcal{X}^{\infty,\alpha}(\Omega(x_0)) \to H_{\mathrm{b}}^{\infty,\alpha}(\Omega(x_0))^{\bullet,-} \text{ for all } x_0 \text{ near } 1,$$

and correspondingly we will have forward solution operators ψ going in the reverse direction, with all relevant constants being uniform in x_0. Looking at the proof of Theorem 5.8 in [42], one only uses the smoothing operator S_{θ_k} with $\theta_k = \theta_0^{(5/4)^k}$ in the k-th step of the iteration, with θ_0 chosen sufficiently large; in our situation, where we have (5.11), we can therefore use the smoothing operator

$$S_{\theta_k} := S_{\theta_k}^{\lambda_k}, \quad \lambda_k = \exp\left(\sum_{j=0}^{k-1} \theta_j^{-1/2}\right)$$

in the k-th iteration step. Note that, for θ_0 large, we have

$$1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\infty = \exp\left(\sum_{j=0}^{\infty} \theta_j^{-1/2}\right) \leq 1 + 2\theta_0^{-1/2}.$$

The solution u to $\phi(u) = 0$, obtained as a limit of an iterative scheme (see [42, Lemma 1]), therefore is an element of $\mathcal{X}^{s,\alpha}(\Omega(\lambda_\infty))$. Taking the hyperbolic nature of the PDE $\phi(u) = 0$ into account once more, it will then, in our concrete setting, be easy to conclude that in fact $u \in \mathcal{X}^{s,\alpha}(\Omega)$.

We now construct the smoothing operators on $\mathbb{R}^n$; the first step of the argument follows the Appendix of [42].

Lemma 5.9. *There is a family $(\tilde{S}_\theta)_{\theta>1}$ of operators on $H^\infty(\mathbb{R}^n)$ satisfying*

$$\|\tilde{S}_\theta v\|_s \leq C_{s,t}\theta^{s-t}\|v\|_t \text{ if } s \geq t \geq 0, \tag{5.12}$$

$$\|v - \tilde{S}_\theta v\|_s \leq C_{s,t}\theta^{s-t}\|v\|_t \text{ if } 0 \leq s \leq t, \tag{5.13}$$

$$\operatorname{supp} \tilde{S}_\theta v \subseteq \{x_1 \leq \theta^{-1/2}\} \tag{5.14}$$

for all $v \in H^\infty(\mathbb{R}^n)$ with $\operatorname{supp} v \subseteq H := \{x_1 \leq 0\}$. Here $\|\cdot\|_s$ denotes the $H^s(\mathbb{R}^n)$-norm, and we write $x = (x_1, x') \in \mathbb{R}^n$.

Proof. Choose $\chi = \chi_1(x_1)\chi_2(x') \in S(\mathbb{R}^n)$ with $\chi_1 \in S(\mathbb{R}), \chi_2 \in S(\mathbb{R}^{n-1})$ so that the Fourier transform $\hat{\chi}$ is identically 1 near 0; put $\chi_\theta(z) = \theta^n\chi(\theta z)$ and define the operator $C_\theta v = \chi_\theta * v$. Then $(C_\theta v)\hat{} = \widehat{\chi_\theta}\hat{v}$ with $\widehat{\chi_\theta}(\xi) = \hat{\chi}(\xi/\theta)$, therefore (5.12) holds for C_θ in place of $\tilde{S}_\theta$ with constants $C'_{s,t}$ since $\hat{\chi}$ decays super-polynomially, and (5.13) holds for C_θ in place of $\tilde{S}_\theta$ with constants $C'_{s,t}$ since $1 - \hat{\chi}(\xi)$ vanishes at $\xi = 0$ with all derivatives.

Next, let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a smooth function depending only on x_1, i.e. $\psi = \psi(x_1)$, so that $\psi(x_1) \equiv 1$ for $x_1 \in (-\infty, 1/2]$, $\psi(x_1) \equiv 0$ for $x_1 \in [1, \infty)$, and $0 \leq \psi \leq 1$. Put $\psi_\theta(x_1, x') = \psi(\theta x_1, x')$, and define

$$\tilde{S}_\theta v := \psi_{\theta^{1/2}} C_\theta v.$$

Condition (5.14) is satisfied by the support assumption on ψ. Let $\varphi = 1 - \psi$ and $\varphi_\theta = 1 - \psi_\theta$. To prove the other two conditions, we use the estimate

$$\|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C''_{s,N}\theta^{-N}\|v\|_{L^2}, \quad \operatorname{supp} v \subset H, \ s, N \geq 0, \tag{5.15}$$

which we will establish below. Taking this for granted, we obtain for v with $\operatorname{supp} v \subset H$:

$$\|\tilde{S}_\theta v\|_s \leq \|C_\theta v\|_s + \|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C'_{s,t}\theta^{s-t}\|v\|_t + C''_{s,0}\|v\|_0$$

for $s \geq t \geq 0$, which is the estimate (5.12); and (5.13) follows from

$$\|v - \tilde{S}_\theta v\|_s \leq \|v - C_\theta v\|_s + \|\varphi_{\theta^{1/2}} C_\theta v\|_s \leq C'_{s,t}\theta^{s-t}\|v\|_t + C''_{s,t-s}\theta^{s-t}\|v\|_0$$

for $0 \leq s \leq t$.

We now prove (5.15) for $s \in \mathbb{N}_0$. For multiindices $\alpha = (\alpha_1, \alpha')$ with $|\alpha| \leq s$, we have for v with $\operatorname{supp} v \subset H$ and for $(x_1, x') \in \operatorname{supp} \varphi_{\theta^{1/2}} C_\theta v$, which in particular implies $x_1 \geq 1/(2\theta^{1/2})$:

$$\partial^\alpha(\varphi_{\theta^{1/2}} C_\theta v)(x_1, x') = \sum_{j=0}^{\alpha_1} \binom{\alpha_1}{j} \theta^{(\alpha_1 - j)/2} \varphi^{(\alpha_1 - j)}(\theta^{1/2} x_1)$$
$$\times \iint_{y_1 \geq 1/(2\theta^{1/2})} \theta^{n+j+|\alpha'|} \chi_1^{(j)}(\theta y_1) \chi_2^{(\alpha')}(\theta y') v(x_1 - y_1, x' - y')\, dy_1\, dy',$$

thus

$$\|\partial^\alpha(\varphi_{\theta^{1/2}} C_\theta v)\|_{L^2} \leq C_s \theta^{n+s} \|\check\chi_\theta\|_{L^1} \|v\|_{L^2},$$

where

$$\check\chi_\theta(x_1, x') = \begin{cases} 0, & x_1 < 1/(2\theta^{1/2}), \\ \sum_{j=0}^{\alpha_1} |\chi_1^{(j)}(\theta x_1) \chi_2^{(\alpha')}(\theta x')| & \text{otherwise.} \end{cases}$$

But $\|\check\chi_\theta\|_{L^1} \leq C_{N,s} \theta^{-N}$ for all N: Indeed, this reduces to the statement that for a fixed $\chi_0 \in S(\mathbb{R})$, one has

$$\int_{1/(2\theta^{1/2})}^\infty |\chi_0(\theta x)|\, dx \leq C_N \int_{\theta^{-1/2}}^\infty (\theta x)^{-2N+1}\, dx = C'_N \theta^{-N}.$$

Hence, we obtain (5.15), and the proof is complete. □

We now combine Theorem 5.7, giving the existence of tame forward solution operators, with Theorem 5.8, in the extended form described above, to solve quasilinear wave equations. We use the space $\mathcal{X}^{s,\alpha}_{\mathbb{R}}$ of real-valued elements of $\mathcal{X}^{s,\alpha}$.

Theorem 5.10. *Let $N \in \mathbb{N}$ and $c_k \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)(M; \mathrm{Sym}^2\, {}^{\mathrm{b}}TM)$ for $1 \leq k \leq N$; define the map $g\colon \mathcal{X}^{s,\alpha} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})(M; \mathrm{Sym}^2\, {}^{\mathrm{b}}TM)$ by $g(u) = \sum_{k=1}^N c_k(u) g_k$ and assume that $\Box_{g(0)}$ satisfies the assumptions of Section 5.1 and of Theorem 5.7. Moreover, let $N' \in \mathbb{N}$ and define*

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} u^{e_j} \prod_{k=1}^{N_j} X_{jk} u, \tag{5.16}$$

where

$$e_j, N_j \in \mathbb{N}_0, N_j \geq 1, N_j + e_j \geq 2, X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}.$$

Then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega; \mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(12,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$\Box_{g(u)} u = f + q(u, {}^{\mathrm{b}}du) \tag{5.17}$$

has a unique solution $u \in \mathcal{X}^{\infty,\alpha}_{\mathbb{R}}$.

If more generally $g(u, {}^{\mathrm{b}}du) = \sum_{k=1}^N c_k(u, X_1 u, \ldots, X_L u)$, where $X_1, \ldots, X_L \in \mathcal{V}_{\mathrm{b}}(M)$ and $c_k \in \mathcal{C}^\infty(\mathbb{R}^{1+L}; \mathbb{R})$, then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega; \mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(14,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$\Box_{g(u, {}^{\mathrm{b}}du)} u = f + q(u, {}^{\mathrm{b}}du) \tag{5.18}$$

has a unique solution $u \in \mathcal{X}^{\infty,\alpha}_{\mathbb{R}}$.

Proof. We write $|\cdot|_s$ for the $\mathcal{X}^{s,\alpha}$-norm and $\|\cdot\|_s$ for the $H_{\mathrm{b}}^{s,\alpha}$-norm.[19] We define the map

$$\phi(u;f) = \Box_{g(u)}u - q(u, {}^{\mathrm{b}}du) - f$$

and check that it satisfies the conditions of Theorem 5.8 with $u_0 = 0$. From the definition of $\Box_{g(u)}$ and the tame estimates for products, reciprocals and compositions, Corollary 3.2 and Propositions 3.4 and 3.7, we obtain

$$\|\phi(u;f)\|_s \leq \|f\|_s + C(|u|_{s_0+2})(1 + |u|_{s+2}), \quad s \geq s_0 > n/2+1,$$

thus the first estimate of (5.9) for $3d \geq s_0 + 2$, $d \geq s_0$, $d \geq 2$. Next, we have $\phi'(u;f)v = \big(\Box_{g(u)} + L(u, {}^{\mathrm{b}}du)\big)v$, where the first order b-differential operator L is of the form

$$L = \sum_{|\beta|\leq 1}\left(\sum_{1\leq|\alpha|\leq 2} a_{\alpha\beta}(u, {}^{\mathrm{b}}du){}^{\mathrm{b}}D^\alpha u\right){}^{\mathrm{b}}D^\beta v + \sum_{|\beta|=1} a_\beta(u, {}^{\mathrm{b}}du)u{}^{\mathrm{b}}D^\beta v, \tag{5.19}$$

with the second sum capturing one term of the linearization of terms $u^{e_j}X_{j1}u$ in q (i.e. terms for which $N_j = 1$). In particular,

$$\phi'(u;f) = P_0(u_0) + \tilde{P}(u, {}^{\mathrm{b}}Du, {}^{\mathrm{b}}D^2u), \tag{5.20}$$

where $P_0 \in \mathrm{Diff}_{\mathrm{b}}^2$ and $\tilde{P} \in H_{\mathrm{b}}^{s-2,\alpha}\mathrm{Diff}_{\mathrm{b}}^2$ for $u \in \mathcal{X}^{s,\alpha}$. Therefore,

$$\|\phi'(u;f)v\|_s \leq C(|u|_{s+2})|v|_{s+2}, \quad s > n/2+1,$$

which gives the second estimate of (5.9) for $2d > n/2+1$ and $3d \geq 2d+2$. Next, we observe that $\phi''(u;f)(v,w)$ is bilinear in v, w, involves up to two b-derivatives of each v and w, and the coefficients depend on up to two b-derivatives of u, thus

$$\|\phi''(u;f)(v,w)\|_s \leq C(|u|_{s+2})|v|_{s+2}|w|_{s+2}, \quad s > n/2+1,$$

which gives the third estimate of (5.9) for $3d > n/2+3$, $3d \geq 2d+2$. In summary, we obtain (5.9) for integer $d > n/2+1$.

Finally, we determine d so that we have the tame estimate (5.10): Given $u \in \mathcal{X}^{s+6,\alpha}$, we can write $\phi'(u;f)$ as in (5.20), with $P_0 \in \mathrm{Diff}_{\mathrm{b}}^2$ and $\tilde{P} \in H_{\mathrm{b}}^{s+4,\alpha}\mathrm{Diff}_{\mathrm{b}}^2$; hence, by Theorem 5.7, we obtain a solution operator

$$\begin{aligned} \psi(u;f)\colon H_{\mathrm{b}}^{s+3,\alpha} &\to \mathcal{X}^{s,\alpha}, \\ |\psi(u;f)v|_s &\leq C(s, |u|_{s_0})(\|v\|_{s+3} + \|v\|_{s_0}|u|_{s+6}), \end{aligned} \tag{5.21}$$

where $s, s_0 > n/2+2$, provided $|u|_{s_0}$ is small enough so that all dynamical and geometric hypotheses hold for $\phi'(u;f)$. Notice that the subprincipal term of $\phi'(u;f)$ can differ from that of $\Box_{g(0)}$ by terms of the form $a(u_0)u_0{}^{\mathrm{b}}D^\beta$, $a \in \mathcal{C}^\infty$, $|\beta| = 1$, see (5.19); however, since such terms eliminate constants, the simple rank 1 resonance at 0 with resonant state 1 does not change; and moreover such terms are *small* because of the factor u_0, hence high energy estimates still hold in a (possibly slightly smaller) strip in the analytic continuation, see the remark below [23, Theorem 1]. Since s_0 is independent of s, we have (5.21) for all $s > n/2+2$, in particular $\psi(u;f)\colon H_{\mathrm{b}}^{\infty,\alpha} \to \mathcal{X}^{\infty,\alpha}$. Now, (5.21) implies that (5.10) holds for $d > n/2+2$, $d \geq 6$, so we need to control $\max(12, n+5)$ derivatives of f.

Thus, we can apply Nash-Moser iteration, Theorem 5.8, to obtain a solution $u \in \mathcal{X}^{s,\alpha}$ of the PDE (5.17), with the caveat that u is a priori supported on a space slightly larger than Ω. However, local uniqueness for quasilinear hyperbolic

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[19] For brevity, we do not specify the underlying set, which, in the notation of Section 5.1, is $\mathfrak{t}_1^{-1}([-\lambda,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty))$ for varying $\lambda \geq 0$.

equations, see e.g. [45, §16.3], implies that u in fact is supported in Ω, and that u is the unique solution of (5.17), finishing the proof of the first part.

The proof of the second part proceeds in the same way, only we need that $d \geq 7$, which makes the control of the stronger $H_{\mathrm{b}}^{\max(14,n+5)}$-norm of f necessary. □

Remark 5.11. In the asymptotically de Sitter setting considered in [29], the above Theorem extends [29, Theorem 8.8], at the cost of requiring the control of more derivatives, since we allow the dependence of the metric $g(u, {}^{\mathrm{b}}du)$ on ${}^{\mathrm{b}}du$ as well.

Remark 5.12. An inspection of the proof of the abstract Nash-Moser theorem 5.8 in [42] shows that there are constants C and s_0, depending only on the 'loss of derivatives' d, such that the following holds: In order to obtain a solution $u \in \mathcal{X}^{s,\alpha}$ for some finite $s \geq s_0$, it is sufficient to take $f \in H_{\mathrm{b}}^{Cs,\alpha}$, still assuming the norm of f in the space indicated in the statement of Theorem 5.10 to be small.

Theorem 5.10 immediately implies the following result on Kerr-de Sitter space:

Corollary 5.13. *Under the assumptions of Theorem 5.10, the quasilinear wave equation (5.17), resp. (5.18), on a 4-dimensional asymptotically Kerr-de Sitter space with $|a| \ll M_\bullet$ has a unique global smooth (i.e. conormal, in the space $\mathcal{X}^{\infty,\alpha}$) solution if the $H_{\mathrm{b}}^{12,\alpha}(\Omega)^{\bullet,-}$-norm, resp. $H_{\mathrm{b}}^{14,\alpha}(\Omega)^{\bullet,-}$-norm, of the forcing term $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)^{\bullet,-}$ is sufficiently small.*

Proof. For a verification of the dynamical assumptions for asymptotically Kerr-de Sitter spaces, we refer the reader to [47, §6]; the resonances on the other hand were computed by Dyatlov [21]. □

5.3. **Solving quasilinear Klein-Gordon equations.** The only difference between wave and Klein-Gordon equations with mass m (which is to be distinguished from the black hole mass $M_\bullet$) is that the resonance of the Klein-Gordon operator $\Box - m^2$ with largest imaginary part, which gives the leading order asymptotics, is no longer at 0 for $m \neq 0$. Thus, if we sort the resonances $\sigma_1, \sigma_2, \ldots$ of $\Box - m^2$ with multiplicity by decreasing imaginary part, assume

$$0 < -\operatorname{Im}\sigma_1 < r < -\operatorname{Im}\sigma_2,$$

and moreover that the high energy estimates for the normal operator family of $\Box - m^2$ hold in $\operatorname{Im}\sigma \geq -r$, the only change in the statement of Theorem 5.6 for Klein-Gordon operators is that the conclusion now is $u \in \mathcal{X}_{\sigma_1}^{s-3,r}$, where $\mathcal{X}_{\sigma_1}^{s-3,r} = \mathbb{C} \oplus H_{\mathrm{b}}^{s-3,r}(\Omega)^{\bullet,-}$, with (c, u') identified with $cx^{i\sigma_1}\chi + u'$ for a smooth cutoff χ near the boundary.[20] We thus obtain the following adapted version of Theorem 5.7:

Theorem 5.14. *In the notation of Section 5.2, under the above assumptions and for $s > n/2 + 2$, define the space*

$$\mathcal{Y}_{\sigma_1}^{s,r} = \{u \in \mathcal{X}_{\sigma_1}^{s,r} \colon Pu \in H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}\}.$$

Then the operator $P \colon \mathcal{Y}^{s,r} \to H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}$ has a continuous inverse S that satisfies the tame estimate

$$\|Sf\|_{\mathcal{X}_{\sigma_1}^{s,r}} \leq C(s, \|v\|_{\mathcal{X}_{\sigma_1}^{s_0,\alpha}})(\|f\|_{H_{\mathrm{b}}^{s+3,r}(\Omega)^{\bullet,-}} + \|f\|_{H_{\mathrm{b}}^{s_0,r}(\Omega)^{\bullet,-}}\|v\|_{\mathcal{X}_{\sigma_1}^{s+4,\alpha}}). \tag{5.22}$$

[20]There are more cases of potential interest: If $r < -\operatorname{Im}\sigma_1$, we obtain $u \in H_{\mathrm{b}}^{s-3,r}(\Omega)^{\bullet,-}$; if $r < 0$, the statement of Theorem 5.6 is unchanged; and if $\operatorname{Im}\sigma_1$ and $\operatorname{Im}\sigma_2$ are close enough together (including the case that σ_1 is a double resonance), one gets two terms in the expansion of u. For brevity, we only explain one scenario here. See also the related discussion in [29, §8.4].

This immediately gives:

Theorem 5.15. *Under the above assumptions and the assumption $\alpha < -2\operatorname{Im}\sigma_1$, let $N, N' \in \mathbb{N}$ and $c_k \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $g_k \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)(M;\mathrm{Sym}^2\,{}^{\mathrm{b}}TM)$ for $1 \leq k \leq N$; define the map $g\colon \mathcal{X}^{s,\alpha}_{\sigma_1} \to (\mathcal{C}^\infty + H_{\mathrm{b}}^{s,\alpha})(M;\mathrm{Sym}^2\,{}^{\mathrm{b}}TM)$ by $g(u) = \sum_{k=1}^N c_k(u)g_k$ and assume that $\Box_{g(0)}$ satisfies the assumptions of Section 5.1 and of Theorem 5.14. Moreover, define*

$$q(u, {}^{\mathrm{b}}du) = \sum_{j=1}^{N'} a_j u^{e_j} \prod_{k=1}^{N_j} X_{jk} u,$$

where

$$e_j, N_j \in \mathbb{N}_0, e_j + N_j \geq 2, a_j \in \mathcal{C}^\infty, X_{jk} \in (\mathcal{C}^\infty + H_{\mathrm{b}}^\infty)\mathcal{V}_{\mathrm{b}}.$$

Then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(12,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$(\Box_{g(u)} - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{5.23}$$

has a unique solution $u \in \mathcal{X}^{\infty,\alpha}_{\sigma_1,\mathbb{R}}$.

If more generally $g(u, {}^{\mathrm{b}}du) = \sum_{k=1}^N c_k(u, X_1u, \ldots, X_Lu)$, where $X_1, \ldots, X_L \in \mathcal{V}_{\mathrm{b}}(M)$ and $c_k \in \mathcal{C}^\infty(\mathbb{R}^{1+L};\mathbb{R})$, then there exists $C_f > 0$ such that for all forcing terms $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega;\mathbb{R})^{\bullet,-}$ satisfying $\|f\|_{H_{\mathrm{b}}^{\max(14,n+5),\alpha}(\Omega)^{\bullet,-}} \leq C_f$, the equation

$$(\Box_{g(u,{}^{\mathrm{b}}du)} - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{5.24}$$

has a unique solution $u \in \mathcal{X}^{\infty,\alpha}_{\mathbb{R}}$.

Together with Theorem 5.10, this proves Theorem 2.

Proof of Theorem 5.15. The proof proceeds as the proof of Theorem 5.10. Notice that we allow the nonlinear term q to be more general, the point being that firstly, any at least quadratic expression in $(u, {}^{\mathrm{b}}du)$ with $u \in \mathcal{X}^{s,\alpha}_{\sigma_1}$ gives an element of $H_{\mathrm{b}}^{s,\alpha}$, and secondly, every element in $\mathcal{X}^{s,\alpha}_{\sigma_1}$ vanishes at the boundary, thus the normal operator family of the linearization of $\Box_{g(u)} - m^2 - q(u, {}^{\mathrm{b}}du) - f$ at any $u \in \mathcal{X}^{s,\alpha}_{\sigma_1}$ is equal to the normal operator family of $\Box_{g(0)} - m^2$, for which one has high energy estimates by assumption. $\Box$

By [31, Lemma 3.5], the assumptions of Theorem 5.15 are satisfied on asymptotically Kerr-de Sitter spaces as long as the mass parameter m is small:

Corollary 5.16. *Under the assumptions of Theorem 5.15 and for a and $m > 0$ sufficiently small, the quasilinear Klein-Gordon equation (5.23), resp. (5.24), on a 4-dimensional asymptotically Kerr-de Sitter space with angular momentum a has a unique global smooth (i.e. conormal, in the space $\mathcal{X}^{\infty,\alpha}_{\sigma_1,\mathbb{R}}$) solution if the $H_{\mathrm{b}}^{12,\alpha}(\Omega)^{\bullet,-}$-norm, resp. $H_{\mathrm{b}}^{14,\alpha}(\Omega)^{\bullet,-}$-norm, of the forcing term $f \in H_{\mathrm{b}}^{\infty,\alpha}(\Omega)^{\bullet,-}$ is sufficiently small.*

5.4. **Proofs of Theorems 3 and 4.** Finally, following the same arguments as used in the previous section, we indicate how to prove the general theorems stated in the introduction. We continue to use, but need to generalize the setting considered in Section 5.1: Namely, generalizing (5.1), we now allow L to be any first order b-differential operator, and correspondingly need information on the skew-adjoint part of P_0; concretely, we define $\hat\beta$ at the (generalized) radial sets $L_\pm$, using the same notation as in (5.2), by

$$\sigma_{\mathrm{b},1}\Big(\frac{1}{2i}(P_0-P_0^*)\Big)\Big|_{L_\pm} = \pm\hat\beta\beta_0\rho. \tag{5.25}$$

Moreover, at the trapped set $\Gamma=\Gamma^-\cup\Gamma^+$, we assume that

$$\mathsf{e}_1|_\Gamma < \nu_{\min}/2, \quad \mathsf{e}_1 = |\sigma|^{-1}\sigma_{\mathrm{b},1}\Big(\frac{1}{2i}(P_0-P_0^*)\Big), \tag{5.26}$$

with $\nu_{\min}$ the minimal normal expansion rate for the Hamilton flow of the principal symbol of P_0, and σ the Mellin dual variable of x after an identification of a collar neighborhood of X in M with $[0,\epsilon')_x\times X$; note that σ is elliptic on Γ. Let r_{th} be the threshold weight for the first part of Theorem 4.4, i.e. $r_{\mathrm{th}}=-\sup\mathsf{e}_1/c_\partial$ with c_∂ as defined in (4.15).

Then Corollary 5.4 holds in the current, more general setting, provided we assume $r<r_{\mathrm{th}}$ and $s'>1+\sup_{L_\pm}(r\tilde\beta-\hat\beta)$. Likewise, we obtain the high energy estimates of Theorem 4.5 under the assumption $s>1/2+\sup_{L_\pm}(\gamma\tilde\beta-\hat\beta)$.

In order to generalize Theorem 5.6, we first choose $0<r_+<1$ such that

$$(\mathsf{e}_1+r_+c_\partial)|_\Gamma<\nu_{\min}/2,$$

which holds for sufficiently small r_+ in view of (5.26) by the compactness of Γ in ${}^{\mathrm{b}}S^*M$. We moreover assume that there are no (nonzero) resonances in $\operatorname{Im}\sigma>-r_+$ in the case of Theorem 3 (Theorem 4), and we assume further that $0<\alpha<r_+$. Then in the proof of Theorem 5.6, ignoring the issue of threshold regularities at radial sets momentarily, we can use the contour shifting argument without loss of derivatives up to, but excluding, the weight r_{th}, corresponding to the contour of integration $\operatorname{Im}\sigma=-r_{\mathrm{th}}$. Shifting the contour further down, we cannot use the non-smooth real principal type estimate at Γ anymore and thus lose 2 derivatives at each step; the total number of additional steps needed to shift the contour down to $\operatorname{Im}\sigma=-\alpha$ is easily seen to be at most

$$N=\max\left(0,\left\lceil\frac{\alpha-r_{\mathrm{th}}}{\alpha}\right\rceil+1\right),$$

hence in order to have the final conclusion that u has an expansion with remainder in $H_{\mathrm{b}}^{s,\alpha}$, we need to assume that u initially is known to have regularity $H_{\mathrm{b}}^{s+2N,r_0}$ for any $r_0\in\mathbb{R}$, which in turn requires $\tilde s\geq s+2N$ and $f\in H_{\mathrm{b}}^{s+2N-1,r_0}$ for the first, lossless, part of the argument to work. Taking the regularity requirements at the radial sets into account, we further need to assume $s\geq s_0>\max(n+1/2,1+\sup(r\tilde\beta-\hat\beta))$. Under these assumptions, the proof of Theorem 5.6 applies, mutatis mutandis, to our current situation, and we obtain a tame solution operator as in Theorem 5.7, which now loses $2N-1$ derivatives.

Thus, we can prove Theorems 3 and 4 using the same arguments which we used in the proof of Theorem 5.10; the 'loss of derivatives' parameter d now needs to satisfy the conditions

$$d\geq 2N+3,\ d>n/2+6,\ d>1+\sup(r\tilde\beta-\hat\beta), \tag{5.27}$$

with the first condition being the actual loss of derivatives, the second one coming from $s>n/2+6$ certainly being a high enough regularity for $\tilde s=s+2N$ to be $>n/2+6$, which is required for the application of the non-smooth microlocal regularity results, and the last condition being the threshold regularity (for the non-smooth estimates) at the radial sets.

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