Instructions:

1. Do not turn anything in.
2. The course staff is happy to discuss the solutions of these exercises with you in office hours or on Piazza.
3. While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 18

In the multicommodity flow problem, the input is a directed graph $G = (V, E)$ with $k$ source vertices $s_1, \ldots, s_k$, $k$ sink vertices $t_1, \ldots, t_k$, and a nonnegative capacity $u_e$ for each edge $e \in E$. An $s_i$-$t_i$ pair is called a commodity. A multicommodity flow if a set of $k$ flows $f^{(1)}, \ldots, f^{(k)}$ such that (i) for each $i = 1, 2, \ldots, k$, $f^{(i)}$ is an $s_i$-$t_i$ flow (in the usual max flow sense); and (ii) for every edge $e$, the total amount of flow (summing over all commodities) sent on $e$ is at most the edge capacity $u_e$. The value of a multicommodity flow is the sum of the values (in the usual max flow sense) of the flows $f^{(1)}, \ldots, f^{(k)}$.

Prove that the problem of finding a multicommodity flow of maximum-possible value reduces in polynomial time to solving a linear program.

Exercise 19

Consider a primal linear program (P) of the form

$$\max c^T x$$

subject to

$$Ax = b$$
$$x \geq 0.$$  

The recipe from Lecture #8 gives the following dual linear program (D):

$$\min b^T y$$

subject to

$$A^T y \geq c$$
$$y \in \mathbb{R}.$$  

Prove weak duality for primal-dual pairs of this form: the (primal) objective function value of every feasible solution to (P) is bounded above by the (dual) objective function value of every feasible solution to (D).\footnote{In Lecture #8, we only proved weak duality for primal linear programs with only inequality constraints (and hence dual programs with nonnegative variables).}
Exercise 20

Consider a primal linear program \( (P) \) of the form
\[
\text{max } c^T x
\]
subject to
\[
Ax \leq b \\
x \geq 0
\]
and corresponding dual program \( (D) \)
\[
\text{min } b^T y
\]
subject to
\[
A^T y \geq c \\
y \geq 0.
\]
Suppose \( \hat{x} \) and \( \hat{y} \) are feasible for \( (P) \) and \( (D) \), respectively. Prove that if \( \hat{x}, \hat{y} \) do not satisfy the complementary slackness conditions, then \( c^T \hat{x} \neq b^T \hat{y} \).

Exercise 21

The minimum-cost bipartite matching problem can be solved by reducing it to the maximum-flow problem. Consider the following linear programming relaxation of the problem:
\[
\text{min } \sum_{e \in E} c_e x_e
\]
subject to
\[
\sum_{e \in \delta(v)} x_e = 1 \quad \text{for all } v \in V \cup W \\
x_e \geq 0 \quad \text{for all } e \in E.
\]
By relaxation we mean that instead of taking values of either 0 or 1, the edge variables can take arbitrary fractional values. In this exercise, we prove that the above LP has an optimal solution that only takes 0-1 values.

(a) By a fractional solution, we mean a feasible solution to the above linear program such that \( 0 < x_e < 1 \) for some edge \( e \in E \). Prove that, for every fractional solution, there is an even cycle \( C \) of edges with \( 0 < x_e < 1 \) for every \( e \in C \).

(b) Prove that, for all \( \epsilon \) sufficiently close to 0 (positive or negative), adding \( \epsilon \) to \( x_e \) for every other edge of \( C \) and subtracting \( \epsilon \) from \( x_e \) for the other edges of \( C \) yields another feasible solution to the linear program.

(c) Show how to transform a fractional solution \( x \) into another fractional solution \( x' \) such that: (i) \( x' \) has fewer fractional coordinates than \( x \); and (ii) the objective function value of \( x' \) is no larger than that of \( x \).

(d) Conclude that the linear programming relaxation above is guaranteed to possess an optimal solution that is 0-1 (i.e., not fractional).
Exercise 22

Consider the following linear programming relaxation of the maximum-cardinality matching problem:

\[
\begin{align*}
\text{max} & \quad \sum_{e \in E} x_e \\
\text{subject to} & \quad \sum_{e \in \delta(v)} x_e \leq 1 \quad \text{for all } v \in V \\
& \quad x_e \geq 0 \quad \text{for all } e \in E,
\end{align*}
\]

where \(\delta(v)\) denotes the set of edges incident to vertex \(v\).

Does this linear program always have an optimal solution that only takes 0-1 values?

Exercise 23

Recall that in the zero-sum game, there are two players whose payoffs are given by an \(m \times n\) matrix \(A\), with \(a_{ij}\) specifying the payoff of the row player and the negative of the payoff of the column player when the former chooses row \(i\) and the latter chooses column \(j\).

Show that the following two linear programs give the utilities of the row and column players respectively, assuming that the other player chooses a strategy to maximize his or her utilities.

\[
\begin{align*}
\text{max} & \quad v \\
\text{subject to} & \quad v - \sum_{i=1}^{m} a_{ij} x_i \leq 0 \quad \text{for all } j = 1, \ldots, n \\
& \quad \sum_{i=1}^{m} x_i = 1 \\
& \quad x_i \geq 0 \quad \text{for all } i = 1, \ldots, m \\
& \quad v \in \mathbb{R},
\end{align*}
\]

and

\[
\begin{align*}
\text{min} & \quad w \\
\text{subject to} & \quad w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \quad \text{for all } i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} y_j = 1 \\
& \quad y_j \geq 0 \quad \text{for all } j = 1, \ldots, n \\
& \quad w \in \mathbb{R},
\end{align*}
\]

Prove that they are both feasible and are dual linear programs. (Note: for those of you who are curious, the solutions of the LPs give the optimal “mixed” strategies for both players, leading to a Nash equilibrium for the zero-sum game.)
Exercise 24

Recall from Problem #6(e) in Problem Set #2 the following linear programming formulation of the $s$-$t$ shortest path problem:

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta^+(S)} x_e \geq 1 \quad \text{for all } S \subseteq V \text{ with } s \in S, t \notin S$$

$$x_e \geq 0 \quad \text{for all } e \in E.$$

Prove that this linear program, while having exponentially many constraints, admits a polynomial-time separation oracle (in the sense of the ellipsoid method, see Lecture #10).