# SEMICLASSICAL RESOLVENT ESTIMATES AT TRAPPED SETS

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Abstract. We extend our recent results on propagation of semiclassical resolvent estimates through trapped sets when a priori polynomial resolvent bounds hold. Previously we obtained non-trapping estimates in trapping situations when the resolvent was sandwiched between cutoffs $\chi$ microlocally supported away from the trapping: $\|\chi R_h(E+i0)\chi\| = \mathcal{O}(h^{-1})$, a microlocal version of a result of Burq and Cardoso-Vodev. We now allow one of the two cutoffs, $\tilde{\chi}$, to be supported at the trapped set, giving $\|\chi R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(\sqrt{a(h)}h^{-1})$ when the a priori bound is $\|\tilde{\chi} R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(a(h)h^{-1})$.

In this brief article we extend the resolvent and propagation estimates of [DaVa10].

Let $(X, g)$ be a Riemannian manifold which is asymptotically conic or asymptotically hyperbolic in the sense of [DaVa10], let $V \in C_0^\infty(X)$ be real valued, let $P = h^2\Delta_g + V(x)$, where $\Delta_g \geq 0$, and fix $E > 0$.

**Theorem 1.** [DaVa10, Theorem 1.2] *Suppose that for any $\chi_0 \in C_0^\infty(X)$ there exist $C_0, k, h_0 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_0]$ we have*

$$\|\chi_0(h^2\Delta_g + V - E - i\varepsilon)^{-1}\chi_0\|_{L^2(X)\to L^2(X)} \leq C_0 h^{-k}. \tag{1}$$

*Let $K_E \subset T^*X$ be the set of trapped bicharacteristics at energy $E$, and suppose that $b \in C_0^\infty(T^*X)$ is identically 1 near $K_E$. Then there exist $C_1, h_1 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_1]$ we have the following nontrapping estimate:*

$$\|\langle r\rangle^{-1/2-\delta}(1-\mathrm{Op}(b))(h^2\Delta_g + V - E - i\varepsilon)^{-1}(1-\mathrm{Op}(b))\langle r\rangle^{-1/2-\delta}\|_{L^2(X)\to L^2(X)} \leq C_1 h^{-1}. \tag{2}$$

Here by bicharacteristics at energy $E$ we mean integral curves in $p^{-1}(E)$ of the Hamiltonian vector field $H_p$ of the Hamiltonian $p = |\xi|^2 + V(x)$, and the trapped ones are those which remain in a compact set for all time. We use the notation $r = r(z) = d_g(z, z_0)$, where $d_g$ is the distance function on $X$ induced by $g$ and $z_0 \in X$ is fixed but arbitrary.

If $K_E = \varnothing$ then (1) holds with $k = 1$. If $K_E \neq \varnothing$ but the trapping is sufficiently 'mild', then (1) holds for some $k > 1$: see [DaVa10] for details and examples. The point is that the losses in (1) due to trapping are removed when the resolvent is cutoff away from $K_E$. Theorem 1 is a more precise and microlocal version of an earlier result of Burq [Bur02] and Cardoso and

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Vodev [CaVo02], but the assumption (1) is not needed in [Bur02, CaVo02]. See [DaVa10] for additional background and references for semiclassical resolvent estimates and trapping.

In this paper we prove that an improvement over the a priori estimate (1) holds even when one of the factors of $(1-\mathrm{Op}(b))$ is removed:

**Theorem 2.** *Suppose that there exist $k>0$ and $a(h)\le h^{-k}$ such that for any $\chi_0\in C_0^\infty(X)$ there exists $h_0>0$ such that for any $\varepsilon>0$, $h\in(0,h_0]$ we have*

$$\|\chi_0(h^2\Delta_g+V-E-i\varepsilon)^{-1}\chi_0\|_{L^2(X)\to L^2(X)}\le a(h)/h. \tag{3}$$

*Suppose that $b\in C_0^\infty(T^*X)$ is identically 1 near $K_E$. Then there exist $C_1,h_1>0$ such that for any $\varepsilon>0$, $h\in(0,h_1]$,*

$$\|\langle r\rangle^{-1/2-\delta}(1-\mathrm{Op}(b))(h^2\Delta_g+V-E-i\varepsilon)^{-1}\langle r\rangle^{-1/2-\delta}\|_{L^2(X)\to L^2(X)}\le C_1\sqrt{a(h)}/h. \tag{4}$$

Note that by taking adjoints, analogous estimates follow if $1-\mathrm{Op}(b)$ is placed to the other side of $(h^2\Delta_g+V-E-i\varepsilon)^{-1}$.

Such results were proved by Burq and Zworski [BuZw04, Theorem A] and Christianson [Chr07, (1.6)] when $K_E$ consists of a single hyperbolic orbit. Theorem 2 implies an optimal semiclassical resolvent estimate for the example operator of [DaVa10, §5.3]: it improves [DaVa10, (5.5)] to

$$\|\chi_0(P-\lambda)^{-1}\chi_0\|\le C\log(1/h)/h.$$

Further, this improved estimate can be used to extend polynomial resolvent estimates from complex absorbing potentials to analogous estimates for damped wave equations; this is a result of Christianson, Schenk, Wunsch and the second author [CSVW].

Theorems 1 and 2 follow from microlocal propagation estimates in a neighborhood of $K_E$, or more generally in a neighborhood of a suitable compact invariant subset of a bicharacteristic flow.

To state the general results, suppose $X$ is a manifold, $P\in\Psi^{m,0}(X)$ a self adjoint, order $m>0$, semiclassical pseudodifferential operator on $X$, with principal symbol $p$. For $I\subset\mathbb{R}$ compact and fixed, denote the characteristic set by $\Sigma=p^{-1}(I)$, and suppose that the projection to the base, $\pi\colon\Sigma\to X$, is proper (it is sufficient, for example, to have $p$ classically elliptic). Suppose that $\Gamma\Subset T^*X$ is invariant under the bicharacteristic flow in $\Sigma$. Define the *forward, resp. backward flowout* $\Gamma_+$, resp. $\Gamma_-$, of $\Gamma$ as the set of points $\rho\in\Sigma$, from which the backward, resp. forward bicharacteristic segments tend to $\Gamma$, i.e. for any neighborhood $O$ of $\Gamma$ there exists $T>0$ such that $-t\ge T$, resp. $t\ge T$, implies $\gamma(t)\in O$, where $\gamma$ is the bicharacteristic with $\gamma(0)=\rho$. Here we think of $\Gamma$ as the trapped set or as part of the trapped set, hence points in $\Gamma_-$, resp. $\Gamma_+$ are backward, resp. forward, trapped. Suppose $V$, $W$ are neighborhoods of $\Gamma$ with $\overline{V}\subset W$, $\overline{W}$ compact. Suppose also that

$$\begin{gathered}\text{If } \rho\in W\setminus\Gamma_+,\text{ resp. } \rho\in W\setminus\Gamma_-,\\ \text{then the backward, resp. forward bicharacteristic from } \rho \text{ intersects } W\setminus\overline{V}.\end{gathered} \tag{5}$$

This means that all bicharacterstics in $V$ which stay in $V$ for all time tend to $\Gamma$.

The main result of [DaVa10], from which the other results in the paper follow, is the following:

**Theorem 3.** [DaVa10, Theorem 1.3] *Suppose that* $\|u\|_{H_h^{-N}} \leq h^{-N}$ *for some* $N \in \mathbb{N}$ *and* $(P-\lambda)u = f$, $\operatorname{Re}\lambda \in I$ *and* $\operatorname{Im}\lambda \geq -\mathcal{O}(h^\infty)$. *Suppose* $f$ *is* $\mathcal{O}(1)$ *on* $W$, $\operatorname{WF}_h(f) \cap \overline{V} = \emptyset$, *and* $u$ *is* $\mathcal{O}(h^{-1})$ *on* $W \cap \Gamma_- \setminus \overline{V}$. *Then* $u$ *is* $\mathcal{O}(h^{-1})$ *on* $W \cap \Gamma_+ \setminus \Gamma$.

Here we say that $u$ is $\mathcal{O}(a(h))$ at $\rho \in T^*X$ if there exists $B \in \Psi^{0,0}(X)$ elliptic at $\rho$ with $\|Bu\|_{L^2} = \mathcal{O}(a(h))$. We say $u$ is $\mathcal{O}(a(h))$ on a set $E \subset T^*X$ if it is $\mathcal{O}(a(h)))$ at each $\rho \in E$.

Note that there is no conclusion on $u$ at $\Gamma$; typically it will be merely $\mathcal{O}(h^{-N})$ there. However, to obtain $\mathcal{O}(h^{-1})$ bounds for $u$ on $\Gamma_+$ we only needed to assume $\mathcal{O}(h^{-1})$ bounds for $u$ on $\Gamma_-$ and nowhere else. Note also that by the propagation of singularities, if $u$ is $\mathcal{O}(h^{-1})$ at one point on any bicharacteristic, then it is such on the whole forward bicharacteristic. If $|\operatorname{Im}\lambda| = \mathcal{O}(h^\infty)$ then the same is true for backward bicharacteristics.

In this paper we show that a (lesser) improvement on the a priori bound holds even when $f$ is not assumed to vanish microlocally near $\Gamma$:

**Theorem 4.** *Suppose that* $\|u\|_{H_h^{-N}} \leq h^{-N}$ *for some* $N \in \mathbb{N}$ *and* $(P-\lambda)u = f$, $\operatorname{Re}\lambda \in I$ *and* $\operatorname{Im}\lambda \geq -\mathcal{O}(h^\infty)$. *Suppose* $f$ *is* $\mathcal{O}(1)$ *on* $W$, $u$ *is* $\mathcal{O}(a(h)h^{-1})$ *on* $W$, *and* $u$ *is* $\mathcal{O}(h^{-1})$ *on* $W \cap \Gamma_- \setminus \overline{V}$. *Then* $u$ *is* $\mathcal{O}(\sqrt{a(h)}h^{-1})$ *on* $W \cap \Gamma_+ \setminus \Gamma$.

In [DaVa10] Theorem 1 is deduced from Theorem 3. Theorem 2 follows from Theorem 4 by the same argument.

*Proof of Theorem 4.* The argument is a simple modification of the argument of [DaVa10, End of Section 4, Proof of Theorem 1.3]; we follow the notation of this proof. Recall first from [DaVa10, Lemma 4.1] that if $U_-$ is a neighborhood of $(\Gamma_- \setminus \Gamma) \cap (\overline{W} \setminus V)$ then there is a neighborhood $U \subset V$ of $\Gamma$ such that if $\alpha \in U \setminus \Gamma_+$ then the backward bicharacteristic from $\alpha$ enters $U_-$. Thus, if one assumes that $u$ is $\mathcal{O}(h^{-1})$ on $\Gamma_-$ and $f$ is $\mathcal{O}(1)$ on $\overline{V}$, it follows that that $u$ is $\mathcal{O}(h^{-1})$ on $U \setminus \Gamma_+$, provided $U_-$ is chosen small enough that $u$ is $\mathcal{O}(h^{-1})$ on $U_-$. Note also that, because $U \subset V$, $f$ is $\mathcal{O}(1)$ on $U$. We will show that $u$ is $\mathcal{O}(\sqrt{a(h)}h^{-1})$ on $U \cap \Gamma_+ \setminus \Gamma$: the conclusion on the larger set $W \cap \Gamma_+ \setminus \Gamma$ follows by propagation of singularities.

Next, [DaVa10, Lemma 4.3] states that if $U_1$ and $U_0$ are open sets with $\Gamma \subset U_1 \Subset U_0 \Subset U$ then there exists a nonnegative function $q \in C_0^\infty(U)$ such that

$$q = 1 \text{ near } \Gamma, \qquad H_p q \leq 0 \text{ near } \Gamma_+, \qquad H_p q < 0 \text{ on } \Gamma_+^{\overline{U_0}} \setminus U_1.$$

Moreover, we can take $q$ such that both $\sqrt{q}$ and $\sqrt{-H_p q}$ are smooth near $\Gamma_+$.

**Remark.** The last paragraph in the proof of [DaVa10, Lemma 4.3] should be replaced by the following: To make $\sqrt{-H_p\tilde{q}}$ smooth, let $\psi(s) = 0$ for $s \leq 0$, $\psi(s) = e^{-1/s}$ for $s > 0$, and assume as we may that $U_\rho \cap \mathcal{S}_\rho$ is a ball with respect to a Euclidean metric (in local coordinates near $\rho$) of radius $r_\rho > 0$ around $\rho$. We then choose $\varphi_\rho$ to behave like $\psi({r'_\rho}^2 - |.|^2)$ with $r'_\rho < r_\rho$ for $|.|$ close to $r'_\rho$, bounded away from 0 for smaller values of $|.|$, and choose $-\chi'_\rho$

to vanish like $\psi$ at the boundary of its support. That sums of products of such functions have smooth square roots follows from [Hö94, Lemma 24.4.8].

The proof of Theorem 4 proceeds by induction: we show that if $u$ is $\mathcal{O}(h^k)$ on a sufficiently large compact subset of $U \cap \Gamma_+ \setminus \Gamma$, then $u$ is $\mathcal{O}(h^{k+1/2})$ on $\Gamma_+^{\overline{U}_0} \setminus U_1$, provided $\sqrt{a(h)}h^{-1} \le Ch^{k+1/2}$.

Now let $U_-$ be an open neighborhood of $\Gamma_+ \cap \operatorname{supp} q$ which is sufficiently small that $H_p q \le 0$ on $U_-$ and that $\sqrt{-H_p q}$ is smooth on $U_-$. Let $U_+$ be an open neighborhood of $\operatorname{supp} q \setminus U_-$ whose closure is disjoint from $\Gamma_+$ and from $T^*X \setminus \overline{U}$. Define $\phi_\pm \in C^\infty(U_+ \cup U_-)$ with $\operatorname{supp} \phi_\pm \subset U_\pm$ and with $\phi_+^2 + \phi_-^2 = 1$ near $\operatorname{supp} q$.

Put

$$b \overset{\text{def}}{=} \phi_- \sqrt{-H_p q^2}, \qquad e \overset{\text{def}}{=} \phi_+^2 H_p q^2.$$

Let $Q, B, E \in \Psi^{-\infty,0}(X)$ have principal symbols $q, b, e$, and microsupports $\operatorname{supp} q$, $\operatorname{supp} b$, $\operatorname{supp} e$, so that

$$\frac{i}{h}[P, Q^*Q] = -B^*B + E + hF,$$

with $F \in \Psi^{-\infty,0}(X)$ such that $\operatorname{WF}'_h F \subset \operatorname{supp} dq \subset U \setminus \Gamma$. But

$$\begin{aligned}\frac{i}{h}\langle [P, Q^*Q]u, u\rangle &= \frac{2}{h}\operatorname{Im}\langle Q^*Q(P-\lambda)u, u\rangle + \frac{2}{h}\langle Q^*Q \operatorname{Im}\lambda u, u\rangle \\ &\ge -2h^{-1}\|Q(P-\lambda)u\|\,\|Qu\| - \mathcal{O}(h^\infty)\|u\|^2 \ge -Ch^{-2}a(h) - \mathcal{O}(h^\infty),\end{aligned}$$

where we used $\operatorname{Im}\lambda \ge -\mathcal{O}(h^\infty)$ and that on $\operatorname{supp} q$, $(P-\lambda)u$ is $\mathcal{O}(1)$. So

$$\|Bu\|^2 \le \langle Eu, u\rangle + h\langle Fu, u\rangle + Ch^{-2}a(h) + \mathcal{O}(h^\infty).$$

But $|\langle Eu, u\rangle| \le Ch^{-2}$ because $\operatorname{WF}'_h E \cap \Gamma_+ = \emptyset$ gives that $u$ is $\mathcal{O}(h^{-1})$ on $\operatorname{WF}'_h E$ by the first paragraph of the proof. Meanwhile $|\langle Fu, u\rangle| \le C(h^{-2} + h^{2k})$ because all points of $\operatorname{WF}'_h F$ are either in $U \backslash \Gamma_+$, where we know $u$ is $\mathcal{O}(h^{-1})$ from the first paragraph of the proof, or on a single compact subset of $U \cap \Gamma_+ \setminus \Gamma$, where we know that $u$ is $\mathcal{O}(h^k)$ by inductive hypothesis. Since $b = \sqrt{-H_p q^2} > 0$ on $\Gamma_+^{\overline{U}_0} \setminus U_1$, we can use microlocal elliptic regularity to conclude that $u$ is $\mathcal{O}(h^{k+1/2})$ on $\Gamma_+^{\overline{U}_0} \setminus U_1$, as desired. □

## References


[Bur02] Nicolas Burq. Lower bounds for shape resonances widths of long range Schrödinger operators. *Amer. J. Math.*, 124(4):677–735, 2002.

[BuZw04] Nicolas Burq and Maciej Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.* 17:2, 443–471, 2004.

[CaVo02] Fernando Cardoso and Georgi Vodev. Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II. *Ann. Henri Poincaré*, 3(4):673–691, 2002.

[Chr07] Hans Christianson. Semiclassical non-concentration near hyperbolic orbits. *J. Funct. Anal.* 246(2):145–195, 2007. Corrigendum. *J. Funct. Anal.* 258(3):1060–1065, 2010.

[CSVW] Hans Christianson, Emmanuel Schenk, András Vasy and Jared Wunsch. From resolvent estimates to damped waves. *In preparation.*

[DaVa10] Kiril Datchev and András Vasy. Propagation through trapped sets and semiclassical resolvent estimates. *Annales de l'Institut Fourier*, to appear.

[Hö94] Lars Hörmander, *The Analysis of Linear Partial Differential Operators. III. Pseudo-Differential Operators,* Springer Verlag, 1994.

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