A Game-theoretic Model of Attention in Social Networks

Ashish Goel [*] Farnaz Ronaghi [†]

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Abstract

We model the economics of producing content in online social networks such as Facebook and Twitter. We propose a game-theoretic model within which we quantify inefficiencies from contributions by strategic users in online environments. Attention and information are assumed to be the main motivation for user contributions. We treat attention as a mechanism for sharing the profit from consuming information and introduce a general framework for analyzing dynamics of contributions in online environments. We analyze the proposed model and identify conditions for existence and efficient computation of pure-strategy Nash equilibrium.

We prove a bicriteria bound on the price of anarchy; in particular we show that the social welfare from central control over level of contribution by users is no larger than the social welfare from strategic agents with twice as large consumption utilities. We then construct and analyze a family of production games that have an arbitrarily large price of anarchy. We also prove non-robustness of the price of anarchy for a particular instance of the introduced family, establishing a distinction between the games studied here and network congestion games.

1 Introduction

Social networking websites allow users to sign up and keep in touch with others by friending them. Users are allowed to post short status updates, photos, videos and links depending on the social network. Despite the differences between these networks, one feature is common to many of them: Users see a linear news feed reflecting a chronologically sorted ordering of posts from friends whenever she logs into each of these sites [BCK+10]. Given that the main advantage of these sites is the convenience of getting a quick update, the balance of information from friends in the feed becomes important in determining the value the user will gain from the update [BCK+10].

A user's news feed can easily get flooded by contributions from an active friend who attracts most of the attention in a social cluster. It has been observed that there is a strong correlation between lack of attention and a user's decision to stop contributing; also, the more a user contributes the more attention she tends to receive. Getting attention paid to one's contributions is a form of value [Fra99] and users are willing to forsake financial gain for it [HLO04]. Attention was also shown to spur further contributions in video sharing [HRW09] and blogging [MY07]; moreover it was introduced as the main ingredient in successful peer production websites [WWH09]. Taking

[*]Department of Management Science and Engineering, Stanford University. Email: ashishg@stanford.edu.This research was supported in part by NSF award IIS-0904325. Part of the research was also sponsored by the Army Research Laboratory and was accomplished under Cooperative Agreement Number W911NF-09-2-0053.

[†]Department of Management Science and Engineering, Stanford University. Email: farnaaz@stanford.edu. This research was supported by a Stanford Engineering Fellowship.

attention and information as the foremost motivation for user activity in social networks, high participation from a subset of friends of user u makes u's attention a scarce resource for other friends of u, which in turn can make these other friends less likely to contribute and hence get even less attention. We model this dynamics of user contributions in online social networks in a game-theoretic setting and quantify the inefficiencies from strategic user participation.

Our Contributions. Our main contribution is proposing and analyzing a game-theoretic model of user participation in online social networks. We refer to the proposed model by "the general production game" throughout the paper. The main elements of our model are users that choose the level/quality of their contributions. Updates from a user are viewed as a bundle in her friends' news feed, ignoring the order in which information arrives. We assume that the network structure is fixed. Users are strategic in selecting their level of contribution but they are not strategic in selecting what information or how much information they consume. They are assumed to be utility-maximizing agents who derive utility from attention and information simultaneously. Moreover they incur a non-negative cost for producing content. Producing more information needs more effort and there is no cost for inaction. Users prefer more information to less but they have limited attention capacity so their utility from consumption has diminishing returns. We formalize these assumptions in section 2 and refer to them as "general assumptions" throughout the paper.

While users consume content, they pay attention to those who generated it. In other words, users share the profit from consuming information with those who produced it in the first place. We introduce a general utility scheme and analyze three instances of it: Proportional, Incremental, and Shapley utility models. The proportional utility model splits user utility from consumption among friends proportional to their contribution. The incremental utility model regards the marginal consumption utility from a user's contribution as the attention she receives. The Shapley utility is similar to the incremental utility; however the incremental utility from a user's information is computed with respect to only the information from "earlier" users in a random permutation [Sha53].

We observe that under general assumptions, a unique pure-strategy Nash equilibrium exists for our proposed game and is characterized via the Karush Kuhn Tucker(KKT) conditions. Our main result states necessary conditions such that the social welfare from central control over user production, obligating users to act according to social interest, is no greater than the social welfare from strategic production when users have twice as large consumption utilities. We have proved that under general assumption both Shapley and incremental utility models satisfy this necessary conditions. Our results suggest that improvements in user experience, such as ease of information discovery and spam detection, can compensate for inefficiencies from strategic behavior in online environments. We also analyze the proposed game for simple interesting cases in section 4.2 and show that the price of anarchy can be arbitrarily large regardless of the number of agents and network structure.

Our main result is similar to the bicriteria bound for network congestion games presented in [RT00], and superficially, it might seem that earlier proof techniques should directly imply our results. However, both atomic and non-atomic network congestion games are proved to have a "robust pure Nash price of anarchy" in the sense of [Rou09]. We show (in section 5) that one particular instance of our proposed production game does not admit a "robust price of anarchy", establishing a distinction between the proposed game and network congestion games.

Related Work. Attention affects the propagation of information in social networks, determining the effectiveness of advertising and viral marketing. Many different approaches have been taken to study attention, including empirical studies of dynamics of attention [WWH09, WH07], empirical and game-theoretic analysis of the impact of attention in marketing [BCK+10, LAH05], and studies on the impact of attention(exposure) in high-quality user-generated content [GM11, JCP09].

Wu et al. [WH07] study dynamics of collective attention for a piece of story on digg.com, and they propose a stochastic model that predicts the amount of attention a story gets by incorporating novelty of the story as a decaying factor. In a follow-up paper [WWH09], the authors study feedback loops of attention in peer production websites such as youtube and digg.com. They empirically show a strong correlation between lack of attention and users' decision to stop contributing.

Borgs et al. [BCK+10] address the asymmetry of online relationships in social networks such as Facebook and Twitter. They model the social network as a complete bi-partite graph where users are either producers or consumers of information, and edges of the graph have non-negative weights that represent the quality of updates from a particular producer (in their setting, an advertiser) to a particular consumer. Users receive update at a rate chosen by advertisers and can adopt two types of behavior in response to an excessively high rate: unfollowing and disengagement. They study the set of ties that are realized and stabilized over time in the followership and engagement models.

Ghosh et al. [GM11] study the problem of high-quality user generated content in online crowd-sourcing websites. They propose a game-theoretic model that incorporates the quality of content and study mechanisms of splitting attention (exposure time) to incentivize high-quality content and maximize user participation. They independently propose a proportional model for splitting the exposure time among contributions, and show that the proportional mechanism elicits both high quality and high participation in equilibrium.

Unlike previous work, we directly model attention as a scarce resource, as well as the cost of creating new and useful pieces of information. Attention is shown to be the foremost motivator in peer production websites such as digg.com and YouTube [WWH09]; it is also one of the main motivators in online social networks such as Facebook and Twitter. Information and attention are duals in social networks hence, instead of modeling information and attention as two separate entities, we model attention as a mechanism for sharing the profit from consuming information. We propose a model in which all agents are strategic and derive utility simultaneously from consuming and producing information. Agents are connected to one another in either a symmetric or asymmetric network, i.e. we don't make any assumptions on network structure. Also we can re-interpret the only decision variable in our model from level of contributions to quality of contributions or rate of updates and our results remain valid.

2 Model

Every user as a member of the social network has friendship relations with at least one other user. User a produces x_a units of information which appear on her friends' feed. She perceives $y_a := \sum_{b\sim a} q_{ba} x_b$ units of information from her feed where $b \sim a$ means a is a friend of b and q_{ba} represents a's interest in b's updates. User a pays attention to user b when she consumes the information produced by her. We model such exchange of attention and information between users of a social network in a utilitarian framework where every user a incurs an increasing cost $c_a(x_a)$ for producing information and derives increasing utility $f_a(y_a)$ from consuming it. Users also derive positive utility from receiving attention. We denote the amount of attention user a receives from her friend b by $t_{a,b}(\vec{x})$. So an arbitrary agent a derives

$$u_a(\vec{x}) = f_a(y_a) - c_a(x_a) + t_a(\vec{x}) \tag{1}$$

utility from her network of friends where $t_a(\vec{x}) = \sum_{b\sim a} t_{a,b}(\vec{x})$ and $\vec{x}$ represents the strategy vector.

We analyze the proposed utility scheme and state conditions under which our bicriteria bound holds. We make three general assumptions throughout the paper. First, we assume that consumption utility $f_a(y_a)$ is a differentiable, concave, and increasing function for every agent and $f_a(0) = 0$. Second, we assume that production cost $c_a(x_a)$ is a differentiable, increasing, and strictly convex function for all agents and $c_a(0) = 0$. Third, we assume $t_{a,b}(\vec{x})$ is increasing in x_a and $t_{a,b}(0, \vec{x}_{-a}) = 0$ where $\vec{x}_{-a}$ is the strategy vector including production level for all users except for user a. We also study three particular instances of our general model denoted by *incremental* utility, *Shapley* utility, and *proportional* utility. Each instance corresponds to a different method of splitting user attention among friends in online environments.

Incremental utility models the amount of attention a user receives by the sum of consumption utility margins she imposes on her friends in the social network. Formally, the incremental utility models the attention user a receives by

$$t_a(\vec{x}) = \sum_{b\sim a} f_b(y_b) - f_b(y_b - q_{ab}x_a). \tag{2}$$

Shapely utility implements the Shapley cost sharing scheme as a profit sharing mechanism to split the profit from consuming information among those who originally created it. Shapley mechanism assigns

$$t_a(\vec{x}) = \sum_{b\sim a} \sum_{\sigma\in\mathbb{S}_{N(b)}} \frac{1}{d_b!} f_b(\sum_{i=1}^{\sigma^{-1}(a)} q_{\sigma(i)b}x_{\sigma(i)}) - f_b(\sum_{i=1}^{\sigma^{-1}(a)-1} q_{\sigma(i)b}x_{\sigma(i)}) \tag{3}$$

units of utility from attention to every user a, where $\mathbb{S}_{N(b)}$ denotes the set of all permutations of b's friends; $N(b)$ is the set of b's friends and $d_b = |N(b)|$. Marginal profit terms inside the second sum are known as *ordered marginals*. The Shapley value is defined as the expectation of *ordered marginals* over a uniform distribution on all arrival permutations [Sha53]. Shapley utility arises as a natural attention sharing mechanism when incoming updates are shown at random order in the feed provided to users.

Proportional utility is an alternate way of splitting user attention among friends in a social network. The amount of attention a user receives is modeled as the weighted sum of friends' consumption utilities where the weights are equal to the proportion of the user's contribution. Formally, the amount of attention a user receives in the network is

$$t_a(\vec{x}) = \sum_{b\sim a} \frac{q_{ab}x_a}{y_b} f_b(y_b). \tag{4}$$

Friends with more high quality updates receive more attention in the proportional mechanism. Similar to the Shapley utility, proportional mechanism arises as a natural profit sharing scheme when updates are viewed in a random order. Position bios is a well-established phenomenon in online social networks; items shown higher in user's update feed have a higher probability of receiving actions. We ignore the impacts of position bios on the distribution of attention throughout the paper.

3 Existence and Computability of Nash Equilibrium

We determine sufficient conditions such that our general utility model admits pure-strategy Nash equilibrium. Moreover we identify exact potential functions for incremental and Shapley utility models; existence of exact potential functions implies convergence of the natural Nash dynamics to a pure-strategy equilibrium.

Strategy vector $\vec{x}$ is a pure Nash equilibrium if every player a chooses her strategy x_a to maximize $u_a(x_a, \vec{x}_{-a})$. Rosen's theorem [Ros65] for concave n-player non-cooperative games establishes existence of a unique pure Nash equilibrium and KKT conditions characterize it.

Proposition 3.1. *Our proposed general production game admits a unique pure-strategy Nash equilibrium if* $|\frac{\partial^2 t_a(\vec{x})}{\partial x_a^2} - \frac{\partial^2 c_a(x_a)}{\partial x_a^2}| > \epsilon$ *for constant* $\epsilon > 0$*, and general assumptions hold. Strategy vector* $\vec{x}$ *is a Nash equilibrium strategy if and only if for every player* a*,*

$$x_a(\frac{\partial t_a(\vec{x})}{\partial x_a} - \frac{\partial c_a(x_a)}{\partial x_a}) = 0. \tag{5}$$

Proof: General assumptions guarantee strict concavity of the utility function for very player a. The strategy space can be reduced to a convex and compact set; since $|\frac{\partial^2 t_a(\vec{x})}{\partial x_a^2} - \frac{\partial^2 c_a(x_a)}{\partial x_a^2}| > \epsilon$ we can define upper bounds on values of x_a. We can apply Rosen's theorem for concave n-player noncooperative games with convex and compact strategy space [Ros65] to conclude existence of Nash equilibria for proposed utility game.

At Nash equilibrium every player solves the following optimization problem:

$$\begin{array}{ll} \text{Maximize} & f_a(y_a) - c_a(x_a) + t_a(\vec{x}) \\ \text{Subject to:} & x_a \geq 0. \end{array}$$

KKT conditions, stated in (5), determine necessary and sufficient conditions for optimality. □

Although Proposition 3.1 proves existence of Nash equilibrium for the proposed production game, it fails to establish Nash equilibrium as naturally arising from user behavior in online social networks. Proposition 3.2 identifies exact potential functions for incremental and Shapley utility models. Existence of exact potential functions alludes that the natural Nash Dynamics, in which players iteratively play best response; converges to a pure Nash Equilibrium for the game although convergence might take exponential time [Ros73].

Proposition 3.2. *The Incremental utility and Shapley utility games admit an exact potential function defined correspondingly as*

$$\Phi^I(\vec{x}) = \sum_a \{f_a(y_a) - c_a(x_a)\}, \tag{6}$$

and

$$\Phi^S(\vec{x}) = \sum_a -c_a(x_a) + \sum_{S \subseteq N(a), s=|S|} \frac{1}{s\binom{d_a}{s}} f_a(\sum_{c \in s} q_{ca} x_c). \tag{7}$$

Proof: Similar to the proof statement we use a superscript of I to denote the incremental model and a superscript of S to denote the Shapley utility model. It is easy to observe that

$$u_a^I(x'_a, \vec{x}_{-a}) - u_a^I(\vec{x}) = \Phi^I(x'_a, \vec{x}_{-a}) - \Phi^I(\vec{x}),$$

so by definition $\Phi^I(\vec{x})$ is an exact potential function for the incremental game.

User Shapley utility $u_a^S(\vec{x}_a)$ can be rewritten as

$$u_a^S(\vec{x}) = f_a(y_a) - c_a(x_a) + \sum_{b\sim a} \sum_{S\subseteq N(b), s=|S|} \frac{1}{s\binom{d_b}{s}} \{f_b(\sum_{c\in S} q_{cb}x_c) - f_b(\sum_{c\in S, c\neq a} q_{cb}x_c)\}. \tag{8}$$

Consider a function of the form

$$\Phi^S(x) \;\; = \;\; \sum_a -c_a(x_a) + \sum_{S\subseteq N(a), s=|S|} \kappa_{a,s} f_a(\sum_{c\in S} q_{ca}x_c)$$

with $\kappa_{a,s} = \frac{1}{s\binom{d_a}{s}}$. This is an exact potential function for the Shapley utility game; if user a switches strategies from x_a to x'_a, then for all of a's friends the consumption utility $f_b(\sum_{c\in S} q_{cb}x_c)$ changes in all subsets containing a and the change in potential function is equal to the change in agent a's utility given in (8). $\square$

While we have not proved convergence bounds with best-response dynamics observe that Under general assumptions proposed potential functions are strictly concave; hence the unique Nash equilibrium of the incremental and Shapley utility games can be computed in polynomial time.

We prove a bicriteria bound on the price of anarchy in the rest of the paper and state conditions under which such bounds hold for the general production games, and apply our bounds to the Shapley and incremental utility models.

4 Analysis of the Price of Anarchy

Price of anarchy quantifies the degradation in the efficiency of a game due to strategic behavior of participating players [RT00]. The pure Nash price of anarchy is defined as the ratio of the welfare for the worst pure Nash equilibrium and the optimum welfare where the welfare function $W(\vec{x})$ is defined as the total utility of all agents. The optimum welfare refers to a setting where a central authority obligates users to behave according to the socially-optimal strategy vector $\vec{x^*} = \arg\max_{\vec{x}} W(x)$. We derive a bicriteria bound on the price of anarchy that compares $W(x^*)$ with equilibrium welfare in an augmented social network. We prove that inefficiencies from players' strategic behavior can be compensated by "doubling" the happiness function. In section 4.2, we analyze the price of anarchy for the simplest class of production games without augmentation. We show that the price of anarchy can be arbitararily big even theough these production games have linear happiness and polynomial cost functions.

4.1 A Bicriteria Bound on The Price of Anarchy

We derive a bicriteria bound on the price of anarchy for the proposed general production game. We compare the optimal social welfare in an online environment against the equilibrium welfare in an augmented environment, where the happiness function is twice as large. We show that the equilibrium welfare in this augmented version is at least as large as the optimal welfare for the original social network. Our result relies on the existence of an exact potential function; it also requires correctness of certain inequalities. We show that both these inequalities hold for Shapley and incremental utility models.

This section requires a more detailed notation since we work with two utility models simultaneously. We introduce the notation first and then state our result formally. Define $g_a(x) = 2f_a(x)$ for all agents a, $W_g(\vec{x})$ denotes the social welfare for the general production game with consumption utilities $g_a(x)$; similarly $W(\vec{x})$ denotes the social welfare for the general production game with consumption utilities $f_a(x)$. We differentiate between the incremental and the Shapley utility models with a superscript of I for the former and superscript of S for the latter. We distinguish the social optimum from the Nash equilibrium by a superscript of $*$ for social optimum and a superscript of e for the Nash equilibrium. For example, $\vec{x}^{I,e}$ represents the equilibrium strategy vector for the incremental game where consumption utility for all agents a is equal to $f_a(x)$ and $\vec{x}_g^{I,*}$ denotes the socially optimal strategy for the same game where consumption utility for all agents a is $g_a(x)$.

We first state our main result in Theorem 4.1; our result compares equilibrium social welfare in an augmented social network against the social optimum in the original one. We consider improved user experience as the source of augmentation in the social network and model it by defining a new happiness function $g_a(x) = 2f_a(x)$. We next show that our result holds for the Shapley and incremental utility models in Proposition 4.3 and Proposition 4.2.

Theorem 4.1. *Let $\vec{x}_g^e$ denote the equilibrium of a general production game with happiness function $g_a(x) = 2f_a(x)$ then $W_g(\vec{x}_g^e) \geq W(\vec{x}^*)$ if:*

1. *the general assumptions hold,*
2. *$\forall a$, $|\frac{\partial^2 t_a(\vec{x})}{\partial x_a^2} - \frac{\partial^2 c_a(x_a)}{\partial x_a^2}| > \epsilon$ for constant $\epsilon > 0$,*
3. *the game admits an exact potential function $\phi(\vec{x})$, and*
4. *for all valid strategy vectors $\vec{x}$, $W_g(\vec{x}) \geq \phi_g(\vec{x})$ and $\phi_g(\vec{x}) \geq W(\vec{x})$.*

Proof: We would like to show that $W_g(\vec{x}_g^e) \geq W(x^*)$. Assumptions one and two guarantee existence of a unique pure-strategy equilibrium $\vec{x}_g^e$. We instantiate assumption four with $\vec{x} = \vec{x}_g^e$ so $W_g(\vec{x}_g^e) \geq \phi_g(\vec{x}_g)$. The equilibrium strategy $\vec{x}_g$ maximizes $\phi_g(\vec{x})$ because $\phi_g(\vec{x})$ is an exact potential function for the general production game, so $\phi_g(\vec{x}_g^e) \geq \phi_g(\vec{x}^*)$. We can instantiate assumption four once more with $\vec{x} = \vec{x}^*$, obtaining $\phi_g(\vec{x}^*) \geq W(\vec{x}^*)$ which concludes the proof.

□

Our bicriteria bound suggests that any inefficiency from user strategic behavior in an online environment can be compensated by improvements in user experience. Every different choice of $t_a(\vec{x})$ corresponds to a different attention sharing mechanism. Not all attention sharing mechanisms admit an exact potential function, e.g. the proportional production game. We identified exact potential functions for the incremental and Shapley utility models in Proposition 3.2; we only need to prove that assumption four from Theorem 4.1 holds.

Proposition 4.2. *Under general assumptions, $W_g^I(\vec{x}) \geq \phi_g^I(\vec{x})$ and $\phi_g^I(\vec{x}) \geq W^I(\vec{x})$ for all valid strategy vectors $\vec{x}$.*

Proof: Potential function for the incremental utility game is given according to (6) and

$$W^I(\vec{x}) = \sum_a \left\{ f_a(y_a) - c_a(x_a) + \sum_{b\sim a} f_b(y_b) - f_b(y_b - q_{ab}x_a) \right\}. \tag{9}$$

It is easy to observe that $W_g^I(\vec{x}) \geq \phi_g^I(\vec{x})$ since $\sum_a \sum_{b\sim a} \{f_b(y_b) - f_b(y_b - q_{ab}x_a)\} \geq 0$.

Two simple observations prove the second part of the proposition. First, $W^I(\vec{x})$ can be expanded and rewritten as:

$$W(\vec{x}) = \sum_a \left\{ f_a(y_a) - c_a(x_a) + \sum_{b\sim a} f_a(y_a) - f_a(y_a - q_{ba}x_b) \right\}$$

Second, since $f_a(x)$ is concave and increasing and $y_a = \sum_{b\sim a} q_{ba}x_b$;

$$f_a(y_a) \geq \sum_{b\sim a} f_a(y_a) - f_a(y_a - q_{ba}x_b).$$

So $\phi_g^I(\vec{x}) \geq W^I(\vec{x})$. □

Similarly we use concavity of user happiness function to show that assumption four also holds for the Shapley utility game.

Proposition 4.3. *Under general assumption, $W_g^S(\vec{x}) \geq \phi_g^S(\vec{x})$ and $\phi_g^S(\vec{x}) \geq W^S(\vec{x})$ for all valid strategy vectors $\vec{x}$.*

Proof: The social welfare function for Shapley utility game with consumption utility $g_a(x)$ is

$$W_g^S(\vec{x}) = \sum_a g_a(y_a) - c_a(x_a) + \sum_{b\sim a} \sum_{S\subseteq N(b), s=|S|} \frac{1}{s\binom{d_b}{s}} \{g_b(\sum_{c\in S} q_{cb}x_c) - g_b(\sum_{c\in S, c\neq a} q_{cb}x_c)\} \tag{10}$$

and

$$\Phi_g^S(\vec{x}) = \sum_a -c_a(x_a) + \sum_{S\subseteq N(a), s=|S|} \frac{1}{s\binom{d_a}{s}} g_a(\sum_{c\in S} q_{ca}x_c).$$

Note that $g_a(\sum_{c\in s} q_{ca}x_c) \leq g_a(y_a)$ since $g_a(x)$ is an increasing function so

$$\Phi_g^S(\vec{x}) \leq \sum_a -c_a(x_a) + \sum_{S\subseteq N(a), s=|S|} \frac{1}{s\binom{d_a}{s}} g_a(y_a). \tag{11}$$

It is now easy to observe that $W_g^S(\vec{x}) \geq \Phi_g^S(\vec{x})$ since the second term in (11) is equal to $g_a(y_a)$ and the last term in (10) is positive.

We next show $\Phi_g^S(\vec{x}) \geq W^S(\vec{x})$. We first expand and rewrite $W^S(\vec{x})$ as

$$W^S(\vec{x}) = \sum_a f_a(y_a) - c_a(x_a) + \sum_{S\subseteq N(a), s=|S|} \frac{1}{s\binom{d_a}{s}} \sum_{b\in S} \{f_a(\sum_{c\in S} q_{ca}x_c) - f_a(\sum_{c\in S, c\neq b} q_{ca}x_c)\}. \tag{12}$$

Since $f_a(x)$ is concave and increasing

$$\sum_{b\in S}\{f_a(\sum_{c\in S} q_{ca}x_c) - f_a(\sum_{c\in S, c\neq b} q_{ca}x_c)\} \leq f_a(\sum_{c\in S} q_{ca}x_c).$$

This is sufficient to show that

$$f_a(y_a) \leq \sum_{S\subseteq N(a), s=|S|} \frac{1}{s\binom{d_a}{s}} f_a(\sum_{c\in S} q_{ca}x_c). \tag{13}$$

One can expand the right hand side (RHS) summation in (13) and rewrite it using the size of subsets as the summation variable as follows

$$RHS = \sum_{s=1}^{d_a} \sum_{S\subseteq N(a), |S|=s} \frac{1}{s\binom{d_a}{s}} f_a(\sum_{c\in S} q_{ca}x_c).$$

We prove (13) using a simple procedure.

1. Take values $f_a(\sum_{c\in S} q_{ca}x_c)$ for all subsets S with original size s, take the set with smallest value.
2. Delete one of its members, x_m.
3. Choose a subset S' that does not contain x_m and add x_m to it.
4. Repeat the procedure until all sets have the same value.

It is easy to observe that the total sum decreases during the procedure so above procedure shows that $\frac{1}{d_a} f_a(y_a) \leq \sum_{S\subseteq N(a), |S|=s} \frac{1}{s\binom{d_a}{s}} f_a(\sum_{c\in S} q_{ca}x_c)$ since all remaining subsets evaluate to $f_a(y_a)$ and there are $\binom{d_a-1}{s-1}$ such subsets after all iterations are over. Summing over all players a we have $\sum_{s=1}^{d_a} \frac{1}{d_a} f_a(y_a) \leq RHS$ and (13) holds so $\Phi_g^S(\vec{x}) \geq W^S(\vec{x})$. □

We showed that the social welfare induced by central control for incremental and Shapley utility models is no larger than the social welfare under strategic contribution when users have twice as large consumption utilities. Although our result does not attribute inefficiencies from user strategic behavior to lack of attention or excessive information, it indicates that improvements in user experience such as spam reduction, easy exploration and improved information discovery compensate for either of the existing inefficiencies from strategic behavior in online environments.

4.2 Simple Games with Unbounded Price of Anarchy

Our bicriteria bound does not exactly quantify the price of anarchy. Although the social gain from central control can be compensated by improvements in user experience, the degradation from strategic behavior can be still unbounded. We introduce a family of general production games that have an arbitrarily large price of anarchy. We consider agents with linear consumption utilities and convex polynomial cost functions. It is worthwhile to note that the incremental, proportional and, Shapley utility functions are equal.

We investigate a family of production games parameterized by γ and a set $\{\alpha_a\}$ of marginal consumption utilities. Agents have linear consumption utility $f_a(y) = \alpha_a y, \alpha_a > 0$. Moreover, they have polynomial cost functions $c_a(x) = \frac{1}{\gamma}x^\gamma, \gamma > 1$. Regardless of network structure the optimal and equilibrium strategy vectors can be characterized via FOC, so we can exactly quantify the price of anarchy.

Theorem 4.4. *Regardless of the network structure, the price of anarchy for the family of production games defined by cost function $c_a(x) = \frac{1}{\gamma}x^\gamma$ and utility functions $f_a(y) = \alpha_a y$, is equal to $(\frac{2\gamma-1}{\gamma-1})2^{\frac{\gamma}{1-\gamma}}$ when $\gamma > 1$ and $\alpha_a > 0$.*

Proof: The utility function for every player a is $u_a(\vec{x}) = \alpha_a y_a - c_a(x_a) + \beta_a x_a$ where $\beta_a = \sum_{b\sim a} q_{ba}\alpha_b$ and the social welfare function is $W(\vec{x}) = \sum_a \{2\beta_a x_a - c_a(x_a)\}$. It is easy to observe that the social welfare and agent utility functions are strictly concave so the first-order optimality conditions characterize the pure Nash equilibrium strategy, $\vec{x}$, and the optimum strategy, $\vec{x^*}$, as

$$\begin{aligned} x_a &= \beta_a^{\frac{1}{\gamma-1}}, \\ x_a^* &= (2\beta_a)^{\frac{1}{\gamma-1}}. \end{aligned}$$

Therefore the social welfare at pure Nash equilibrium $W(\vec{x}) = \frac{2\gamma-1}{\gamma}\sum_a \beta_a^{\frac{\gamma}{\gamma-1}}$ and the optimal social welfare $W(\vec{x^*}) = \frac{\gamma-1}{\gamma}2^{\frac{\gamma}{\gamma-1}}\sum_a \beta_a^{\frac{\gamma}{\gamma-1}}$. The pure-Nash price of anarchy is defined as the ratio of the pure Nash equilibrium social welfare divided by the optimum social welfare and is equal to $(\frac{2\gamma-1}{\gamma-1})2^{\frac{\gamma}{1-\gamma}}$. □

Exact analysis of the price of anarchy provides us with more predictive power over the existing inefficiencies due to strategic behavior. We now distinguish games from the introduced family with an arbitrarily large and an arbitrarily close-to-one price of anarchy.

Corollary 4.5. *Regardless of network structure, the price of anarchy for the family of production games defined by cost function $c_a(x) = \frac{1}{\gamma}x^\gamma$ and utility functions $f_a(y) = \alpha_a y, \alpha_a > 0$ can be arbitrarily large when γ gets arbitrarily close to one and the price of anarchy can be arbitrarily close to one when γ gets arbitrarily large.*

Corollary 4.5 identifies utility games with almost linear cost functions and linear consumption utilities as instances with unbounded price of anarchy. It also identifies utility games, that have polynomial cost with large coefficient and linear consumption utilities, as instances with almost no inefficiencies from strategic behavior. Non-existence of inefficiencies from strategic behavior in the latter example is mainly due to infinitesimal production at the pure Nash equilibrium and the social optimum for all users.

5 Robust Analysis of the Price of Anarchy

Smooth analysis of games with sum objectives identifies a sufficient condition for an upper bound on the price of anarchy of pure Nash equilibria and encodes a canonical proof template for deriving such bounds [Rou09]. Canonical bounds extend automatically to more general notions of equilibria such as mixed Nash equilibrium, correlated equilibrium and no-regret sequences. A utility maximization game is (λ, μ)-smooth if for every two outcomes $\vec{x}$ and $\vec{x^*}$,

$$\sum_a u_a(x_a^*, \vec{x}_{-a}) \geq \lambda W(\vec{x}^*) - \mu W(\vec{x}). \tag{14}$$

Roughgarden [Rou09] defines robust price of anarchy as the best lower bound on the price of anarchy that is provable using a smoothness argument. The robust price of anarchy for a utility maximization game is

$$\sup\left\{\frac{\lambda}{1+\mu} \quad : \quad (\lambda, \mu) \text{ s.t. game is } (\lambda, \mu)\text{-smooth}\right\}. \tag{15}$$

Congestion games with cost functions restricted to a fixed set are proved to be tight; meaning that the canonical price of anarchy is also robust. In particular network routing games are (λ, μ)-smooth with robust price of anarchy of $\frac{4}{3}$ for non-atomic flows and a price of anarchy of $\frac{5}{2}$ for atomic flows. We focus on production games with linear consumption utility and quadratic cost functions($\gamma = 2$). Theorem 4.4 quantifies price of anarchy of $\frac{3}{4}$ for this class of games. On the other hand, we show that the robust price of anarchy for the same class of games is at most 0.098, meaning that the robust price of anarchy is not tight which is a strong distinction between production games and network congestion games.

Theorem 5.1. *Robust price of anarchy for a production game with consumption utilities $f_a(y) = \alpha_a y, \alpha > 0$ and production cost $c_a(x) = 0.5x^2$ is at most 0.098.*

Proof: Agent utility for the proposed family of games is

$$u_a(\vec{x}) = \alpha_a y_a - \frac{x_a^2}{2} + \beta_a x_a, \tag{16}$$

where $\beta_a = \sum_{b \sim a} q_{ba}\alpha_b$. Also the social welfare function can be summarized into

$$W(\vec{x}) = \sum_a \left\{2\beta_a x_a - \frac{x_a^2}{2}\right\}. \tag{17}$$

Consider two strategy vectors $\vec{x}$ and $\vec{x^*}$ where for all users a, $\vec{x_a} = d\beta_a$ and $\vec{x_a^*} = c\beta_a$ for constants c and d ($c \neq d$). Smoothness conditions in (14) can be written for $\vec{x}$ and $\vec{x^*}$ as

$$\sum_a (d + c - \frac{c^2}{2})\beta_a^2 \geq \sum_a (2\lambda c - \lambda\frac{c^2}{2} - 2\mu d + \mu\frac{d^2}{2})\beta_a^2.$$

So (λ, μ)-smoothness requires

$$d + c - \frac{c^2}{2} \geq 2\lambda c - \lambda\frac{c^2}{2} - 2\mu d + \mu\frac{d^2}{2}. \tag{18}$$

Robust price of anarchy is defined as the supremum of $\frac{\lambda}{1+\mu}$ over all pairs of (λ, μ) for which the smoothness conditions hold. Equation (18) requires $\mu \geq \frac{2\lambda c - \lambda\frac{c^2}{2} - d - c + \frac{c^2}{2}}{2d - \frac{d^2}{2}}$ where $2d - \frac{d^2}{2} > 0$. Supremum of $\frac{\lambda}{1+\mu}$ takes place at the smallest value of μ, so we can set

$$\mu = \frac{2\lambda c - \lambda\frac{c^2}{2} - d - c + \frac{c^2}{2}}{2d - \frac{d^2}{2}}. \tag{19}$$

After substituting μ from (19), the canonical price of anarchy bound from (14) will be equal to

$$\frac{(2d - \frac{d^2}{2})\lambda}{(2c - \frac{c^2}{2})\lambda + 2d - \frac{d^2}{2} - d - c - \frac{c^2}{2}.} \tag{20}$$

Equation (20) is a hyperbolic function in λ and its supremum is equal to

$$\frac{4d - d^2}{4c - c^2} \tag{21}$$

when $2c - \frac{c^2}{2} > 0$ and $2d - \frac{d^2}{2} - d - c - \frac{c^2}{2} > 0$. Our choice of values for c and d that satisfy above inequalities determines an upper bound on the robust price of anarchy. Values $c = 2.2$ and $d = 0.1$ generate an upper bound of 0.098 for robust price of anarchy using (21) and satisfy above inequalities with valid values for λ and μ. □

Theorem 5.1 shows that the robust price of anarchy is not equal to the pure-Nash price of anarchy so any bound from the canonical analysis of the game is not tight. This is in contrast with Roughgarden's result about network routing games where canonical bounds are tight even for the smallest class of equilibria, i.e. pure Nash equilibria [Rou09]. Theorem 5.1 does not prove smoothness of the production game but in a sense it shows that canonical analysis and robust price of anarchy are not the right tool for analyzing proposed production games. Although our bicriteria bound is very similar to that of Tardos et al. in [RT00], but Theorem 5.1 proves a strong distinction between routing games and the games studies in this paper.

6 Discussion

We introduced a game-theoretic framework to analyze inefficiencies from strategic behavior in online social networks. We proved that the degradation in efficiency of the proposed game resulting from strategic user participation can be compensated by improvements in the online environment.

Although we are unable to give a closed-form solution for the pure-equilibrium strategy in the general production game, we can characterize a closed-form solution in d-regular graphs. We are also able to find the pure-strategy Nash equilibrium numerically in general networks. Equilibrium utility is higher for larger values of d in a d-regular graph. Our numerical analysis on several random networks shows that the equilibrium utility, the utility from attention, and the utility from information are significantly correlated with the number of friends a user has in the social network. There are a number of interesting directions for future work within the proposed framework; we conclude this paper by explaining some of these directions.

A natural direction to explore is to study the proposed general production game as a network formation game where strategic players can add and drop links in the social network. Because the proposed general production game does not include any cost for information overload, dense network structures are more likely to form. Empirical evidence indicates that users appreciate new information less as they consume more information. We implicitly incorporate cost of information overload by modeling user happiness as a concave function. An interesting direction is to model an explicit cost for information overload. Our preliminary results show that unfortunately our bicriteria bound on the price of anarchy does not hold in this setting.

Different mechanisms of viewing information on user news feed in online social networks can be modeled as a different attention sharing mechanism within our general production framework. An interesting future direction is to compare available attention sharing schemes e.g. chronological sorting, collaborative filtering, and etc in the proposed game-theoretic framework. This is very similar to the mechanism-design approach that Ghosh et al have taken [GM11][GH11].

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