Network Simplification: the Gaussian Diamond Network with Multiple Antennas

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***Abstract*—We consider the N-relay Gaussian diamond network when the source and the destination have $n_s \geq 2$ and $n_d \geq 2$ antennas respectively. We show that when $n_s = n_d = 2$ and when the individual MISO channels from the source to each relay and the SIMO channels from each relay to the destination have the same capacity, there exists a two relay sub-network that achieves approximately all the capacity of the network. To prove this result, we establish a simple relation between the joint entropies of three Gaussian random variables, which is not implied by standard Shannon-type entropy inequalities.**[1]

I. Introduction

Consider a source communicating to a destination with the help of wireless relays. With network simplification we refer to the problem of removing a number of wireless relays while maintaining a desired fraction of the wireless network capacity. Our ultimate goal is to understand by how much and how we can prune an arbitrary wireless network; here we present results for special classes of Gaussian diamond networks.

Our first results in this direction, presented in [1], assumed that all nodes have single antennas. The source is connected to the relays through a broadcast channel, while the relays are connected to the destination through a multiple-access channel, as depicted in Fig. 1. In this paper, we take the next natural step, and consider the case where the source has n_s transmit antennas while the destination has n_d receive antennas.

We start by formulating the N-relay network simplification problem as a combinatorial problem, similar to our approach in [1]. To do so, we provide a simplification result for the point-to-point MIMO channel. We show that if we have n_t transmit and $n_r \geq n_t$ receive antennas, there exists a subset of n_t receive antennas which alone achieve the capacity of the original $(n_t \times n_r)$ MIMO channel within a gap of $n_t \log((n_r - n_t + 1)n_t) + n_t \log n_t$ bits/s/Hz. An analogous result holds when $n_t \geq n_r$.

However, to proceed from this point, combinatorial arguments similar to [1] (where $n_s = n_d = 1$), cannot be directly applied. This is because multiple antennas introduce more degrees of freedom in the network. The channel from the source to each relay (or from each relay to the destination node) is no longer characterized by a single coefficient (the channel gain), but by a vector of coefficients. Therefore, it

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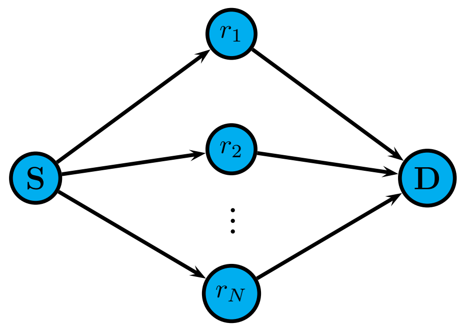

Fig. 1. The Gaussian N-relay diamond network. The source with n_s transmit antennas is connected to N single antenna relays through a broadcast channel; the relays are connected to the destination which has n_d receive antennas through a multiple-access channel.

is not only the configuration of the channel gains, which corresponds to the magnitudes of these vectors, but also the orientation of the vectors that can lead to "small" capacities for the subnetworks.

Our main result is to show that, when $n_s = n_d = 2$ and the individual multiple-input-single output (MISO) channels from the source to each relay and the single-input-multiple-output (SIMO) channels from each relay to the destination have the same capacity, there exist two relays that together approximately achieve the whole capacity of the network. That is, to understand the new dimension of channel orientations that comes into play, we focus on the case where the magnitudes of the channel coefficient vectors are assumed to be equal while their orientations are arbitrary. In this case, for $n_s = n_d = 2$, it is clearly necessary to use at least two relays to approach capacity, since we have two degrees of freedom in the system. We show that this is also sufficient, which is non-trivial since arbitrary orientations of the channel vectors could potentially lead to small capacities for all 2-relay subnetworks.

The main ingredient of our proof comes from establishing a simple relation between the joint entropies of three Gaussian random variables, which is not implied by standard Shannon-type entropy inequalities, and might be of interest in itself. We show that if X, Y, and Z are three jointly Gaussian random variables with $I(X;Y)$ smaller than both $I(X;Z)$ and $I(Y;Z)$, then $\min(I(X;Z), I(Y;Z)) \leq I(X;Y)+2$ bits. Intuitively, if X and Y give little information about each other, Z can not simultaneously give a lot of information about both X and Y.

We finally show that, our result does not hold if we remove

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the restriction of equal capacities for the individual links. We provide an example configuration for the N-relay diamond network with $n_s = n_d = 2$, where all 2-relay subnetworks can at most achieve half the capacity of the whole network. A natural question in this case is whether we can always find 2 relays that would allow us to achieve half the network capacity. We are able to answer this question positively for the case when the diamond network contains 3 or 4 relay nodes.

The paper is organized as follows. Section II provides our model; Section III summarizes our main result; Section IV presents a simplification result for the MIMO channel; Section V formulates the combinatorial problem for the N-relay diamond network; Section VI proves our main result and Section VII considers arbitrary diamond networks with $N \leq 4$.

II. Model

We consider the Gaussian N-relay diamond network depicted in Fig. 1 where the source node S wants to communicate to the destination node D with the help of N relay nodes, denoted $\mathcal{N}$. Assume that the source node is equipped with n_s transmit antennas and the destination node is equipped with n_d receive antennas, while each relay has a single transmit and receive antenna. We assume that $N \geq \max(n_s, n_d)$, typically $N \gg \max(n_s, n_d)$. Let $X_s[t]$ and $X_i[t]$ denote the signals transmitted by the source node S and the relay node $i \in \mathcal{N}$ respectively at time instant $t \in \mathbb{N}$. Similarly, $Y_d[t]$ and $Y_i[t]$ denote the signals received by the destination node d and the relay node i respectively. The transmitted signal $X_i[t]$ by relay i is a causal function of its received signal $Y_i[t]$. We have

$$Y_i[t] = h_{is} X_s[t] + Z_i[t],$$

$$Y_d[t] = \sum_{i=1}^{N} h_{di}^{\dagger} X_i[t] + Z[t],$$

where h_{is} denotes the $1 \times n_s$ complex channel vector between the n_s transmit antennas at the source node and the relay node i and $h_{di}^{\dagger}$ denotes $n_d \times 1$ complex channel vector between the relay node i and the n_d receive antennas at destination node. Note that X_s and Y_d are vectors of length n_s and n_d respectively, while X_i and Y_i are scalars. $Z_i[t]$, $i \in \mathcal{N}$ are independent and identically distributed white Gaussian random processes of power spectral density $N_0/2$ Watts/Hz. Similarly $Z[t]$ is a length n_d circularly symmetric Gaussian vector of identity covariance matrix and power spectral density $N_0/2$ Watts/Hz. All nodes are subject to an average power constraint P and the narrow-band system is allocated a bandwidth of W. Note that the equal power constraint assumption is without loss of generality as the channel coefficients are arbitrary. We assume that the channel coefficients are known at all the nodes.

We denote the capacity of the multiple-input single-output channel between the source node and the relay i by α_i, i.e.,

$$\alpha_i = \log(1 + \text{SNR}\,||h_{is}||^2),$$

where $\text{SNR} = \frac{P}{N_0 W}$. Similarly, the capacity of the individual SIMO channel from the relay node i to the destination node is given by

$$\beta_i = \log(1 + \text{SNR}\,||h_{id}||^2).$$

III. Main Result

The following theorem summarizes our main result.

Theorem 3.1: Consider the Gaussian N-relay diamond network with capacity C, where $n_s = 2$, $n_d = 2$ and $\alpha_i = \alpha$ and $\beta_i = \beta$ for all $i \in \mathcal{N}$. Then, there exists a 2-relay diamond subnetwork such that its capacity C_2 satisfies

$$C_2 \geq C - G,$$

where $G = 18 + 4\log(N-1)$ is a universal constant independent of the channel configurations and the operating SNR.

When the α_i's and β_i's are not equal, there exist configurations of the N-relay diamond network with $n_s = n_d = 2$ such that every 2-relay sub-network provides at most half the capacity. We provide such an example in Section VII. In the same section, we also prove the following proposition.

Proposition 3.1: In every Gaussian N-relay diamond network with capacity C, where $n_s = 2, n_d = 2$ and $N \leq 4$, there exists a 2-relay diamond subnetwork such that its capacity C_2 satisfies

$$C_2 \geq \frac{1}{2} C - G,$$

where $G = 11 + 2\log(N-1)$ is a universal constant independent of the channel configurations and the operating SNR.

IV. MIMO Channel Simplification

Consider a MIMO channel with n_t transmit and n_r receive antennas and the $n_r \times n_t$ channel matrix denoted by H. We have $Y = HX + Z$. The capacity of this channel is well known to be [4]

$$C_{n_r \times n_t} = \max_{Q \geq 0,\, tr(Q) \leq \text{SNR}} \log\det(I + HQH^{\dagger}) \quad (1)$$

where Q is a positive semidefinite matrix, $tr(Q)$ is the trace of the matrix Q, $\text{SNR} = \frac{P}{N_0 W}$ and $H^{\dagger}$ denotes the conjugate transpose of H.

Theorem 4.1: Consider an $n_r \times n_t$ MIMO channel with capacity $C_{n_r \times n_t}$ in (1) and assume $n_r \geq n_t$. Let $\mathcal{R}$ be the set of receive antennas. Let

$$C_{n_t \times n_t} = \max_{\substack{A \subseteq \mathcal{R} \\ |A| = n_t}} \log\det(I + \text{SNR} H_A H_A^{\dagger}),$$

where H_A is the sub-MIMO channel from the n_t transmit antennas to the n_t receive antennas in the set $A \subseteq \mathcal{R}$. We have

$$C_{n_t \times n_t} - G_0 \leq C_{n_r \times n_t} \leq C_{n_t \times n_t} + G_1 \quad (2)$$

where $G_0 = n_t \log n_t$ and $G_1 = n_t \log((n_r - n_t + 1)n_t)$.

The above theorem suggests that the capacity loss incurred by using a subset of the receive antennas in an $n_r \times n_t$ MIMO channel, with $n_r \geq n_t$, is bounded by a universal constant independent of the channel gains and the operating SNR, if the number of selected receive antennas is larger than or equal to n_t. This implies that in the high capacity regime using only $\min(n_t, n_r)$ antennas on both sides of the channel is approximately optimal. Antenna selection for the MIMO channel has been extensively studied in the literature [7], [8],

[9], [10]. A similar result to Theorem 4.1 appears in [2]. The proof of the theorem is given in [12].

An analogous result to Theorem 4.1 holds for the case of $n_t > n_r$: Let $C'_{n_r \times n_t}$ be the capacity of a MIMO channel given in (4.1) but with a total power constraint $n_t P$ instead of P. [2] Let

$$C'_{n_t \times n_t} = \max_{\substack{A \subseteq \mathcal{T} \\ |A|=n_r}} \log\det(I + \mathrm{SNR}\, H_A H_A^\dagger),$$

where $\mathcal{T}$ is the set of n_t transmit antennas. We have

$$C'_{n_t \times n_t} \le C'_{n_r \times n_t} \le C'_{n_t \times n_t} + G_2 \tag{3}$$

where $G_2 = n_r \log((n_t - n_r + 1)n_r) + n_r \log n_r$.

V. Relay Networks with Multiple Antennas

In this section, we use the MIMO channel simplification result of Section IV, to derive lower and upper bounds on the capacity of the diamond relay network. These simple lower and upper bounds allow us to pose the question of interest as a purely combinatorial problem. We provide solutions to this combinatorial problem in certain special cases in Sections VI and VII. The flow of our analysis is analogous to [1].

A. Approximate Capacity of a Diamond Relay Network

Consider a subset $\Gamma \subseteq \mathcal{N}$ of the relay nodes, such that $|\Gamma| = k$. For a subset $\Lambda \subseteq \Gamma$, define $\overline{\Lambda} = \Gamma \setminus \Lambda$. Let C_Γ be the capacity of the k-relay diamond sub-network (assuming the remaining $N-k$ relay nodes are not used for the S-D communication). Then

$$\tilde{C}_\Gamma - G_4 \le C_\Gamma \le \tilde{C}_\Gamma + G_3, \tag{4}$$

$$\text{where} \quad \tilde{C}_\Gamma = \min_{\Lambda \subseteq \Gamma} \Big(\max_{\substack{A \subseteq \bar{\Lambda} \\ |A| \le n_s}} \log\det(I + \mathrm{SNR}\, H_{AS} H_{AS}^\dagger) + \max_{\substack{A \subseteq \Lambda \\ |A| \le n_d}} \log\det(I + \mathrm{SNR}\, H_{DA} H_{DA}^\dagger) \Big), \tag{5}$$

and $G_4 = 3k + n_s \log n_s$ and $G_3 = n_s \log((k - n_s + 1)n_s) + n_d \log((k - n_d + 1)n_d) + n_d \log n_d$. H_{AS} is the cooperative MIMO channel between the source and a subset $A \subseteq \Gamma$ of the relay nodes, with columns $h_{is}, i \in A$. H_{DA} is analogously defined. To prove (4) we combine the information theoretic cutset upper bound on the capacity of the k-relay diamond network with the lower and upper bounds on MIMO capacity in (2) and (3). To obtain the lower bound in (4), we also refer to the result of [6] that the cutset upper bound is achievable within $3k$ bits/s/Hz.

B. A Combinatorial Problem

In the previous section, we have seen that up to a total gap of $G_4 + G_3$, the capacity of a k-relay diamond network behaves approximately like (5). In the rest of the discussion we will concentrate on this approximate form of the capacity. Note that if we establish a result for the approximate capacity, we can translate it to a constant gap result for the actual capacity: Let C denote the capacity of the network with all relays, i.e. $C = C_{\mathcal{N}}$, and let C_k be the capacity of the k-relay subnetwork with the largest capacity, i.e.,

$$C_k = \max_{\substack{\Gamma \subseteq \mathcal{N} \\ |\Gamma| = k}} C_\Gamma. \tag{6}$$

Let $\tilde{C} = \tilde{C}_{\mathcal{N}}$ and $\tilde{C}_k = \max_{\substack{\Gamma \subseteq \mathcal{N} \\ |\Gamma|=k}} \tilde{C}_\Gamma$ be the corresponding approximate capacities. If we can show that

$$\tilde{C}_k \ge r_k \tilde{C} - \gamma_k, \tag{7}$$

using the lower and upper bounds bounds in (4) yields

$$C_k \ge r_k (C - G_3) - \gamma_k - G_4. \tag{8}$$

Let us introduce the notation

$$\begin{aligned} \alpha(A) &= \log\det(I + \mathrm{SNR} H_{AS} H_{AS}^\dagger), \\ \beta(A) &= \log\det(I + \mathrm{SNR} H_{DA} H_{DA}^\dagger), \end{aligned} \tag{9}$$

where $A \subseteq \mathcal{N}$. $\alpha : 2^n \to \mathbb{R}^+$ and $\beta : 2^n \to \mathbb{R}^+$ are two positive set functions defined on subsets of $\mathcal{N}$. $\tilde{C}_\Gamma$ can be rewritten in terms of these set functions as

$$\tilde{C}_\Gamma = \min_{\Lambda \subseteq \Gamma} \Big(\max_{\substack{A \subseteq \bar{\Lambda} \\ |A| \le n_s}} \alpha(A) + \max_{\substack{A \subseteq \Lambda \\ |A| \le n_d}} \beta(A) \Big).$$

In the rest of the paper, we will aim to establish a universal lower bound on r_k and a universal upper bound on γ_k in (7), independent of the particular channel configurations and the operating SNR, by using the properties of the set functions α and β. By (8), this translates to a worst case guarantee on the fraction of the capacity we can achieve with k relays within a constant additive gap. Since in the rest of the discussion, we only work in terms of the approximate capacities $\tilde{C}_\Gamma$, we simply refer to it as the capacity of the subnetwork Γ.

The set functions in (9) can be associated with the joint entropies of the random variables

$$Y_{si} = h_{si} \sqrt{\mathrm{SNR}}\, X_s + Z_{si}, \;\; Y_{id} = h_{di} \sqrt{\mathrm{SNR}}\, X_d + Z_{di},$$

where X_s and X_d are circularly symmetric Gaussian random vectors of length 2, zero mean and identity covariance matrix. Z_{si} and Z_{di} are independent circularly symmetric Gaussian random variables of unit variance. We have $\alpha(A) = H(Y_{si}, i \in A) - |A| \log(2\pi e)$ and $\beta(A) = H(Y_{id}, i \in A) - |A| \log(2\pi e)$. Therefore these set functions should satisfy certain properties satisfied by the joint entropies of a set of Gaussian random variables. In particular, they have to satisfy the following submodularity properties, also called the Shannon inequalities for entropy:

(i) $\alpha(A_1) \le \alpha(A_2)$ if $A_1 \subseteq A_2$.
(ii) $\alpha(A_1 \cup A_2) \le \alpha(A_1) + \alpha(A_2) - \alpha(A_1 \cap A_2)$.

Similar relations hold for the function β.

However, the above properties are not sufficient to prove the result in Theorem 3.1; below we establish an additional property we will need. The proof is provided in Appendix A.

[2]We use this result to simplify the cooperative MIMO channel between the n_t relay nodes and the destination node in the next section. Since every relay node has power P, the cooperative MIMO channel has total power $n_t P$.

Lemma 5.1: Let X, Y, Z be jointly Gaussian random variables. Let $I(X,Y) = \min(I(X,Y), I(X,Z), I(Y,Z))$. Then $\min(I(X,Z), I(Y,Z)) \leq I(X,Y) + 2\,\text{bits}$.

When $H(X) = H(Y) = H(Z)$, the lemma reduces to the following: If $H(X,Y) = \max(H(X,Y), H(X,Z), H(Y,Z))$, then

$$H(X,Y) \leq \max(H(X,Z), H(Y,Z)) + 2. \tag{10}$$

Intuitively, the lemma suggests that if the mutual information between X and Y is small, i.e. X and Y are close to be independent, then Z can not give a lot of information simultaneously about both of them.

VI. Proof of Theorem 3.1

Let us write α_i for $\alpha(\{i\})$ and $\alpha_{i,j}$ for $\alpha(\{i,j\})$. When $\alpha_i = \alpha$ for all $i \in \mathcal{N}$, (10) implies the following relation for any three relays $i, j, k \in \mathcal{N}$: Let $\alpha_{i,j} = \max(\alpha_{i,j}, \alpha_{j,k}, \alpha_{i,k})$, then

$$\alpha_{i,j} \leq \max(\alpha_{j,k}, \alpha_{i,k}) + 2. \tag{11}$$

Similar relations hold for β. Using this property, we will prove that $\tilde{C}_2 \geq \tilde{C} - 4$ in this case.

Note that when $\alpha_i = \alpha$ and $\beta_i = \beta$, by the property (ii), any $\alpha_{i,j} \leq 2\alpha$ and $\beta_{k,l} \leq 2\beta$. Therefore, for any $\Gamma \subseteq \mathcal{N}$,

$$\tilde{C}_\Gamma = \min(\max_{i,j \in \Gamma} \alpha_{i,j}, \max_{k,l \in \Gamma} \beta_{k,l}). \tag{12}$$

That is the min cut is either $\Lambda = \emptyset$ or $\Lambda = \Gamma$, since the value of any other cut is at least $\alpha + \beta \geq 2\min(\alpha, \beta)$.

Let $1, 2 \in \mathcal{N}$ be the pair of relays with the largest α value, i.e. $\alpha_{1,2} = \max_{i,j \in \mathcal{N}} \alpha_{i,j}$ and similarly $\beta_{3,4} = \max_{k,l \in \mathcal{N}} \beta_{k,l}$. We assume that the pairs with maximum α and β values are disjoint since this is the most difficult case to deal with. Note that by (12), $\alpha_{1,2} \geq \tilde{C}$ and $\beta_{3,4} \geq \tilde{C}$. Below we argue that there exists a two relay subnetwork $\Gamma \in \{1,2,3,4\}$ such that $\tilde{C}_\Gamma \geq \tilde{C} - 4$. We assume that $\beta_{1,2} < \tilde{C} - 4$ and $\alpha_{3,4} < \tilde{C} - 4$, since otherwise we are done.

- By applying (11) to $\alpha(\{1,2,3\})$, either $\alpha_{1,3} \geq \tilde{C} - 2$ or $\alpha_{2,3} \geq \tilde{C} - 2$. W.l.o.g, assume $\alpha_{1,3} \geq \tilde{C} - 2$.
- Then $\beta_{1,3} < \tilde{C} - 4$, otherwise $\tilde{C}_{\{1,3\}} \geq \tilde{C} - 4$.
- By applying (11) to $\beta(\{1,3,4\})$, we have $\beta_{1,4} \geq \tilde{C} - 2$.
- Then $\alpha_{1,4} < \tilde{C} - 4$, otherwise $\tilde{C}_{\{1,4\}} \geq \tilde{C} - 4$.
- By applying (11) to $\alpha(\{1,2,4\})$, we have $\alpha_{2,4} \geq \tilde{C} - 2$.
- By applying (11) to $\alpha(\{2,3,4\})$, we have $\alpha_{2,3} \geq \tilde{C} - 4$.
- Also, $\beta_{2,4} < \tilde{C} - 4$, otherwise $\tilde{C}_{\{2,4\}} \geq \tilde{C} - 4$.
- By applying (11) to $\beta(\{2,3,4\})$, we have $\beta_{2,3} \geq \tilde{C} - 2$. Combined with $\alpha_{2,3} \geq \tilde{C} - 4$, this yields $\tilde{C}_2 \geq \tilde{C} - 4$.

VII. Arbitrary Diamond Networks with $n_s = 2, n_d = 2$ and $N \leq 4$

When the equality constraint on the individual capacities $\alpha_i = \alpha$ and $\beta_i = \beta$ is removed, the assertion in Theorem 3.1 does not hold anymore. Consider the 4-relay network in Fig 2, and assume that, as illustrated also in the figure, $\alpha_1 = \alpha_2 = \delta$, $\alpha_3 = \alpha_4 = 2\delta$, $\beta_1 = \beta_2 = 2\delta$, $\beta_3 = \beta_4 = \delta$. These are the capacities of the individual channels from the source to each relay, and from each relay to the destination.

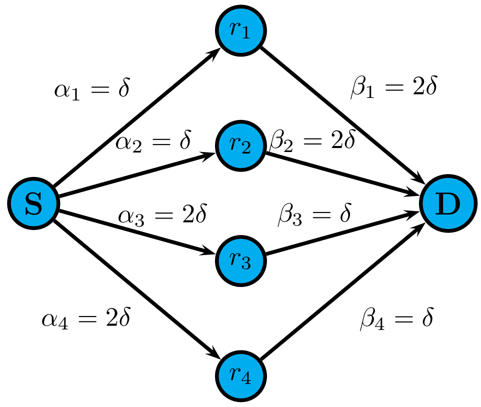

Fig. 2. An instantiation of a diamond network with 4 relays.

Now assume $n_s = n_d = 2$, and moreover that $\alpha_{1,2} = 2\delta$, $\alpha_{1,3} = 2\delta$, $\alpha_{1,4} = 3\delta$, $\alpha_{2,3} = 3\delta$, $\alpha_{2,4} = 3\delta$, $\alpha_{3,4} = 4\delta$. Symmetrically, assume $\beta_{1,2} = 4\delta$, $\beta_{1,3} = 3\delta$, $\beta_{1,4} = 2\delta$, $\beta_{2,3} = 2\delta$, $\beta_{2,4} = 3\delta$, $\alpha_{3,4} = 2\delta$. (One can easily verify that these assignments satisfy the conditions in (i) and (ii) and Lemma 5.1. Indeed, we can also specify the channel vectors that would lead to the set functions given above. Consider for example $h_{1s} = [a\ 0]$, $h_{2s} = [0\ a]$, $h_{3s} = [a^2\ 0]$, $h_{4s} = [0\ a^2]$ and $h_{d1} = [a^2\ 0]$, $h_{d2} = [0\ a^2]$, $h_{d3} = [0\ a]$, $h_{d4} = [a\ 0]$ when $a \gg 1$.) Then the capacity of this network becomes 4δ. By using a single relay we can at most achieve δ, a fraction $1/4$ of the total capacity. This fact illustrates that the conclusions of [1] do not extend to the case of multiple antennas. On the other hand every two-relay subnetwork has capacity 2δ. Therefore $r_k = 1/2$ for this configuration and the conclusion of Theorem 3.1 also does not hold.

When $n_s = n_d = 2$, we can show that this example corresponds to a worst case configuration, i.e. $r_k \geq \frac{1}{2}$ when $N \leq 4$ for any configuration of the channels. Below we prove Proposition 3.1 for $N = 3$. The proof follows similar lines for $N = 4$.

Proof of Proposition 3.1: We will rely on a weaker version of the properties (i) and (ii) for the set functions α and β. Namely,

$$\max\{\alpha_i, \alpha_j\} \leq \alpha_{i,j} \leq \alpha_i + \alpha_j. \tag{13}$$

Assume we have a 3-relay network with capacity $\tilde{C}$, and its every two-relay subnetwork has capacity $< \frac{\tilde{C}}{2}$. Then:

- There exists $i \in \{1,2,3\}$ such that $\alpha_i \geq \tilde{C}/2$, w.l.o.g. assume it is $i = 1$: Otherwise the cut $\Lambda = \emptyset$, yields a value strictly smaller than $\tilde{C}$ contradicting with the assumption that the capacity of the network is $\tilde{C}$.
- $\beta_1 < \tilde{C}/2$: Otherwise, $\tilde{C}_1 \geq \tilde{C}/2$, leading to a contradiction.
- Since the capacity of the 2-relay subnetwork with relays $\{2,3\}$ is $< \tilde{C}/2$, we either have $\alpha_{2,3} < \tilde{C}/2$ or $\beta_{2,3} < \tilde{C}/2$ or $\alpha_2 + \beta_3 < \tilde{C}/2$ or $\beta_2 + \beta_3 < \tilde{C}/2$. All these cases combined with $\beta_1 < \tilde{C}/2$ and the condition (13) lead to a cut for the 3-relay network with value $< \tilde{C}$ which contradicts with the fact that the capacity of the network is $\tilde{C}$. This concludes the proof of the proposition.

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APPENDIX A
PROOF OF LEMMA 5.1

The joint entropy of the three jointly Gaussian random variables is given by

$$H(X,Y,Z) = \log\det((2\pi e)K)$$

where K is the 3×3 covariance matrix of the three variables. Note that since K is positive semidefinite, it can be written as $K = SS^\dagger$. Let $s_1, s_2, s_3 \in \mathbb{C}^3$ denote the rows of S. The joint entropy of a subset A of these random variables X, Y, Z is given by $H(A) = \log\det((2\pi e)K_A)$ where K_A is the corresponding submatrix of K.

Let $I(X,Y) = \min(I(X,Y), I(X,Z), I(Y,Z))$ and let X, Y correspond to the upper left 2×2 submatrix of K. We have

$$I(X,Y) = h(X) + h(Y) - h(X,Y)$$
$$= \log(||s_1||^2) + \log(||s_2||^2) - \log(||s_1||^2||s_2||^2 - |< s_1, s_2 >|^2)$$
$$= -\log(1-\cos^2(f(s_1,s_2))),$$

where

$$f(s_1,s_2) = \arccos\left(\frac{|< s_1, s_2 >|}{||s_1||||s_2||}\right).$$

Note that for $I(X,Y)$ to be minimal we have $f(s_1,s_2) = \max(f(s_1,s_2), f(s_1,s_3), f(s_2,s_3))$. Below in Proposition A.3, we prove that for any three vectors s_1, s_2, s_3 in $\mathbb{C}^n$, we have

$$f(s_1,s_2) \le f(s_1,s_3) + f(s_2,s_3).$$

Therefore $\max(f(s_1,s_3), f(s_2,s_3)) \ge f(s_1,s_2)/2$. We have

$$\min(I(X,Z), I(Y,Z)) \le -\log(1-\cos^2(f(s_1,s_2)/2))$$
$$= -\log(\sin^2(f(s_1,s_2)/2)) \le I(X,Y) + 2,$$

since $\log(\sin^2(f(s_1,s_2))) - \log(\sin^2(f(s_1,s_2)/2)) \le 2$. This concludes the proof of the lemma. □

For two complex vectors $a, b \in \mathbb{C}^n$, the quantity $f(a,b)$ is roughly like the angle between these vectors. Below we prove that this quantity satisfies the triangle inequality:

Note that $0 \le f(a,b) \le \pi/2$ and $f(a,b) = f(\lambda a, b) = f(a, \mu b) = f(Ma, Mb)$ where λ, μ are nonzero complex numbers and M is an arbitrary unitary matrix. In particular, $f(a,b) = f(-a,b) = f(a,-b)$.

Proposition A.1: Let a, b be two n dimensional complex vectors and P a complex plane containing the vector a. Let b_P be the projection of the vector b on the plane P. We have $f(a,b) \ge f(a,b_P)$.

Proof: Since cosine is a decreasing function, it suffices to show that $\frac{|\langle a,b\rangle|}{|a||b|} \le \frac{|\langle a,b_P\rangle|}{|a||b_P|}$. Without loss of generality, we may assume that $|a| = |b| = 1$. Let $b'_P = b - b_P$. By definition, b'_P is orthogonal to the plane P and in particular it is orthogonal to both a and b_P. So we have:

$$\frac{|\langle a,b\rangle|}{|a||b|} = |\langle a,b\rangle| = |\langle a, b_P + b'_P\rangle| = |\langle a,b_P\rangle| \le \frac{|\langle a,b_P\rangle|}{|b_P|}.$$

Proposition A.2: If a, b, c are three vectors in $\mathbb{C}^2$ then $f(a,b) \le f(a,c) + f(b,c)$.

Proof: We can assume that a, b form a basis for $\mathbb{C}^2$, since otherwise $f(a,b) = 0$ and the assertion is trivial. W.l.og. we may assume $|a| = |b| = 1$ and also $c = (1,0)$. The last equality is due to the fact that we can scale c to make its length equal to 1 and then we multiply all the vectors a, b, c by an appropriate unitary matrix M. Let $a = (a_1+ib_1, a_2+ib_2), c = (c_1+id_1, c_2+id_2)$. The assertion then is equivalent to the following inequality: $\text{Arccos}(\sqrt{a_1^2+b_1^2}) + \text{Arccos}(\sqrt{c_1^2+d_1^2}) \ge \text{Arccos}(\sqrt{X^2+Y^2})$, in which $X = a_1c_1 + a_2c_2 + b_1d_1 + b_2d_2$, $Y = b_1c_1 + b_2c_2 - a_1d_1 - a_2d_2$. Since $\cos(x)$ is a decreasing function on the interval $[0,\pi]$, by applying the cosine function on both sides of the inequality we can equivalently prove that:

$$\sqrt{a_1^2+b_1^2}\sqrt{c_1^2+d_1^2} - \sqrt{a_2^2+b_2^2}\sqrt{c_2^2+d_2^2} \le \sqrt{X^2+Y^2}$$

The proof uses basic calculations, and is omitted.

Proposition A.3: If a, b, c are three vectors in $\mathbb{C}^n$ then $f(a,b) \le f(a,c) + f(b,c)$.

Proof: First notice that the vectors a, b, c span a subspace of $\mathbb{C}^n$ whose dimension is at most 3. So, we need only consider the case that $a, b, c \in \mathbb{C}^3$. If a, b are on the same direction then $f(a,b) = 0$ and there is nothing to prove. Otherwise, let P be the plane generated by a, b.

Let c_P be the projection of c on the plane P. By Proposition A.1 we know that $f(a,c) \ge f(a,c_P)$ and $f(b,c) \ge f(b,c_P)$. So, $f(a,c) + f(b,c) \ge f(a,c_P) + f(b,c_P)$. Since a, b, c_P are all laying on the 2-dimensional plane P, by Proposition A.2 we know that $f(a,c_P) + f(b,c_P) \ge f(a,b)$. Combining these two inequalities we conclude that $f(a,b) \le f(a,c) + f(b,c)$.