MS&E 319/CS 369X: Topics in Network Algorithms. Winter 2005-06
Course URL: `http://www.stanford.edu/~ashishg/network-algorithms`.
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4.1 Adversarial Queueing Model (contd.)

The following protocols are universally stable (definition given in last lecture) in the adversarial queueing model:

1. LIS (Longest in system). The packet which has been injected earliest in the network has the highest priority.
2. SIS (Shortest in system). The packet which has been injected last in the network has the highest priority.
3. FTG (Furthest to go). The packet which has the highest number of hops left to reach its destination gets the highest priority.

The following protocols are unstable at arbitrarily low injection rates [2][3]:

1. FIFO (First in first out). Packet which enters the queue first has the highest priority. The instability of FIFO was shown in the class.
2. NTG (Nearest to go). The packet which has the least number of hops left to reach its destination gets the highest priority.

The results in this paper were proved in [1].

4.2 Shortest in System is Universally Stable

The following lemma bounds the time taken by a particular packet p to cross an edge e, once p is in the queue of e. However, the bound is in terms a parameter.

Lemma 4.1 *Suppose packet p is waiting to cross edge e, and there are $(k-1)$ packets in the network that have priority over p and need to eventually cross e. Then p crosses e in at most $\frac{k+b}{\epsilon}$ time steps, where $\epsilon = 1 - r$.*

Proof: Let the current time be t_0. Consider the time interval $I = [t_0, t_0 + \frac{k+b}{\epsilon}]$.

Since $r = 1 - \epsilon$, the number of packets injected during I which need to cross $e \leq \left(\frac{k+b}{\epsilon}\right)(1-\epsilon) + b$ $= \frac{k+b}{\epsilon} - k$.

The number of packets which may traverse e before p during $I \leq \frac{k+b}{\epsilon} - 1$ (packets injected during

I and old packets which had a higher priority).

However, the length of the interval is $\frac{k+b}{\epsilon}$. Therefore, p is certainly going to traverse e during the interval I. ■

Now we bound the parameter of the last lemma. Let $k_1 = b$ and k_{j+1} be inductively defined as $\frac{k_j+b}{\epsilon}$.

Lemma 4.2 *Let the path of packet p be $e_1, \ldots, e_n$. When packet p arrives at edge e_j, there are at most $k_j - 1$ packets in the network which need to cross e_j and have priority over p.*

Proof: Proof by induction.

Base case, $j = 1$. Since e_1 is the first edge in the path of p, the packets which would have a higher priority than p would be those which are injected simultaneously with p, which is at most b (the burst parameter).

Induction case: Suppose the claim is true for all edges at and before e_j. Then by the last lemma p will arrive in the queue of e_{j+1} in another $\frac{k_j+b}{\epsilon}$ time steps, during which time at most another $(1-\epsilon)\frac{k_j+b}{\epsilon}$ packets requiring any edge e arrive with priority over p. Thus, when packet p arrives at edge e_{j+1}, there are at most $k_j + (1-\epsilon)\frac{k_j+b}{\epsilon} \leq k_{j+1}$ packets in the network which need to cross e_{j+1} and have priority over p. ■

Theorem 4.3 *SIS is universally stable.*

Proof: We use the approximation $k_j \leq b\left(\frac{2}{\epsilon}\right)^{j-1}$. If d is the number of hops in the path of packet p, then the total time p takes to reach its destination is $\leq k_2 + k_3 + \ldots k_{d+1} \leq \frac{2}{\epsilon} b \left[\sum_{j=1}^{d-1} \left(\frac{2}{\epsilon}\right)^j\right] \leq \left(\frac{2}{\epsilon}\right)^{d+1} b$. ■

4.3 Longest in System is Universally Stable

Let's fix packet p and assume it goes through $e_1, \ldots, e_d$. Suppose p arrives e_j at time T_{j-1}, and traverses e_j at time T_j. At time t, define c_t as the age of the oldest packet in the system. Let $c = \max_t\{c_t\}$.

Lemma 4.4 $T_d - T_0 \leq c(1-\epsilon^d) + \frac{b}{1-\epsilon}$.

Proof: Packets which have higher priority over p would have been injected during the time interval $[t - c_t, T_0)$.

$$\begin{aligned}
T_{j+1} - T_j &\leq (T_0 - T_j + c_{T_j})(1-\epsilon) + b \\
&\leq (T_0 - T_j + c)(1-\epsilon) + b \\
T_{j+1} &\leq b + (T_0 + c)(1-\epsilon) + \epsilon T_j \\
&\leq \epsilon^{j+1} T_0 + (b + (T_0 + c)(1-\epsilon)) \sum_{i=0}^{j} \epsilon^i \\
T_d &\leq \epsilon^d T_0 + (b + (T_0 + c)(1-\epsilon)) \left(\frac{1-\epsilon^d}{1-\epsilon} \right) \\
&\leq \left(\epsilon^d + (1-\epsilon) \frac{1-\epsilon^d}{1-\epsilon} \right) T_0 + c(1-\epsilon^d) + b \frac{1-\epsilon^d}{1-\epsilon} \\
T_d - T_0 &\leq c(1-\epsilon^d) + \frac{b}{1-\epsilon}
\end{aligned}$$

■

The following lemma bounds c.

Lemma 4.5 *Let d_p be the number of edges for packet p and $d_{\max} = \max_p\{d_p\}$. Then $c \leq \left(\frac{b}{1-\epsilon}\right)\left(\frac{1}{\epsilon^{d_{\max}}}\right)$.*

Proof:

$$\begin{aligned}
c &\leq \max_p (T_{d_p} - T_0) \\
&\leq T_{d_{\max}} - T_0 \\
&\leq c(1-\epsilon^{d_{\max}}) + \frac{b}{1-\epsilon}.
\end{aligned}$$

Hence we get $c \leq \left(\frac{b}{1-\epsilon}\right)\left(\frac{1}{\epsilon^{d_{\max}}}\right)$. ■

Therefore, the delay is bounded and we get the following theorem.

Theorem 4.6 *LIS is universally stable.*

References

[1] M. Andrews, B. Awerbuch, A. Fernandez, T. Leighton, Z. Liu, J. Kleinberg. *Universal-Stability Results and Performance Bounds for Greedy Contention-Resolution Protocols.* Journal of the ACM, Vol 48, No. 1, 2001 p39-69.

[2] R. Bhattacharjee, A. Goel, Z. Lotker. *Instability of FIFO at Arbitrarily Low Rates in the Adversarial Queueing Model.* SIAM J. Comput. 34(2): 318-332 (2004).

[3] T. Tsaparas. *Stability in adverarial queueing theory.* M.Sc Thesis, Department of Computer Science, University of Toronto, 1997.