Stability of the Maximum Size Matching

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Abstract

Among scheduling algorithms used in input queued switches, it is known that the Maximum Size Matching (MSM) algorithm maximizes the instantaneous throughput. Although there have been results on the stability of some specific types of MSMs for scheduling traffic [5], the question of how it performs under uniform traffic remains open. Simulations suggest the stability of MSM algorithms under uniform traffic but there have been no analytical results proving the same. In this article, we introduce a new Lyapunov function in order to prove that under uniform arrival traffic MSM algorithm achieves 100% throughput and also to find a stability region for the arrival rates.

1 Introduction

Input queued (IQ) switch architectures have a low memory bandwidth requirement compared to output queued (OQ) or shared memory switches. This property has made IQ switches more suitable for implementation in internet routers and ATM switches. It is well known that IQ switches have throughput limitations due to the head of line (HoL) blocking which reduces the throughput to about 58% [3]. To overcome HoL blocking, separate virtual output queues (VOQ) were introduced at each input, one VOQ for the cells at each input destined to each of the N outputs. The use of VOQs and the maximum weight matching algorithm for scheduling packets from inputs to outputs was then shown to yield 100% throughput, or equally, to ensure the stability of the switch [1, 2, 6].

In this paper we will study the performance of the Maximum Size Matching (MSM) algorithm. Previous work [1] has shown that that a 3×3 switch is unstable under MSM (when ties are broken randomly[1]). This instability result is surprising because, by definition, the MSM algorithm transfers the maximum possible work in each time slot; i.e. it maximizes the instantaneous throughput. However, the counter-example of [1] shows that this myopic action does not lead to the maximum (long-term) throughput. Later on, with some extra work, [4] proved that the MSM algorithm is unstable even for 2×2 switches.

These results raise the following natural question:

[1]That is, when there is more than one matching with the maximum size, the MSM algorithm chooses one of them uniformly at random. We shall also use this tie-breaking rule for all MSM algorithms in this paper.

- Does the MSM algorithm with random tie breaking have a nonempty stability region? That is, is there an admissible arrival rate vector so that the MSM algorithm is stable? What is the stability region?

One approach to answer this question is to study the stability of the MSM algorithm under the uniform arrival rate matrix; i.e. where the average arrival rate at input i for output j equals α/N (for $\alpha < 1$) for every i and j. This is the most benign type of input matrix, because under this input even the Random policy (which chooses one of the $N!$ matchings uniformly at random, and independently of the state of the switch) is stable. This is easy to see: Under the uniform arrival rate matrix and with Bernoulli i.i.d. arrival distribution a packet arrives at each input i with probability α in each time slot. An arriving packet is destined to one of the N outputs with probability $1/N$. Thus, focussing on any one VOQ, we see that the arrivals to the VOQ are Bernoulli with probability α/N. Under the Random policy the VOQ is served with probability $1/N$ in each time slot. Thus, the size of the VOQ is a birth-death chain with birth rate strictly smaller than the death rate, proving the stability of the Random policy under uniform arrivals. Clearly, the MSM algorithm is better than the Random policy, so its stability under uniform arrivals ought to be easy to establish. However, this statement seems surprisingly difficult to prove.

In this paper, we introduce a new Lyapunov function for addressing the stability of the MSM. We use this Lyapunov function to prove the stability of the MSM algorithm for switches of size 2 and 3 under uniform arrivals. In fact, we provide a stability region which is larger and includes the uniform arrival rate vector. Our work takes a first step towards understanding the complete stability region of the MSM algorithm. Given that the MSM algorithm is not stable at speedup 1, this in turn could help determine the smallest speedup such that the MSM algorithm is stable. Such results are interesting not just for our theoretical understanding of switch algorithms, but they could also help us understand how to design better algorithms at speedups bigger than 1.

2 Coefficient Rate of Random Processes

In the theory of dynamical systems the method of Lyapunov functions is a way to show the stability of solutions of differential equations. This method has been developed and applied to Markov Chains to show the stability (Foster's criterion), as it has been used to show the stability of MWM [6, 1, 2].

To begin consider an $N \times N$ switch. Let the number of packets waiting at input i for output j at time n be denoted by $q_{ij}(n)$ and assume the arrival traffic is uniform with rate $\lambda = \alpha/N$. During this section we represent the switch by a bipartite graph with inputs and outputs as vertices and nonempty queues as edges.

Definition 1 At every time slot let $L(n)$ be the following quadratic form

$$L(n) = \sum_{i,j} q_{ij}(n)^2 + \frac{2}{3} \sum_{i,j,r,s} f_{ijrs} q_{ij}(n) q_{rs}(n) - \frac{2}{3} \sum_{i,j,r,s} g_{ijrs} q_{ij}(n) q_{rs}(n) \tag{1}$$

where

$$f_{ijrs} = \begin{cases} 1 & \text{if edges } \{i,j\} \text{ and } \{r,s\} \text{ have a common vertex.} \\ 0 & \text{otherwise} \end{cases}$$

and $g_{ijrs} = 1 - f_{ijrs}$.

Remark The quadratic form which is often used to show the stability of IQ switches is $\hat{L}(n) = \sum_{i,j} q_{ij}(n)^2$. For example, it has been used to show the stability of the MWM [6, 1, 2]. But, it does not show the stability of MSM algorithm. The function $L(n)$, defined in (1) consists of $\hat{L}(n)$ and two extra terms. First one is 2/3 times sum of the terms $q_{ij}q_{rs}$ where q_{ij} and q_{rs} share either their inputs or their outputs and the second term in (1) is $-2/3$ times sum of the terms $q_{ij}q_{rs}$ where q_{ij} and q_{rs} have no common input or output, for all possible i, j, r and s. Since number of the negative terms is of $O(N^4)$ and number of positive terms is of $O(N^3)$ for large N the quadratic form is not even positive and cannot be a Lyapunov function. Hence our result is only for $N < 4$.

For simplicity, let's write the quadratic form in matrix format: $L(n) = Q(n)^t P Q(n)$ with P the appropriate $N^2 \times N^2$ matrix and $Q(n)$ an $N^2 \times 1$ vector with entries $q_{ij}(n)$. For example if $N = 2$ and $Q(n) = [q_{11}(n), q_{12}(n), q_{21}(n), q_{22}(n)]^t$ then:

$$P = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & 1 & \frac{-1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} & 1 & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

Theorem 1 Consider an $N \times N$ IQ switch with uniform arrival traffic. If MSM algorithm with random tie breaking is applied. For $N < 4$, there exists an $\epsilon > 0$ and a constant $c(N)$ depending only on N such that (Foster's Criteria)[2]:

$$\mathbb{E}(L(n+1) - L(n)|Q(n)) \leq -\epsilon \left(\sum_{ij} q_{ij}(n) \right) + c(N) \tag{2}$$

PROOF: First, for all i, j, n, let $\pi_{ij}(n)$ be the random variable defined as:

$$\pi_{ij}(n) = \begin{cases} 1 & \text{if MSM algorithm uses edge } \{i, j\} \text{ at the } n^{th} \text{ time slot} \\ 0 & \text{otherwise} \end{cases}$$

If the arrival and departure vectors are denoted by $A(n)$ and $D(n)$ respectively, then for the drift of $L(n)$ we have:

$$\begin{aligned} \mathbb{E}\Big(L(n+1) - L(n)|Q(n)\Big) &= \mathbb{E}\Big(Q(n+1)^t P Q(n+1) - Q(n)^t P Q(n)|Q(n)\Big) \\ &= \mathbb{E}\Big(2Q(n)^t P A(n) + A(n)^t P A(n) + D(n)^t P D(n) \\ &\quad -2D(n)^t P A(n) - 2Q(n)^t P D(n)|Q(n)\Big) \\ &= \mathbb{E}\Big(2Q(n)^t P A(n) - 2Q(n)^t P D(n)|Q(n)\Big) \\ &\quad +\mathbb{E}\Big(A(n)^t P A(n) + D(n)^t P D(n) - 2D(n)^t P A(n)|Q(n)\Big) \end{aligned}$$

[2]This condition is the same as the Foster's Criteria because if at least one of $q_{ij}(n)$ becomes very large then right hand side of (2) will become negative.

The second term (i.e. $\mathbb{E}\big(A(n)^tPA(n)+D(n)^tPD(n)-2D(n)^tPA(n)|Q(n)\big)$) is bounded by a constant called $c(N)$, because the vectors $A(n)$ and $D(n)$ have entries at most equal to one and P is a fixed matrix.[3]. Hence, from now on let's focus on the first term which is denoted by $\bigstar$.

Independence of the arrivals and queue sizes plus the fact that $\mathbb{E}(A_{ij}(n))=\lambda$ results in:

$$\begin{aligned}
\mathbb{E}(2Q(n)^tPA(n)|Q(n)) &= 2\left(\sum_{ij} q_{ij}(n)\right)\Big(\frac{\text{sum of any row or column of P}}{N}\Big) \\
&= 2\left(\sum_{ij} q_{ij}(n)\right)\left(1+\frac{2(N-1)}{3}-\frac{(N-1)^2}{3}\right)\lambda \\
&= 2\left(\sum_{ij} q_{ij}(n)\right)\left(\frac{N(4-N)}{3}\right)\lambda
\end{aligned}$$

Let p_{ijrs} be the entry in row ij and column rs of the matrix P then:

$$\begin{aligned}
\mathbb{E}(2Q(n)^tPD(n)|Q(n)) &= 2\sum_{ij} q_{ij}(n)\sum_{rs} p_{ijrs}\mathbb{E}(D_{rs}(n)) \\
&= 2\sum_{ij} q_{ij}(n)\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))
\end{aligned}$$

Hence $\bigstar$ reduces to:

$$2\sum_{ij} q_{ij}(n)\left(\left(\frac{N(4-N)}{3}\right)\lambda-\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))\right)$$

Let $0<\epsilon\leq\frac{2N(4-N)}{3}(\frac{1}{N}-\lambda)$. The term $\frac{2N(4-N)}{3}(\frac{1}{N}-\lambda)$ is positive by admissibility of the arrivals and assumption $N<4$. Proving the following for all i and j proves (2):

$$q_{ij}(n)\left(\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))\right) \;\geq\; q_{ij}(n)\left(\frac{4-N}{3}\right) \tag{3}$$

Note that this holds when $q_{ij}(n)=0$. Since (3) is symmetric with respect to all q_{ij}'s, it is enough to show that it holds when $i=j=1$. So in the rest of the proof assume $q_{11}(n)>0$ and the goal is to show that:

$$\sum_{rs} p_{11rs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n)) \;\geq\; \frac{4-N}{3} \tag{4}$$

For the simplicity of notation let $Y_{11}=\sum_{rs} p_{11rs}\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}$. Now partition all the edges of the graph other than q_{11} in two different groups:

[3]Other than the fact that arrival is uniform, it is also assumed that if at time slot n the arrival to q_{ij} is denoted by $A_{ij}(n)$ then $\mathbb{E}(A^2_{ij}(n))<\infty$.

- $\mathbf{A} = \{$all edges that are adjacent to $q_{11}\}$ (i.e. all q_{ij} such that $f_{11ij} = 1$)
- $\mathbf{B} = \{$all edges that are non-adjacent to $q_{11}\}$ (i.e. all q_{ij} such that $g_{11ij} = 1$)

Since $q_{11}(n) > 0$ there are only three types of maximum size matchings:

- **a)** q_{11} and k edges from $\mathbf{B}$ for $0 \leq k \leq N-1$
- **b)** *Two* edges from the set $\mathbf{A}$ and k edges from $\mathbf{B}$ for $0 \leq k \leq N-2$
- **c)** *One* edge from the set $\mathbf{A}$ and k edges from $\mathbf{B}$ for $0 \leq k \leq N-2$

For matchings of type **a**, $Y_{11} = 1 - \frac{k}{3} \geq 1 - \frac{N-1}{3} = \frac{4-N}{3}$ and for matchings of type **b**, $Y_{11} = \frac{2}{3} - \frac{k}{3} \geq \frac{2}{3} - \frac{N-2}{3} = \frac{4-N}{3}$ hence in both cases the inequality $Y_{11} \geq \frac{4-N}{3}$ holds. Type **c** matchings need to be treated differently. If a type **c** matching is a maximum size matching it has one edge from $\mathbf{A}$. Replacing this edge with q_{11} results in a type **a** maximum size matching. Since MSM algorithm chooses one matching at random among the set of all possible maximum size matchings, the new constructed type **a** matching can be chosen instead of the original type **c** with equal probability.

If the size of the maximum size matching is $k+1$, a matching of size k from the set $\mathbf{B}$ can participate in exactly one type **a** and at most $N-k-1$ type **c** matchings. That is because the vertices already used by edges in $\mathbf{B}$ cannot be used for $\mathbf{A}$ edges and also all the participated $\mathbf{A}$ edges have to use only one of the endpoints of q_{11}, otherwise a matching of size $k+2$ will be constructed which is a contradiction. Hence:

$$\mathbb{E}(Y_{11}|Q(n)) \geq \frac{1 - \frac{k}{3} + (N-k-1)(\frac{1}{3} - \frac{k}{3})}{N-k} \geq \frac{4-N}{3}$$

And the last inequality holds for all values of $N = 2, 3$ and $0 < k+1 \leq N$. ■

Next step is to show that $L(n)$ is positive for $N < 4$ which added to Theorem 1 shows the stability of MSM algorithm under uniform arrival traffic for 2×2 and 3×3 input-queued switches.

Theorem 2 The function $L(n)$ which is defined in (1) is positive when N =2 and 3.

PROOF: Let $R(N, x, y, z)$ be the $N^2 \times N^2$ matrix whose entries r_{ijrs} are as follows:

$$r_{ijrs} = \begin{cases} x & \text{if edges } f_{ijrs} = 1 \\ y & \text{if edges } g_{ijrs} = 1 \\ z & \text{if } \{i,j\} = \{r,s\} \end{cases}$$

For all N: $P = R(N, \frac{1}{3}, -\frac{1}{3}, 1)$. When $N = 2$, let $R_1 = R(2, \frac{1}{3}, 0, \frac{2}{3})$ and $R_2 = R(2, 0, -\frac{1}{3}, \frac{1}{3})$. Using matlab R_2 is positive definite matrix and since entries of R_1 are non-negative then by the fact that queue sizes are not negative numbers then

$$Q(n)^t P Q(n) = Q(n)^t R_1 Q(n) + Q(n)^t R_2 Q(n) \geq 0$$

Similarly for $N = 3$: $R_2 = R(3, \frac{1}{6}, -\frac{1}{3}, 1)$ is positive definite and $R_1 = R(3, \frac{1}{6}, 0, 0)$ has non-negative entries.

■

So the following is proved:

Theorem 3 In 2×2 and 3×3 IQ switches with uniform arrival traffic, MSM algorithm with random tie breaking achieves 100% throughput.

3 Generalization and further work

The current function $L(n)$ as mentioned in the last section, can be negative for large N. In fact it is only positive for $N < 4$. Hence at this time we can only prove the stability for N =2 and 3. However we are at this time working on finding a more general quadratic form which does not have these limitations, to extend the result to all N: i.e. proving the stability of any $N \times N$ switch under uniform traffic.

Furthermore using the same Lyapunov function we can find a larger stability region which includes the uniform traffic for MSM algorithm. This is a step towards finding the complete stability region of the MSM algorithm. For example, when $N = 2$ the function $L(n)$ shows the stability of the the switch using MSM algorithm and following arrival matrix:

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

where $\lambda_1 + \lambda_2 < 1$. To show this, rewriting (4) for the new arrival traffic:

$$\begin{aligned} \sum_{rs} p_{11rs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n)) &\geq \lambda_1 + \frac{2}{3}\lambda_2 - \frac{1}{3}\lambda_1 \\ &= \frac{2}{3}(\lambda_1 + \lambda_2) \end{aligned}$$

which holds for $\lambda_1 + \lambda_2 < 1$.

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