THE SPECTRAL PROJECTIONS AND THE RESOLVENT FOR SCATTERING METRICS

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ABSTRACT. In this paper we consider a compact manifold with boundary X equipped with a scattering metric g as defined by Melrose [9]. That is, g is a Riemannian metric in the interior of X that can be brought to the form $g = x^{-4}\,dx^2 + x^{-2}h'$ near the boundary, where x is a boundary defining function and h' is a smooth symmetric 2-cotensor which restricts to a metric h on ∂X. Let $H = \Delta + V$ where $V \in x^2\mathcal{C}^\infty(X)$ is real, so V is a 'short-range' perturbation of Δ. Melrose and Zworski started a detailed analysis of various operators associated to H in [11] and showed that the scattering matrix of H is a Fourier integral operator associated to the geodesic flow of h on ∂X at distance π and that the kernel of the Poisson operator is a Legendre distribution on $X \times \partial X$ associated to an intersecting pair with conic points. In this paper we describe the kernel of the spectral projections and the resolvent, $R(\sigma \pm i0)$, on the positive real axis. We define a class of Legendre distributions on certain types of manifolds with corners, and show that the kernel of the spectral projection is a Legendre distribution associated to a conic pair on the b-stretched product X^2_{b} (the blowup of X^2 about the corner, $(\partial X)^2$). The structure of the resolvent is only slightly more complicated.

As applications of our results we show that there are 'distorted Fourier transforms' for H, ie, unitary operators which intertwine H with a multiplication operator and determine the scattering matrix; and give a scattering wavefront set estimate for the resolvent $R(\sigma \pm i0)$ applied to a distribution f.

1. INTRODUCTION

In this paper we study the basic analytic operators associated to short range Schrödinger operators H on a manifold with boundary, X, with scattering metric. The analytic operators of interest are the Poisson operator, scattering matrix, spectral projections and resolvent family. These will be defined and described in detail later in the introduction. The first two operators were analyzed rather completely by Melrose and Zworski [11] and we will use their analysis extensively in this paper.

A scattering metric g on a manifold with boundary X^n is a smooth Riemannian metric in the interior of X which can be brought to the form

$$g = \frac{dx^2}{x^4} + \frac{h'}{x^2} \tag{1.1}$$

near ∂X for some choice of a boundary defining function x and for some smooth symmetric 2-cotensor h' on X which restricts to a metric h on ∂X. We note that this fixes x up to $x^2\mathcal{C}^\infty(X)$. To understand the geometry that such a metric endows, suppose that for some choice of product coordinates (x, y) near the boundary, g

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takes the warped product form

$$\frac{dx^2}{x^4} + \frac{h(y)}{x^2} \quad \text{for } x \le x_0.$$

Then changing variable to $r = 1/x$, this looks like $dr^2 + r^2 h(y)$ for $r \ge r_0$, which is the 'large end of a cone'. A general scattering metric may be considered a 'short range' perturbation of this geometry. A particularly important example is $\mathbb{R}^n$, with the flat metric, which after radial compactification and after introduction of the boundary defining function $x = 1/r$ takes the form (1.1) with h the standard metric on the sphere.

Let Δ be the (positive) Laplacian of g, and let $V \in x^2\mathcal{C}^\infty(X)$ be a real valued function. We will study operators of the form $H = \Delta + V$ which we consider a 'short range Schrödinger operator' by analogy with the Euclidean situation. The case $V \equiv 0$ is already of considerable interest. Since the interior of X is a complete Riemannian manifold, H is essentially self-adjoint on $L^2(X)$ ([13], chapter 8).

One of the themes of this paper is that the operator $\Delta + V$ on $\mathbb{R}^n$ is very typical of a general H as above. Melrose and Zworski [11] showed that the scattering-microlocal structure of the Poisson operator and scattering matrix can be described purely in terms of geodesic flow on the boundary ∂X. This is equally true of the spectral projections and resolvent, and looked at from this point of view the case of $\mathbb{R}^n$ is a perfect guide to the general situation.

Our results are phrased in terms of Legendrian distributions introduced by Melrose and Zworski. We need to extend their definitions to the case of manifolds with corners of codimension 2, since the Schwartz kernels of the spectral projections and resolvent are defined on X^2 which has codimension 2 corners. One of our motivations for writing this paper is to understand Legendrian distributions on manifolds with corners, in the belief that similar techniques may be used to do analogous constructions for N body Schrödinger operators. For the three body problem, Legendrian techniques have already proved useful — see [1].

The structure of generalized eigenfunctions of a scattering metric has been described by Melrose [9]. He showed that for any $\lambda \in \mathbb{R} \setminus \{0\}$ and $a \in \mathcal{C}^\infty(\partial X)$ there is a unique $u \in \mathcal{C}^{-\infty}(X)$ which satisfies $(H - \lambda^2)u = 0$ and which is of the form

$$u = e^{-i\lambda/x} x^{(n-1)/2} v_- + e^{i\lambda/x} x^{(n-1)/2} v_+, \ v_\pm \in \mathcal{C}^\infty(X), \ v_-|_{\partial X} = a. \tag{1.2}$$

Of course, such a representation has been known for $\Delta + V$ on $\mathbb{R}^n$ for a long time. Note that changing the sign of λ reverses the role of v_+ and v_-, and correspondingly the role of the 'boundary data' $v_+|_{\partial X}$ and $v_-|_{\partial X}$. Thus, if we wish to take $\lambda > 0$, this statement says that to get a generalized eigenfunction u of H of this form, we can either prescribe $v_+|_{\partial X}$ or $v_-|_{\partial X}$ freely, but the prescription of either of these determines u uniquely. In particular, prescribing $v_+|_{\partial X}$ determines $v_-|_{\partial X}$ and vice versa.

The Poisson operator is defined as the map from boundary data to generalized eigenfunctions of H [11]. Thus, with the notation of (1.2), for $\lambda \in \mathbb{R} \setminus \{0\}$, $P(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^{-\infty}(X)$ is given by

$$P(\lambda)a = u.$$

Moreover, the scattering matrix $S(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^\infty(\partial X)$ is the map

$$S(\lambda)a = v_+|_{\partial X}. \tag{1.3}$$

Thus, $S(\lambda)v_-|_{\partial X} = v_+|_{\partial X}$, i.e. the scattering matrix links the parametrizations of generalized eigenfunctions u of H by the boundary data $v_-|_{\partial X}$ and $v_+|_{\partial X}$. We remark that the normalization of $P(\lambda)$ and $S(\lambda)$ here is the opposite of the one used in [9] (i.e. the roles of $S(\lambda)$ and $S(-\lambda)$ are interchanged); it matches that of [11].

The kernels of these operators are distributions on $X \times \partial X$ and $\partial X \times \partial X$ respectively. Both kernels were analyzed in [11] using scattering microlocal analysis (described in the next section). The authors showed that the kernel of the Poisson operator is a conic Legendrian pair which can be described in terms of geodesic flow on ∂X and that the kernel of the scattering matrix is a Fourier integral operator whose canonical relation is geodesic flow at time π on ∂X.

Melrose also showed that the resolvent family $R(\sigma) = (H - \sigma)^{-1}$ is in the scattering calculus (a class of pseudodifferential operators on X defined in [9]) for $\operatorname{Im}\sigma \neq 0$. We are interested in this paper in understanding the behaviour of the resolvent kernel as $\operatorname{Im}\sigma \to 0$ when $\operatorname{Re}\sigma \neq 0$. This is related to the spectral measure of H via Stone's theorem

$$\text{s-}\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_a^b R(\sigma + i\epsilon) - R(\sigma - i\epsilon)\, d\sigma = \frac{1}{2}\Big(E_{(a,b)} + E_{[a,b]} \Big). \tag{1.4}$$

We will show in Lemma 5.2 that in fact

$$R(\lambda^2 + i0) - R(\lambda^2 - i0) = \frac{i}{2\lambda} P(\lambda) P(\lambda)^* \tag{1.5}$$

as operators $\dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$. Modulo a few technicalities (see Lemma 5.1) this shows that the spectral measure $dE(\sigma)$ of H is differentiable as a map from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$, and we may write

$$(dE)(\lambda^2) \equiv 2\lambda \operatorname{sp}(\lambda)\, d\lambda = \frac{1}{2\pi} P(\lambda) P(\lambda)^* d\lambda, \quad \lambda > 0$$

for some operator $\operatorname{sp}(\lambda) : \dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$ which we call the generalized spectral projection, or (somewhat incorrectly) just the spectral projection at energy $\lambda^2 > 0$. Thus $\operatorname{sp}(\lambda) = (4\pi\lambda)^{-1} P(\lambda) P(\lambda)^*$. Our approach in analyzing the kernel of the resolvent is to understand the microlocal nature of the composition $P(\lambda)P(\lambda)^*$ and then to construct the resolvent using a regularization of the formal expression

$$R(\lambda^2 \pm i0) = \int_{-\infty}^{\infty} (\sigma - \lambda^2 \mp i0)^{-1} dE(\sigma). \tag{1.6}$$

Our results on the spectral projection and the resolvent are stated precisely in Theorems 7.2 and Theorem 8.2. Here we will give an informal description. Let X_b^2 denote the blow-up of $X^2 = X \times X$ about its corner $(\partial X)^2$ (see Section 2 for a detailed description of the space, and Figure 1 for a picture). Then we show

Main Results. *(1) The kernel of the spectral projection* $\operatorname{sp}(\lambda)$ *is a Legendre distribution associated to an intersecting pair* $(L(\lambda), L^\sharp(\lambda))$ *with conic points. Here* $L(\lambda)$ *and* $L^\sharp(\lambda)$ *are two Legendrians contained in the boundary of a 'stretched cotangent bundle' over* X_b^2 *which can be defined purely geometrically (see equations* (7.10) *and* (7.18)*).*

(2) The kernel of the boundary value $R(\lambda^2 + i0)$ *of the resolvent of* H *on the positive real axis is of the form* $R_1 + R_2 + R_3$*, where* R_1 *is in the scattering calculus (and thus qualitatively similar to the resolvent off the real axis),* R_2 *is an intersecting*

Legendre distribution, supported near $\partial\,\mathrm{diag}_b$ *and* R_3 *is a Legendre distribution associated to* $(L(\lambda), L^\sharp(\lambda))$.

The paper is organized as follows. In section 2 we review elements of the scattering calculus, including Legendrian distributions. Sections 3 and 4 are devoted to the definition and parametrization of Legendre distributions on manifolds with codimension two corners with a 'stratification' of the boundary, in a sense made precise there. In section 5 we recall Melrose and Zworski's results on the Poisson operator and prove (1.5). Sections 7 and 8 contain the statements and proof of the main results, Theorems 7.2 and 8.2. This is preceded by section 6 which discusses the Euclidean spectral projection and resolvent in Legendrian terms, and turns out to be a useful guide to the general situation.

The final two sections contain applications of the main results. In section 9, we show that the Poisson operator can be viewed as a 'distorted Fourier transform' for H. Let us define the operators $P_\pm^* : \mathcal{C}_c^\infty(X) \to \mathcal{C}^{-\infty}(\partial X \times \mathbb{R}_+)$ by the formula

$$(P_\pm^* u)(y, \lambda) = (2\pi)^{-1/2} (P(\pm\lambda)^* u)(y). \tag{1.7}$$

Then $P_\pm^*$ extends to an isometry from $H_{ac}(H)$, the absolutely continuous spectral subspace of H, to $L^2(\partial X \times \mathbb{R}_+)$, with adjoint

$$(P_\pm f)(z) = (2\pi)^{-1/2} \int_0^\infty P(\pm\lambda)(f(\cdot, \lambda))\, d\lambda. \tag{1.8}$$

If the operator S is defined on $\mathcal{C}^\infty(\partial X \times \mathbb{R}_+)$ by

$$(Sf)(y, \lambda) = (S(\lambda) f(\cdot, \lambda))(y), \tag{1.9}$$

then we find that $S = P_- P_+^*$, analogous to a standard formula in scattering theory for the scattering matrix in terms of the distorted Fourier transforms.

In section 10, we derive a bound for the scattering wavefront set of $R(\lambda^2 \pm i0) f$ in terms of the scattering wavefront set of f. This is not a new result, since Melrose proved it in [9] using positive commutator estimates. However, we wish to emphasize that it follows from a routine calculation, once one understands the Legendrian structure of the resolvent.

In future work, we plan to outline a symbol calculus for these types of distributions which can be used to construct the resolvent kernel 'directly', that is, without going via the spectral projections. We also plan to use our results here to get wavefront set bounds on the Schrödinger kernel e^{itH}, and the wave kernel $e^{it\sqrt{H}}$, applied to suitable distributions f.

Notation and conventions. On a compact manifold with boundary, X, we use $\dot{\mathcal{C}}^\infty(X)$ to denote the class of smooth functions, all of whose derivatives vanish at the boundary, with the usual topology, and $\mathcal{C}^{-\infty}(X)$ to denote its topological dual. On the radial compactification of $\mathbb{R}^n$ these correspond to the space of Schwartz functions and tempered distributions, respectively. The Laplacian Δ is taken to be positive. The space $L^2(X)$ is taken with respect to the Riemannian density induced by the scattering metric g, thus has the form $a\, dx dy / x^{n+1}$ near the boundary, where a is smooth.

The outgoing resolvent $R(\lambda^2 + i0)$, $\lambda > 0$, is such that application to a $\dot{\mathcal{C}}^\infty(X)$ function yields a function of the form

$$R(\lambda^2 + i0) = x^{(n-1)/2} e^{i\lambda/x} a(x, y), \quad a \in \mathcal{C}^\infty(X).$$

This corresponds to τ being *negative* in the scattering wavefront set.

If M is a manifold with corners and S is a p-submanifold (that is, the inclusion of S in M is smoothly modelled on $\mathbb{R}^{n',k'} \subset \mathbb{R}^{n,k}$), then $[M;S]$ denotes the blowup of S inside M (see [8]). The new boundary hypersurface thus created is called the 'front face'.

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2. Scattering Calculus and Legendrian distributions

In this section we describe very briefly elements of the scattering calculus. Further information can be found in [9].

A scattering metric g on a manifold with boundary X determines a class of 'sc-vector fields', denoted $\mathcal{V}_{\mathrm{sc}}(X)$, namely those $\mathcal{C}^\infty$ vector fields which are of finite length with respect to g. Locally this consists of all $\mathcal{C}^\infty$ vector fields in the interior of X, and near the boundary, is the $\mathcal{C}^\infty(X)$-span of the vector fields $x^2\partial_x$ and $x\partial_{y_i}$, as can be seen from (1.1). The set $\mathcal{V}_{\mathrm{sc}}(X)$ is in fact a Lie Algebra. From it we may define $\mathrm{Diff}^k_{\mathrm{sc}}(X)$, the set of differential operators which can be expressed as a finite sum of products of at most k sc-vector fields with $\mathcal{C}^\infty(X)$-coefficients.

In [9] Melrose constructed a natural 'microlocalization' $\Psi_{\mathrm{sc}}(X)$ of $\mathrm{Diff}_{\mathrm{sc}}(X)$, ie an algebra of operators which extends the differential operators in a similar way to that in which pseudodifferential operators on a closed manifold extend the differential operators. The operators are described by specifying their distributional kernels. Since pseudodifferential operators on a closed manifold have kernels conormal to the diagonal, elements of $\Psi_{\mathrm{sc}}(X)$ should be conormal to the diagonal in the interior of X^2; what remains to be specified is their behaviour at the boundary. For this purpose, X^2 has rather singular geometry, since the diagonal intersects the boundary in a non-normal way. This difficulty is overcome by blowing up X^2 to resolve the singularities. There are two blowups to be performed in order to describe the kernels of $\Psi_{\mathrm{sc}}(X)$. First we form the b-double space

$$X^2_{\mathrm{b}} = [X^2; (\partial X)^2].$$

The diagonal lifts to a product-type submanifold $\mathrm{diag}_{\mathrm{b}}$. We next blow up the boundary of this submanifold, obtaining the scattering double space X^2_{sc}:

$$X^2_{\mathrm{sc}} = [X^2_{\mathrm{b}}; \partial\, \mathrm{diag}_{\mathrm{b}}].$$

The lift of the diagonal to X^2_{sc} is denoted $\mathrm{diag}_{\mathrm{sc}}$. The boundary hypersurfaces are denoted lb, rb, bf and sc which arise from the boundaries $\partial X \times X$, $X \times \partial X$, the first blowup and the second blowup, respectively (see figure).

It is not hard to check that $\mathrm{Diff}_{\mathrm{sc}}(X)$ corresponds precisely to all distributions on X^2_{sc} conormal to, and supported on, $\mathrm{diag}_{\mathrm{sc}}$, which are smooth up to the boundary (using the Riemannian density to define integration against functions). Melrose defines $\Psi^{k,0}_{\mathrm{sc}}(X; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ to be the class of kernels on X^2_{sc} of the form $\kappa|d\mu|^{1/2}$ where κ is conormal of order k to $\mathrm{diag}_{\mathrm{sc}}$, smooth up to sc, and rapidly decreasing at lb, rb and bf, and $|d\mu|$ is the product of Riemannian densities from the left and right factors of X^2, lifted to X^2_{sc}. These kernels act naturally on half-densities on X. One can show that they map $\dot{\mathcal{C}}^\infty$ half-densities to $\dot{\mathcal{C}}^\infty$ half-densities, and thus,

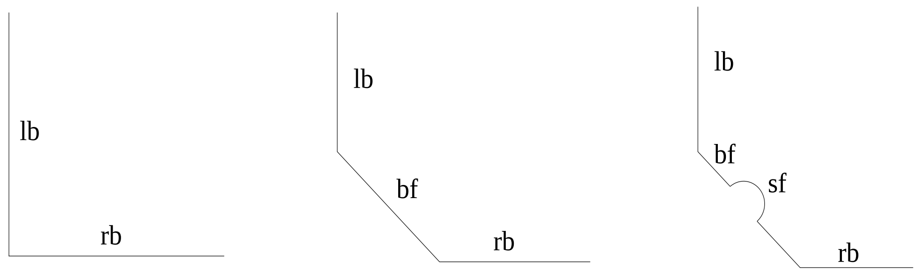

FIGURE 1. Blowing up to produce the double scattering space

by duality, also map $\mathcal{C}^{-\infty}$ half-densities to $\mathcal{C}^{-\infty}$ half-densities. The space $\Psi_{\rm sc}^{k,l}(X)$ is defined to be $x^l\,\Psi_{\rm sc}^{k,0}(X)$. See [9] for more details, or [3] for a more detailed description where X is the compactification of $\mathbb{R}^n$.

We next describe the symbols of these operators. First, we need to consider a rescaled version of the cotangent bundle which matches the degeneration of the sc-vector fields at the boundary. Since $\mathcal{V}_{\rm sc}(X)$ can always be expressed locally as the $\mathcal{C}^\infty(X)$-span of n local sections, there is a vector bundle ${}^{\rm sc}TX$ whose smooth sections are exactly the sc-vector fields: $\mathcal{V}_{\rm sc}(X) = \mathcal{C}^\infty(X; {}^{\rm sc}TX)$. The dual bundle of ${}^{\rm sc}TX$ is denoted by ${}^{\rm sc}T^*X$, is called the scattering cotangent bundle, and a basis of its smooth sections is given by $\frac{dx}{x^2}$, $\frac{dy_j}{x}$, $j = 1, \dots, N-1$, in the local coordinates described above. We can write a covector $v \in {}^{\rm sc}T_p^*M$ as $v = \tau\frac{dx}{x^2} + \mu\cdot\frac{dy}{x}$, and thereby we have coordinates (x, y, τ, μ) on ${}^{\rm sc}T^*X$ near ∂X. Notice that $\tau = x^2\xi$ and $\mu = x\eta$ in terms of the dual cotangent variables (ξ, η) to (x, y), so we will sometimes refer to (τ, μ) as rescaled cotangent variables. The half-density bundle induced by ${}^{\rm sc}T^*X$ is denoted by ${}^{\rm sc}\Omega^{\frac12}X$.

We may alternatively think of sc-vector fields as smoothly varying linear forms on ${}^{\rm sc}T^*X$. Since $\mathcal{V}_{\rm sc}(X)$ is a Lie Algebra, there is a well-defined notion of the kth degree part of an operator $P \in \mathrm{Diff}_{\rm sc}^k(X)$; its symbol is the corresponding homogeneous polynomial of degree k on the fibres of ${}^{\rm sc}T^*X$. This is the interior principal symbol of P, denoted $\sigma_{\rm int}(P)$. The symbol map extends to $\sigma_{\rm int} : \Psi_{\rm sc}^{k,0}(X) \to S^k({}^{\rm sc}T^*X)/S^{k-1}({}^{\rm sc}T^*X)$, which is multiplicative,

$$\sigma_{\rm int}(AB) = \sigma_{\rm int}(A)\cdot\sigma_{\rm int}(B),$$

and such that there is an exact sequence:

$$0 \longrightarrow \Psi_{\rm sc}^{k-1,0}(X) \longrightarrow \Psi_{\rm sc}^{k,0}(X) \longrightarrow S^k({}^{\rm sc}T^*X)/S^{k-1}({}^{\rm sc}T^*X) \longrightarrow 0.$$

There is also a boundary principal symbol map which carries more information than just the restriction of $\sigma_{\rm int}$ to the scattering cotangent bundle over the boundary. Since $[\mathcal{V}_{\rm sc}(X), \mathcal{V}_{\rm sc}(X)] \subset x\mathcal{V}_{\rm sc}(X)$, there is also a well-defined notion of the order x^0 part of $P \in \mathrm{Diff}_{\rm sc}^k(X)$ at ∂X. The *full* symbol of this order zero part is a (in general nonhomogeneous) polynomial of degree k on ${}^{\rm sc}T^*_{\partial X}X$, denoted $\sigma_\partial(P)$, whose order k part agrees with $\sigma_{\rm int}(P)$ restricted to the boundary. This symbol map extends to $\sigma_\partial : \Psi_{\rm sc}^{k,0}(X) \to S^k({}^{\rm sc}T^*_{\partial X}X)$ which is also multiplicative,

$$\sigma_\partial(AB) = \sigma_\partial(A)\cdot\sigma_\partial(B),$$

and gives another exact sequence

$$0 \longrightarrow \Psi_{\rm sc}^{k,1}(X) \longrightarrow \Psi_{\rm sc}^{k,0}(X) \longrightarrow S^k({}^{\rm sc}T^*_{\partial X}X) \longrightarrow 0.$$

Surprisingly at first sight, perhaps, this symbol map is totally well defined, i.e., there is no ambiguity modulo symbols of order $k-1$ (expressed by the exact sequence above). This is because we already defined the symbol map as only taking values at $x=0$; the analogous procedure for the interior symbol would be to divide by $|\xi|^k$ and restrict to the sphere at infinity, whereupon (at least for classical operators) the interior symbol would also become completely well defined.

The combinination of the two principal symbols $\sigma_{\text{int}}(A)$ and $\sigma_\partial(A)$, called the joint symbol $j_{\text{sc}}(A)$ in [9], thus determines $A \in \Psi_{\text{sc}}^{k,0}(X)$ modulo $\Psi_{\text{sc}}^{k-1,1}(X)$. In particular, if $A \in \Psi_{\text{sc}}^{0,0}(X)$ and both $\sigma_{\text{int}}(A)=0$ and $\sigma_\partial(A)=0$ then A is compact on $L^2(X)$. In order to talk about both symbols of A at the same time, it is useful to introduce

$$C_{\text{sc}}X = {}^{\text{sc}}S^*X \cup {}^{\text{sc}}T^*_{\partial X}X$$

where ${}^{\text{sc}}S^*X$ is the scattering cosphere bundle, i.e. the quotient of ${}^{\text{sc}}T^*X \setminus 0$ by the natural $\mathbb{R}^+$-action. Thus, $j_{\text{sc}}(A)$ is a function on $C_{\text{sc}}X$; it is given by $\sigma_\partial(A)$ on ${}^{\text{sc}}T^*_{\partial X}(X)$ and the rescaled $\sigma_{\text{int}}(A)$ on ${}^{\text{sc}}S^*X$ (we divide out by $|\xi|^k$, so for $k \neq 0$, $j_{\text{sc}}(A)$ over ${}^{\text{sc}}S^*X$ is invariantly defined as a section of a trivial line bundle).

The bundle ${}^{\text{sc}}T^*X$, restricted to ∂X, plays a fundamental role in the scattering calculus since scattering microlocal analysis takes place on it. We now briefly explain these ideas. For notational simplicity let us denote the restriction of ${}^{\text{sc}}T^*X$ to the boundary by K. Given local coordinates (x,y) near the boundary of ∂X, there are coordinates (y,τ,μ) induced on K as above.

We have already seen that the joint symbol $j_{\text{sc}}(A)$ of $A \in \Psi_{\text{sc}}^{k,0}(X)$ is a function on $C_{\text{sc}}X$. We will say that A is elliptic at $q \in C_{\text{sc}}X$ if $j_{\text{sc}}(A)(q) \neq 0$ and define the characteristic variety of A, $\Sigma(A)$, to be the set of non-elliptic points of $C_{\text{sc}}X$. This notion of ellipticity allows us to define the 'scattering wavefront set' $\mathrm{WF}_{\text{sc}}(u)$ of a distribution $u \in \mathcal{C}^{-\infty}(X)$. Namely, we define it to be the subset of $C_{\text{sc}}X$ whose complement is

$$\begin{aligned} \mathrm{WF}_{\text{sc}}(u)^{\complement} = \{q \in C_{\text{sc}}X \mid \exists\ A \in \Psi_{\text{sc}}^{0,0}(X) \text{ such that} \\ A \text{ is elliptic at } q \text{ and } Au \in \dot{\mathcal{C}}^\infty(X)\}. \end{aligned} \tag{2.1}$$

For example, the scattering wavefront set of $e^{i/x}$ is contained in $\{\tau=-1, \mu=0\}$. This is easy to see since the function $e^{i/x}$ is annihilated by $-ix^2\partial x - 1$ and $-i\partial_{y_i}$ whose boundary symbols are $-\tau-1$ and μ_i, respectively. Note that the restriction of $\mathrm{WF}_{\text{sc}}(u)$ to ${}^{\text{sc}}S^*_{X^\circ}X$ is just the usual wave front set $\mathrm{WF}(u)$.

The scattering wavefront set of u is empty if and only if $u \in \dot{\mathcal{C}}^\infty(X)$; so its part in K may be regarded as a measure of nontrivial behaviour of the distribution at the boundary, where 'trivial' means 'all derivatives vanish at the boundary'. This is useful in solving equations such as $(\Delta-\lambda^2)u=0$. The interior symbol of $\Delta-\lambda^2$ is elliptic, hence for any such u, $\mathrm{WF}_{\text{sc}}(u) \subset K$; in particular u must be $\mathcal{C}^\infty$ in the interior. On the other hand, the boundary symbol, $\tau^2+h(y,\mu)-\lambda^2$ is not elliptic when $\lambda>0$ and u can then be expected to have nontrivial scattering wavefront set in K. For this reason, we will concentrate on the geometry and analysis associated K. We remark that the corresponding statements on ${}^{\text{sc}}S^*_{X^\circ}X$ are the subject of standard microlocal analysis (see e.g. [5]) and are very similar in nature; the two are connected by (a localized version of) the Fourier transform, see [9, 11].

The manifold K carries a natural contact structure. To see this, recall that ${}^{\text{sc}}T^*X$ is canonically isomorphic to the usual cotangent bundle over the interior, so the symplectic form may be pulled back to ${}^{sc}T^*X$, though it blows up as x^{-2} at the boundary then. Contracting with the vector $x^2\partial_x$, we obtain a smooth one-form whose restriction to K we denote χ. In local coordinates (y, τ, μ), it takes the form

$$\chi \equiv i_{x^2\partial_x}(\omega) = d\tau + \mu \cdot dy. \tag{2.2}$$

Thus, χ is a contact form, i.e. it is nondegenerate. Changing the boundary defining function to $\tilde{x} = ax$, the form $\tilde{\chi} = a\chi$ changes by a nonzero multiple, so the contact structure on K is totally well-defined.

Using the contact structure we define the Hamilton vector field of $A \in \Psi^{k,0}_{\mathrm{sc}}(X)$ to be the Hamilton vector field of its boundary symbol, and the bicharacteristics of A to be the integral curves of its Hamilton vector field. The bicharacteristics turn out to play a similar role in the scattering calculus that they do in standard microlocal analysis. For example, Melrose [9] showed that

Theorem 2.1. *Suppose $A \in \Psi^{k,0}_{sc}(X)$ has real boundary symbol. Then for $u \in \mathcal{C}^\infty(X^o) \cap \mathcal{C}^{-\infty}(X)$, we have $\mathrm{WF}_{sc}(Au) \subset \mathrm{WF}_{sc}(u)$, $\mathrm{WF}_{sc}(u) \setminus \mathrm{WF}_{sc}(Au) \subset \Sigma(A)$, and $\mathrm{WF}_{sc}(u) \setminus \mathrm{WF}_{sc}(Au)$ is a union of maximally extended bicharacteristics of A inside $\Sigma(A) \setminus \mathrm{WF}_{sc}(Au)$.*

Next, we discuss Legendrian distributions. A Legendre submanifold of K is a smooth submanifold G of dimension $n-1$ such that χ vanishes on G. The simplest examples of Legendre submanifolds are given by projectable Legendrians, that is, those Legendrians G for which the projection to the base, ∂X, is a diffeomorphism, and therefore coordinates y on the base lift to smooth coordinates on G. It is not hard to see, using the explicit form (2.2) for the contact form, that G is then given by the graph of a differential $\phi(y)/x$ where $\tau = -\phi(y)$ on G. Conversely, any function ϕ gives rise to a projectable Legendrian. We say that ϕ is a phase function which parametrizes G. A Legendrian distribution associated to a projectable Legendrian G is by definition a function of the form

$$u = x^m e^{i\phi(y)/x} a(x, y),$$

where ϕ parametrizes G and $a \in \mathcal{C}^\infty(X)$. It is quite easy to show that $\mathrm{WF}_{\mathrm{sc}}(u) \subset G$.

In the general case, where G is not necessarily projectable, we say that a function $\phi(y, w)/x$, $w \in \mathbb{R}^k$, is a non-degenerate phase function parametrizing G, locally near $q = (y_0, \tau_0, \mu_0) \in G$, if for some w_0

$$q = (y_0, d_{(x,y)}(\phi(y_0, w_0)/x)) \ \text{ and } \ d_w \phi(y_0, w_0) = 0,$$

ϕ satisfies the non-degeneracy hypothesis

$$d_{(y,w)}\Big(\frac{\partial \phi}{\partial w_i}\Big) \text{ are linearly independent at } (y_0, w_0), \tag{2.3}$$

and locally near q, G is given by

$$G = \{(y, d_{(x,y)}(\phi/x)) : \ (y, w) \in C_\phi\} \text{ where } C_\phi = \{(y, w) : \ d_w \phi(y, w) = 0\}. \tag{2.4}$$

Notice that non-degeneracy implies that C_ϕ is a smooth $(n-1)$-dimensional submanifold of (y, w)-space and that the correspondence in (2.4) is $\mathcal{C}^\infty$.

A scattering half-density valued Legendre distribution $u \in I^m_{sc}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$ associated to G is a distribution in $\mathcal{C}^{-\infty}(X; {}^{\mathrm{sc}}\Omega^{\frac12})$ such that

$$u = u_0 + \sum_{j=1}^{J} u_j \cdot \nu_j, \tag{2.5}$$

where $u_0 \in \dot{\mathcal{C}}^\infty(X; {}^{\mathrm{sc}}\Omega^{\frac12})$, $\nu_j \in \mathcal{C}^\infty(X; {}^{\mathrm{sc}}\Omega^{\frac12})$, and u_j are given in local coordinates by integrals

$$u_j(x,y) = x^{m-\frac{k}{2}+\frac{n}{4}} \int_{\mathbb{R}^k} e^{i\phi_j(y,w)/x}\, a_j(x,y,w)\, dw, \tag{2.6}$$

where ϕ_j are non-degenerate phase functions parametrizing G, $a_j \in \mathcal{C}^\infty_c([0,\epsilon)_x \times U_y \times U'_w)$. Then u is smooth in the interior of X, and as in the projectable case, the scattering wavefront of w is contained in G. Note that the definition of Legendre distributions is such that for $A \in \mathrm{Diff}^k_{\mathrm{sc}}(X)$, $u \in I^m_{sc}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$ we have $Au \in I^m_{sc}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$, i.e. Legendre distributions are preserved under the application of scattering differential operators. More generally, this statement remains true for scattering pseudo-differential operators $A \in \Psi^{k,0}_{\mathrm{sc}}(X)$; see [11].

We also need to discuss two different types of distributions associated to the union of two Legendre submanifolds. The simpler type is the class of intersecting Legendrian distributions defined in [1] and modelled directly on the intersecting Lagrangian distributions of Melrose and Uhlmann [10]. A pair (G_0, G_1) is an intersecting pair of Legendre submanifolds if G_0 is a smooth Legendre submanifold and G_1 is a smooth Legendre submanifold with boundary, meeting G_0 transversally and such that $\partial G_1 = G_0 \cap G_1$. A nondegenerate parametrization near $q \in \partial G_1$ is a phase function

$$\phi(y,w,s)/x = (\phi_0(y,w) + s\phi_1(y,w,s))/x, \quad w \in \mathbb{R}^k,$$

such that

$$q = (y_0, d_{(x,y)}\phi(y_0,w_0,0)/x), \quad d_{w,s}\phi(y,w,0) = 0$$

and such that ϕ_0 is a nondegenerate parametrization of G_0 near q, ϕ parametrizes G_1 for $s > 0$ with

$$ds,\ d\frac{\partial \phi}{\partial v_i} \ \text{ and } \ d\frac{\partial \phi}{\partial s} \ \text{ linearly independent at } q.$$

A scattering half-density valued Legendre distribution $u \in I^m_{sc}(X, (G_0, G_1); {}^{\mathrm{sc}}\Omega^{\frac12})$ associated to the pair (G_0, G_1) is a distribution in $\mathcal{C}^{-\infty}(X; {}^{\mathrm{sc}}\Omega^{\frac12})$ such that

$$u = u_0 + \sum_{j=1}^{J} u_j \cdot \nu_j, \tag{2.7}$$

where $u_0 \in I^{m+1/2}_{sc}(X, G_0; {}^{\mathrm{sc}}\Omega^{\frac12}) + I^m_{sc,c}(X, G_1; {}^{\mathrm{sc}}\Omega^{\frac12})$, $\nu_j \in \mathcal{C}^\infty(X; {}^{\mathrm{sc}}\Omega^{\frac12})$, and u_j are given by integrals

$$u_j(x,y) = x^{m-\frac{k+1}{2}+\frac{n}{4}} \int_{\mathbb{R}^k} \int_0^\infty e^{i\phi_j(y,w,s)/x}\, a_j(x,y,w,s)\, ds\, dw, \tag{2.8}$$

where the ϕ_j are non-degenerate phase functions parametrizing (G_0, G_1), and the $a_j \in \mathcal{C}^\infty_c([0,\epsilon)_x \times U_y \times U'_w) \times [0,\infty)_s$. The subscript c in $I^m_{sc,c}(X, G_1; {}^{\mathrm{sc}}\Omega^{\frac12})$ above indicates that the microlocal support is in the interior of G_1. The order of u on G_0

is always equal to that on G_1 plus one half (to see this, integrate by parts once in s). As before, u is smooth in the interior of X, and the scattering wavefront of u is contained in $G_0 \cup G_1$.

Following [11], we also need to discuss certain pairs of Legendre submanifolds with conic singularities and distributions corresponding to them. So suppose first that $\tilde{x}$ is any boundary defining function and $G^\sharp$ is a smooth Legendre submanifold of ${}^{sc}T^*_{\partial X}X$ which is given by the graph of the differential $\lambda\, d(1/\tilde{x})$. Suppose also that G is a Legendre submanifold of K and that its closure satisfies

$$\mathrm{cl}(G) \setminus G \subset G^\sharp, \tag{2.9}$$

and $\mathrm{cl}(G) \setminus G$ is the site of an at most conic singularity of $\mathrm{cl}(G)$. That is, $\mathrm{cl}(G)$ is the image, under the blow-down map, in ${}^{sc}T^*_{\partial X}X$ of a smooth (closed embedded) manifold with boundary, $\hat{G}$, of

$$[{}^{sc}T^*_{\partial X}X; \mathrm{span}\{d(1/\tilde{x})\}] \tag{2.10}$$

which intersects the front face of the blow-up transversally. This means in local coordinates generated by $(\tilde{x}, y)$, (writing $\mu = r\hat{\mu} = |\mu|\hat{\mu}$) that

$$\hat{G} = \{(y, \tau, r, \hat{\mu}) :\ r \geq 0,\ |\hat{\mu}| = 1,\ \tau = T(y, r, \hat{\mu}),\ g_j(y, r, \hat{\mu}) = 0\}, \tag{2.11}$$

$$\mathrm{cl}(G) = \{(y, \tau, \mu) :\ \tau = T(y, |\mu|, \hat{\mu}),\ g_j(y, |\mu|, \hat{\mu}) = 0\},\ \hat{\mu} = \mu/|\mu| \tag{2.12}$$

where the n functions g_j are such that $d_{(y,\hat{\mu})}g_j$, $j = 1, \ldots, n$, are independent at the base point $(y_0, \hat{\mu}_0, 0)$. In particular, $r = |\mu|$ has a non-vanishing differential on $\hat{G}$. Then the pair $\tilde{G} = (G, G^\sharp)$ is called an intersecting pair of Legendre submanifolds with conic points.

Remark 2.2. To avoid confusion, let us emphasize at this point that we do not assume, as was done in [11], that $\tilde{x}$ is the boundary defining function distinguished by a choice of scattering metric. That assumption was relevant to the analysis of [11], but not to a description of the geometry of intersecting pairs of Legendre submanifolds with conic points, which is independent of any choice of metric structure. In this paper, the boundary defining function which describes $G^\sharp$ will *not* always be that which is given by the scattering metric.

Remark 2.3. It is equivalent, in specifying the meaning of the conic singularity of G, to require that $\mathrm{cl}\, G$ is the image of a smooth manifold with boundary in the slightly different blown up space

$$[{}^{sc}T^*_{\partial X}X; \lambda d(1/\tilde{x})],$$

i.e., we only need to blow up the graph of $\lambda d(1/\tilde{x})$ (which is precisely $G^\sharp$), rather than its linear span.

For later use we will write down the parametrization of a conic pair with respect to a given boundary defining function x, not necessarily equal to $\tilde{x}$. Then $G^\sharp$ is parametrized by a function $\phi_0(y)$. Blowing up the span of $G^\sharp$, we have coordinates $y, \tau, r = |\mu - d_y\phi_0|$ and $\nu = \widehat{\mu - d_y\phi_0} \in S^{n-2}$. We say that the function ϕ/x, with $\phi \in \mathcal{C}^\infty(U_y \times [0, s_0)_s \times U'_u)$ gives a non-degenerate parametrization of the pair $\tilde{G}$ near a singular point $q \in \partial\hat{G}$ if $U \times [0, s_0) \times U'$ is a neighborhood of $q' = (y_0, 0, w_0)$ in $\partial X \times [0, \infty) \times \mathbb{R}^k$, ϕ is of the form

$$\phi(y, s, w) = \phi_0(y) + s\psi(y, s, w), \quad \psi \in \mathcal{C}^\infty(U \times [0, \infty) \times U'), \tag{2.13}$$

$$q = (y_0, -\phi(y_0), 0, \widehat{d_y\psi(q')})$$

in the coordinates (y, τ, r, ν), the differentials

$$d_{(y,w)}\psi \text{ and } d_{(y,w)}\frac{\partial\psi}{\partial w_j} \quad j = 1, \dots, k, \tag{2.14}$$

are independent at $(y_0, 0, w_0)$, and locally near q,

$$\hat{G} = \{(y, -\phi, s|d_y\psi|, \widehat{d_y\psi}) \mid (y, s, w) \in C_\phi\}, \quad \text{where} \tag{2.15}$$

$$C_\phi = \{(y, s, w) : \ \partial_s\phi(y, s, w) = 0, \ \partial_w\psi(y, s, w) = 0\}. \tag{2.16}$$

It follows that C_ϕ is diffeomorphic to the desingularized submanifold $\hat{G}$.

A Legendre distribution $u \in I^{m,p}_{sc}(X, \tilde{G}; {}^{\mathrm{sc}}\Omega^{\frac12})$ associated to such an intersecting pair $\tilde{G}$ is a distribution $u \in \mathcal{C}^{-\infty}(X; {}^{\mathrm{sc}}\Omega^{\frac12})$ such that

$$u = u_0 + u^\sharp + \sum_{j=1}^{J} u_j \cdot \nu_j, \tag{2.17}$$

where $u_0 \in I^m_{sc}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$, $u^\sharp \in I^p_{sc}(X, G^\sharp; {}^{\mathrm{sc}}\Omega^{\frac12})$, $\nu_j \in \mathcal{C}^\infty(X; {}^{\mathrm{sc}}\Omega^{\frac12})$ and the u_j are given by integrals

(2.18)

$$u_j(x, y) = \int\limits_{[0,\infty)\times\mathbb{R}^k} e^{i\phi_j(y,s,w)/x} a_j(x, y, x/s, s, w) \left(\frac{x}{s}\right)^{m+\frac{n}{4}-\frac{k+1}{2}} s^{p+\frac{n}{4}-1}\, ds\, dw$$

with $a_j \in \mathcal{C}^\infty_c([0, \epsilon)_x \times U_y \times [0, \infty)_{x/s} \times [0, \infty)_s \times U'_w)$ and ϕ_j phase functions parametrizing $\tilde{G}$. Note that $k + 1$ appears in place of k in the exponent of x (present as x/s) since there are $k + 1$ parameters, u and s.

3. Legendre geometry on manifolds with codimension 2 corners

In this section we will describe a class of Legendre submanifolds associated to manifolds M with codimension 2 corners which are endowed with certain extra structure that we will discuss in detail. The aim of this section together with the next is to describe a class of distributions that will include, in particular, the spectral projections $\mathrm{sp}(\lambda)$ on X^2_{b}. Since there are several cases of interest, we will describe the situation in reasonable generality, but before doing so let us consider the setting of primary interest, $M = X^2_{\mathrm{b}}$, to see what the ingredients of the setup should be.

Recall that $X^2_{\mathrm{b}} = [X^2; (\partial X)^2]$ is the blowup of the double space X^2 about the corner. The boundary hypersurfaces are labelled lb, rb and bf according as they arise from $(\partial X) \times X$, $X \times (\partial X)$ or $(\partial X)^2$. It has two codimension two corners, $\mathrm{lb} \cap \mathrm{bf}$ and $\mathrm{rb} \cap \mathrm{bf}$; note that $\mathrm{lb} \cap \mathrm{rb} = \emptyset$. Let $\beta_{\mathrm{b}} : X^2_{\mathrm{b}} \to X^2$ be the blow-down map and $\pi^2_{\mathrm{b,L}}$ and $\pi^2_{\mathrm{b,R}}$ be the left and right stretched projections $X^2_{\mathrm{b}} \to X$,

$$\pi^2_{\mathrm{b,L}} = \pi_L \circ \beta_{\mathrm{b}} : X^2_{\mathrm{b}} \to X, \ \pi^2_{\mathrm{b,R}} = \pi_R \circ \beta_{\mathrm{b}} : X^2_{\mathrm{b}} \to X. \tag{3.1}$$

If z or (x, y) are local coordinates on X, in the interior or near the boundary, respectively, then we will denote by z' or (x', y') the lift of these coordinates to X^2_{b} from the left factor (ie, pulled back by ${\pi^2_{\mathrm{b,L}}}^*$) and by z'' or (x'', y'') the lift from the right factor. Write $\sigma = x'/x''$. Then (x'', σ, y', y'') are coordinates on M near $\beta_{\mathrm{b}}^{-1}(p', p'') \setminus \mathrm{rb} \subset \mathrm{bf}$ where $\sigma = \frac{x'}{x''}$. We have a similar result, with the role of x'

and x'' interchanged, near $\beta_{\mathrm{b}}^{-1}(p', p'') \setminus \mathrm{lb}$. Away from bf, β_{b} is a diffeomorphism, so near the interior of lb, rb, or in the interior of X_{b}^2, we can use the product coordinates from X^2 directly.

We need to identify a Lie Algebra of vector fields, $\mathcal{V}$, on X_{b}^2 that plays the role of $\mathcal{V}_{\mathrm{sc}}$ for the scattering calculus. The structure algebra $\mathcal{V}$ will generate a rescaled cotangent bundle which will play the role of K in the previous section. It is not hard to guess what $\mathcal{V}$ should be. Recall from the introduction that the spectral projection $\mathrm{sp}(\lambda) = (2\pi)^{-1} P(\lambda) P(\lambda)^*$; heuristically, then, $(2\pi)\,\mathrm{sp}(\lambda)(z, z') = \int P(z', y)\overline{P(z'', y)}dy$ and as a function of z, Melrose and Zworski showed that $P(\lambda)$ is a conic Legendrian pair, so associated with $\mathcal{V}_{\mathrm{sc}}(X)$ acting on the z variable. Thus, we may confidently predict that the structure algebra is generated by lifting $\mathcal{V}_{\mathrm{sc}}(X)$ to X_{b}^2 from the left and the right:

$$\mathcal{V} = (\pi_{\mathrm{b,L}}^2)^* \mathcal{V}_{\mathrm{sc}}(X) \oplus (\pi_{\mathrm{b,R}}^2)^* \mathcal{V}_{\mathrm{sc}}(X). \tag{3.2}$$

Let us consider the qualitative properties of this Lie Algebra. Near the interior of bf, using coordinates (x'', σ, y', y''), we have the vector fields

$$x''^2 \partial_{x''} - x''\sigma\partial_\sigma, \quad x''\partial_{y''_i}, \quad x''\sigma\partial_{y'_j}, \quad x''\sigma^2\partial_\sigma. \tag{3.3}$$

Away from lb and rb, σ is a smooth nonvanishing function, so this generates the scattering Lie algebra on the non-compact manifold with boundary $X_{\mathrm{b}}^2 \setminus (\mathrm{lb} \cup \mathrm{rb})$, i.e. they generate $\mathcal{V}_{\mathrm{sc}}(X_{\mathrm{b}}^2 \setminus (\mathrm{lb} \cup \mathrm{rb}))$ as a $\mathcal{C}^\infty(X_{\mathrm{b}}^2 \setminus (\mathrm{lb} \cup \mathrm{rb}))$-module. Near the interior of lb, using coordinates (x', y', z''), we have the vector fields

$$x'^2\partial_{x'}, \quad x'\partial_{y'_i}, \quad \partial_{z''_j}.$$

This is the fibred-cusp algebra developed by Mazzeo and Melrose [7]: the vector fields restricted to lb are tangent to the fibers of a fibration, in this case the projection from $\mathrm{lb} = (\partial X) \times X \to \partial X$, and in addition, for a distinguished boundary defining function, here x'', $Vx'' \in x''^2\mathcal{C}^\infty(X_{\mathrm{b}}^2)$ if $V \in \mathcal{V}$. A similar property holds for the right boundary rb. Near one of the corners, say $\mathrm{lb} \cap \mathrm{bf}$, we have the vector fields (3.3). NB: at $\sigma = 0$, the vector fields $x''^2\partial_{x''}$ and $x''\sigma\partial_\sigma$ are not separately in this Lie algebra, only their difference is. This is important, say, in checking (4.2).

The structure algebra $\mathcal{V}$ may be described globally in terms of a suitably chosen total boundary defining function x (that is, a product of boundary defining functions for each hypersurface). We define x by the Euclidean-like formula

$$x^{-2} = x'^{-2} + x''^{-2}. \tag{3.4}$$

Note that here, x', say is an arbitrary defining function for X, lifted to X_{b}^2 via the left projection, but then x'' is the *same* function lifted via the right projection. We have fibrations ϕ_{lb} and ϕ_{rb} on lb and rb, the projections down to ∂X, and the trivial fibration $\phi_{\mathrm{bf}} = \mathrm{id} : \mathrm{bf} \to \mathrm{bf}$. Then

$$\mathcal{V} = \{V \in \mathcal{V}_{\mathrm{b}}(X_{\mathrm{b}}^2) \mid Vx \in x^2\mathcal{C}^\infty(X_{\mathrm{b}}^2), V \text{ tangent to } \phi_{\mathrm{lb}}, \phi_{\mathrm{rb}}, \phi_{\mathrm{bf}}\}. \tag{3.5}$$

Now we consider a generalization of the structure just described. Let M be a manifold with corners of codimension 2, $N = \dim M$, and let $M_1(M)$ denotes the set of its boundary hypersurfaces. Suppose that for each boundary hypersurface H of M we have a fibration

$$\phi_H : H \to Z_H. \tag{3.6}$$

We also impose the following compatibility conditions between the fibrations. We assume that there is one boundary hypersurface of M (possibly disconnected, but which we consider as one, and regard as a single element of $M_1(M)$) which we denote by mf ('main face'), such that

$$\phi_{\mathrm{mf}} = \mathrm{id}, \tag{3.7}$$

and that for all other $H, H' \in M_1(M)$, not equal to mf, then either $H = H'$ or $H \cap H' = \emptyset$, and if $H \neq \mathrm{mf} \in M_1(M)$ then the base of the fibration Z_H is a compact manifold without boundary, $H \cap \mathrm{mf} \neq \emptyset$ and the fibers of ϕ_H intersect mf transversally. Additionally, we suppose that a distinguished total boundary defining function x is given.

In terms of local coordinates the conditions given above mean that there are local coordinates near $p \in \mathrm{mf} \cap H$ of the form $(\tilde{x}, x'', y', y'')$ (with $y' = (y'_1, \ldots, y'_l)$, $y'' = (y''_1, \ldots, y''_m)$) such that $x'' = 0$ defines mf, $\tilde{x} \equiv x/x'' = 0$ defines H, and the fibers of ϕ_H are locally given by

$$\{(0, x'', y', y'') : \ y' = \mathrm{const}\}. \tag{3.8}$$

Near $p \in \mathrm{int}(\mathrm{mf})$ we simply have coordinates of the form $(x, \bar{y})$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{N-1})$ (the fibration being trivial there), while near $p \in \mathrm{int}(H)$, $H \in M_1(X)$, $H \neq \mathrm{mf}$, we have coordinates (x, y', z'') (with $y' = (y'_1, \ldots, y'_l)$, $z'' = (z''_1, \ldots, z''_{m+1})$) and the fibers of ϕ_H are locally given by

$$\{(0, y', z'') : \ y' = \mathrm{const}\}. \tag{3.9}$$

We let Φ be the collection of the boundary fibration-maps:

$$\Phi = \{\phi_H : \ H \in M_1(M)\} \tag{3.10}$$

which we think of as a 'stratification' of the boundary of M. It is also convenient to introduce the non-compact manifold with boundary $\tilde{M}$ given by removing all boundary hypersurfaces but mf from M:

$$\tilde{M} = M \setminus \bigcup \{H \in M_1(M) : \ H \neq \mathrm{mf}\}. \tag{3.11}$$

Definition 3.1. The fibred-scattering structure on M, with respect to the stratification Φ and total boundary defining function x, is the Lie algebra $\mathcal{V}_{\mathrm{s}\Phi}(M)$ of smooth vector fields satisfying

(3.12)
$$V \in \mathcal{V}_{\mathrm{s}\Phi}(M) \text{ iff } V \in \mathcal{V}_{\mathrm{b}}(M), \ Vx \in x^2\mathcal{C}^\infty(M), \ V \text{ is tangent to the fibres of } \Phi.$$

It is straightforward to check that $\mathcal{V}_{\mathrm{s}\Phi}(M)$ is actually well-defined if x is defined up to multiplication by positive functions which are constant on the fibers of Φ. Note that (3.12) gives local conditions and near the interior of mf, hence in $\tilde{M}$, where Φ is trivial, they are equivalent to requiring that $V \in \mathcal{V}_{\mathrm{sc}}(\tilde{M})$. Moreover, we remark that in the interior of each boundary hypersurface H, the structure is just a fibred cusp structure (though on a non-compact manifold) which is described in [7].

We write $\mathrm{Diff}_{\mathrm{s}\Phi}(M)$ for the differential operator algebra generated by $\mathcal{V}_{\mathrm{s}\Phi}(M)$. One can check in local coordinates (as was done explicitly for X^2_{b} above) that locally these are precisely the $\mathcal{C}^\infty(M)$-combinations of N independent vector fields, and therefore such vector fields are the space of all smooth sections of a vector bundle, ${}^{\mathrm{s}\Phi}TM$. We denote the dual bundle by ${}^{\mathrm{s}\Phi}T^*M$. It is spanned by one-forms of the form $d(f/x)$ where $f \in \mathcal{C}^\infty(M)$ is constant on the fibers of Φ (which is of course no restriction in the interior of mf). Since we are more interested in ${}^{\mathrm{s}\Phi}T^*M$, we will write down explicit basis in local coordinates for ${}^{\mathrm{s}\Phi}T^*M$; the corresponding bases for ${}^{\mathrm{s}\Phi}TM$ are easily found by duality. Near $p \in \mathrm{mf} \cap H$, $H \in M_1(M)$, we choose the local coordinates $(x'', \tilde{x}, y', y'')$ as discussed above. Then a basis of ${}^{\mathrm{s}\Phi}T^*M$ is given by

$$\frac{dx}{x^2},\ \frac{d\tilde{x}}{x},\ \frac{dy'_j}{x},\ \frac{dy''_k}{x''}\ (j = 1, \dots, l,\ k = 1, \dots, m). \tag{3.13}$$

Alternatively, a basis is given by

$$d\left(\frac{1}{x}\right),\ d\left(\frac{1}{x''}\right),\ \frac{dy'_j}{x},\ \frac{dy''_k}{x''}\ (j = 1, \dots, l,\ k = 1, \dots, m). \tag{3.14}$$

Corresponding to these bases, we introduce coordinates on ${}^{\mathrm{s}\Phi}T^*M$. Thus for $p \in \mathrm{bf}$ we write $q \in {}^{\mathrm{s}\Phi}T^*_pM$ as

$$q = \tau\frac{dx}{x^2} + \eta\frac{d\tilde{x}}{x} + \mu' \cdot \frac{dy'}{x} + \mu'' \cdot \frac{dy''}{x''}, \tag{3.15}$$

giving coordinates $(x'', \tilde{x}, y'_j, y''_k; \tau, \eta, \mu'_j, \mu''_k)$ on ${}^{\mathrm{s}\Phi}T^*M$, or alternatively,

$$q = \tilde{\tau}\frac{dx}{x^2} + \tau''\frac{dx''}{(x'')^2} + \mu' \cdot \frac{dy'}{x} + \mu'' \cdot \frac{dy''}{x''}, \tag{3.16}$$

giving coordinates $(x'', \tilde{x}, y'_j, y''_k; \tilde{\tau}, \tau'', \mu'_j, \mu''_k)$. The relation between the two is given by $\tau = \tilde{\tau} + \tilde{x}\tau''$, $\eta = -\tau''$. In the interior of H where we have coordinates (x, y', z''), z'' being coordinates on an appropriate coordinate patch in the interior of X (considered as the right factor in X^2), a basis is given by

$$\frac{dx}{x^2},\ \frac{dy'_j}{x}\ (j = 1, \dots, l),\ dz''_k\ (k = 1, \dots, m + 1). \tag{3.17}$$

If $M = X^2_{\mathrm{b}}$ with x given by (3.4) and the boundary fibration by $\Phi = (\phi_{\mathrm{lb}}, \phi_{\mathrm{rb}}, \phi_{\mathrm{bf}})$, we can replace x in all these equations by x' since near lb we have $x = x'(1+\sigma^2)^{-1/2}$ where σ can be used as the coordinate $\tilde{x}$ in our general notation. Therefore,

$${}^{\mathrm{s}\Phi}T^*X^2_{\mathrm{b}} = (\pi^2_{\mathrm{b,L}})^{*\mathrm{sc}}T^*X \oplus (\pi^2_{\mathrm{b,R}})^{*\mathrm{sc}}T^*X \tag{3.18}$$

and correspondingly for the structure bundle ${}^{\mathrm{s}\Phi}TX^2_{\mathrm{b}}$ we have (3.2) as desired.

We have already discussed how the b-double space X^2_{b} fits into this framework. Another example is given by the following setup. Let Y be a manifold with boundary and let C_j, $j = 1, \dots, s$, be disjoint closed embedded submanifolds of ∂Y. Let x be a boundary defining function of Y. Let $M = [Y; \cup_j C_j]$ be the blow up of Y along the C_j. The pull-back of x to M by the blow-down map $\beta : M \to Y$ gives a total boundary defining function, and β restricted to the boundary hypersurfaces of M gives the fibrations. Thus, the lift of ∂Y to M is the main face, mf, with the trivial fibration, and if $H_j = \beta^{-1}(C_j)$ is the front face corresponding to the blow-up of C_j, then $\phi_{H_j} : H_j \to C_j$. These data define a stratification on ∂M as discussed above. Such a setting is a geometric generalization of the three-body problem and it has been discussed in [14], see also [15], where the structure bundle was denoted by ${}^{3\mathrm{sc}}TM$. We note that ${}^{\mathrm{s}\Phi}T^*M$ can be identified as $\beta^{*\mathrm{sc}}T^*Y$; this is particularly easy to see by considering that ${}^{\mathrm{s}\Phi}T^*M$ is spanned by $d(f/x)$ with f constant on the fibers of β. We remark that any other boundary defining function of Y is of the form ax where $a \in \mathcal{C}^\infty(Y)$, $a > 0$, so $\beta^*a \in \mathcal{C}^\infty(M)$ is constant on the fibers of ϕ_H, i.e. on those of β, so this structure does not depend on the choice of the boundary defining function of Y as long as we use its pull-back as total bondary defining function on M.

Returning to the general setting, for each $H \in M_1(M)$, $H \neq \mathrm{mf}$, there is a natural subbundle of ${}^{\mathrm{s}\Phi}T^*_H M$ spanned by one-forms of the form $d(f/x)$ where f *vanishes* on H. Restricted to each fiber F of H, this is isomorphic to the scattering cotangent bundle of that fiber, and we denote this subbundle by ${}^{\mathrm{sc}}T^*(F;H)$. Indeed, if f vanishes on H then $f = \tilde{x}\tilde{f}$, $\tilde{f} \in \mathcal{C}^\infty(M)$, so modulo terms vanishing at H, $d(f/x) = d(\tilde{f}/x'')$ is equal to $-\tilde{f}\, dx''/(x'')^2 + d_{y''}\tilde{f}/x''$, so locally a basis of ${}^{\mathrm{sc}}T^*(F;H)$ is given by $dx''/(x'')^2$, dy''_j/x'', $j = 1, \dots, m-1$. The quotient bundle, ${}^{\mathrm{s}\Phi}T^*_H M/{}^{\mathrm{sc}}T^*(F;H)$, is the pull back of a bundle ${}^{\mathrm{s}\Phi}N^*Z_H$ from the base Z_H of the fibration ϕ_H, and ${}^{\mathrm{s}\Phi}N^*Z_H$ is isomorphic to ${}^{\mathrm{sc}}T^*_{Z_H\times\{0\}}((Z_H)_{y'} \times [0,\epsilon)_x)$. If $M = X^2_{\mathrm{b}}$, ${}^{\mathrm{s}\Phi}N^*Z_{\mathrm{lb}} = {}^{\mathrm{s}\Phi}N^*\partial X$ is simply ${}^{\mathrm{sc}}T^*_{\partial X}X$, and similarly for rb. The induced map ${}^{\mathrm{s}\Phi}T^*_H M \to {}^{\mathrm{s}\Phi}N^*Z_H$ will be denoted by $\tilde{\phi}_H$; its fibres are isomorphic to ${}^{\mathrm{sc}}T^*F$. In terms of the coordinates $(x'', y', y'', \tau, \eta, \mu', \mu'')$ on ${}^{\mathrm{s}\Phi}T^*_H M$, the fibers are given by y', τ, μ' being constant.

It will be very often convenient to realize $\phi_H^*({}^{\mathrm{s}\Phi}N^*Z_H)$ as a subbundle of ${}^{\mathrm{s}\Phi}T^*_H M$ by choosing a subbundle complementary to ${}^{\mathrm{sc}}T^*(F;H)$. Such a subbundle arises naturally if we have a smooth fiber metric on ${}^{\mathrm{s}\Phi}T^*_H M$ which allows us to choose the orthocomplement of ${}^{\mathrm{sc}}T^*(F;H)$. It can also be realized very simply if the total space of the fibration (H in our case) has a product structure. In particular, for $M = X^2_{\mathrm{b}}$, the bundle decomposition (3.2) gives ${}^{\mathrm{sc}}T^*(F;\mathrm{lb}) = (\pi^2_{\mathrm{b,R}})^*_{\mathrm{lb}}\,{}^{\mathrm{sc}}T^*X$ and $\phi^*_{\mathrm{lb}}\,{}^{\mathrm{s}\Phi}N^*\partial X = (\pi^2_{\mathrm{b,L}})^*_{\mathrm{lb}}\,{}^{\mathrm{sc}}T^*X$.

We wish to define Legendre distributions and more generally Legendre distributions with conic points, associated to our structure. In the interior of the main face, we have a scattering structure, so we will consider Legendre submanifolds of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}M$ satisfying certain conditions at the boundary of mf. Before we do so, we need to consider the behaviour of the contact form on ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}M$ at the boundary. Using the coordinates (3.15), we find that the contact form is

$$\chi = i_{x^2\partial_x}(\omega) = d\tau + \eta\, d\tilde{x} + \mu' \cdot dy' + \tilde{x}\mu'' \cdot dy''.$$

This degenerates at the boundary. In fact it is linear on fibres and vanishes identically on ${}^{\mathrm{sc}}T^*(F;H)$ at the corner, and therefore induces a one-form on the quotient bundle ${}^{\mathrm{s}\Phi}T^*_H M/{}^{\mathrm{sc}}T^*(F;H)$ over the corner. In local coordinates, this is $d\tau + \mu' \cdot dy'$, and is the pullback of a non-degenerate one-form on ${}^{\mathrm{s}\Phi}N^*Z_H$, thus giving ${}^{\mathrm{s}\Phi}N^*Z_H$ a natural contact structure.

The choice of local coordinates $(x'', \tilde{x}, y', y'')$ and the corresponding basis of ${}^{\mathrm{s}\Phi}T^*M$ given in (3.16) induces the one-form

$$d\tau'' + \mu'' \cdot dy'' \tag{3.19}$$

on the fibers of the restriction of $\tilde{\phi}_H$ to ${}^{\mathrm{s}\Phi}T^*_{H\cap\mathrm{mf}}M$. A simple computation shows that under a smooth change of coordinates

$$\tilde{x} \to a\tilde{x}, \quad x'' \to bx'', \quad y' \to Y'(y') + \tilde{x}Y'_1, \quad y'' \to Y'',$$

such that $x = \tilde{x}x''$ changes according to $x \to x(g(y') + \tilde{x}g_1)$, this is a well-defined one-form on each fibre of $\tilde{\phi}_H$ up to multiplication by the smooth positive factor a.

Thus, there is a natural contact structure on the fibres of $\tilde{\phi}_H$ as well. All three contact structures play a role in the following definition.

Definition 3.2. A Legendre submanifold G of ${}^{s\Phi}T^*M$ is a Legendre submanifold of ${}^{s\Phi}T^*_{\mathrm{mf}}M$ which is transversal to ${}^{s\Phi}T^*_{H\cap\mathrm{mf}}M$ for each $H \in M_1(M)\backslash\{\mathrm{mf}\}$, for which the map $\tilde{\phi}_H$ induces a fibration $\tilde{\phi}'_H : G \cap {}^{s\Phi}T^*_{H\cap\mathrm{mf}}M \to G_1$ where G_1 is a Legendre submanifold of ${}^{s\Phi}N^*Z_H$ whose fibers are Legendre submanifolds of ${}^{sc}T^*_{\partial F}F$.

We will say that a function $\phi(\tilde{x}, y', y'', v, w)/x$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^l$, is a non-degenerate parametrization of G locally near $q \in {}^{s\Phi}T^*_{\mathrm{mf}\cap H}M \cap G$ which is given in local coordinates as $(\tilde{x}_0, y'_0, y''_0, \tau_0, \eta_0, \mu'_0, \mu''_0)$ with $\tilde{x}_0 = 0$, if ϕ has the form

$$\phi(\tilde{x}, y', y'', v, w) = \phi_1(y', v) + \tilde{x}\phi_2(\tilde{x}, y', y'', v, w) \tag{3.20}$$

such that ϕ_1, ϕ_2 are smooth on neighborhoods of (y'_0, v_0) (in $\partial X \times \mathbb{R}^k$) and $q' = (0, y'_0, y''_0, v_0, w_0)$ (in $\mathrm{bf} \times \mathbb{R}^{k+k'}$) respectively with

$$(0, y'_0, y''_0, d_{(x,\tilde{x},y',y'')}(\phi/x)(q')) = q,\ d_v\phi(q') = 0,\ d_w\phi_2(q') = 0, \tag{3.21}$$

ϕ is non-degenerate in the sense that

$$d_{(y',v)}\frac{\partial\phi_1}{\partial v_j},\ d_{(y'',w)}\frac{\partial\phi_2}{\partial w_{j'}}\ j = 1, \dots, k,\ j' = 1, \dots, k' \tag{3.22}$$

are independent at (y'_0, v_0) and q' respectively, and locally near q, G is given by

$$G = \{(\tilde{x}, y', y'', d_{(x,\tilde{x},y',y'')}(\phi/x)) :\ (\tilde{x}, y', y'', v, w) \in C_\phi\} \tag{3.23}$$

where

$$C_\phi = \{(\tilde{x}, y', y'', v, w) :\ d_v\phi = 0,\ d_w\phi_2 = 0\}. \tag{3.24}$$

Note that these assumptions indeed imply that in ${}^{s\Phi}T^*_{\mathrm{int(mf)}}M$, but in the region where C_ϕ parametrizes G, ϕ/x is a non-degenerate phase function (in the usual sense) for the Legendre submanifold G. Also, ϕ_1 parametrizes G_1 and for fixed $(y', v) \in C_{\phi_1}$, $\phi_2(0, y', \cdot, v', \cdot)$ parametrizes the corresponding fiber of $\tilde{\phi}'_H$.

Proposition 3.3. *Let G be a Legendre submanifold. Then for any point $q \in \partial G$ there is a non-degenerate parametrization of G in some neighbourhood of q.*

Proof. The existence of such a phase function can be established using [11, Proposition 5] and the transversality condition (G to ${}^{s\Phi}T^*_{H\cap\mathrm{mf}}M$). Indeed, this proposition shows that there exists a splitting $(y'_\flat, y'_\sharp)$ and a corresponding splitting $(\mu'_\flat, \mu'_\sharp)$ such that $(y'_\sharp, \mu'_\flat)$ give coordinates on G_1. It also shows that there exists a splitting $(y''_\flat, y''_\sharp)$ such that $(y''_\sharp, \mu''_\flat)$ give coordinates on the fibers of $G \cap {}^{s\Phi}T^*_{\mathrm{lb}\cap\mathrm{bf}}M \to G_1$. Taking into account the transversality condition, we deduce that $(\tilde{x}, y'_\sharp, y''_\sharp, \mu'_\flat, \mu''_\flat)$ give coordinates on G. We write

$$\begin{aligned} \tau &= T_1(y'_\sharp, \mu'_\flat) + \tilde{x}\,T_2(\tilde{x}, y'_\sharp, y''_\sharp, \mu'_\flat, \mu''_\flat), \\ y'_\flat &= Y'_{\flat,1}(y'_\sharp, \mu'_\flat) + \tilde{x}\,Y'_{\flat,2}(\tilde{x}, y'_\sharp, y''_\sharp, \mu'_\flat, \mu''_\flat), \\ y''_\flat &= Y''_\flat(\tilde{x}, y'_\sharp, y''_\sharp, \mu'_\flat, \mu''_\flat). \end{aligned} \tag{3.25}$$

Here the special form of τ and $y'_\flat$ at $\tilde{x} = 0$ follows from the fibration requirement. We also write $T = T_1 + \tilde{x}\,T_2$, and similarly for $Y'_\flat$. Then the argument of [11, Proposition 5] shows that

$$\phi/x = (y'_\flat \cdot \mu'_\flat - T - Y'_\flat \cdot \mu'_\flat + \tilde{x}\,(y''_\flat \cdot \mu''_\flat - Y''_\flat \cdot \mu''_\flat))/x \tag{3.26}$$

gives a desired parametrization. Indeed, the proof of that proposition guarantees that ϕ/x parametrizes G in the interior of ${}^{s\Phi}T^*_{\mathrm{mf}}M$ but near q, and the fact that G is transversal to ${}^{s\Phi}T^*_{H\cap\mathrm{mf}}M$, hence it is the closure of its intersection with ${}^{s\Phi}T^*_{\mathrm{int(mf)}}M$ proves that ϕ/x gives a desired parametrization of the Legendre submanifold G of ${}^{s\Phi}T^*M$. In particular,

$$\phi_1(y', \mu'_\flat) = y'_\flat \cdot \mu'_\flat - T_1(y'_\sharp, \mu'_\flat) - Y'_{\flat_1}(y'_\sharp, \mu'_\flat) \cdot \mu'_\flat. \tag{3.27}$$

□

Next, we define pairs of Legendre submanifolds with conic points. Again it will be an intersecting pair with conic points, in the sense of Melrose-Zworski, in the interior of mf with certain specified behaviour at the boundary of mf.

Definition 3.4. A Legendre pair with conic points, $(G, G^\sharp)$, in ${}^{s\Phi}T^*M$ consists of two Legendre submanifolds G and $G^\sharp$ of ${}^{s\Phi}T^*M$ which form an intersecting pair with conic points in ${}^{sc}T^*_{\tilde M}\tilde M$ such that for each $H \in M_1(M)\backslash\{\mathrm{mf}\}$ the fibrations $\tilde\phi'_H$, $\tilde\phi^\sharp_H$ have the same Legendre submanifold G_1 of ${}^{s\Phi}N^*Z_H$ as base and for which the fibres are intersecting pairs of Legendre submanifolds with conic points of ${}^{sc}T^*_{\partial F}F$.

Let us unravel some of the complexities in this definition. First of all, $G^\sharp$ is by definition a global section, and consequently, G_1 is too. Thus, $G^\sharp$ is parametrized by a phase function of the form $(\phi_1(y') + \tilde x\phi_2(\tilde x, y', y''))/x$ near $\tilde x = 0$. Then, G_1 is parametrized by $\phi_1(y')/x$. Second, the fibres of $\tilde\phi^\sharp_H$ are pairs with conic points $(G(y'), G^\sharp(y'))$, which we may parametrize by y' since G_1 is projectable. It is not hard to see then that $G^\sharp(y')$ is parametrized by $\phi_2(0, y', y'')/x''$.

As before, we can simplify the coordinate expressions by changing to a total boundary defining function $\overline{x}$ so that $G^\sharp$ is parametrized by $\lambda/\overline{x}$. Let us temporarily redefine x to be $\overline{x}$. Then G_1 is parametrized by λ/x, and $G^\sharp(y')$ by zero. Since $(G(y'), G^\sharp(y'))$ form a conic pair in the fibre labelled by y', then $\mathrm{cl}\, G(y')$ is the image under blowdown of a smooth submanifold $\hat G(y')$, when $\{\tau'' = 0, \mu'' = 0\}$ is blown up in the fibre (cf. remark 2.3). Since $|d\mu''| \neq 0$ on the boundary of $\hat G(y')$, it is also the case that each $\mathrm{cl}\, G(y')$ is the image under blowdown of a smooth submanifold when we blow up

$$[{}^{s\Phi}T^*_{\mathrm{mf}}(M); \{\mu'' = 0, \tau'' = 0, \mu' = 0\}], \tag{3.28}$$

i.e., the span of $d(1/x)$ over mf. In the interior of mf, this is precisely the blowup that desingularizes G, so we see that the conic geometry of $(G, G^\sharp)$ inside ${}^{s\Phi}T^*_{\mathrm{mf}}M$ holds uniformly at the boundary of mf.

Let us return to our original total boundary defining function x. Then $G^\sharp$ is parametrized by a function on mf of the form $\phi_0/x = (\phi_1(y') + \tilde x\phi_2(\tilde x, y', y''))/x$ near the boundary with H. To desingularize G, we blow up the span of $G^\sharp$ inside ${}^{s\Phi}T^*_{\mathrm{mf}}M$. Coordinates on the blown-up space are

$$(\tilde x, y', y'', \tau, \frac{(\tau'' + \phi_2)}{r}, \frac{(\mu' - d_{y'}\phi_0)}{r}, r = |\mu''|, \nu = \hat\mu'')$$

near the boundary of $\hat G$, the lift of G to this blown-up space which is a smooth manifold with boundary. By a non-degenerate parametrization of $(G, G^\sharp)$, as above, near a point $q \in \partial\hat G \cap {}^{s\Phi}T^*_{\mathrm{mf}\,\cap H}M$ is meant a function ϕ/x, where

$$\phi(\tilde x, y', y'', s, w) = \phi_1(y') + \tilde x\phi_2(\tilde x, y', y'') + s\tilde x\psi(\tilde x, y', y'', s, w), \quad w \in \mathbb{R}^k,$$

is a smooth function defined on a neighborhood of $q' = (0, y_0', y_0'', 0, w_0)$ with

$$q = (0, y_0', y_0'', -\phi_1, -\frac{\psi}{|d_{y''}\psi|}, \frac{d_{y'}\psi}{|d_{y''}\psi|}, 0, \widehat{d_{y''}\psi})(q'), \quad d_s\phi(q') = 0, \quad d_w\psi(q') = 0, \tag{3.29}$$

ϕ non-degenerate in the sense that

$$d_{(y'',w)}\frac{\partial \psi}{\partial w_j},\ d_{(y'',w)}\psi, \quad j = 1, \dots, k, \tag{3.30}$$

are independent at q' and locally near q, $\hat{G}$ is given by

$$\hat{G} = \{(\tilde{x}, y', y'', -\phi_1, -\frac{\psi}{|d_{y''}\psi|}, \frac{d_{y'}\psi}{|d_{y''}\psi|}, s|d_{y''}\psi|, \widehat{d_{y''}\psi}) \mid (\tilde{x}, y', y'', s, w) \in C_\phi\}, \tag{3.31}$$

$$C_\phi = \{(\tilde{x}, y', y'', s, w) :\ d_s\phi \equiv \psi + s\psi_s = 0,\ d_w\psi = 0,\ s \geq 0\}. \tag{3.32}$$

Then $\hat{G}$ is locally diffeomorphic to C_ϕ.

Proposition 3.5. *Let $(G, G^\sharp)$ be as above. Then there is a non-degenerate parametrization of $(G, G^\sharp)$ near any point $q \in \partial\hat{G} \cap {}^{s\Phi}T^*_{\mathrm{mf}\cap H}M$.*

Proof. It is easiest to parametrize the pair using the total boundary defining function x for which $G^\sharp$ is parametrized by λ/x, as above, and then change back to the original coordinates. Using the parametrization argument of [11, Proposition 6] for the conic pairs of Legendre submanifolds $(G(y_0'), G^\sharp(y_0'))$, we see that there are coordinates y'' and a splitting $(y_\sharp'', y_\flat'')$, which we can take to be $y_\sharp'' = (y_1'', \dots, y_k'')$, $y_\flat'' = (y_{k+1}'', \dots, y_m'')$, of these such that after the span of $d(1/x)$ is blown up inside ${}^{s\Phi}T^*_{\mathrm{mf}}M$ the image of the base point q is $y'' = 0$, $\tau''/|\mu_m''| = 0$, $\hat{\mu}_1'' = \ldots = \hat{\mu}_{m-1}'' = 0$, $\hat{\mu}_m'' = 1$ where $\hat{\mu}_j'' = \mu_j''/\mu_m''$. Then $(y_\sharp'', \hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}'', \mu_m'')$ are coordinates on $\hat{G}(y_0')$, and therefore, on $\hat{G}(y')$ for y' in a neighbourhood of y_0'. Using the transversality, we conclude that $(\tilde{x}, y', y_\sharp'', \hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}'', \mu_m'')$ give coordinates on $\hat{G}$. Then, following [11, Equation (7.5)],

$$\phi/x = (-T + \tilde{x}(sy_\flat'' \cdot w - sY_\flat''(\tilde{x}, y', y_\sharp'', s, w) \cdot w + sy_m'' - sY_m''(\tilde{x}, y', y_\sharp'', s, w)))/x, \tag{3.33}$$

where on $\hat{G}$

$$\begin{aligned} \tau &= T(\tilde{x}, y', y_\sharp'', \mu_m'', \hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}'') \\ &= \lambda + \mu_m'' T_2(\tilde{x}, y', y_\sharp'', \mu_m'', \hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}''), \\ y_\flat'' &= Y_\flat''(\tilde{x}, y', y_\sharp'', \mu_m'', \hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}'') \end{aligned} \tag{3.34}$$

(so $s = \mu_m''$, $w = (\hat{\mu}_{k+1}'', \dots, \hat{\mu}_{m-1}'')$), gives a parametrization in our sense. Notice that, taking into account $sw = (\mu_{k+1}, \dots, \mu_{m-1})$, this is of the same form as (3.26) for $s > 0$ (with no $y_\flat'$ variables). □

4. Legendre distributions on manifolds with codimension 2 corners

We now define a class of distributions on a manifold M with stratification Φ as discussed above. These will be modelled on oscillatory functions whose phase is compatible with Φ. Thus, an example of a Legendre distribution is an oscillatory function

$$u = x^q e^{i\phi/x} a,\ a, \phi \in \mathcal{C}^\infty(M),\ \phi \text{ constant on the fibers of } \Phi. \tag{4.1}$$

As mentioned in the previous section, the differentials $d(\phi/x)$ for such functions ϕ span ${}^{\mathrm{s}\Phi}T^*M$, so the definition of ${}^{\mathrm{s}\Phi}T^*M$ is precisely designed so that

$$(4.2)\quad P \in \mathrm{Diff}_{\mathrm{s}\Phi}(M),\ u = x^q e^{i\phi/x} a \text{ with } \phi,\ a \text{ as in (4.1)} \\ \Longrightarrow\quad Pu = x^q e^{i\phi/x}\tilde a, \quad \tilde a \in C^\infty(M).$$

In general, we need to consider superpositions of such oscillatory functions. Thus, near the corner $H \cap \mathrm{mf}$, $H \in M_1(M) \setminus \{\mathrm{mf}\}$, we consider distributions of the form

$$(4.3)\quad u(x'', \tilde x, y', y'') = \int e^{i\phi(\tilde x, y', y'', v, w)/x} a(x'', \tilde x, y', y'', v, w) \\ (x'')^{m-(k+k')/2+N/4}\tilde x^{r-k/2+N/4-f/2}\, dv\, dw$$

with $N = \dim M$, $a \in \mathcal{C}_c^\infty([0,\epsilon) \times U \times \mathbb{R}^{k+k'})$, U open in mf, f the dimension of the fibres of H and ϕ a phase function parametrizing a Legendrian G on U. On the other hand, in the interior of H we consider distributions of the form

$$(4.4)\qquad u'(x, y', z'') = \int e^{i\phi_1(y', w)/x} a(x, y', z'', w) x^{r-k/2+N/4-f/2}\, dw$$

with $N = \dim M$, $a \in \mathcal{C}_c^\infty([0,\epsilon) \times U \times \mathbb{R}^k)$, U open in H, ϕ_1 is a phase function parametrizing the Legendrian G_1 (associated to G as in Definition 3.2) on $\phi_H(U)$. Note that distributions of the form (4.3) are automatically of the form (4.4) in the interior of H since $\phi = \phi_1 + \tilde x\phi_2$ as in (3.20), so $e^{i\phi/x}$ is equal to $e^{i\phi_1/x}$ times a smooth function in this region.

We let ${}^{\mathrm{s}\Phi}\Omega^{\frac12}M$ be the half-density bundle induced by ${}^{\mathrm{s}\Phi}T^*M$; near the interior of mf this is just ${}^{\mathrm{sc}}\Omega^{\frac12}\tilde M$. For $M = X_{\mathrm{b}}^2$, (3.2) implies that

$$(4.5)\qquad {}^{\mathrm{s}\Phi}\Omega^{\frac12}X_{\mathrm{b}}^2 = (\pi_{\mathrm{b,L}}^2)^{*\mathrm{sc}}\Omega^{\frac12} \otimes (\pi_{\mathrm{b,R}}^2)^{*\mathrm{sc}}\Omega^{\frac12},$$

so its smooth sections are spanned by $(\pi_{\mathrm{b,L}}^2)^*|dg|^{\frac12} \otimes (\pi_{\mathrm{b,R}}^2)^*|dg|^{\frac12}$ over $\mathcal{C}^\infty(X_{\mathrm{b}}^2)$. Notice that in the blow-up situation $[Y; \cup_j C_j]$ of the previous section, we have ${}^{\mathrm{s}\Phi}\Omega^{\frac12}[Y; \cup_j C_j] = \beta^{*\mathrm{sc}}\Omega^{\frac12}Y$.

To simplify the notation and allow various orders at the various hypersurfaces H, we introduce an 'order family', $\mathcal{K}$, which assigns a real number to each hypersurface. Thus, $\mathcal{K}$ is a function

$$(4.6)\qquad \mathcal{K} : M_1(M) \to \mathbb{R}.$$

Here we only consider one-step polyhomogeneous ('classical') symbols a, but the discussion could be easily generalized to arbitrary polyhomogeneous symbols.

Definition 4.1. A Legendrian distribution $u \in I^{\mathcal{K}}_{\mathrm{s}\Phi}(M, G; {}^{\mathrm{s}\Phi}\Omega^{\frac12})$ associated to a Legendre submanifold G of ${}^{\mathrm{s}\Phi}T^*M$ is a distribution $u \in \mathcal{C}^{-\infty}(M; {}^{\mathrm{s}\Phi}\Omega^{\frac12})$ with the property that for any $H \in M_1(M) \setminus \{\mathrm{mf}\}$ and for any $\psi \in \mathcal{C}_c^\infty(M)$ supported away from $\bigcup\{H' \in M_1(M) :\ H' \neq H, H' \neq \mathrm{mf}\}$, ψu is of the form

$$(4.7)\qquad \psi u = u_0 + u_1 + \sum_{j=1}^{J} w_j \cdot \nu_j + \sum_{j=1}^{J'} w_j' \cdot \nu_j',$$

where $u_0 \in \dot{\mathcal{C}}^\infty(M; {}^{\mathrm{s}\Phi}\Omega^{\frac12})$, $u_1 \in I^m_{\mathrm{sc,c}}(\tilde M, G; {}^{\mathrm{s}\Phi}\Omega^{\frac12})$ (c meaning compact support in $\tilde M$), $m = \mathcal{K}(\mathrm{mf})$, $\nu_j, \nu_j' \in \mathcal{C}^\infty(M; {}^{\mathrm{s}\Phi}\Omega^{\frac12})$, and $w_j, w_j' \in \mathcal{C}^{-\infty}(M)$ are given by oscillatory integrals as in (4.3) and (4.4) respectively with $m = \mathcal{K}(\mathrm{mf})$, $r = \mathcal{K}(H)$.

Remark 4.2. Sometimes, when convenient, we use the notation $I^{m,r}_{\mathrm{s}\Phi}(M, G, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ when we localize a fibred Legendre distribution near a hypersurface H so that only the order m at mf and r at H are relevant.

Remark 4.3. The reason for the appearance of the fiber-dimension f in our order convention, i.e. in (4.3) – (4.4), can be understood from the blow-up example $[Y; C]$ of the previous section. Namely, suppose that C is a closed embedded submanifold of ∂Y, Y being a manifold with boundary. Let G be the scattering conormal bundle ${}^{\mathrm{sc}}N^*C$ of C; this is simply the annihilator of $\mathcal{V}_{\mathrm{sc}}(Y; C) = x\mathcal{V}_{\mathrm{b}}(Y; C)$, $\mathcal{V}_{\mathrm{b}}(Y; C)$ denoting the set of vector fields in $\mathcal{V}_{\mathrm{b}}(Y)$ tangent to C in the ordinary sense. If C is given by $x = 0$, $y' = 0$, in local coordinates (x, y), $y = (y', y'')$, and the dual coordinates are (τ, μ', μ''), then G is given by $x = 0$, $y' = 0$, $\tau = 0$, $\mu'' = 0$. Suppose that $u \in I^m_{sc}(Y, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$. Thus, u is a sum of oscillatory integrals of the form

$$x^{m-\frac{k}{2}+\frac{n}{4}} \int e^{iy'\cdot w/x} a(x, y, w)\, dw,$$

$n = \dim Y$, k the codimension of C in ∂Y, and a smooth and compactly supported. Pulling back u to $[Y; C]$ by the blow-down map β gives a function of the form

$$\beta^* u = x^{m-\frac{k}{2}+\frac{n}{4}} b, \quad b \in \mathcal{C}^\infty([Y; C]),$$

that vanishes to infinite order on mf. Indeed $Y' = y'/x$ is a coordinate in the interior of ff, so above we simply took the inverse Fourier transform of a in w. Thus, taking G' to be the zero section of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}[Y; C]$, we see that $\beta^* u \in I_{\mathrm{s}\Phi}([Y; C], G', {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with order ∞ on mf and order m on ff. The agreement of the Legendrian order in these two ways of looking at u justifies the order convention in the definition.

For this class of distributions to be useful, we need to know that it is independent of the choice of parametrizations used. This follows from

Proposition 4.4. *Let L be a Legendrian as in Definition 3.2, let $q \in L$, and let ϕ, $\tilde\phi$ be two parametrizations of a neighbourhood $U \subset L$ containing q. Then if A is smooth and has support sufficiently close to q, the function*

$$u = x^q \tilde{x}^r \int e^{i\phi(y', y'', \tilde{x}, v, w)/x} a(y', y'', \tilde{x}, x'', v, w)\, dv\, dw$$

can be written

$$u = u_0 + x^{\tilde q} \tilde{x}^{\tilde r} \int e^{i\tilde\phi(y', y'', \tilde{x}, \tilde v, \tilde w)/x} \tilde{a}(y', y'', \tilde{x}, x'', \tilde v, \tilde w)\, d\tilde v\, d\tilde w$$

for some $u_0 \in \dot{\mathcal{C}}^\infty(X)$, $\tilde q$, $\tilde r$, and smooth $\tilde a$.

Proof. The heart of the proof is an adaptation of Hörmander's proof of equivalence of phase functions ([5], Theorem 3.1.6) to this setting. If q lies above an interior point of mf, then the result is proved in [11], so we only need to consider q lying above a point on the boundary of mf. Using coordinates as in (3.20), we will prove the following Lemma.

Lemma 4.5. *Assume that the phase functions*

$$\phi(\tilde{x}, y', y'', v, w) = \phi_1(y', v) + \tilde{x}\phi_2(\tilde{x}, y', y'', v, w)$$

and

$$\tilde\phi(\tilde{x}, y', y'', \tilde v, \tilde w) = \tilde\phi_1(y', \tilde v) + \tilde{x}\tilde\phi_2(\tilde{x}, y', y'', \tilde v, \tilde w)$$

parametrize a neighbourhood U of L containing

$$\begin{aligned}(4.8)\quad q &= (0, y_0', y_0'', d_{(x,\tilde{x},y',y'')}(\tilde{\phi}/x)(q')) = (0, y_0', y_0'', d_{(x,\tilde{x},y',y'')}(\tilde{\phi}/x)(\tilde{q}')),\\ q' &= (0, y_0', y_0'', v_0, w_0) \in C_\phi, \quad \tilde{q}' = (0, y_0', y_0'', \tilde{v}_0, \tilde{w}_0) \in C_{\tilde{\phi}}.\end{aligned}$$

If $\dim v = \dim \tilde{v}$, $\dim w = \dim \tilde{w}$, *and the signatures of* $d^2_{vv}\phi_1$ *and* $d^2_{ww}\phi_2$ *at* (y_0', v_0) *and* $(0, y_0', y_0'', v_0, w_0)$ *are equal to the corresponding signatures for* $\tilde{\phi}$, *then the two phase functions are equivalent near* q. *That is, there is a coordinate change* $\tilde{v}(\tilde{x}, y', y'', v, w)$ *and* $\tilde{w}(\tilde{x}, y', y'', v, w)$ *mapping* q' *to* $\tilde{q}'$ *such that*

$$\tilde{\phi}(\tilde{x}, y', y'', \tilde{v}(\tilde{x}, y', y'', v, w), \tilde{w}(\tilde{x}, y', y'', v, w)) = \phi(\tilde{x}, y', y'', v, w)$$

in a neighbourhood of q'.

Proof. To prove this we follow the structure of Hörmander's proof. The first step is to show that $\tilde{\phi}$ is equivalent to a phase function $\psi(\tilde{x}, y', y'', v, w)$ that agrees with ϕ at the set C_ϕ, given by (3.24) to second order. This is accomplished by pulling $\tilde{\phi}$ back with a diffeomorphism of the form

$$(\tilde{x}, y', y'', v, w) \mapsto \big(\tilde{x}, y', y'', V(\tilde{x}, y', y'', v, w), W(\tilde{x}, y', y'', v, w)\big)$$

that restricts to the canonical diffeomorphism from C_ϕ to $C_{\tilde{\phi}}$. This diffeomorphism is given by the identifications of C_ϕ and $C_{\tilde{\phi}}$ with L. The mapping F may be constructed as in the first part of Hörmander's proof.

In the second step, we first note that by the equivalence result for Legendrians on manifolds with boundary ([11], Proposition 5), there is a change of variables $v' = v'(y', v)$ near (y_0', v_0) such that

$$\psi(\tilde{x}, y', y'', v', w) = \phi_1(y', v) + O(\tilde{x}).$$

Therefore, we may assume that $\phi_1 = \psi_1$. If that is so, then $\phi - \psi$ is $O(\tilde{x})$, in addition to vanishing to second order at $C_\phi = \{\partial_v \phi = 0, \partial_w \phi_2 = 0\}$. Therefore, one can express

$$(4.9)\qquad \psi - \phi = \frac{\tilde{x}}{2}\Big(\sum \frac{\partial\phi}{\partial v_j}\frac{\partial\phi}{\partial v_k} b^{11}_{jk} + 2\sum \frac{\partial\phi}{\partial v_j}\frac{\partial\phi_2}{\partial w_k} b^{12}_{jk} + \sum \frac{\partial\phi_2}{\partial w_j}\frac{\partial\phi_2}{\partial w_k} b^{22}_{jk}\Big)$$

for some smooth functions b^{ab}_{jk} of all variables.

We look for a change of variables of the form

$$(4.10)\qquad \begin{aligned}(\tilde{v} - v)_j &= \tilde{x}\sum \frac{\partial\phi}{\partial v_k} a^{11}_{jk} + \tilde{x}\sum \frac{\partial\phi_2}{\partial w_k} a^{12}_{jk}\\ (\tilde{w} - w)_j &= \sum \frac{\partial\phi}{\partial v_k} a^{21}_{jk} + \sum \frac{\partial\phi_2}{\partial w_k} a^{22}_{jk}.\end{aligned}$$

Using Taylor's theorem, we may write

$$\begin{aligned}(4.11)\quad \phi(\tilde{x}, y', y'', \tilde{v}, \tilde{w}) - \phi(\tilde{x}, y', y'', v, w) &= \sum (\tilde{v} - v)_j \frac{\partial\phi}{\partial v_j} + \tilde{x}\sum (\tilde{w} - w)_j \frac{\partial\phi_2}{\partial w_j}\\ + \sum (\tilde{v} - v)_j (\tilde{v} - v)_k c^{11}_{jk} + \tilde{x}\sum (\tilde{v} - v)_j &(\tilde{w} - w)_k c^{12}_{jk} + \tilde{x}\sum (\tilde{w} - w)_j (\tilde{w} - w)_k c^{22}_{jk}\end{aligned}$$

for some smooth functions c^{ab}_{jk}. We wish to equate ψ and $\phi(\cdot, \tilde{v}, \tilde{w})$. Substituting (4.10) into (4.11) and equating with (4.9) gives an expression which is quadratic in the $\partial_v \phi$ and $\partial_w \phi_2$. Matching coefficients gives the matrix equation

$$A = B + Q(A, C, \tilde{x})$$

to be solved, where

$$A = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix},$$

and similarly for the others. The expression $Q(A, C, x)$ is a polynomial in the entries which is homogeneous of degree two in A.

By the inverse function theorem, this has a solution $A = A(B, C, \tilde{x})$ if B is sufficiently small. The proof is completed by showing that there is a path ψ_t of phase functions parametrizing L that interpolate between ϕ and ψ. We define ψ_t by considering a path of b^{ab}_{jk}'s and using (4.9).

Recall that the nondegeneracy hypothesis is that

$$\left.\begin{aligned} &d_{y',v}\frac{\partial\psi}{\partial v_i} \text{ are independent,} \\ &d_{y'',w}\frac{\partial\psi_2}{\partial w_i} \text{ are independent} \end{aligned}\right\} \quad \text{at } q'. \tag{4.12}$$

At this point, by (4.9),

$$d^2_{vv}\psi = d^2_{vv}\phi \quad \text{and} \quad d^2_{y'v}\psi = d^2_{y'v}\phi$$

regardless of the value of B, so the first condition is automatically satisfied. The other derivatives are

$$\begin{aligned} d^2_{ww}\psi &= d^2_{ww}\phi + (d^2_{ww}\phi)(b^{22})(d^2_{ww}\phi) \\ d^2_{y''w}\psi &= d^2_{y''w}\phi + (d^2_{y''w}\phi)(b^{22})(d^2_{ww}\phi) \end{aligned} \tag{4.13}$$

where b^{22} is evaluated at q'. Thus the second nondegeneracy condition at q' is that $\mathrm{Id} + (b^{22})(d^2_{ww})\phi$ is nondegenerate. As shown in the last part of Hörmander's proof, the condition that the signatures of $d^2_{ww}\phi$ and $d^2_{ww}\psi$ are equal is precisely the condition that interpolation between ϕ and ψ is possible by nondegenerate phase functions. This completes the proof of the lemma. □

Returning to the proof of the proposition, if the dimension and signature hypotheses of the lemma are satisfied, then the phases are equivalent in some neighbourhood of q and thus, by a change of variables, the function u can be written with respect to $\tilde{\phi}$. In the general case, one modifies ϕ and $\tilde{\phi}$ by adding a nondegenerate quadratic form $\langle Cv', v'\rangle + \tilde{x}\langle Dw', w'\rangle$ in extra variables to ϕ, and a similar expression to $\tilde{\phi}$, so as to satisfy the hypotheses of the Lemma. This requires a compatibility mod 2 between the difference of the dimension of v and w and the signature of the corresponding Hessian, but this is always satisfied by Theorem 3.1.4 of [5]. The function u can be written with respect to the modified ϕ since the effect of the quadratic term is just to multiply by a constant times powers of x and $\tilde{x}$. By the Lemma, u can be written with respect to the modified $\tilde{\phi}$, and therefore, with respect to $\tilde{\phi}$ itself. □

Legendrian distributions with conic points are defined analogously by oscillatory integrals of the form

$$\begin{aligned} w(x'', \tilde{x}, y', y'') = \int_{[0,\infty)\times\mathbb{R}^{k'}} &e^{i\phi(\tilde{x}, y', y'', s, w)/x} a(x'', \tilde{x}, y', y'', x''/s, s, w) \\ &\left(\frac{x''}{s}\right)^{m-(k'+1)/2+N/4} s^{p-1+N/4}\tilde{x}^{r+N/4-f/2}\, ds\, dw \end{aligned} \tag{4.14}$$

with $N = \dim M$, $a \in \mathcal{C}_c^\infty([0,\epsilon) \times U \times [0,\infty) \times [0,\infty) \times \mathbb{R}^{k'})$, U open in mf, ϕ is a phase function parametrizing a Legendrian with conic points $(G, G^\sharp)$ on U. Note that $k'+1$ appears instead of k' since there are $k'+1$ parameters: w and s. The integral converges absolutely if $x'' > 0$ and $\tilde{x} > 0$ (so $x > 0$ and we are in the interior of M) giving a smooth function in the interior of M. Indeed, if $x'' > c > 0$ and the support of a is in $x''/s < c'$, then on the support of a, $s > c/c'$, so there is no problem with the s integral at 0. Also, if a is supported in $s > 0$, then this gives a parametrization completely analogous to (4.3) (without the v variables, since we are assuming that we do not need them), i.e. in this case we simply have a distribution in $I_{\mathrm{s}\Phi}(M, G^\sharp; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with order m on mf and r on H. On the other hand, if a is supported in $x''/s > 0$, then we can change variables $\rho = s/x''$ (in place of s) allows us to write the integral as one analogous to (4.3) with parameters (ρ, v), so the result is in $I_{\mathrm{s}\Phi}(M, G, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with order p on mf and order r on H. In addition, if we just keep x'' away from 0 (but possibly let $\tilde{x}$ to 0), i.e. in the interior of H, the previous argument showing $s > 0$ on the support of a works whereby we conclude that the oscillatory integral in (4.14) reduces to an oscillatory section of the form (4.4) without any v variables to integrate out. In particular, localized in the interior of H, the integral gives an element of $I_{\mathrm{s}\Phi}(M, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ of order r on H (and infinite order vanishing at mf) (or of $I_{\mathrm{s}\Phi}(M, G^\sharp; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with order r at H).

Since the order on $H \neq \mathrm{mf}$ is forced to be the same for both $G^\sharp$ and G, the order of a distribution associated to a Legendrian pair with conic points can be captured by a single order family $\mathcal{K}$, which we take to give the orders on G, and an additional real number p to give the order on $G^\sharp$ at mf.

Definition 4.6. A Legendrian distribution $u \in I_{\mathrm{s}\Phi}^{\mathcal{K},p}(M, \tilde{G}; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ associated to a Legendre pair with conic points $\tilde{G} = (G, G^\sharp)$ of ${}^{\mathrm{s}\Phi}T^*M$ (with $\mathcal{K}$ as in (4.6)) is a distribution $u \in \mathcal{C}^{-\infty}(M; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with the property that for any $H \in M_1(M) \setminus \{\mathrm{mf}\}$ and for any $\psi \in \mathcal{C}_c^\infty(M)$ supported away from $\bigcup\{H' \in M_1(M) :\ H' \neq \mathrm{mf},\ H' \neq H\}$, ψu is of the form

$$\psi u = u_0 + u_1 + u_2 + \sum_{j=1}^{J} w_j \cdot \nu_j \tag{4.15}$$

where $u_0 \in I_{\mathrm{s}\Phi}^{\mathcal{K}'}(M, G^\sharp; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ with $\mathcal{K}'(H) = \mathcal{K}(H)$ if $H \neq \mathrm{mf}$, $\mathcal{K}'(\mathrm{mf}) = p$, $u_1 \in I_{\mathrm{s}\Phi}^{\mathcal{K}}(M, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$, $u_2 \in I_{\mathrm{sc,c}}^{\mathcal{K}(\mathrm{mf}),p}(\tilde{M}, \tilde{G}; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ (c meaning compact support in $\tilde{M}$), $\nu_j \in \mathcal{C}^\infty(M; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$, and $w_j \in \mathcal{C}^{-\infty}(M)$ are given by oscillatory integrals as in (4.14).

Remark 4.7. Again, for the sake of convenience, we occasionally use the notation $I_{\mathrm{s}\Phi}^{m,p,r}(M, \tilde{G}, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ when we localize a Legendre distribution, associated to an intersecting pair $\tilde{G}$ with conic points, near a hypersurface H so that only the order m on G and p on $G^\sharp$ at mf, and the (common) order r at H are relevant.

Again, for this class of distributions to be useful, we need to know that it is independent of the choice of parametrizations. This follows in the same way as above once we have the following result about equivalence of phase functions parametrizing Legendrian pairs with conic points.

Lemma 4.8. *Let $(G, G^\sharp)$ be a Legendrian pair with conic points and*

$$\phi(\tilde{x}, y', y'', s, w) = \phi_1(y') + \tilde{x}(\phi_2(\tilde{x}, y', y'') + s\psi(y', s, w))$$

and

$$\tilde\phi(\tilde x, y', y'', \tilde s, \tilde w) = \phi_1(y') + \tilde x(\phi_2(\tilde x, y', y'') + \tilde s\tilde\psi(\tilde x, y', y'', \tilde s, \tilde w))$$

two phase functions parametrizing a neighbourhood U of L containing

$$q = (0, y_0', y_0'', -\phi_1, -\frac{\psi}{|d_{y''}\psi|}, \frac{d_{y'}\psi}{|d_{y''}\psi|}, 0, \widehat{d_{y''}\psi})(q') \tag{4.16}$$

$$= (0, y_0', y_0'', -\phi_1, -\frac{\tilde\psi}{|d_{y''}\tilde\psi|}, \frac{d_{y'}\tilde\psi}{|d_{y''}\tilde\psi|}, 0, \widehat{d_{y''}\tilde\psi})(\tilde q'), \tag{4.17}$$

$$q' = (0, y_0', y_0'', 0, w_0) \in C_\phi, \quad \tilde q' = (0, y_0', y_0'', 0, \tilde w_0) \in C_{\tilde\phi}. \tag{4.18}$$

If $\dim w = \dim \tilde w$, *and the signatures of* $d^2_{ww}\psi$ *and* $d^2_{\tilde w\tilde w}\tilde\psi$ *at* $(0, y_0', y_0'', 0, w_0)$ *are equal, then the two phase functions are locally equivalent.*

Proof. Again we follow the structure of Hörmander's proof. First, we may assume that $\tilde\phi$ agrees with ϕ at the set C_ϕ, given by (3.32) to second order. As before, this is possible if we can find a diffeomorphism of the form

$$(\tilde x, y', y'', s, w) \mapsto \big(\tilde x, y', y'', S(\tilde x, y', y'', s, w), W(\tilde x, y', y'', s, w)\big)$$

that restricts to the canonical diffeomorphism from C_ϕ to $C_{\tilde\phi}$. This is simply a version of the result from [11], Proposition 7, with y' an extra parameter.

In the second step, we fix y' and $\tilde x = 0$ and use the result of Melrose and Zworski for Legendrian pairs with conic points, which says that there is a coordinate change mapping $\psi(0, y', y'', s, w)$ to $\tilde\psi(0, y', y'', \tilde s, \tilde w)$. We can deduce the same result, with the same proof, letting y' vary parametrically. Thus we can assume that $\psi = \tilde\psi$ at $\tilde x = 0$. Consequently, $\tilde\phi - \phi = O(\tilde x^2 s)$. Thus, we may write

$$\tilde\phi - \phi = \frac{\tilde x^2 s}{2}\Big((\psi + s\psi_s)^2 b_{00} + 2\sum(\psi + s\psi_s)(\psi_{w_j})b_{0j} + \sum(\psi_{w_j})(\psi_{w_k})b_{jk}\Big). \tag{4.19}$$

We look for a change of variables of the form

$$\begin{aligned} (\tilde s - s) &= s\big((\psi + s\psi_s)a_{00} + \sum \psi_{w_j}a_{0j}\big) \\ (\tilde w - w)_j &= \sum(\psi + s\psi_s)a_{j0} + \sum \psi_{w_k}a_{jk} \end{aligned} \tag{4.20}$$

Using Taylor's theorem, we may write

$$\begin{aligned} \phi(\tilde x, y', y'', \tilde s, \tilde w) - \phi(\tilde x, y', y'', s, w) = \tilde x\Big((\tilde s - s)(\psi + s\psi_s) + s\sum(\tilde w - w)_j\psi_{w_j} \\ + \sum(\tilde s - s)^2 c_{00} + 2s\sum(\tilde s - s)(\tilde w - w)_k c_{0j} + s\sum(\tilde w - w)_j(\tilde w - w)_k c_{jk}\Big) \end{aligned} \tag{4.21}$$

for some smooth functions $c_{\alpha k}$. Equating $\tilde\phi$ and $\phi(\cdot, \tilde s, \tilde w)$ gives the matrix equation

$$A = \tilde x B + Q(A, C, s)$$

where Q is homogeneous of degree two in A. This always has a solution for small $\tilde x$ by the inverse function theorem. This completes the proof of the Lemma. □

5. Poisson operators and spectral projections

In this section we will prove a statement asserted in the introduction, namely, that the spectral projection at energy $\lambda^2 > 0$ can be expressed in terms of the Poisson operators.

We briefly recall the structure of the Poisson operator from the paper of Melrose and Zworski [11]. We also remark that a somewhat different approach has been presented in [16]; in this approach the form (2.18) of Legendre distributions associated to intersecting pairs with conic points arises very explicitly. Following [11], we often make $P(\pm\lambda)$ a map from half-densities to half-densities to simplify some of the notation. The correspondence between the smooth functions and half-densities is given by the trivialization of the half-density bundles by the Riemannian densities; so for example we have

$$P(\pm\lambda)(a|dh|^{1/2}) = (P(\pm\lambda)a)\,|dg|^{1/2}, \quad a \in \mathcal{C}^\infty(\partial X). \tag{5.1}$$

With this normalization the kernel of $P(\pm\lambda)$ will be a section of the kernel density bundle

$$\mathrm{KD}_{\mathrm{sc}}^{\frac{1}{2}} = \pi_L^* \, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}} X \otimes \pi_R^* \, \Omega^{\frac{1}{2}} \, \partial X. \tag{5.2}$$

where $\pi_L : X \times \partial X \to X$ and $\pi_R : X \times \partial X \to \partial X$ are the projections. We remark that smooth sections of $\mathrm{KD}_{\mathrm{sc}}^{\frac{1}{2}}$ are of the form $a|dg|^{\frac{1}{2}}|dh|^{\frac{1}{2}}$, $a \in \mathcal{C}^\infty(X \times \partial X)$, while smooth sections of ${}^{\mathrm{sc}}\Omega^{\frac{1}{2}}(X \times \partial X)$ are of the form $a|dg|^{\frac{1}{2}}|dh|^{\frac{1}{2}}x^{-(n-1)/2}$. Melrose and Zworski constructed the kernel of $P(\lambda)$ in [11] as a section of ${}^{\mathrm{sc}}\Omega^{\frac{1}{2}}(X \times \partial X)$, essentially by identifying $\mathrm{KD}_{\mathrm{sc}}^{\frac{1}{2}}$ with ${}^{\mathrm{sc}}\Omega^{\frac{1}{2}}(X \times \partial X)$ (via the mapping $a|dg|^{\frac{1}{2}}|dh|^{\frac{1}{2}} \mapsto a|dg|^{\frac{1}{2}}|dh|^{\frac{1}{2}}x^{-(n-1)/2}$), given a choice of boundary defining function x.

We introduce the following Legendre submanifolds of ${}^{\mathrm{sc}}T^*_{\partial X \times \partial X}(X \times \partial X)$:

$$G^\sharp(\lambda) = \{(y, y', \tau, \mu, \mu') : \ \mu = 0, \ \mu' = 0, \ \tau = -\lambda\}, \tag{5.3}$$

and for $\lambda > 0$ let

$$\begin{aligned} G(\lambda) =&\{(y, y', \tau, \mu, \mu') : \ (y, \hat\mu) = \exp(sH_{\frac{1}{2}h})(y', \hat\mu'), \ \tau = \lambda\cos s, \\ &\mu = \lambda(\sin s)\hat\mu, \ \mu' = -\lambda(\sin s)\hat\mu', \ s \in [0, \pi), \ (y', \hat\mu') \in S^*\partial X\}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} G(-\lambda) =&\{(y, y', \tau, \mu, \mu') : \ (y, \hat\mu) = \exp((s-\pi)H_{\frac{1}{2}h})(y', \hat\mu'), \ \tau = \lambda\cos s, \\ &\mu = \lambda(\sin s)\hat\mu, \ \mu' = -\lambda(\sin s)\hat\mu', \ s \in (0, \pi], \ (y', \hat\mu') \in S^*\partial X\}. \end{aligned} \tag{5.5}$$

We recall that $G(\lambda)$ is actually smooth at $s = 0$, and $G(-\lambda)$ is smooth at $s = \pi$. We introduce notation for the Legendre pair

$$\tilde G(\lambda) = (G(\lambda), G^\sharp(\lambda)), \quad \tilde G(-\lambda) = (G(-\lambda), G^\sharp(-\lambda)). \tag{5.6}$$

For later use, we also define $G_{y_0} = G_{y_0}(\lambda)$ to be the Legendrian in ${}^{\mathrm{sc}}T^*_{\partial X} X$

$$G_{y_0}(\lambda) = \{(y', \tau, \mu') \mid (y_0, y', \tau, \mu, \mu') \in G(\lambda) \text{ for some } \mu\}. \tag{5.7}$$

Thus, Melrose and Zworski construct the kernel of $P(\pm\lambda)$, $\lambda > 0$, as a Legendre distribution associated to a Legendre pair with conic points:

$$P(\pm\lambda) \in I_{sc}^{-(2n-1)/4, -1/4}(X \times \partial X, \tilde G(\pm\lambda), \mathrm{KD}_{\mathrm{sc}}^{\frac{1}{2}}). \tag{5.8}$$

To express the spectral projection in terms of the Poisson operator we first need to note the following technical result.

Lemma 5.1. *The resolvent $R(\sigma)$ extends to a continuous family of operators from $x^{1/2+\epsilon}L^2$ to $x^{-1/2-\epsilon}L^2$ on each of the closed sets*

$$\{\sigma \in \mathbb{C} \mid \operatorname{Re}\sigma \geq \epsilon \text{ and } \pm \operatorname{Im}\sigma \geq 0\}$$

for any $\epsilon > 0$.

Proof. (sketch) This result follows readily from [9] (see also [16, Proposition 2.5] for more details) by noting that the estimates of Propositions 8 and 9, which are uniform scattering wavefront set estimates on $R(\sigma+i\epsilon)$ as $\epsilon \to 0$, $\sigma \in \mathbb{R}$, can also be shown to be uniform on a small σ-interval not containing zero. Then the argument in the proof of Proposition 14 extends to show joint continuity of $R(\sigma+i\epsilon)$ in both σ and ϵ. □

This allows us to obtain the desired formula:

Lemma 5.2. *The generalized spectral projection at energy λ is*

$$\operatorname{sp}(\lambda) = (4\pi\lambda)^{-1}P(\lambda)P(\lambda)^* = (4\pi\lambda)^{-1}P(-\lambda)P(-\lambda)^*.$$

Proof. Regarding the spectral projection $E_{(a^2,b^2)}$ as a map $\dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$, and using Lemma 5.1 we may take the pointwise limit as $\epsilon \to 0$ in the integral (1.4) to obtain

$$E_{(a^2,b^2)} = \frac{1}{2\pi i}\int_a^b \Big(R(\lambda^2+i0) - R(\lambda^2-i0)\Big)2\lambda d\lambda.$$

Thus it remains to show that

$$R(\lambda^2+i0) - R(\lambda^2-i0) = \frac{i}{2\lambda}P(\lambda)P(\lambda)^* = \frac{i}{2\lambda}P(-\lambda)P(-\lambda)^* \tag{5.9}$$

as operators $\dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$. To derive (5.9), first recall that $R(\lambda^2 \pm i0)f$, $f \in \dot{\mathcal{C}}^\infty(X)$, has an asymptotic expansion similar to (1.2). Namely,

$$R(\lambda^2 \pm i0)f = \pm e^{\pm i\lambda/x}x^{(n-1)/2}v_\pm,\ v_\pm \in \mathcal{C}^\infty(X). \tag{5.10}$$

Thus,

$$R(\lambda^2+i0)f - R(\lambda^2-i0)f = P(\lambda)(v_-|_{\partial X}), \tag{5.11}$$

since both sides are generalized eigenfunctions of H, have an expansion as in (1.2) and the leading coefficients of the $e^{-i\lambda/x}$ term agree, namely they are $v_-|_{\partial X}$. Thus, we only have to determine $v_-|_{\partial X}$ in (5.10).

For this purpose, recall the boundary pairing formula from Melrose's paper [9, Proposition 13]. Thus, suppose that

$$u_j = e^{-i\lambda/x}x^{(n-1)/2}v_{j,-} + e^{i\lambda/x}x^{(n-1)/2}v_{j,+},\ v_{j,\pm} \in \mathcal{C}^\infty(X), \tag{5.12}$$

and $f_j = (H-\lambda^2)u_j \in \dot{\mathcal{C}}^\infty(X)$. Let $a_{j,\pm} = v_{j,\pm}|_{\partial X}$. Then

$$2i\lambda \int_{\partial X} (a_{1,+}\,\overline{a_{2,+}} - a_{1,-}\,\overline{a_{2,-}})\,dh = \int_X (u_1\,\overline{f_2} - f_1\,\overline{u_2})\,dg. \tag{5.13}$$

We apply this result with $u_1 = -R(\lambda^2-i0)f$, $u_2 = P(\lambda)a$, $a \in \mathcal{C}^\infty(\partial X)$. Thus, $f_1 = -f$, $f_2 = 0$, $a_{2,-} = a$, $a_{1,+} = 0$ and $a_{1,-} = v_-|_{\partial X}$ with the notation of (5.10). We conclude that

$$2i\lambda \int_{\partial X} v_-|_{\partial X}\,\overline{a}\,dh = -\int_X f\,\overline{P(\lambda)a}\,dg. \tag{5.14}$$

Thus, $v_-|_{\partial X} = \frac{i}{2\lambda}P(\lambda)^*f$. Combining this with (5.11), we deduce the first equality in (5.9). The second equality can be proved similarly. □

6. An example

Before we discuss the general case, let us consider the case of the resolvent of the Laplacian on Euclidean space. Using the Fourier transform, it is not hard to get expressions for the kernels of the spectral projections and resolvent.

The spectral projection at energy λ^2 is

$$\frac{\lambda^{(n-2)}}{2(2\pi)^n} \int_{S^{n-1}} e^{i(z'-z'')\cdot\lambda\omega} \, d\omega \, |dz'dz''|^{\frac{1}{2}}$$

Since $z' - z''$ is a homogeneous function of degree one on $\mathbb{R}^{2n}$, this is a Legendrian distribution on the radial compactification of $\mathbb{R}^{2n}$; denote the Legendre submanifold, which will be described in detail below, by L. Let us see what sort of object this is on X^2_{b}, where here X is the radial compactification of $\mathbb{R}^n$. In fact, as noted in [3],

$$X^2_{\mathrm{b}} = [\overline{\mathbb{R}^{2n}}; S', S'']$$

where S' is the submanifold of the boundary of $\overline{\mathbb{R}^{2n}}$ which is the intersection of the plane $z' = 0$ with the boundary, and similarly for S''. Let S be the intersection of the plane $z' = z''$ with the boundary. Away from S, $|z' - z''|$ is a large parameter, and stationary phase in ω shows that the Legendrian L is the union of two projectable Legendrians $L_\pm$ parametrized by $\pm\lambda|z' - z''|$. At S, stationary phase is no longer valid, and L is no longer projectable. In fact the part of L lying above S is

$$\{\hat{z}' = \hat{z}'', |z'|/|z''| = 1, \mu' = -\mu'', \tau = 0, \eta^2 + 2|\mu'|^2 = 2\lambda^2\} \tag{6.1}$$

in the coordinates (7.5), which is codimension one in L. It is the intersection of ${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$ with $\Sigma(\Delta - 2\lambda^2)$, where ${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$ is the boundary of the closure of $N^* \operatorname{diag}_{\mathrm{b}}$ in ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}^{2n}}$,

$${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}} = \{y' = y'', \sigma = 1, \mu' = -\mu'', \tau = 0\},$$

and $\Delta = \Delta_L + \Delta_R$ is the Laplacian in $2n$ variables. Notice that $(\Delta - 2\lambda^2)\operatorname{sp}(\lambda) = 0$, so we know that $L \subset \Sigma(\Delta - 2\lambda^2)$ and the bicharacteristic flow of $\Delta - 2\lambda^2$ is tangent to L. Under bicharacteristic flow, $\dot{\tau} = -2(|\mu'|^2 + |\mu''|^2 + \eta^2)$, which is strictly negative on (6.1), so $L_\pm$ are in fact the flowouts from $L \cap {}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$ by backward/forward bicharacteristic flow.

Next consider the structure of $\operatorname{sp}(\lambda)$ near lb (the case of rb is similar). After blowup of say S'', creating the boundary hypersurface lb, we have smooth coordinates $x'' = 1/|z''|$, $\sigma = |z''|/|z'|$, $\hat{z}'$ and $\hat{z}''$. Let $x' = x''\sigma$. Then near the corner $x'' = \sigma = 0$, the phase function takes the form

$$\lambda|z' - z''| = \lambda(|z' - z''|^2)^{1/2} = \lambda(1 - \sigma\hat{z}' \cdot \hat{z}'' + O(\sigma^2))/x',$$

which parametrizes a Legendrian distribution on X^2_{b} near $\mathrm{lb} \cap \mathrm{bf}$. In fact we can read off that the Legendre submanifold G_1 is $G^\sharp(\lambda)$, and the Legendre submanifold lying in the fibre above $\hat{z}'$ is $G_{\hat{z}'}(\lambda)$.

If there were a potential V present, $H = \Delta + V$, say $V \in \dot{\mathcal{C}}^\infty(\mathbb{R}^n)$, then the lifts of V from the left and right factor to $\overline{\mathbb{R}^{2n}}$ would be singular, and in fact Legendrian, at S' and S'' respectively. Then the analysis of [1] suggests that there would be another Legendrian propagating out from S' and S''. This Legendrian may be considered to be the origin of the second Legendre submanifold, denoted $L^\sharp$, that we shall see is present in the spectral projection in the general case.

Next consider the resolvent $R(\lambda^2 + i0)$. This has kernel

$$\int e^{i(z'-z'')\cdot\zeta}(|\zeta|^2 - \lambda^2 - i0)^{-1}\,d\zeta\ |dz'dz''|^{\frac{1}{2}}.$$

To understand this kernel, we need to cut it into pieces. First we introduce a cutoff in ζ, $1 = \psi(\zeta) + (1 - \psi(\zeta))$ where ψ vanishes in a neighbourhood of $|\zeta| = \lambda$ and $1 - \psi$ has compact support. Then

$$\int e^{i(z'-z'')\cdot\zeta}(|\zeta|^2 - \lambda^2 - i0)^{-1}\psi(\zeta)\,d\zeta\ |dz'dz''|^{\frac{1}{2}}$$

is (the kernel of) a constant coefficient pseudodifferential operator on $\mathbb{R}^n$, and hence lies in the scattering calculus. To deal with the other piece, we write

$$(|\zeta|^2 - \lambda^2 - i0)^{-1} = (|\zeta| + \lambda)^{-1}(|\zeta| - \lambda - i0)^{-1} = \frac{i}{|\zeta| + \lambda}\int_0^\infty e^{-i(|\zeta|-\lambda)\overline{t}}d\overline{t}, \tag{6.2}$$

change variable to $t = \overline{t}|z|$ and substitute into the previous expression. The result is

$$\int\int_0^\infty dt\ e^{i(z'-z'')\cdot\zeta-(|\zeta|-\lambda)t|z|}(1 - \psi(\zeta))\frac{i|z|}{|\zeta| + \lambda}\,|\zeta|^{n-1}d|\zeta|d\hat{\zeta}\ |dz'dz''|^{\frac{1}{2}}. \tag{6.3}$$

The phase $\Phi = (z' - z'')\cdot\zeta - (|\zeta| - \lambda)t|z|$ now parametrizes the pair of intersecting Legendrian submanifolds $(L_+, {}^{\mathrm{sc}}N^*\operatorname{diag}_{\mathrm{b}})$ (see section 2). Notice that away from S, ie, away from ${}^{\mathrm{sc}}N^*\operatorname{diag}_{\mathrm{b}}$, Φ is an exact nondegenerate quadratic in $(|\zeta|, t)$ and the rest of the integral is independent of t. Therefore, the top order part of the stationary phase approximation is accurate to all orders, and we find that microlocally in this region, the resolvent is $2\pi i$ times the spectral projection. We have ignored some analytical details here: namely, the fact that the integrand in (6.3) is not compactly supported in t, and we should retain a factor $e^{-\epsilon\overline{t}}$ in the integral (6.2) where ϵ is later sent to zero. These details will be taken care of when we treat the general situation.

As with the spectral projection, we expect to get an extra Legendrian whenever we perturb the Laplacian by a potential.

In the rest of this paper we show that when we look at these kernels on $\mathbb{R}^{2n}$ from the point of scattering microlocal analysis they become a rather good guide to the general situation.

7. Legendre structure of the spectral projections

In this section we will show that the spectral projection $\mathrm{sp}(\lambda)$ is a Legendre distribution associated to an intersecting pair $(L = L(\lambda), L^\sharp = L^\sharp(\lambda) \cup L^\sharp(-\lambda))$ of Legendre submanifolds with conic points, on $M = X^2_{\mathrm{b}}$.

We now introduce the Legendre submanifolds L and $L^\sharp$. First note that the another coordinate system near bf, different from that discussed at the beginning of Section 3, which is sometimes more convenient, is given by replacing σ by $\theta = \arctan\sigma$. Thus, (3.13) shows that we can write $\alpha \in {}^{\mathrm{s}\Phi}T^*_pM$, $p = (x'', \theta, y', y'')$ near $\mathrm{lb}\cap\mathrm{bf}$, as

$$\alpha = \tau\frac{dx}{x^2} + \eta\frac{d\theta}{x} + \tilde{\mu}'\cdot\frac{dy'}{x} + \tilde{\mu}''\cdot\frac{dy''}{x''}. \tag{7.1}$$

Note that we have $x = x''\sin\theta$, so we should take $\tilde{x} = \sin\theta$ in (3.15). Comparison with (3.15) also shows that η is defined slightly differently here, but this should not cause any trouble. Also, we use the notation $\tilde{\mu}'$, $\tilde{\mu}''$ in place of μ', μ'', since these will be reserved from the coordinates coming from the bundle decomposition (3.18).

We note that an attractive feature of the coordinates $(x, \theta, y', y'', \tau, \eta, \tilde{\mu}', \tilde{\mu}'')$ induced by (7.1) is that in these coordinates the product metric

$$\tilde{g} = (\pi^2_{\mathrm{b,L}})^* g + (\pi^2_{\mathrm{b,R}})^* g \tag{7.2}$$

becomes

$$\tilde{g} = \frac{dx^2}{x^4} + \frac{d\theta^2 + \cos^2\theta\,(\pi^2_{\mathrm{b,L}})^* h + \sin^2\theta\,(\pi^2_{\mathrm{b,R}})^* h}{x^2} \tag{7.3}$$

which is of the form

$$\tilde{g} = \frac{dx^2}{x^4} + \frac{\tilde{h}}{x^2}, \tag{7.4}$$

so $\tilde{g}$ is a scattering metric on $\tilde{M}$.

Corresponding to the direct sum decomposition of ${}^{\mathrm{s}\Phi}T^*M$ given by the pull-backs of ${}^{\mathrm{sc}}T^*X$ by the stretched projections, (3.18), we can also use coordinates $(x'', \sigma, y', y'', \tau', \tau'', \mu', \mu'')$ on ${}^{\mathrm{sc}}T^*\tilde{M}$ where we write $q \in {}^{\mathrm{sc}}T^*_p\tilde{M}$, $p = (x'', \sigma, y', y'')$, as

$$q = \tau' \frac{dx'}{(x')^2} + \tau'' \frac{dx''}{(x'')^2} + \mu' \cdot \frac{dy'}{x'} + \mu'' \cdot \frac{dy''}{x''}; \tag{7.5}$$

we may of course replace σ by $\theta = \arctan\sigma$. A reason why (7.5) is often convenient is that it matches the structure of the composition $P(\lambda)P(\lambda)^*$. The relationship between (7.5) and (7.1) is given by

$$\tau = \tau'\cos\theta + \tau''\sin\theta,\ \ \eta = \tau'\sin\theta - \tau''\cos\theta,\ \ \tilde{\mu}' = (\cos\theta)\mu',\ \ \tilde{\mu}'' = (\sin\theta)\mu''. \tag{7.6}$$

Now, the Laplacian of the product metric $\tilde{g}$, dicussed in (7.3), is

$$\Delta_{\tilde{g}} = \Delta_L + \Delta_R, \tag{7.7}$$

Δ_L and Δ_R being the Laplacians of g lifted from either factor of X using the product $X \times X$, and lifting further to the blown up space X^2_{b}. Similarly lifting the perturbation V, and using that $(\Delta + V - \lambda^2)P(\lambda) = 0$, we see that

$$(\Delta_{\tilde{g}} + V_L + V_R - 2\lambda^2)\,\mathrm{sp}(\lambda) = 0, \tag{7.8}$$

the density factors being trivialized by the Riemannian density. Thus, the kernel of $\mathrm{sp}(\lambda)$, also denoted by $\mathrm{sp}(\lambda)$, is a generalized eigenfunction of

$$\tilde{H} = \Delta_{\tilde{g}} + V_L + V_R = \Delta_L + \Delta_R + V_L + V_R \in \mathrm{Diff}^2_{\mathrm{s}\Phi}(M). \tag{7.9}$$

Away from $\mathrm{lb}\cup\mathrm{rb}$, i.e. on $\tilde{M}$, $\tilde{H}$ is a scattering differential operator (just as every other operator in $\mathrm{Diff}_{\mathrm{s}\Phi}(M)$), $\tilde{H} \in \mathrm{Diff}^2_{\mathrm{sc}}(\tilde{M})$, so based on the paper of Melrose and Zworski [11], we can expect the appearance of Legendrian submanifolds of ${}^{\mathrm{sc}}T^*\tilde{M}$ so that $\mathrm{sp}(\lambda)$ becomes a (singular) Legendre distribution. It is thus natural to expect that the full analysis, including the behavior of the kernel up to $\mathrm{lb}\cup\mathrm{rb}$ would involve Legendre distributions on the manifold M with fibred boundary faces.

We first define the smooth Legendre submanifold L of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}M$ by using the parametrization that will appear in the composition $P(\lambda)P(\lambda)^*$. Thus, we let

(7.10)
$$\begin{aligned}
L = L(\lambda) =&\{(\theta, y', y'', \tau', \tau'', \mu', \mu'') : \ \exists (y, \hat\mu) \in S^*\partial X, \ \ s, s' \in [0, \pi], \\
&(\sin s)^2 + (\sin s')^2 > 0, \ \text{s.t.} \\
&\sigma = \tan\theta = \frac{\sin s'}{\sin s}, \ \tau' = \lambda\cos s', \ \tau'' = -\lambda\cos s, \\
&(y', \mu') = \lambda\sin s' \exp(s' H_{\frac12 h})(y, \hat\mu), (y'', \mu'') = -\lambda\sin s \exp(s H_{\frac12 h})(y, \hat\mu)\} \\
&\cup \{(\theta, y, y, \lambda, -\lambda, 0, 0) : \ \theta \in [0, \pi/2], \ y \in \partial X\} \\
&\cup \{(\theta, y, y, -\lambda, \lambda, 0, 0) : \ \theta \in [0, \pi/2], \ y \in \partial X\}.
\end{aligned}$$

(Note that the requirement $(\sin s)^2 + (\sin s')^2 > 0$ just means that s and s' cannot take values in $\{0, \pi\}$ at the same time.) Although it is not immediately obvious, with this definition L is a smooth submanifold of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}M$.

Let us show this. Note that to get points away from rb and away from the last two sets in (7.10), we must have $\sin s > 0$, so we can restrict the parameter s to $(0, \pi)$ in this region. Thus, τ'', σ and $(y'', \mu''/|\mu''|)$ (note that $\mu'' \neq 0$ as $\sin s \neq 0$) give coordinates on L away from rb and the last two sets. A similar argument works near rb if we replace $\sigma = \tan\theta$ by $\cot\theta$. To see the smoothness near the last two sets in the definition of L above, note that near $\tau' = \lambda$, $\tau'' = -\lambda$, $\sigma \in [0, C)$ where $C > 1$, L is given by

$$\begin{aligned}
&\{(\sigma, y', y'', \tau', \tau'', \mu', \mu'') : \ \exists (y, \mu) \in T^*\partial X, \ |\mu| < C^{-1}, \ \sigma \in [0, C) \ \text{s.t.} \\
&\tau' = \lambda(1 - |\sigma\mu|^2)^{1/2}, \ \tau'' = -\lambda(1 - |\mu|^2)^{1/2}, \\
&(y', \mu') = \lambda\exp(f(\sigma\mu)H_{\frac12 h})(y, \sigma\mu), \ (y'', \mu'') = -\lambda\exp(f(\mu)H_{\frac12 h})(y, \mu)\}
\end{aligned} \tag{7.11}$$

where

$$f(\mu) = \frac{\arcsin(|\mu|)}{|\mu|} \tag{7.12}$$

is smooth and $f(0) = 1$. Thus, the differential of the map

$$(y, \mu) \mapsto -\lambda\exp(f(\mu)H_{\frac12 h})(y, \mu) = (y'', \mu'') \tag{7.13}$$

is invertible near $\mu = 0$, so it gives a diffeomorphism near $|\mu| = 0$. Hence, σ and (y'', μ'') give coordinates on L in this region, so L is smooth here. Near rb, but still with τ' near λ, $\tau'' = -\lambda$, we replace $\sigma = \tan\theta$ by $\cot\theta$ and deduce that $\cot\theta$ and (y', μ') give coordinates on L here. Near $\tau' = -\lambda$, $\tau'' = \lambda$ a similar argument works if we replace s by $\pi - s$, s' by $\pi - s'$, $(y, \hat\mu)$ by $\exp(\pi H_{\frac12 h})(y, \hat\mu)$ in the definition of L above. Notice that the in the first set on the right hand side of (7.10) (and also in the other pieces)

$$(y'', \hat\mu'') = \exp((s - s')H_{\frac12 h})(y', \hat\mu') \tag{7.14}$$

for some appropriate $\hat\mu'$, $\hat\mu''$, so the distance of y' and y'' is at most π. Also notice that with the notation of (7.1), τ is given by

$$\tau = \lambda(\cos s' \cos\theta - \cos s \sin\theta) \tag{7.15}$$

so that $\tau \in (-\sqrt{2}\lambda, \sqrt{2}\lambda)$ on L, i.e. L is disjoint from the 'radial set' in ${}^{\mathrm{sc}}T^*_{\partial\tilde M}\tilde M$ induced by $\Delta_{\tilde g} - 2\lambda^2$.

Now we discuss the 'ends' of L with the goal of understanding the composition $P(\lambda)P(\lambda)^*$ in mind. The closure of L in ${}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}M$ is

$$\operatorname{cl} L = L \cup F_\lambda \cup F_{-\lambda} \tag{7.16}$$

where

$$F_\lambda = \{(\theta, y', y'', -\lambda, -\lambda, 0, 0) : \ \exists \hat\mu \in S^*_{y'}\partial X \text{ s.t. } \exp(\pi H_{\frac12 h})(y', \hat\mu) \in S^*_{y''}\partial X\}. \tag{7.17}$$

The condition in the definition of F_λ just amounts to saying that there is a geodesic of length π between y' and y''; it arises if we let $s \to 0$, $s' \to \pi$ (or vice versa) in (7.10). We also define

$$L^\sharp(\lambda) = \{(\theta, y', y'', -\lambda, -\lambda, 0, 0) : \ y', y'' \in \partial X, \ \theta \in [0, \pi/2]\}, \tag{7.18}$$

so $L^\sharp(\lambda)$ is a Legendrian submanifold of ${}^{\mathrm{sc}}T^*_{\mathrm{bf}}\tilde M$ for each λ and

$$\operatorname{cl} L \cap L^\sharp(\pm\lambda) = F_{\pm\lambda}. \tag{7.19}$$

Proposition 7.1. *The pair*

$$\tilde L = \tilde L(\lambda) = (L(\lambda), L^\sharp(\lambda) \cup L^\sharp(-\lambda)) \tag{7.20}$$

is a pair of intersecting Legendre manifolds with conic points.

Proof. First we show near $\mathrm{bf} \cap \mathrm{lb}$ that L and $L^\sharp(\pm\lambda)$ satisfy the conditions of Definition 3.2. This is clear for $L^\sharp(\pm\lambda)$, which projects under $\tilde\phi_{\mathrm{lb}}$ to $G^\sharp(\pm\lambda)$ (see (5.3)) at the corner with fibres $G^\sharp(\pm\lambda)$. To see this for L, notice that at the corner, i.e. when $\sigma = 0$, then by (7.10) we have

$$\begin{aligned} L \cap {}^{\mathrm{s}\Phi}T^*_{\mathrm{lb}\cap\mathrm{bf}}X^2_{\mathrm{b}} &= \{(0, y', y', \pm\lambda, \mp\lambda, 0, 0)\} \\ &\cup \{(0, y', y'', \pm\lambda, -\lambda\cos s, 0, \mu'') \mid (y'', \mu'') = \mp\lambda \sin s \exp(H_{\frac12 h})(y', \hat\mu')\} \end{aligned} \tag{7.21}$$

which also projects to $G^\sharp(\lambda) \cup G^\sharp(-\lambda)$, with fibre above $(y', \pm\lambda, 0)$ equal to $G_{y'}(\pm\lambda)$. It is shown in [11] that $(G_{y'}(\pm\lambda), G^\sharp(\lambda) \cup G^\sharp(-\lambda))$ form an intersecting pair with conic points. Since $\tilde L$ is invariant under the transformation $(z', z'') \to (z'', z')$, $\tau' \to -\tau''$, $\tau'' \to -\tau'$, $\mu' \to -\mu''$, $\mu'' \to -\mu'$, this is also the case near rb. It remains to show that in the interior of bf that they form an intersecting pair with conic points.

To see this, notice that using the description (7.10) of L, near $L^\sharp(\lambda)$, s' is near π and s is near 0 there, so we may write, using (7.10), $|\mu''| = \lambda \sin s$, $|\mu'| = \lambda \sin s' = \sigma|\mu''|$, $s' = \pi - \arcsin(\sigma|\mu''|/\lambda)$ (arcsin takes values in $[-\pi/2, \pi/2]$ as usual), $(y, \hat\mu) = \exp(-sH_{\frac12 h})(y'', -\hat\mu'')$, and then near $L^\sharp(-\lambda)$, L is given by

$$\begin{aligned} &\{(\sigma, y', y'', \tau', \tau'', \mu', \mu'') : \ \mu'' = |\mu''|\hat\mu'', \\ &\quad \tau' = -(\lambda^2 - \sigma^2|\mu''|^2)^{1/2}, \ \tau'' = -(\lambda^2 - |\mu''|^2)^{1/2}, \\ &\quad (y', \mu') = \sigma|\mu''| \exp((\pi - \arcsin(\sigma|\mu''|/\lambda) - \arcsin(|\mu''|/\lambda))H_{\frac12 h})(y'', -\hat\mu'')\}, \end{aligned} \tag{7.22}$$

i.e. the $N-1$ variables $\sigma, y'', |\mu''|, \hat\mu''$ give coordinates on L near $L^\sharp(-\lambda)$. This means that L is the image of a smooth manifold in μ''-polar coordinates. However, notice also that $|\mu'| = \sigma|\mu''|$, i.e. away from rb it is a bounded multiple of $|\mu''|$, and similarly $\tau' - \tau''$ is a bounded multiple of $|\mu''|$, so L is the image of a smooth manifold in $(\tau' - \tau'', \mu', \mu'')$-polar coordinates. A similar argument works near rb with the role of μ' and μ'' interchanged, so we can conclude that L is the image of a closed embedded submanifold $\hat{L}$ of

$$[{}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}M; \mathrm{span}\{\frac{dx'}{(x')^2} + \frac{dx''}{(x'')^2}\}] = [{}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}M; \mathrm{span}\{d((\sigma+1)/x')\}], \tag{7.23}$$

under the blow-down map. Also $\hat{L}$ is transversal to the front face of the blow-up (since away from rb resp. lb, $|\mu''|$ resp. $|\mu'|$ is a coordinate on $\hat{G}_\lambda$). These, together with corresponding arguments near $L^\sharp(-\lambda)$, establish the result. □

Finally we come to one of the main results of this paper.

Theorem 7.2. *The kernel of the spectral projection at energy λ^2 is a Legendre distribution associated to the intersecting pair with conic points, $\tilde{L} = \tilde{L}(\lambda) = (L(\lambda), L^\sharp(\lambda) \cup L^\sharp(-\lambda))$. More precisely, we have*

$$\mathrm{sp}(\lambda) = (4\pi\lambda)^{-1} P(\lambda)P(\lambda)^* \in I^{\mathcal{K}, \frac{n-2}{2}}_{\mathrm{s}\Phi}(X^2_b, \tilde{L}, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}) \tag{7.24}$$

with $\mathcal{K}(\mathrm{lb}) = \mathcal{K}(\mathrm{rb}) = \frac{n-1}{2}$, $\mathcal{K}(\mathrm{mf}) = -\frac{1}{2}$.

Proof. We prove the result by microlocalizing in each factor of the composition $P(\lambda)P(\lambda)^* = P(-\lambda)P(-\lambda)^*$ and showing quite explicitly that we get a Legendre conic pair associated to $(L, L^\sharp)$. Thus, let Q_i and R_j be pseudodifferential operators on X, $\sum_i Q_i = \sum_j R_j = \mathrm{Id}$, associated to a microlocal partition of unity, such that each $Q_i P(\lambda)$, $Q_i P(-\lambda)$, $R_j P(\lambda)$ and $R_j P(-\lambda)$ (which are all Legendrian conic pairs associated to $(G(\lambda), G^\sharp(\lambda))$ or $(G(-\lambda), G^\sharp(-\lambda))$) can be parametrized by a single phase function. Then we write

$$\mathrm{sp}(\lambda) = \sum_{i,j} Q_i P(\lambda)P(\lambda)^* R_j^*$$

and analyze each term separately. There are three qualitatively different types of parametrizations that can occur: (1) if Q_i is microlocalized near $G^\sharp(-\lambda)$ then $Q_i P(\lambda)$ has a phase function of the form $-\cos d(y', y)/x'$; (2) if Q_i is microlocalized where $-\lambda < \tau < \lambda$ then $Q_i P(\lambda)$ has phase function $\phi(y', y, v)/x'$ parametrizing $G(\lambda)$; and (3) if Q_i is localized near $G^\sharp(\lambda)$ then $Q_i P(\lambda)$ has phase function $(1 + s\psi(y', y, s, w))/x'$ parametrizing $(G(\lambda), G^\sharp(\lambda))$. Similarly for $Q_i P(-\lambda)$, $(R_j P(\lambda))^*$, etc. A subcase of the last case is $\psi \equiv 0$, i.e. the phase function parametrizes only $G^\sharp(\lambda)$.

Let us denote by (a, b) the case that Q_i falls into case a and R_j into case b. Then there are nine possibilities in all, but we can eliminate several of them. First, writing $Q_i P(\lambda)P(\lambda)^* R_j^*$ as $Q_i P(-\lambda)P(-\lambda)^* R_j^*$ this has the effect of 'switching $G(\lambda)$ and $G(-\lambda)$' so that case $(3, a)$ is the same as case $(1, 4-a)$ with the sign of λ reversed. Thus we can neglect cases $(3,1), (3,2), (3,3)$. Similarly, case $(2,3)$ is the same as $(2,1)$. Thus we have only to consider cases $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ and $(1,3)$. Also, since $\mathrm{sp}(\lambda)$ is formally self-adjoint in the sense that

$$\mathrm{sp}(\lambda)(z', z'') = \overline{\mathrm{sp}(\lambda)(z'', z')},$$

we may restrict attention to a neighbourhood of lb and the interior of bf.

First consider case $(2,2)$. In this case, the composition is an integral of the form

$$\begin{aligned}(7.25)\qquad &\int e^{i\phi/x} x'^{-k/2} a'(y',y,v,x') x''^{-k'/2} a''(y,y'',w,x'')\,dv\,dw\,dy,\\ &\phi(y',y'',\theta,v,w,y) = \cos\theta\,\phi'(y',y,v) - \sin\theta\,\phi''(y,y'',w)\end{aligned}$$

where $v \in \mathbb{R}^k$, $w \in \mathbb{R}^{k'}$ and ϕ' and ϕ'' both parametrize $G(\lambda)$ in region two. (We will omit the $s\Phi$-half-density factors for notational simplicity.) The integral is rapidly decreasing in x unless $d_y\phi = 0$. This implies that

$$\sigma = \tan\theta = \frac{|d_y\phi'|}{|d_y\phi''|},$$

but in case $(2,2)$, $d_y\phi_i = \mu_i$ is never zero, so the integral is rapidly decreasing at lb and rb. Thus we only need to analyze (7.25) in the interior of bf.

Let us show that ϕ is a nondegenerate phase function in the interior of bf, with phase variables (v,w,y). Consider the phase function ϕ'. With y held fixed, ϕ' parametrizes G_y, the Legendrian defined in (5.7). (It is a non-degenerate parametrization since the coordinate functions y_i are linearly independent on $G(\lambda)$, hence on C_ϕ, $d(y_i)$ and $d_{(y',y,v)}\partial_{v_j}\phi'$ are independent. That is equivalent to independence of $d_{(y',v)}\partial_{v_j}\phi'$ when $d_v\phi' = 0$, which is to say that ϕ with y held fixed parametrizes G_y nondegenerately.) Note that on G_y, the rank of the $(n-1)$ differentials $d(\mu_i)$ is everywhere at least $(n-2)$, by (5.4). Therefore, when $d_v\phi' = 0$, the rank of the $k+n-1$ differentials $d_{y',v}\partial_{y_i}\phi'$ and $d_{y',v}\partial_{v_j}\phi'$ is at least $k+n-2$. Next, note that

$$d_\theta\partial_y\phi = -\sin\theta\frac{\partial\phi'}{\partial y} - \cos\theta\frac{\partial\phi''}{\partial y},$$

and when $d_y\phi = 0$, we have $\cos\theta\, d_y\phi' = \sin\theta\, d_y\phi''$, so this is

$$d_\theta\phi = -\frac{1}{\cos\theta}(\sin^2\theta + \cos^2\theta) d_y\phi'' \neq 0.$$

Finally, by the same reasoning as above, the differentials $d_{(y'',w)}\partial_{w_k}\phi''$ are independent when $d_w\phi'' = 0$. Putting these facts together we have shown that ϕ is a nondegenerate phase function.

It is not hard to see that the Legendrian parametrized by ϕ is L. Setting $d_v\phi = d_w\phi = d_y\phi = 0$ we see that $(\sigma, y', y'', \tau', \tau'', \mu', \mu'')$ is on the Legendrian only if there are points $(y, y', \tau', \mu_1, \mu')$ and $(y, y'', \tau'', \mu_2, \mu'')$ in $G(\lambda)$ with y', y'' and y on the same geodesic, with y not the middle point, with $\sigma = |\mu_1|/|\mu_2|$. With $|\mu_1| = \sin s'$ and $|\mu_2| = \sin s$ this is the same as (7.10). Thus (7.25) is in $I^m_{s\Phi}(X^2_b, L)$ with $m = -\frac{1}{2}$ by (4.3).

Next consider case $(1,1)$. In fact we can treat $(1,1)$ and $(1,2)$ simultaneously, by considering the integral of the form

(7.26)

$$x^{-k'}\int e^{i\left(-\cos\theta\cos d(y',y) - \sin\theta\,\phi''(y,y'',w)\right)/x} a'(y',y',x') a''(y,y'',x'',w)\;dy\;dw$$

where $w \in \mathbb{R}^{k'}$ and a' is supported where y' is close to y. In the interior of bf the phase function ϕ is non-degenerate since the differentials $d_{y'}\partial_{y_i}\phi$ are linearly independent when y' is close to y, and $d_{(y'',v)}\partial_{v_j}\phi''$ are linearly independent as above. The same reasoning as above shows that ϕ parametrizes L in the interior of bf. Near lb the phase is of the form $-\cos d(y',y) - \theta\phi'' + O(\theta^2)$, which has the form (3.20) with $\phi_1 = -\cos d(y',y)$ and $\phi_2 = -\phi''$. It is a non-degenerate parametrization of L near the corner since $d_{y'}\partial_{y_i}\phi_1$ and $d_{(y'',v)}\phi_2$ are independent. Since ϕ_1 parametrizes $G^\sharp(-\lambda)$ and for fixed $y = y'$, ϕ_2 parametrizes $G_{y'}(-\lambda)$, we see that (7.26) is in $I_{\mathrm{s}\Phi}^{-\frac{1}{2},\frac{n-1}{2}}(X_{\mathrm{b}}^2, \tilde{L})$.

Near the interior of lb, we have an integral expression

$$\int e^{-i\lambda \cos d(y',y)/x'} a'(y', y, x')a''(y, z'')dy. \tag{7.27}$$

The second factor is a smooth function in this region since z'' lies in the interior of X. The phase is nondegenerate at y, so performing stationary phase we get

$$e^{-i\lambda/x'} x'^{(n-1)/2} c(x', y', z'') \tag{7.28}$$

which is an expression of the form (4.4) with $r = (n-1)/2$.

The case $(2, 1)$ is easily treated, because in this case we have an integral

$$x^{-k} \int e^{i\left(\cos\theta\phi'(y',y,v)+\sin\theta\cos d(y,y'')\right)/x} a'(y', y', \theta, x')a''(y, y'', \theta, x'')\; dy \tag{7.29}$$

where $|d_y\phi'| \neq 0$ whenever $d_v\phi' = 0$. In this case, we get nontrivial contributions when $d_y\phi = d_v\phi = 0$, which implies that

$$\sigma = \frac{|\sin d(y', y)|}{|d_y\phi'|} \text{ and } d_v\phi' = 0$$

there. But this means that $\sigma \geq c$, so we only get contributions at the interior of bf (since we are neglecting a neighbourhood of rb), and in this region the argument is similar to that of case $(2, 2)$.

Finally we have case $(1, 3)$. There the composite operator takes the form

$$\begin{aligned} &\int_0^\infty ds \int e^{i\lambda\left(-\cos\theta\cos d(y',y)-\sin\theta(1+s\psi(y,y'',s,w))\right)/x} \\ &\qquad a'(y', y, x')a''(y, y'', w, s, x''/s)\big(\frac{x''}{s}\big)^{-(k'+1)/2} s^{(n-3)/2}\, dw\, dy \end{aligned} \tag{7.30}$$

where a'' is supported where s is small. When $s = 0$, the Hessian of the phase function in y is nondegenerate, with a critical point at $y = y'$, so for small s we get a critical point $y = f(y', y'', s, w)$. When $\theta = 0$ the critical point is also $y' = y$ so the critical value takes the form $y = y' + s\theta g(y', y'', \theta, s, w)$. Using the stationary phase lemma we may rewrite the integral as

$$\begin{aligned} &\int_0^\infty ds \int e^{i\lambda(-\cos\theta-\sin\theta+s\theta\tilde{\psi}(y',y'',\theta,s,w))/x} a(y', y'', \theta, s, x/s, w) \\ &\qquad \big(\frac{x''}{s}\big)^{(n-1)/2-(k'+1)/2} s^{(2n-3)/2}\theta^{(n-1)/2}\, dy\, dw \end{aligned} \tag{7.31}$$

where $\tilde{\psi}(y', y'', 0, 0, w) = \psi(y', y'', 0, 0, w)$. This is a nondegenerate parametrization of $(L, L^\sharp(\lambda))$ both near the interior of bf and near the corner lb $\cap$ bf with orders $-\frac{1}{2}$ at L, $(n-2)/2$ at $L^\sharp$ and $(n-1)/2$ at lb, so the integral is in $I_{\mathrm{s}\Phi}^{-\frac{1}{2},\frac{n-2}{2},\frac{n-1}{2}}(X_{\mathrm{b}}^2, \tilde{L})$. The argument in the interior of lb is the same as in case $(1, 2)$. This completes the proof of the theorem. $\square$

8. The resolvent

In this section we identify the structure of the resolvent $R(\lambda^2 \pm i0)$ on the real axis. The resolvent is a little more complicated than the spectral projection $\mathrm{sp}(\lambda)$ since it has interior singularities along the diagonal. However we can cut the resolvent into three pieces, each of which has a structure which has been described. One piece lies in the scattering calculus; this takes care of the interior singularities. A second piece is an intersecting Legendrian distribution microsupported in the interior of bf. The third piece is microlocally identical to the spectral projection in the region $\tau < 0$ (for $R(\lambda^2 + i0)$) or $\tau > 0$ (for $R(\lambda^2 - i0)$). See section 6 for a discussion of the resolvent on $\mathbb{R}^n$.

In preparation for the description of the second piece, we describe the intersection properties of the two Legendrians L and ${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$, the boundary of the closure of $N^* \operatorname{diag}_{\mathrm{b}}$ in ${}^{\mathrm{sc}}T^* \tilde{M}$:

$$ {}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}} = \{\sigma = 1, y' = y'', \mu' = -\mu'', \tau = 0\}. $$

The following proposition shows that L and ${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$ have the correct geometry for an intersecting Legendrian distribution.

Proposition 8.1. *The intersection $L \cap {}^{sc}N^* \operatorname{diag}_b$ is codimension one in L. In fact, the function τ, restricted to L, has non-vanishing differential at $\tau = 0$, and $L \cap \{\tau = 0\}$ is precisely equal to $L \cap {}^{sc}N^* \operatorname{diag}_b$. Thus L is divided by ${}^{sc}N^* \operatorname{diag}_b$ into two pieces, $L_+ = \{\tau < 0\}$ and $L_- = \{\tau > 0\}$.*

Proof. Clearly $L \cap {}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}} \subset L \cap \{\tau = 0\}$, so we prove the reverse inclusion. Setting $\tau = 0$ in (7.10) and using (7.6) we see that

$$ \sigma = \frac{\cos s'}{\cos s} = \frac{\sin s'}{\sin s}, $$

so $s = s'$ on the first piece of $L \cap \{\tau = 0\}$ in (7.10) and then we have $\sigma = 1$, $y = y'$, $\mu' = -\mu''$. It is easy to check on the second and third pieces using (7.6).

Since L is contained in the characteristic variety of $\Delta_{\tilde{g}} - 2\lambda^2$ (see (7.7)) which is $\{\tau^2 + |\tilde{\mu}'|^2 + |\tilde{\mu}''|^2 + \eta^2 = 2\lambda^2\}$, it is tangent to the bicharacteristic flow of $\Delta_{\tilde{g}}$. Under this flow τ evolves by $\dot{\tau} = -2(|\tilde{\mu}'|^2 + |\tilde{\mu}''|^2 + \eta^2)$, so on $\Sigma(\Delta_{\tilde{g}} - 2\lambda^2) \cap \{\tau = 0\}$, $\dot{\tau} = -4\lambda^2$ and hence $d\tau \neq 0$. □

Theorem 8.2. *The boundary value $R(\lambda^2 + i0)$ of the resolvent of H on the positive real axis is of the form $R_1 + R_2 + R_3$, where R_1 is a scattering pseudodifferential operator of order $(-2, 0)$, R_2 is an intersecting Legendrian distribution of order $-\frac{1}{2}$ associated to $({}^{sc}N^* \operatorname{diag}_b, L_+)$ and R_3 is a Legendrian distribution associated to the conic pair $(L_+, L^\sharp(\lambda))$, which is microlocally identical to $\mathrm{sp}(\lambda)$ for $\tau < 0$, and in particular has the same orders as $\mathrm{sp}(\lambda)$ (see Theorem 7.2).*

Proof. We restrict our attention to $R(\lambda^2 + i0)$ for the sake of definiteness. Similar results hold for $R(\lambda^2 - i0)$ with some changes of signs; in particular, L_+ changes to L_- in the statement above.

We start by defining R_1. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near λ^2 and supported in a small neighborhood of λ^2, and let

$$ R_1 = (\mathrm{Id} - \psi(H))R(\lambda^2 + i0), \;\; R' = \psi(H)R(\lambda^2 + i0). \tag{8.1} $$

Then R_1 is the distributional limit of $(1 - \psi(H))R(\lambda^2 + i\epsilon) = \tilde{\psi}_\epsilon(H)$ where $\tilde{\psi}_\epsilon(t) = (1 - \psi(t))(t - \lambda^2 - i\epsilon)^{-1}$, so $\tilde{\psi}_\epsilon \in S^{-1}(\mathbb{R})$ is a 'classical' symbol for all $\epsilon \geq 0$, as $\lambda^2 \notin \operatorname{supp}(1-\psi)$. Thus, by the symbolic functional calculus of [3] $\tilde\psi_\epsilon(H) \in \Psi_{sc}^{-2,0}(X)$ for all $\epsilon \geq 0$; this means that its kernel is conormal to diag_{sc} on X^2_{sc}, polyhomogeneous at sf, and vanishes to infinite order on all other boundary hypersurfaces. Since $\sup|\tilde\psi_\epsilon - \tilde\psi_0| \to 0$, we have $\tilde\psi_\epsilon(H) \to \tilde\psi_0(H)$ as bounded operators on $L^2(X)$, hence certainly as distributions on X^2_b. Hence, $R_1 = \tilde\psi_0(H)$ is a scattering pseudodifferential operator. Note that if ψ_1 is another cutoff function with the same properties and $\tilde\psi_1$ is defined similarly to $\tilde\psi$ then $\tilde\psi - \tilde\psi_1 \in \mathcal{C}^\infty_c(\mathbb{R})$, so $\tilde\psi(H) - \tilde\psi_1(H) \in \Psi_{sc}^{-\infty,0}(X)$ by the symbolic functional calculus, i.e. its kernel is polyhomogeneous on X^2_{sc} (has no singularity at diag_{sc}) and vanishes to infinite order an all other boundary hypersurfaces. Thus, the splitting $R_1 + R'$ is in fact well defined modulo such terms, and it separates the diagonal singularity (which is really a high energy phenomenon) represented by R_1 and the singularity due to the existence of generalized eigenfunctions of energy λ^2, represented by R'.

To analyze R', note that the kernel of R' is the distributional limit of $\psi(H)(H - \lambda^2 - i\epsilon)^{-1}$ which is a function of H supported on the positive real axis; hence we can use our knowledge of the spectral projections for $\lambda > 0$ to compute this term as

$$\int_0^\infty \psi(\rho)(\rho^2 - \lambda^2 - i\epsilon)^{-1} \operatorname{sp}(\rho) 2\rho\, d\rho.$$

The term $(\rho^2 - \lambda^2 - i\epsilon)^{-1}$ can be expressed as

$$(\rho^2 - \lambda^2 - i\epsilon)^{-1} = \frac{i}{\rho+\lambda} \int_0^\infty e^{-i(\rho-\lambda)\bar t} e^{-\epsilon \bar t} d\bar t \tag{8.2}$$

(presently we will change variable to $t = \bar t/x$). We have shown in the proof of Theorem 7.2 that the kernel of $\operatorname{sp}(\lambda)$ is a sum of expressions of the form

$$x^q \int e^{i\lambda\phi/x} a \tag{8.3}$$

where ϕ parametrizes some part of $(L(1), L^\sharp(1))$ and a is smooth on X^2_b and in the phase variables. We need to prove smoothness of a in λ as well; for ease of exposition this is left until the end of the section. We may suppose that we have broken up $\operatorname{sp}(\lambda)$ in such a way that on each local expression, ϕ is either strictly less than zero, strictly greater than zero or takes values in $(-\delta, \delta)$ for some small δ. Thus to understand R' we must understand the expressions

(8.4)
$$\int\int_0^\infty dt\, \frac{2i\rho}{x(\rho+\lambda)} \psi(\rho) e^{-i(\rho-\lambda)t/x} e^{-\epsilon t/x} x^q e^{i\rho\phi(y',y'',\theta,v)/x} a(y',y'',\theta,x',v,\rho)\, d\rho\, dv$$

as $\epsilon \to 0$. (Note that ϕ will also depend on an s variable if it is parametrizing $(L, L^\sharp)$ near at a conic point; this is not indicated in notation for simplicity.) We define the phase function $\Phi = -(\rho-\lambda)t + \rho\phi$.

It is inconvenient that (8.4) is not compactly supported in t, so first we insert $\chi(t) + (1-\chi(t))$ into the integral, where χ is identically equal to one on $[0,2]$ and is compactly supported. The phase function Φ is stationary in σ when $t = \phi$ which is always less than 2. We may also produce any negative power t^{-k} in the integrand by writing $e^{-i(\rho-\lambda)t/x} = -ixt^{-1}\partial_\rho e^{-i(\rho-\lambda)t/x}$ and repeatedly integrating the ρ derivative by parts. Thus, by stationary phase, when $(1-\chi(t))$ is inserted the integral is smooth on X^2_b and rapidly decreasing as $x \to 0$, uniformly as $\epsilon \to 0$. Hence we may ignore this term and insert the factor $\chi(t)$ into (8.4).

Let us consider (8.4) at $\epsilon = 0$ with the term $\chi(t)$ inserted, that is,

(8.5)
$$\int\int_0^\infty \frac{2i\rho}{x(\rho+\lambda)} dt\ \psi(\rho) e^{-i(\rho-\lambda)t/x} \chi(t) x^q e^{i\rho\phi(y',y'',\theta,v)/x} a(y',y'',\theta,x',v,\rho) d\rho dv.$$

when the phase ϕ is negative. In this case there are no critical points of Φ since $d_\rho \Phi = 0$ requires $t = \phi$ and t is nonnegative. Thus in this case, again, the expression is smooth and rapidly decreasing as $x \to 0$ uniformly in ϵ.

When the phase is positive, we get critical points when ϕ parametrizes $(L, L^\sharp)$, $\rho = \lambda$ and $t = \phi$. We can ignore the fact that the range of integration of t has a boundary since $d_\rho \Phi \neq 0$ there. The phase function Φ is nondegenerate in (t, ρ) and exactly quadratic, and moreover the rest of the expression is independent of t. It follows that the first term of the stationary phase expansion is accurate to all orders, so that up to a smooth function which is rapidly decreasing in x, (8.5) is equal to

$$(2\pi i) \int x^q e^{i\lambda\phi(y',y'',\theta,v)/x} a(y',y'',\theta,x',v,\lambda) dv$$

which is $2\pi i$ times the corresponding piece of $\mathrm{sp}(\lambda)$. We take R_3 to be the sum of the contributions from all terms with $\phi > 0$ in our decomposition. Since $\tau < 0$ when $\phi > 0$, we see that $R_3 \in I^{\mathcal{K}, \frac{n-2}{2}}_{\mathrm{s}\Phi}(X^2_{\mathrm{b}}, (L_+(\lambda), L^\sharp(\lambda)), {}^{\mathrm{s}\Phi}\Omega^{\frac12})$ with $\mathcal{K}(\mathrm{lb})$ as in Theorem 7.2, and is (up to a numerical factor) microlocally identical to $\mathrm{sp}(\lambda)$ there.

When the phase changes sign, then we have an intersecting Legendrian distribution associated to $(L_+, {}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}})$. To see this, note that when $t > 0$, $\partial_t \Phi = 0$ implies $\rho = \lambda$, $\partial_\rho \Phi = 0$ implies $t = \phi$ and $d_v \Phi = 0$ implies that ϕ parametrizes $L(1)$. Thus we get the Legendrian $L(\lambda)$ for $t > 0$, subject to the restriction $\tau = -\phi < 0$ which gives us L_+. The phase is non-degenerate since ϕ is nondegenerate and the Hessian of Φ in (t, ρ) is nondegenerate. In addition, $dt \neq 0$ at $t = 0$ on C_Φ. If $\Phi_0 = \rho\phi$ denotes Φ restricted to $t = 0$, then $d_\rho \Phi_0 = 0$ implies that $\phi = 0$ and $d_v \Phi_0 = 0$ implies that ϕ parametrizes $L(1)$. The Legendrian parametrized is then

$$\{(y', y'', \theta, \tau, \mu', \mu'', \eta) \mid \tau = 0, (y', y'', \theta, 0, \rho\mu', \rho\mu'', \rho\eta) \in L(1)\}$$

which is ${}^{\mathrm{sc}}N^* \operatorname{diag}_{\mathrm{b}}$. Thus in this case (8.4) is an intersecting Legendrian distribution when $\epsilon = 0$. We define R_2 to the sum of all the contributions when ϕ changes sign. It is easy to see that as $\epsilon \to 0$ we get convergence in distributions to R_2, since by multiplying by a suitable power of x, we have convergence in L^1 by the Dominated Convergence Theorem; therefore the expression converges in some weighted L^2 space.

We have now written $R(\lambda^2 + i0)$ as a sum $R_1 + R_2 + R_3 + \tilde{R}$, where R_i have the required properties and $\tilde{R}$ is a smooth kernel vanishing with all derivatives at the boundary, hence trivially fits any of the descriptions of the R_i. Thus, together with the proof of smoothness of (8.3) in λ, the proof of the theorem is complete. $\square$

Finally we show smoothness of (8.3) in λ. To do this it is enough to show smoothness of the symbol of $P(\lambda)$ in λ. The Poisson operator $P(\lambda)$ is constructed in [11] from a parametrix $K(\lambda)$, which is explicit and easily seen to be smooth in λ, plus an correction term, $U(\lambda) = R(\lambda^2 + i0)(\Delta - \lambda^2)K$. Here K is constructed so that $(\Delta - \lambda^2)K(\lambda)$ is in $\dot{\mathcal{C}}^\infty(X \times \partial X)$. Therefore, the following lemma completes the proof.

Lemma 8.3. *The resolvent $R(\lambda^2 + i0)$ acting on $f \in \dot{\mathcal{C}}^\infty(X)$ is of the form*
$$x^{(n-1)/2} e^{i\lambda/x} a(x, y, \lambda),$$
where a is smooth in x, y and λ.

Proof. Smoothness in x and y is proved in [9], so it remains to prove smoothness in λ. Let $\sigma = \lambda^2$. One can show, as in [9], proof of Proposition 14, that for $|\operatorname{Re}\sigma| \geq \epsilon$, $R(\sigma)^2 f$ has a limit in $x^{-3/2-\epsilon}L^2$ as σ approaches the real axis from $\operatorname{Im}\sigma > 0$, and the limit is continuous in σ. (Indeed, this is completely analogous to Jensen's proof of the corresponding statement in the Euclidean setting, see [6].) Moreover, it is the unique solution to $(\Delta - \sigma)^2 u = f$ satisfying the radiation condition $\tau < 0$ on $\mathrm{WF}_{\mathrm{sc}}(u)$. Let us denote the limit by $R(\sigma + i0)^k f$. Since $\partial_\sigma R(\sigma) = -R(\sigma)^2$, we see that both $R(\sigma + i\epsilon) f$ and $\partial_\sigma R(\sigma + i\epsilon) f$ have limits in $x^{-3/2-\epsilon}L^2$ as $\epsilon \to 0$. Hence, we have
$$\partial_\sigma R(\sigma + i0) f = -R(\sigma + i0)^2 f.$$
Since $a = x^{-(n-1)/2} e^{-i\lambda/x} R(\lambda^2 + i0) f$, we see that the λ derivative of a exists in $x^{-3/2-\epsilon}L^2$.

If we calculate
$$\frac{d}{d\lambda}(\Delta - \lambda^2)(e^{i\lambda/x} a) = (\Delta - \lambda^2)\big(e^{i\lambda/x} \frac{\partial a}{\partial \lambda}\big) + (\Delta - \lambda^2)\big(\frac{i}{x} e^{i\lambda/x} a\big) - 2\lambda(e^{-i\lambda/x} a) = 0,$$
we see that
$$\begin{aligned} (\Delta - \lambda^2)\big(e^{i\lambda/x}\frac{\partial a}{\partial\lambda}\big) &= (\Delta - \lambda^2)\big(\frac{i}{x} e^{i\lambda/x} a\big) - 2\lambda(e^{-i\lambda/x} a) \\ &= \frac{i}{x} f - 2i(i\lambda) e^{i\lambda/x} a + 2i e^{i\lambda/x}(x^2\partial_x) a - (n-1) i x e^{i\lambda/x} a - 2\lambda(e^{-i\lambda/x} a) \\ &= \frac{i}{x} f + 2ix e^{i\lambda/x}\big(x\partial_x - \frac{n-1}{2}\big) a \\ &= e^{i\lambda/x} x^{(n+3)/2} b(x, y, \lambda) + \frac{i}{x} f, \quad b \in \mathcal{C}^\infty(X). \end{aligned} \tag{8.6}$$

We can find a solution to
$$(\Delta - \lambda^2) v_1 = e^{i\lambda/x} x^{(n+3)/2} b \mod \dot{\mathcal{C}}^\infty$$
of the form $v_1 = e^{i\lambda/x} a_1$, with $a_1 \in x^{(n+1)/2}\mathcal{C}^\infty(X)$ and varying continuously with λ (since b depends on a only through taking x and y derivatives, not λ derivatives). Therefore,
$$(\Delta - \lambda^2)\big(e^{i\lambda/x}\big(\frac{\partial a}{\partial \lambda} + a_1\big)\big) \in \dot{\mathcal{C}}^\infty(X), \tag{8.7}$$
with the right hand side continuous in λ. It follows from Proposition 12 of [9], and Lemma 5.1 that $\partial_\lambda a$ is a smooth function on X which is continuous in λ.

This implies that actually the right hand side of (8.7) is $\mathcal{C}^1$ in λ. Thus, we can repeat the argument with $e^{i\lambda/x}(\partial_\lambda a + a_1)$ in place of $e^{i\lambda/x} a$. Inductively we see that a is $\mathcal{C}^k$ in λ for any k, proving the lemma. □

Remark 8.4. In this paper we analyzed the kernel of the resolvent at the real axis by using the Poisson operator. Conversely, the resolvent determines, in a simple way, the other analytic operators associated to H. We have already seen that the generalized spectral projection is $\mathrm{sp}(\lambda) = (R(\lambda^2 + i0) - R(\lambda^2 - i0))/(2\pi i)$, i.e. essentially the difference of the limits of the resolvent taken from either side of the real axis. Next, the Poisson operator $P(\pm\lambda)$ is a multiple of the principal symbol of the kernel of $R(\lambda^2 \pm i0)$ at either lb or rb. (We did not define the principal symbol for Legendre distributions on manifolds with corners because we did not need this notion, but for oscillatory functions as in (4.1), it is, up to a constant multiple, the restriction of the amplitude a to $x = 0$. Note that the kernel of $R(\lambda^2 \pm i0)$ is indeed such an oscillatory function at lb and rb since $L^\sharp(\pm\lambda)$ are global sections of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}}X^2_{\mathrm{b}}$.) Finally, $S(\lambda)$ is a multiple of the principal symbol of $R(\lambda^2 \pm i0)$ along $L^\sharp(\pm\lambda)$ in bf.

These statements follow from the proof of Theorem 7.2, together with the fact that $R(\lambda^2 \pm i0)$ is microlocally identical to $\mathrm{sp}(\lambda)$ for $\tau < 0$ and $\tau > 0$, respectively. For example, in (7.27), $a'(y, y, 0)$ is a constant (it is the initial data for the transport equation for the principal symbol of $P(\lambda)$, see [11]), and $a''(y, z'')$ is a multiple of the kernel of $P(\lambda)^*$, so the stationary phase argument yields (7.28) with $c(0, y', z'')$ being a multiple of the kernel of $P(\lambda)^*$. The statement about $S(\lambda)$ follows similarly, but it can also be deduced from the fact that its kernel is the principal symbol of $P(\lambda)$ along $G^\sharp(\lambda)$.

9. Distorted Fourier Transform

Here we interpret some of our results in terms of a generalized 'Distorted Fourier transform'. Namely, we show that the Poisson operators determine two isometries $P_\pm$ from the absolutely continuous spectral subspace of H to $L^2(\partial X \times \mathbb{R}^+)$ (cf. [4] in the more complicated N-body case). The coordinate on $\mathbb{R}^+$ should be thought of as the square root of the energy, λ. We will show that they determine the scattering matrix through equation (9.4), which is analogous to a standard formula in scattering theory (see [13], chapter 9, section 3, or [12], section XI.4).

Proposition 9.1. *The maps $P^*_\pm$, defined on $\mathcal{C}^\infty_c(X) \cap H_{ac}(H)$ by*

$$(P^*_\pm u)(y, \lambda) = (2\pi)^{-1/2}(P(\pm\lambda)^* u)(y), \tag{9.1}$$

map into $L^2(\partial X \times \mathbb{R}^+)$ and extend to unitary maps from $H_{ac}(H)$ to $L^2(\partial X \times \mathbb{R}^+)$, with adjoints

$$(P_\pm f)(z) = (2\pi)^{-1/2} \int_0^\infty P(\pm\lambda)(f(\cdot, \lambda))\, d\lambda. \tag{9.2}$$

They intertwine the Hamiltonian with multiplication by λ^2:

$$P^*_\pm H = \lambda^2 P^*_\pm.$$

Proof. By Lemma 5.1, for $0 < \epsilon < a < b < \infty$, and a dense set of $f \in L^2$, namely, $f \in x^{1/2+\epsilon}L^2$, the function $\langle f, R(\lambda)f\rangle$ is continuous on $|\operatorname{Re}\lambda| \geq \epsilon$ and $\pm \operatorname{Im}\lambda \geq 0$. It follows that for such f, and $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_a^b \langle f, R(\lambda^2 + i\epsilon)f\rangle - \langle f, R(\lambda^2 - i\epsilon)f\rangle\, 2\lambda d\lambda \\ &= \int_a^b i\langle f, P(\lambda)P(\lambda)^* f\rangle\, d\lambda \quad \text{by equation (5.9)} \\ &= \int_a^b i\left|P(\lambda)^* f\right|^2 d\lambda. \end{aligned} \tag{9.3}$$

Sending $a \to 0$ and $b \to \infty$, we see that

$$\langle E_{(0,\infty)}f, f\rangle = \frac{1}{2\pi}\int_0^\infty \left|P(\lambda)^* f\right|^2 d\lambda$$

for a dense set of f. Since $H_{ac}(H) = E_{(0,\infty)}$, this shows that P_+^* extends to an isometry from $H_{ac}(H)$ to $L^2(\partial X \times \mathbb{R}^+)$. The same reasoning shows that P_-^* also extends as an isometry. It is easy to see that the adjoint of $P_\pm^*$ is $P_\pm$.

The intertwining property follows from $(H - \lambda^2)P(\pm\lambda) = 0$ by multiplying on the left and right by $P_\pm^*$. ☐

Proposition 9.2. *If the operator S is defined by*

$$(Sf)(y, \lambda) = (S(\lambda)f(\cdot, \lambda)(y)),$$

then we have

$$S = P_-^* P_+. \tag{9.4}$$

Proof. Multiplying both sides of the equation on the left by P_-, it suffices to establish $P_- S = P_+$. Since each operator acts fibrewise in λ, it is sufficient to establish $P(-\lambda)S(\lambda) = P(\lambda)$. Recall that for $v \in \mathcal{C}^\infty(\partial X)$, $P(\lambda)v$ is the solution to $(H - \lambda^2)u = 0$ with incoming boundary data v. Denote by w the outgoing boundary data of u, ie, $w = S(\lambda)v$. Then $P(-\lambda)w$ is also equal to u. But this proves $P(-\lambda)S(\lambda) = P(\lambda)$, so the proof is complete. ☐

Remark 9.3. For $H = \Delta + V$ on $\mathbb{R}^n$, our normalization for the two distorted Fourier transforms is different from the standard normalization, which would have the kernels of the two being $c \cdot e^{-i\lambda z \cdot \theta}$ plus incoming/outgoing correction term. This makes the scattering matrix for the free Laplacian equal to the identity operator. Here, the two distorted Fourier transforms have the form $c \cdot e^{\pm i\lambda z \cdot \theta}$ plus incoming/outgoing correction, and the scattering matrix for the free Laplacian is $i^{-n+1}R$, where R is the antipodal map on the $(n-1)$-sphere. In [2] these two operators are called the analytic and geometric scattering matrices, respectively. Notice that only the geometric scattering matrix has an analogue for a general scattering metric.

10. Wavefront set relation

Here we will use the microlocal information about the resolvent from Theorem 8.2 to find a bound on the scattering wavefront set of $R(\lambda + i0)f$ in terms of the scattering wavefront set of f. The result is not new, since Melrose proved it in [9] using positive commutator estimates. However, it is quite illuminating to see how the result arises from the structure of the resolvent kernel itself. The fundamental tool is the pushforward theorem of [2].

Theorem 10.1. *If $f \in \mathcal{C}^\infty(X^o) \cap \mathcal{C}^{-\infty}(X)$, and $\mathrm{WF}_{sc}(f)$ does not intersect $G^\sharp(-\lambda)$, then $\mathrm{WF}_{sc}(R(\lambda^2 + i0)f)$ is contained in the union of $\mathrm{WF}_{sc}(f)$, $G^\sharp(\lambda)$ and bicharacteristic rays of $H - \lambda^2$ that start at $\mathrm{WF}_{sc}(f) \cap \Sigma(H - \lambda^2)$ and propagate in the direction of decreasing τ. That is,*

(10.1)

$$\begin{gathered}\mathrm{WF}_{sc}(R(\lambda^2 + i0)f) \subset \mathrm{WF}_{sc}(f) \cup G^\sharp(\lambda) \cup \\ \{(y, \tau, \mu) \mid \exists\, (y', \tau', \mu') \in \mathrm{WF}_{sc}(f) \ \textit{with}\ \tau' = \lambda \cos s,\ \mu' = \lambda \sin s \hat{\mu}',\ |\hat{\mu}'| = 1, \\ \textit{and}\ (y, \mu) = \lambda \sin(s + t) \exp(tH_{\frac{1}{2}h})(y', \hat{\mu}'), \tau = \lambda \cos(s + t)\ \textit{for some}\ s\ \textit{and}\ t > 0.\}\end{gathered}$$

Proof. We write $R(\lambda^2 + i0) = R_1 + R'$ as in the proof of Theorem 8.2. Then R_1 is a scattering pseudodifferential operator, so $\mathrm{WF}_{\mathrm{sc}}(R_1 f) \subset \mathrm{WF}_{\mathrm{sc}}(f)$. Thus we may restrict attention to $R'f$. The action of R' on f may be represented by pulling f up to X_{b}^2 from the left factor, multiplying by the kernel of R' and then pushing forward

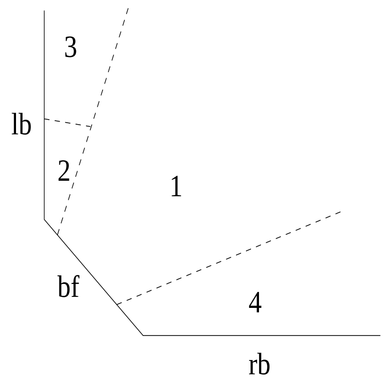

FIGURE 2. The approximate supports of R_i' on X_b^2

to the right factor of X (see figure). We will apply the pushforward theorem to this pushforward to obtain scattering wavefront set bounds.

Recall that the pushforward theorem from [2] applies in the situation when we have a map $\pi : \tilde{X} \to X$ between manifolds with boundary, which is locally of the form $(x, y, z) \mapsto (x, y)$ in local coordinates where x is a boundary defining function for X, and $\pi^* x$ is one for $\tilde{X}$. Corresponding coordinates on the two scattering cotangent bundles over the boundary are then (y, z, τ, μ, ζ) and (y, τ, μ). Then the pushforward theorem states that for densities $u \in \mathcal{C}^\infty(\tilde{X}^o) \cap \mathcal{C}^{-\infty}(\tilde{X})$,

$$\mathrm{WF}_{\mathrm{sc}}(\pi_* u) \subset \{(y, \tau, \mu) \mid \exists\ (y, z, \tau, \mu, 0) \in \mathrm{WF}_{\mathrm{sc}}(u)\}. \tag{10.2}$$

Unfortunately there is no pushforward theorem currently available on manifolds with corners. Since X_b^2 has codimension 2 corners, we cannot apply the theorem directly, but must cut off the resolvent kernel away from the corners and treat the part near the corners by a different argument. It turns out that nothing special happens at the corners and the bound on the wavefront set is just what one would expect from a consideration of the boundary contributions alone.

To localize the kernel of R', choose a partition of unity $1 = \chi_1 + \chi_2 + \chi_3 + \chi_4$, where χ_1 is zero in a neighbourhood of lb and rb, χ_2 is supported near $\mathrm{lb} \cap \mathrm{bf}$, χ_3 is zero on bf and rb and χ_4 is supported close to rb. Correspondingly decompose

$$R' = R_1' + R_2' + R_3' + R_4'.$$

We investigate each piece separately.

The easiest piece to deal with is $R_3' f$. The product of the two distributions on X_b^2 is localized near lf, and f is smooth there. We know that R_3' is a Legendrian associated to $\{\tau' = -1, \mu' = 0\} = G^\sharp(\lambda)$ and is smooth in z'' (where z'' is a set of coordinates in the interior of the right factor of X). Thus if ζ'' is a rescaled dual variable to z'', the wavefront set of the product is contained in $\{\tau' = -1, \mu' = 0, \zeta'' = 0\}$ and by (10.2), the result has scattering wavefront set in $G^\sharp(\lambda)$.

Next we treat $R_1' f$. This is localized near the interior of bf. We will see that the contribution here is the part that propagates inside the characteristic variety of $H - \lambda^2$. To apply (10.2), we must first choose compatible coordinates on X_b^2 and X. Let us take x', y', y'' and $\sigma = x'/x''$ near the interior of bf, and let $\overline{\tau}'$, $\overline{\mu}'$, $\overline{\mu}''$ and $\overline{\nu}$ be the corresponding rescaled cotangent coordinates. They are related to coordinates (7.5) by

$$\overline{\mu}' = \mu', \quad \overline{\mu}'' = \sigma\mu'', \quad \overline{\tau}' = \tau' + \tau''\sigma, \quad \overline{\nu} = -\tau''. \tag{10.3}$$

The scattering wavefront set of the lift of f to X_{b}^2 is

$$\{(y', y_0'', \sigma, \overline{\tau}' = \sigma\tau_0'', \overline{\mu}' = 0, \overline{\mu}'' = \sigma\mu_0'', \overline{\nu} = -\tau_0'') \mid (y_0'', \tau_0'', \mu_0'') \in \mathrm{WF}_{\mathrm{sc}}(f)\}.$$

The wavefront set of R_1' is contained in the union of $N^*_{\mathrm{sc}}(\mathrm{diag_b})$ and L_+. In our present coordinates this is the union of

$$\{(y'' = y', \sigma = 1, \tau' = 0, \overline{\mu}' = -\overline{\mu}'', \overline{\nu} = 0)\}, \tag{10.4}$$

corresponding to $N^*_{\mathrm{sc}}(\mathrm{diag_b})$, and

$$\begin{gathered}\{(y', y'', \sigma, \overline{\tau}', \overline{\mu}', \overline{\mu}'', \overline{\nu}) : \exists (y, \hat{\mu}) \in S^*\partial X,\ s, s' \in [0, \pi],\ (\sin s)^2 + (\sin s')^2 > 0, \\ s + s' > 0 \ \text{ such that } \ \sigma = \frac{\sin s'}{\sin s}, \quad \overline{\tau}' = \lambda\cos s' + \sigma\lambda\cos s, \quad \overline{\nu} = -\lambda\cos s, \\ (y', \overline{\mu}') = \lambda\sin s' \exp(s' H_{\frac{1}{2}h})(y, \hat{\mu}), \quad (y'', \overline{\mu}'') = -\sigma\lambda\sin s\exp(sH_{\frac{1}{2}h})(y, \hat{\mu})\} \\ \cup \{(y, y, \sigma, \lambda - \sigma\lambda, 0, 0, \lambda) : \ \sigma \in [1, \infty),\ y \in \partial X\} \\ \cup \{(y, y, \sigma, -\lambda + \sigma\lambda, 0, 0, -\lambda) : \ \sigma \in (0, 1],\ y \in \partial X\}\end{gathered} \tag{10.5}$$

by (7.10) and (10.3) (the condition $s + s' > 0$, and the restrictions on σ in the last two lines, gives the correct 'half' of the Legendrian for $R(\lambda^2 + i0)$ since this is precisely the part of (7.10) where $\tau < 0$). It is not hard to see that (10.4), associated to $N^*_{\mathrm{sc}}(\mathrm{diag_b})$, gives us $\mathrm{WF}_{\mathrm{sc}}(f)$ back again, so consider the second part. The first piece can be re-expressed as

$$\begin{gathered}\{(y', y'', \sigma, \overline{\tau}', \overline{\mu}', \overline{\mu}'', \overline{\nu}) : \exists\ s, s' \in [0, \pi],\ (\sin s)^2 + (\sin s')^2 > 0,\ s + s' > 0 \\ \text{such that } \ \sigma = \frac{\sin s'}{\sin s}, \quad \overline{\tau}' = \lambda\cos s' + \sigma\lambda\cos s, \quad \overline{\nu} = -\lambda\cos s, \\ \overline{\mu}' = \lambda\sin s'\hat{\eta}', \ |\hat{\eta}'| = 1, \quad \overline{\mu}'' = \lambda\sigma\sin s\hat{\eta}'', \ |\hat{\eta}''| = 1, \\ (y', \eta') = \exp((s' - s)H_{\frac{1}{2}h})(y'', \eta'')\}.\end{gathered} \tag{10.6}$$

The scattering wavefront set of the product is in each fibre the setwise sum of the scattering wavefront sets of the factors, ie,

$$\begin{gathered}\{(y', y'', \sigma, \overline{\tau}', \overline{\mu}', \overline{\mu}'', \overline{\nu}) : \exists\ s, s' \in [0, \pi],\ (\sin s)^2 + (\sin s')^2 > 0,\ s + s' > 0 \ \text{ with} \\ \sigma = \frac{\sin s'}{\sin s}, \quad \overline{\tau}' = \lambda\cos s' + \sigma\lambda\cos s + \sigma\tau_0'', \quad \overline{\nu} = -\lambda\cos s - \tau_0'' \\ \overline{\mu}' = \lambda\sin s'\hat{\eta}', \ |\hat{\eta}'| = 1, \quad \overline{\mu}'' = \lambda\sigma\sin s\hat{\eta}'' + \sigma\mu_0'', \ |\hat{\eta}''| = 1, \\ (y', \eta') = \exp((s' - s)H_{\frac{1}{2}h})(y'', \eta'') \mid (y_0'', \tau_0'', \mu_0'') \in \mathrm{WF}_{\mathrm{sc}}(f)\} \\ \cup \{(y, y, \sigma, \lambda - \sigma\lambda + \sigma\tau_0'', \sigma\mu_0'', \lambda - \tau_0'') : \ \sigma \in [1, \infty),\ y \in \partial X\} \\ \cup \{(y, y, \sigma, -\lambda + \sigma\lambda + \sigma\tau_0'', \sigma\mu_0'', -\lambda - \tau_0'') : \ \sigma \in (0, 1],\ y \in \partial X\}\end{gathered} \tag{10.7}$$

By (10.2), we only get a contribution to the scattering wavefront set of the push-forward when $\overline{\nu} = \overline{\mu}'' = 0$, ie, when

$$\tau_0'' = -\lambda\cos s, \quad \mu_0'' = -\lambda\sin s\hat{\eta}'',$$

ie when the corresponding point of $\mathrm{WF}_{\mathrm{sc}}(f)$ is in the characteristic variety, and then we get points

$$\begin{aligned}(10.8)\quad &\{(y', \tau', \mu') \mid \tau' = \lambda \cos s', (y', \mu') = \lambda \sin s' \exp((s' - s)H_{\frac{1}{2}h})(y_0'', \hat{\eta}''),\ s' > s\} \\ &\cup \{(y, \lambda, 0) \mid (y, \lambda, 0) \in \mathrm{WF}_{\mathrm{sc}}(f)\}.\end{aligned}$$

The first set is precisely the bicharacteristic ray from $(y_0'', \tau_0'', \mu_0'')$ in the direction in which τ decreases, and the second set is the intersection of the scattering wavefront set of f with $G^\sharp(\lambda)$. This shows that the contribution from the interior of bf is contained in the set specified in the theorem.

Next we consider the corner contributions. We use a rather crude method here to bound the scattering wavefront set contributions near the corner; it is not optimal, but as the support of χ_2 shrinks to the corner, we recover the optimal result. First we treat R_2'. Notice that $\overline{\tau}' = -1$ at the corner, so, by taking the support of χ_2 sufficiently small, we may suppose that the phase function, ϕ, of the kernel of R_2' takes values arbitrarily close to 1.

If we localize in the coordinate y', then via a diffeomorphism that takes a neighbourhood of the boundary of X to a neighbourhood to a boundary point of $\overline{\mathbb{R}^n}$ (the radial compactification of $\mathbb{R}^n$) we may suppose that $R_2' f$ is a function on $\mathbb{R}^n$. Then we may exploit the following characterization of the scattering wavefront set on $\mathbb{R}^n$ (see [2]):

$$\begin{aligned}(10.9)\quad \mathrm{WF}_{\mathrm{sc}}(u)^{\complement} = \{(\hat{z}_0, \zeta_0) \mid \exists \mathcal{C}^\infty \text{ function } \phi(x, \hat{z}) \\ \text{with } \phi(0, \hat{z}_0) \neq 0, \widehat{\phi u}(\zeta) \text{ smooth near } \zeta = \zeta_0\}.\end{aligned}$$

Here, ζ is related to the (τ, μ) coordinates by $\tau = -\zeta \cdot \hat{z}$ and $\mu = \zeta + (\zeta \cdot \hat{z})\hat{z}$, ie, $-\tau$ and μ are the components of ζ with respect to $\hat{z}$. Thus, we need to investigate where

$$\begin{aligned}(10.10)\quad \int e^{-iz\cdot\zeta} dz \int \frac{dx''\, dy''}{x''^{n+1}} \int dv\, dw\, e^{i\lambda\phi(\hat{z}', v, y'', \sigma, w)/x'} f(x'', y'')|z'|^{-k} \\ \chi_2(z, y'', \sigma)\tilde{\chi}(z) a(z, y'', \sigma, v, w)\end{aligned}$$

is smooth in ζ. Here $\tilde{\chi}$ is a function that localizes the integral in z. We have multiplied by a large negative power of $|z'|$ to make the integral convergent; this does not affect the scattering wavefront set.

Note that $f \cdot \chi_2 \cdot \tilde{\chi} \cdot a$ has stable regularity under repeated application of the vector fields $z_i \partial_{z_j}$, since f is independent of z and $\chi_2, \tilde{\chi}, a$ are symbolic in z. Integrating by parts, and expressing z_i as $i\partial_{\zeta_i}$, we find that the integral has stable regularity under the application of

$$(\zeta_j + \hat{z}_j' \lambda \phi + \lambda \hat{z}_j' \sigma \partial_\sigma \phi)\partial_{\zeta_i}.$$

But ϕ is close to 1, and σ is close to zero, so for some small $\delta > 0$, if $|\zeta + \lambda \hat{z}'| > \delta$ then the factor multiplying the vector field is always nonzero, showing smoothness in ζ. Thus,

$$\mathrm{WF}_{\mathrm{sc}}(R_2'(f)) \subset \{|\zeta + \lambda \hat{z}'| \leq \delta\}.$$

As the support of χ_2 shrinks to the corner, δ shrinks to zero, so we see that the contribution of the corner to the scattering wavefront set is contained in $\{\zeta' = -\lambda \hat{z}\}$, which corresponds under (10.9) to $G^\sharp(\lambda)$. This is just what one would expect by taking the result for the left boundary or the b-face and assuming that it were valid uniformly to the corner.

Finally we come to R_4'. To deal with this, we use a duality argument and exploit the fact that the adjoint of R_4' is R_2' for $R(\lambda^2 - i0)$. We use the characterization

Lemma 10.2. *Let $q \in K$ and $f \in \mathcal{C}^\infty(X^o) \cap \mathcal{C}^{-\infty}(X)$. Then $q \notin \mathrm{WF}_{sc}(f)$ if there exists $B \in \Psi_{sc}^{0,0}(X)$, elliptic at q, and a sequence $(f_n) \in \dot{\mathcal{C}}^\infty(X)$ converging to f in $\mathcal{C}^{-\infty}(X)$ such that $\langle u, Bf_n \rangle$ converges for all $u \in \mathcal{C}^{-\infty}(X)$.*

To prove the lemma, note that the condition on the sequence Bf_n implies that Bf_n converges weakly in every weighted Sobolev space, hence by compact embedding theorems, strongly in every weighted Sobolev space, and thus in $\dot{\mathcal{C}}^\infty(X)$. Also, we know that $Bf_n \to Bf$ in $\mathcal{C}^{-\infty}(X)$, so this implies that $Bf \in \dot{\mathcal{C}}^\infty(X)$. Since B is elliptic at q this shows that $q \notin \mathrm{WF}_{\mathrm{sc}}(f)$.

Now given q that is not in the set (10.1), choose a self-adjoint pseudodifferential operator B elliptic at q and with $\mathrm{WF}'_{\mathrm{sc}}(B)$ disjoint from the set (10.1). Since (10.1) is invariant under the forward bicharacteristic flow of $H - \lambda^2$, one may assume that $\mathrm{WF}'_{\mathrm{sc}}(B)$ is invariant under the backward bicharacteristic flow. Also, since by assumption $\mathrm{WF}_{\mathrm{sc}}(f)$ does not meet $G^\sharp(-\lambda)$, we may assume that B is elliptic on $G^\sharp(-\lambda)$. Choose a zeroth order self-adjoint pseudodifferential operator A with is microlocally trivial on (10.1) and microlocally equal to the identity on $\mathrm{WF}'_{\mathrm{sc}}(B)$.

Since $Af \in \dot{\mathcal{C}}^\infty(X)$, it is possible to choose $f_n \in \dot{\mathcal{C}}^\infty(X)$ converging to f in $\mathcal{C}^{-\infty}(X)$, such that $Af_n \to Af$ in $\dot{\mathcal{C}}^\infty(X)$. Consider the distributional pairing

$$\langle BR_4' f_n, u \rangle$$

for $u \in \mathcal{C}^{-\infty}(X)$. This is equal to

$$\begin{aligned} &\langle BR_4' Af_n, u \rangle + \langle BR_4' (\mathrm{Id} - A) f_n, u \rangle \\ &= \langle Af_n, {R_4'}^* Bu \rangle + \langle f_n, (\mathrm{Id} - A){R_4'}^* Bu \rangle \end{aligned} \tag{10.11}$$

Note that ${R_4'}^*$ is defined on any distribution since the kernel is smooth in the interior and vanishes at rb (hence there is no integral down to any boundary hypersurface which would lead to convergence problems). Since Af_n converges in $\dot{\mathcal{C}}^\infty(X)$, the first term converges. As for the second term, the scattering wavefront set of Bu is contained in $\mathrm{WF}'_{\mathrm{sc}}(B)$ which is invariant under the *backward* bicharacteristic flow and contains $G^\sharp(-\lambda)$, so taking into account that ${R_4'}^*$ is essentially the term R_2' for the incoming resolvent $R(\lambda^2 - i0)$, the above results show that the scattering wavefront set of ${R_4'}^*(Bu)$ is contained in $\mathrm{WF}'_{\mathrm{sc}}(B)$ and is disjoint from the essential support of $\mathrm{Id} - A$. Thus $(\mathrm{Id} - A){R_4'}^* Bu \in \dot{\mathcal{C}}^\infty(X)$, and so the second term converges. This establishes that $q \notin \mathrm{WF}_{\mathrm{sc}}(f)$ by the Lemma above, and completes the proof of the proposition. ☐

Remark 10.3. Although the proof is quite long, the important point is that the main, propagative part of the scattering wavefront set comes from the contribution from bf and is obtained by a routine computation based on the Pushforward theorem.

Remark 10.4. The condition that the scattering wavefront set of f not meet $G^\sharp(-\lambda)$ cannot be removed. For example, on $\mathbb{R}^n$, the resolvent $R(\lambda^2 + i0)$ is conjugated via Fourier transform to the operation of multiplication by $(|\zeta|^2 - \lambda^2 - i0)^{-1}$. This is undefined on certain functions, for example $(|\zeta|^2 - \lambda^2 + i0)^{-1}$, whose inverse Fourier transforms f have scattering wavefront set intersecting $G^\sharp(-\lambda)$.

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