On Low-rank Trace Regression under General Sampling Distribution

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Abstract

A growing number of modern statistical learning problems involve estimating a large number of parameters from a (smaller) number of noisy observations. In a subset of these problems (matrix completion, matrix compressed sensing, and multi-task learning) the unknown parameters form a high-dimensional matrix $B^*$, and two popular approaches for the estimation are convex relaxation of rank-penalized regression or non-convex optimization. It is also known that these estimators satisfy near optimal error bounds under assumptions on rank, coherence, or spikiness of the unknown matrix.

In this paper, we introduce a unifying technique for analyzing all of these problems via both estimators that leads to short proofs for the existing results as well as new results. Specifically, first we introduce a general notion of low-rank and spikiness for $B^*$ and consider a general family of estimators (including the two estimators mentioned above) and prove non-asymptotic error bounds for the their estimation error. Our approach relies on a generic recipe to prove restricted strong convexity for the sampling operator of the trace regression. Second, and most notably, we prove similar error bounds when the regularization parameter is chosen via $K$-fold cross-validation. This result is significant in that existing theory on cross-validated estimators (Satyen Kale and Vassilvitskii [26], Kumar et al. [17], Abou-Moustafa and Szepesvari [1]) do not apply to our setting since our estimators are not known to satisfy their required notion of stability. Third, we study applications of our general results to four subproblems of (1) matrix completion, (2) multi-task learning, (3) compressed sensing with Gaussian ensembles, and (4) compressed sensing with factored measurements. For (1), (3), and (4) we recover matching error bounds as those found in the literature, and for (2) we obtain (to the best of our knowledge) the first such error bound. We also demonstrate how our frameworks applies to the exact recovery problem in (3) and (4).

1 Introduction

We consider the problem of estimating an unknown parameter matrix $B^* \in \mathbb{R}^{d_r \times d_c}$ from $n$ noisy observations

$$Y_i = \text{tr}(B^*X_i^\top) + \varepsilon_i,$$

for $i = 1, \ldots, n$ where each $\varepsilon_i \in \mathbb{R}$ is a zero mean noise and each $X_i \in \mathbb{R}^{d_r \times d_c}$ is a known measurement matrix, sampled independently from a distribution $\Pi$ over $\mathbb{R}^{d_r \times d_c}$. We also assume the estimation problem is high-dimensional (when $n \ll d_r \times d_c$).

Over the last decade, this problem has been studied for several families of distributions $\Pi$ that span a range of applications. It is constructive to look at the following four special cases of the problem:

- **Matrix-completion:** Let $\Pi$ be uniform distribution on canonical basis matrices for $\mathbb{R}^{d_r \times d_c}$, the set of all matrices that have only a single non-zero entry which is equal to 1. In this case we recover the

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well-known matrix completion problem, that is estimating $\mathbf{B}^\ast$ when $n$ noisy observations of (uniformly randomly) selected entries are available \[4, 5, 12, 13\]. A more general version of this problem is when $\Pi$ is a non-uniform probability distribution over the basis matrices \[27, 21, 14\].

- **Multi-task learning:** When support of $\Pi$ is only matrices that have a single non-zero row, then the problem reduces to the multi-task learning problem. Specifically, when we have $n$ observations of $d_{r}$ different supervised learning tasks, represented by $d_{r}$ linear regression models with unknown $d_{r}$-dimensional parameters $B_{i}^{\ast}, \ldots, B_{d_{r}}^{\ast}$ respectively, that form rows of $\mathbf{B}^\ast$. Equivalently, when the $i_{r}$-th row of $\mathbf{X}_{i}$ is non-zero, we can assume $Y_{i}$ is a noisy observation for the $i_{r}$-th task, with feature vector equal to the $i_{r}$-th row of $\mathbf{X}_{i}$. In multi-task learning the goal is to learn the parameters (matrix $\mathbf{B}^\ast$), leveraging structural properties (similarities) of the tasks \[7\].

- **Compressed sensing via Gaussian ensembles:** If we view the matrix as a high-dimensional vector of size $d_{r}d_{c}$, then the estimation problem can be viewed as an example of the compressed sensing problem, given certain structural assumptions on $\mathbf{B}^\ast$. In this literature it is known that Gaussian ensembles, when each $\mathbf{X}_{i}$ is a random matrix with entries filled with i.i.d. samples from $\mathcal{N}(0, 1)$, are a suitable family of measurement matrices \[3\].

- **Compressed sensing via factored measurements:** Consider the previous example. One drawback of the Gaussian ensembles is the need to store $n$ large matrices that requires memory of size $O(n d_{r} d_{c})$. \[24\] propose factored measurements to reduce this memory requirement. They suggest to use rank 1 matrices $\mathbf{X}_{i}$ of the form $\mathbf{U} \mathbf{V}^\top$, where $\mathbf{U} \in \mathbb{R}^{d_{r}}$ and $\mathbf{V} \in \mathbb{R}^{d_{c}}$ are random vectors which reduces the memory requirement to $O(nd_{r} + nd_{c})$.

A popular estimator, using observations (1.1), is given by solution of the following convex program,

$$
\min_{\mathbf{B} \in \mathcal{S}} \frac{1}{n} \sum_{i=1}^{n} \left[ Y_{i} - \text{tr}(\mathbf{BX}_{i}^\top) \right]^2 + \lambda \|\mathbf{B}\|_{*},
$$

where $\mathcal{S} \subseteq \mathbb{R}^{d_{r} \times d_{c}}$ is an arbitrary convex set of matrices with $\mathbf{B}^\ast \in \mathcal{S}$, $\lambda$ is a regularization parameter, and $\|\mathbf{B}\|_{*}$ is the trace-norm of $\mathbf{B}$ (defined in §2) which favors low-rank matrices. This type of estimator was initially introduced by Candes and Recht \[4\] for the noise-free version of the matrix completion problem and has been later studied in more general cases. An admittedly incomplete list of follow up work is \[6, 20, 10, 23, 25, 15, 21, 22, 14\]. Another class of estimators, studied by \[28, 12, 13\], changes the variable $\mathbf{B}$ in (1.2) to $\mathbf{UV}^\top$ where $\mathbf{U}$ and $\mathbf{V}$ are explicitly low-rank matrices, and replaces the trace-norm penalty by a ridge type penalty term on entries of $\mathbf{U}$ and $\mathbf{V}$, see (3.3) in §3.2 for details. These two bodies of literature provide tail bounds for the corresponding estimators, under certain assumptions on rank, coherence (or spikiness) of $\mathbf{B}^\ast$ for a few classes of sampling distributions $\Pi$. We defer a detailed discussion of this literature to \[9, 11\] and references therein.

**Contributions.** Our paper extends the above literature, and makes the following contributions:

(i) We introduce a general notion of spikiness and rank for $\mathbf{B}^\ast$, and construct error bounds (building on analysis of \[14\]) for the estimation error of a large family of estimators. Our main contribution is a general recipe for proving the well-known restricted strong convexity (RSC) condition, defined in §3.

(ii) Next, we prove the first (to the best of our knowledge) error bound for the cross-validated version of our family of estimators. Specifically, all bounds in the literature for the matrix estimation, as well as our bounds in (i), require the regularization parameter $\lambda$ to be larger than a constant multiple of $\|\sum_{i=1}^{n} \mathbf{X}_{i}\|_{op}$, which is not feasible in practice due to lack of access to $\{\varepsilon_{i}\}_{i=1}^{n}$. In fact, instead of using these “theory-inspired” estimators, practitioners select $\lambda$ via cross-validation. We prove that this cross-validated estimator satisfies similar error bounds as the ones in (i). We also show, via simulations, that the cross-validated estimator outperforms the “theory-inspired” estimators, and is nearly as good as the oracle estimator that chooses $\lambda$ by having knowledge of $\mathbf{B}^\ast$.
We note that the literature on analysis of the cross-validated estimators [20, 17, 1] does not apply to our setting since they require the estimation algorithm enjoy certain stability criteria. However, establishing this criteria for our case is highly non-trivial for two reasons: (a) we are studying a family of algorithm and not a specific algorithm, and (b) stability is unknown to hold even for a single low-rank matrix estimation method (including both convex or non-convex optimization)\(^1\).

(iii) We apply our results from (i) to the four classes of problems discussed above. While for matrix completion and both cases of compressed sensing (with Gaussian ensembles and with factored measurements) we obtain matching error bounds as the ones in the existing literature ([22, 14], [3], and [2] respectively), we prove (to the best of our knowledge) the first such error bounds for the multi-task learning problem. We note that [25, 21] also consider the trace regression problem under general sampling distributions. However, they only provide error bounds for the estimation error, when the corresponding sampling operator satisfies restricted isometry property (RIP) or RSC. However, none of these papers proves whether these conditions hold for the multi-task learning problem. In fact, [25] state their analysis cannot prove RIP for the multi-task learning problem. We indeed prove that RSC holds for all four classes of problems, leveraging our unifying method of proving the RSC condition.

For Gaussian ensembles and factored measurements, when there is no noise, our results also demonstrate that \(B^+\) can be exactly recovered, when the number of observations is above a certain threshold. Our recovery thresholds match the ones in [4] and [2] respectively.

**Organization of the paper.** We introduce additional notation and state the precise formulation of the problem in \(\S 2\). Then in \(\S 3\) we introduce a family of estimators and prove tail bounds on their estimation error. \(\S 4\) contains our results for the cross-validated estimator and corresponding numerical simulations. Application of our main error bounds to the aforementioned four classes of problems is given in \(\S 5\), and exact recovery results are given in \(\S 6\). Details of the proofs are discussed in \(\S A-B\).

# 2 Notation and Problem Formulation

We use bold caps notation (e.g., \(A\)) for matrices and non-bold capital letters for vectors (e.g., \(V\)). For any positive integer \(m\), \(e_1, e_2, \ldots, e_m\) denotes the standard basis for \(\mathbb{R}^m\), and \(I_m\) is the \(m \times m\) identity matrix. The *trace inner product* of matrices \(A_1\) and \(A_2\) with the same dimensions is defined as

\[
\langle A_1, A_2 \rangle := \text{tr}(A_1 A_2^\top).
\]

For \(d_r \times d_c\) matrices \(X_1, X_2, \ldots, X_n\), let the sampling operator \(\mathcal{X} : \mathbb{R}^{d_r \times d_c} \rightarrow \mathbb{R}^n\) be given by

\[
[\mathcal{X}(B)]_i := \langle B, X_i \rangle \quad \text{for all } i \in [n],
\]

where by \([k]\), we denote the set \(\{1, 2, \ldots, k\}\). For any two real numbers \(a\) and \(b\), \(a \lor b\) and \(a \land b\) denotes \(\max(a, b)\) and \(\min(a, b)\) respectively. Also, a real valued random variable \(z\) is \(\sigma\)-sub-Gaussian, if \(\mathbb{E}[\exp(\eta z)] \leq \exp(\sigma^2 z^2/2)\) for all \(\eta \in \mathbb{R}\).

For a norm\(^2\) \(\mathcal{N} : \mathcal{X} \rightarrow \mathbb{R}^+ \cup \{0\}\) defined on the vector space \(\mathcal{X}\), let \(\mathcal{N}^* : \mathcal{X} \rightarrow \mathbb{R}^+ \cup \{0, \infty\}\) be its dual norm defined as

\[
\mathcal{N}^*(X) = \sup_{\mathcal{N}(Y) \leq 1} \langle X, Y \rangle \quad \text{for all } X \in \mathcal{X}.
\]

In this paper, we use several different matrix norms. A brief explanation of these norms is brought in the following. Let \(B\) be a matrix with \(d_r\) rows and \(d_c\) columns,

\(^1\)One may be able to analyze the cross-validated estimator for the convex relaxation case by extending the analysis of [8] which is for LASSO. But even if that would be possible, it would be only for a single estimator based on convex relation, and not the larger family of estimators we study here. In addition, it would be a long proof (for LASSO it is over 30 pages), however our proof is only few pages.

\(^2\)\(\mathcal{N}\) can also be a semi-norm.
1. \( L^\infty \)-norm is defined by \( \| B \|_\infty := \max_{(i,j) \in [d_r] \times [d_c]} \{ |B_{ij}| \} \).

2. Frobenius norm is defined by \( \| B \|_F := \sqrt{\sum_{(i,j) \in [d_r] \times [d_c]} B_{ij}^2} \).

3. Operator norm is defined by \( \| B \|_{op} := \sup_{\| V \|_2 = 1} \| BV \|_2 \). An alternative definition of the operator norm is given by using the singular value decomposition (SVD) of \( B = UV^\top \), where \( D \) is a diagonal matrix and \( r \) denotes the rank of \( B \). In this case, it is well-known that \( \| B \|_{op} = D_{11} \).

4. Trace norm is defined by \( \| B \|_* := \sum_{i=1}^r D_{ii} \).

5. \( L_{p,q} \)-norm, for \( p, q \geq 1 \), is defined by \( \| B \|_{p,q} := \left( \sum_{c=1}^{d_c} \left( \sum_{r=1}^{d_r} |B_{rc}|^p \right)^{q/p} \right)^{1/q} \).

6. \( L^2(\Pi) \)-norm is defined by \( \| B \|_{L^2(\Pi)} := \sqrt{\mathbb{E}[(B, X)^2]} \), when \( X \) is sampled from a probability measure \( \Pi \) on \( \mathbb{R}^{d_r \times d_c} \).

7. Exponential Orlicz norm is defined for any \( p \geq 1 \) and probability measure \( \Pi \) on \( \mathbb{R}^{d_r \times d_c} \) as

\[
\| B \|_{\psi_p(\Pi)} := \| (B, X) \|_{\psi_p} = \inf \left\{ t > 0 : \mathbb{E} \left[ \left( e^{(\|B\|_{\psi_p})^p} - 1 \right) \right] \leq 1 \right\},
\]

where \( X \) has distribution \( \Pi \).

Now, we will state the main trac regression problem that is studied in this paper.

**Problem 2.1.** Let \( B^* \) be an unknown \( d_r \times d_c \) matrix with real-valued entries that is also low-rank, specifically, \( r \ll \min(d_r, d_c) \). Moreover, assume that \( \Pi \) is a distribution on \( \mathbb{R}^{d_r \times d_c} \) and \( X_1, X_2, \ldots, X_n \) are i.i.d. samples from \( \Pi \), and their corresponding sampling operator is \( \mathcal{X} : \mathbb{R}^{d_r \times d_c} \to \mathbb{R}^n \). Our regression model is given by

\[
Y = \mathcal{X}(B^*) + E,
\]

where observation \( Y \) and noise \( E \) are both vectors in \( \mathbb{R}^n \). Elements of \( E \) are denoted by \( \varepsilon_1, \ldots, \varepsilon_n \) where \( \{\varepsilon_i\}_{i=1}^n \) is a sequence of independent mean zero random variables with variance at most \( \sigma^2 \). The goal is to estimate \( B^* \) from the observations \( Y \).

We also use the following two notations: \( \Sigma := \frac{1}{n} \sum_{i=1}^n \varepsilon_i X_i \) and \( \Sigma_R := \frac{1}{n} \sum_{i=1}^n \zeta_i X_i \) where \( \{\zeta_i\}_{i=1}^n \) is an i.i.d. sequence with Rademacher distribution.

## 3 Estimation Method and Corresponding Tail Bounds

This section is dedicated to the tail bounds for the trace regression problem. The results and the proofs in this section are based on (with slight generalizations) those found in Klop et al. [14]. For the sake of completeness, the proofs are reproduced (adapted) for our setting and are presented in §B.

### 3.1 General notions of rank and spikiness

It is a well-known fact that, in Problem 2.1, the low-rank assumption is not sufficient for estimating \( B^* \) from the observations \( Y \). For example, changing one entry of \( B^* \) increases the rank of the matrix by (at most) 1 while it would be impossible to distinguish between these two cases unless the modified single entry is observed. To remedy this difficulty, Candes and Recht [4], Keshavan et al. [12] propose an incoherence assumption. If singular value decomposition (SVD) of \( B^* \) is \( UV \), then the incoherence assumption roughly means that all rows of \( U \) and \( V \) have norms of the same order. Alternatively, Negahban and Wainwright [22] studied the problem under a different (and less restrictive) assumption, which bounds the spikiness of
the matrix $B^*$. Here, we define a general notion of spikiness and rank for a matrix that includes the one by Negahban and Wainwright [22] as a special case. We define the spikiness and low-rankness of a matrix $B \in \mathbb{R}^{d_r \times d_c}$ as

$$\text{spikiness of } B := \frac{\mathcal{N}(B)}{\|B\|_F} \quad \text{and} \quad \text{low-rankness of } B := \frac{\|B\|_*}{\|B\|_F}.$$ 

The spikiness used in Negahban and Wainwright [22] can be recovered by setting $\mathcal{N}(B) = \sqrt{d_r d_c} \|B\|_\infty$. This choice of norm, however, is not suitable for many distributions for $X_i$’s (e.g., see §5.2 and §5.4). Instead, we use exponential Orlicz norm to guide selection of norm $\mathcal{N}$, depending on the tail of a given distribution.

### 3.1.1 Intuition on Selecting $\mathcal{N}$

Here we provide some intuition on the use of exponential Orlicz norm for selecting $\mathcal{N}$. [21] shows how an error bound can be obtained from the RSC condition (defined in §3.3 below) on a suitable set of matrices. The condition roughly requires that $\|X(B)\|_2^2 / n \geq \alpha \|B\|_F^2$ for a constant $\alpha$. Assuming that random variables $\langle X_i, B \rangle$ are not heavy-tailed, then $\|X(B)\|_2^2 / n$ concentrates around its mean, $\|B\|_{L^2(\Pi)}$. Orlicz norm, which measures how heavy-tailed a distribution is, helps us construct a suitable “constraint” set of matrices where the aforementioned concentration holds simultaneously.

To be more concrete, consider the multi-tasking example (studied in §5.2) for the simpler case of $d_r = d_c = d$. In particular, $X_i = e_i X_i^\top$ where $e_i$ is a vector with all entries equal to zero except for one of its entries which is equal to one, and the location of this entry is chosen uniformly at random in the set $[d]$. Also, $X_i$ is a vector of length $d$ whose entries are iid $\mathcal{N}(0,d)$ random variables.

In this example, we have $\|B\|_{L^2(\Pi)} = \|B\|_2^2$ for all $B \in \mathbb{R}^{d \times d}$. Now, if a fixed $B$ is such that $\langle X_i, B \rangle$ has a light-tail, one can show that for sufficiently large $n$, due to concentration, $\|X(B)\|_2^2 / n$ is at least $\|B\|_F^2 / 2$. Next, we investigate matrices $B$ with this property and without loss of generality we assume that $\|B\|_F = 1$. Now, consider two extreme cases: let $B_1$ be the matrix whose first row has $\ell_2$-norm equal to one and the other entries are zero, and $B_2$ be a matrix whose all rows have $\ell_2$-norm equal to $1/\sqrt{d}$. Intuitively, $\|X(B_1)\|_2^2$ has a heavier tail than $\|X(B_2)\|_2^2$, because in the first case, $\langle X_i, B \rangle$ is zero most of the times, but it is very large occasionally, whereas in the other one, almost all values $\langle X_i, B \rangle$ are roughly of the same size. This intuition implies that matrices whose rows have almost the same size are more likely to satisfy RSC than the other ones. However, since $X_i^\top$ is invariant under rotation, one can see that the only thing that matters for RSC, is the norm of the rows. Indeed, Orlicz norm verifies our intuition and after doing the computation (see §5.2 for details), one can see that $\|B\|_{\psi_2(\Pi)} = O(\sqrt{d} \|B\|_{2,\infty})$. We will later see in §5.2 that $\mathcal{N}(B)$ defined to be a constant multiple of $\sqrt{d} \|B\|_{2,\infty}$ would be a suitable choice. Note that, for the matrix completion application, one can argue similarly that, in order for a matrix to satisfy RSC condition with high probability, it cannot have a few very large rows. However, the second argument does not apply, as the distribution is not invariant under rotation, and actually a similar argument as the former implies that each row cannot also have a few very large entries. Therefore, all the entries should be roughly of the same size, which would match the spikiness notion of [22].

### 3.2 Estimation

Before introducing the estimation approach, we state our first assumption.

**Assumption 3.1.** Assume that $\mathcal{N}(B^*) \leq b^*$ for some $b^* > 0$.

Note that in Assumption 3.1, we only require a bound on $\mathcal{N}(B^*)$ and *not* the general spikiness of $B^*$.

Our theoretical results enjoy a certain notion of algorithm independence. To make this point precise, we start by considering the trace-norm penalized least squares loss functions, also stated in a different format in
\[ \mathcal{L}(B) := \frac{1}{n} \| Y - \mathcal{X}(B) \|_2^2 + \lambda \| B \|_* . \]  

However, we do not necessarily need to find the global minimum of (3.1). Let \( S \subseteq \mathbb{R}^{d_r \times d_c} \) be an arbitrary convex set of matrices with \( B^* \in S \). All of our bounds are stated for any any \( \hat{B} \) that satisfies

\[ \hat{B} \in S \quad \text{and} \quad \mathcal{R}(\hat{B}) \leq b^* \quad \text{and} \quad \mathcal{L}(\hat{B}) \leq \mathcal{L}(B^*) . \tag{3.2} \]

While the global minimizer, \( \arg \min_{B \in S, \mathcal{R}(B) \leq b^*} \mathcal{L}(B) \), satisfies (3.2), we can also achieve this condition by using other loss minimization problems. A notable example would be to use the alternating minimization approach which aims to solve

\[ (\hat{U}, \hat{V}) = \arg \min_{U \in \mathbb{R}^{d_r \times r}, V \in \mathbb{R}^{d_c \times r}, UV^T \in S} \frac{1}{n} \| Y - \mathcal{X}(UV^T) \|_2^2 + \frac{\lambda}{2} \left( \| U \|_F^2 + \| V \|_F^2 \right), \tag{3.3} \]

where \( r \) is a pre-selected value for the rank. If we find the minimizer of (3.3), then it is known that \( \hat{B} = \hat{U} \hat{V}^T \) satisfies (3.2) (see for example [12] or [20]).

### 3.3 Restricted Strong Convexity and the Tail Bounds

**Definition 3.1** (Restricted Strong Convexity Condition). The upper bound that we will state relies on the restricted strong convexity (RSC) condition which will be proven to hold with high probability. For a constraint set \( C \subseteq \mathbb{R}^{d_r \times d_c} \), we say that \( \mathcal{X}(\cdot) \) satisfies RSC condition over the set \( C \) if there exists constants \( \alpha(\mathcal{X}) > 0 \) and \( \beta(\mathcal{X}) \) such that

\[ \frac{\| \mathcal{X}(B) \|_F^2}{n} \geq \alpha(\mathcal{X}) \| B \|_F^2 - \beta(\mathcal{X}) , \]

for all \( B \in C \).

For the upper bound, we need the RSC condition to hold for a specific family of constraint sets that are parameterized by two positive parameters \( \nu, \eta \). Define \( C(\nu, \eta) \) as:

\[ C(\nu, \eta) := \left\{ B \in \mathbb{R}^{d_r \times d_c} \mid \mathcal{R}(B) = 1, \| B \|_F^2 \geq \nu, \| B \|_* \leq \sqrt{\eta} \| B \|_F \right\} . \tag{3.4} \]

Next result (proved in §B.1) provides the upper bound on the estimation error, when \( \lambda \) is large enough and the RSC condition holds on \( C(\nu, \eta) \) for some constants \( \alpha \) and \( \beta \).

**Theorem 3.1.** Let \( B^* \) be a matrix of rank \( r \) and define \( \eta := 72r \). Also assume that \( \mathcal{X}(\cdot) \) satisfies the RSC condition for \( C(\nu, \eta) \) defined as in Definition 3.1 with constant \( \alpha = \alpha(\mathcal{X}) \) and \( \beta = \beta(\mathcal{X}) \). In addition, assume that \( \lambda \) is chosen such that

\[ \lambda \geq 3 \| \Sigma \|_{op} , \tag{3.5} \]

where \( \Sigma = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i X_i \). Then, for any matrix \( \hat{B} \) satisfying (3.2), we have

\[ \| \hat{B} - B^* \|_F^2 \leq \left( \frac{100 \lambda^2 r}{3 \alpha^2} + \frac{8 b^2 \beta}{\alpha} \right) \vee 4 b^* \nu . \tag{3.6} \]

Note that even though the assumptions of Theorem 3.1 involve the noise and the observation matrices \( X_i \), no distributional assumption is required and the result is deterministic. However, we would employ probabilistic results later to show that the assumptions of the theorem hold. Specifically, condition (3.5) for \( \lambda \) is guaranteed to hold with high probability, using a version of Bernstein tail inequality for the operator norm of matrix martingales. This is stated as Proposition B.1 in §B.2 that also appears as Proposition 11 in [14].

The other condition in Theorem 3.1, RSC for \( C(\nu, \eta) \), will be shown to hold with high probability via Theorem 3.2 below. Before stating this result, we need two distributional assumptions for \( \mathcal{X}(\cdot) \). Recall that the distribution (over \( \mathbb{R}^{d_r \times d_c} \)) from which our observation matrices \( \{X_i\}_{i=1}^{n} \) are sampled is denoted by \( \Pi \).
Assumption 3.2. For constants $\gamma_{\min}, \gamma_{\max} > 0$, the following inequalities hold:
\[
\gamma_{\min} \|B\|_F^2 \leq \|B\|_{L^2(\Pi)}^2 \leq \gamma_{\max} \|B\|_F^2 \quad \text{for all } B \in \mathbb{R}^{d_x \times d_e}.
\]

Assumption 3.3. There exists $\varepsilon > 0$, such that
\[
E\left[ (X, B)^2 \cdot I(|(X, B)| \leq \varepsilon) \right] \geq \frac{1}{2} E\left[ (X, B)^2 \right],
\]
for all $B$ with $\mathcal{H}(B) \leq 1$, where the expectations are with respect to $\Pi$.

Remark 3.1. We will show later (Corollary A.1 of §A) that whenever $\text{Var}((X, B)) \approx 1$ uniformly over $B$ with $\mathcal{H}(B) = 1$, then $\varepsilon$ is a small constant that does not depend on the dimensions.

Next, result shows that a slightly more general form of the RSC condition holds with high probability.

Theorem 3.2 (Restricted Strong Convexity). Define
\[
C'(\theta, \eta) := \left\{ A \in \mathbb{R}^{d_x \times d_e} \mid \mathcal{H}(A) = 1, \|A\|_{L^2(\Pi)}^2 \geq \theta, \|A\|_F \leq \sqrt{\eta} \|B\|_F \right\}.
\]
If Assumptions 3.2-3.3 hold, then the inequality
\[
\frac{1}{n} \|X(A)\|_2^2 \geq \frac{1}{4} \|A\|_{L^2(\Pi)}^2 - \frac{93\eta c^2}{\gamma_{\min}} E\left[ \|\Sigma_R\|_{op} \right]^2 \quad \text{for all } A \in C'(\theta, \eta)
\]
holds with probability greater than $1 - 2 \exp\left( - \frac{Cn\theta}{c^2} \right)$ where $C > 0$ is an absolute constant, provided that $Cn\theta > c^2$, and $\Sigma_R := \frac{1}{n} \sum_{i=1}^{n} \xi_i X_i$ with $\{\xi_i\}_{i=1}^{n}$ is an i.i.d. sequence with Rademacher distribution.

Note that Theorem 3.2 states RSC holds for $C'(\theta, \eta)$ which is slightly different than the set $C(\nu, \eta)$ defined in (3.4). But, using Assumption 3.2, we can see that
\[
C(\nu, \eta) \subseteq C'(\gamma_{\min} \nu, \eta).
\]
Therefore, the following variant of the RSC condition holds.

Corollary 3.1. If Assumptions 3.2-3.3 hold, with probability greater than $1 - 2 \exp\left( - \frac{Cn\gamma_{\min} \nu}{c^2} \right)$, the inequality
\[
\frac{1}{n} \|X(A)\|_2^2 \geq \frac{\gamma_{\min}}{4} \|A\|_F^2 - \frac{93\eta c^2}{\gamma_{\min}} E\left[ \|\Sigma_R\|_{op} \right]^2
\]
holds for all $A \in C(\nu, \eta)$, where $C > 0$ is an absolute constant, provided that $Cn\gamma_{\min} \nu > c^2$, and $\Sigma_R := \frac{1}{n} \sum_{i=1}^{n} \xi_i X_i$ with $\{\xi_i\}_{i=1}^{n}$ is an i.i.d. sequence with Rademacher distribution.

We conclude this section by stating the following corollary. This corollary puts together the RSC condition (the version in Corollary 3.1) and the general deterministic error bound (Theorem 3.1) to obtain the following probabilistic error bound.

Corollary 3.2. If Assumptions 3.1-3.3 hold, and let $\lambda$ be larger than $C E\left[ \|\Sigma_R\|_{op} \right] b^* \varepsilon$ where $C$ is an arbitrary (and positive) constant. Then,
\[
\|\hat{B} - B^*\|_F^2 \leq \frac{C'\lambda^2 r}{\gamma_{\min}^2},
\]
holds with probability at least $1 - \mathbb{P}\left( \lambda < 3\|\Sigma\|_{op} \right) - 2 \exp\left( - \frac{C'' n \lambda^2 r}{c^2 b^* \gamma_{\min}^2} \right)$ for numerical constants $C', C'' > 0$. 

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Proof. First, we denote the threshold for \( \lambda \) by \( \lambda_1 := C \mathbb{E} \left[ \| \Sigma_R \|_{\text{op}} \right] b^r c. \) Now, by defining
\[
\alpha := \frac{\gamma_{\min}}{4}, \quad \beta := \frac{6696 \cdot r^2 \gamma_{\min}}{\gamma_{\min}} \mathbb{E} \left[ \| \Sigma_R \|_{\text{op}} \right]^2, \quad \text{and} \quad \nu := \frac{\lambda^2 r}{\gamma_{\min}^2 b^2},
\]
we observe that
\[
\left( \frac{100 \lambda^2 r}{3 \alpha^2} + \frac{8 b^2 \beta}{\alpha} \right) \vee \left( \frac{2 b^2 r}{\alpha} \right) \leq \frac{C' \lambda^2 r}{\gamma_{\min}},
\]
for sufficiently large constant \( C' > 0 \) (in fact, we would need \( C' \geq 534 + 2.15 \times 10^5 \times C^{-2} \)). The rest follows immediately from Theorem 3.1 and Corollary 3.1. We note that the only condition of Corollary 3.1 can be shown to hold by taking \( C'' \) such that \( C'' n \lambda^2 r > c^2 b^2 \gamma_{\min} \).

\[\square\]

**Remark 3.2 (Optimality).** We will see in §5 that Corollary 3.2 provides the same upper bound as in Corollary 1 of [22], for the matrix completion problem, and the same as in Theorem 2.4 of [3], for the compressed sensing case. In both of these papers the bounds are shown to be optimal.

**Remark 3.3.** While Corollary 3.2 relies on two conditions for \( \lambda \), namely \( \lambda \geq \lambda_1 \) and that \( \mathbb{P} \left( \lambda < 3 \| \Sigma \|_{\text{op}} \right) \) is small, only the latter condition is important to obtain a tail bound like (3.6) in Theorem 3.1. The additional condition \( \lambda \geq \lambda_1 \) is only helping to make the upper bound simpler, i.e., \( C' \lambda^2 r/\gamma_{\min}^2 \) instead of the right hand side of (3.6).

## 4 Tail Bound for the Cross-Validated Estimator

One of the assumptions required for the tail bounds of §3 for \( \hat{B} = \hat{B}(\lambda) \) is that \( \lambda \) should be larger than \( 3 \| \Sigma \|_{\text{op}} \). However, in practice we do not have access to the latter which relies on the knowledge of noise values \( \{ \varepsilon_i \}_{i=1}^n \). Therefore, practitioners often use cross-validation to tune parameter \( \lambda \). In this section, we prove that if \( \lambda \) is selected via cross-validation, \( \hat{B}(\lambda) \) enjoys similar tail bounds as in §3. This would provide theoretical backing for selection of \( \lambda \) via cross-validation for our family of estimators.

Let \( \{ (X_i, y_i) \}_{i=1}^n \) be a set of observations and denote by \( K \) the number of cross-validation folds. Let \( \{ I_k \}_{k \in [K]} \) be a set of disjoint subsets of \( [n] \) where \( \cup_{k \in [K]} I_k = [n] \). Also, we define \( I_{-k} := [n] \setminus I_k \). Letting \( n_k := |I_k| \), we have \( n = n_1 + \cdots + n_K \). Let \( X_k(\cdot) \) and \( \hat{X}_k(\cdot) \) be sampling operators for \( \{ X_i \}_{i \in I_k} \) and \( \{ X_i \}_{i \in I_{-k}} \), respectively. Similarly, \( Y_k \) and \( Y_{-k} \) denote the response vectors corresponding to \( \hat{X}_k(B^*) + E_k \) and \( \hat{X}_{-k}(B^*) + E_{-k} \) respectively. In our analysis, we assume that each partition contains a large fraction of all samples, namely, we assume that \( n_k \geq N/(2K) \) for all \( k \in [K] \).

Also, throughout this section we assume that, for any \( \lambda > 0 \), the estimators \( \hat{B}_{-k}(\lambda) \), satisfy (3.2) for observations \( (\hat{X}_{-k}(B^*), Y_{-k}) \) for each \( k \in [K] \). Define
\[
\hat{E}(\lambda) := \sum_{k=1}^{K} \left\| Y_k - \hat{X}_k(\hat{B}_{-k}(\lambda)) \right\|_2^2.
\]
For any fixed \( \lambda \), it can be observed that \( \left\| Y_k - \hat{X}_k(\hat{B}_{-k}(\lambda)) \right\|_2^2 \) is an unbiased estimate of the prediction error for \( \hat{B}_{-k}(\lambda) \). For every \( \lambda \) we also define the estimator \( \hat{B}_{cv}(\lambda) \) as follows:
\[
\hat{B}_{cv}(\lambda) := \sum_{k=1}^{K} \frac{n_k}{n} \hat{B}_{-k}(\lambda).
\]
Cross-validation works by starting with a set \( \Lambda = \{ \lambda_1, \lambda_2, \cdots, \lambda_L \} \) of potential (positive) regularization parameters and then choosing \( \lambda_{cv} \in \arg \min_{\lambda \in \Lambda} \hat{E}(\lambda) \). Then, then the K-fold cross-validated estimator with respect to \( \Lambda \) is \( \hat{B}_{cv}(\lambda_{cv}) \).

In the remaining of this section, we state two main results. First, in Theorem 4.1, we show a bound for \( \hat{B}_{cv}(\lambda_{cv}) \) where \( \lambda_{cv} \) can be any value in \( \Lambda \). Then, in Theorem 4.2, we combine Theorem 4.1 with Corollary 3.2, to obtain the main result of this section which is an explicit tail bound for \( \hat{B}_{cv}(\lambda_{cv}) \).

**Theorem 4.1.** Let \( \Lambda = \{ \lambda_1, \lambda_2, \cdots, \lambda_L \} \) be a set of positive regularization parameters, \( \hat{B}_{cv} \) be defined as in (4.1), and \( \hat{\lambda} \) be a random variable such that \( \hat{\lambda} \in \Lambda \) almost surely. Moreover, define 

\[
\bar{\sigma}^2 = \frac{1}{n} \sum_{i \in [n]} \text{Var}(\varepsilon_i),
\]

and assume that \( (\varepsilon_i)_{i=1}^n \) are independent mean-zero \( \sigma^2 \)-sub-Gaussian random variables. Then, for any \( t > 0 \), we have 

\[
\| \hat{B}_{cv}(\hat{\lambda}) - B^* \|_{L^2(\Pi)}^2 \leq \bar{\sigma}^2 + t,
\]

with probability at least 

\[
1 - 6KL \exp \left[ -C \min \left( \frac{t^2}{\sigma^4 \vee b^*4}, \frac{t}{\sigma^2 \vee b^*2} \right) \cdot \frac{n}{K} \right],
\]

where \( C > 0 \) is a numerical constant.

In order to prove Theorem 4.1, we need to state and prove the following lemma.

**Lemma 4.1.** Let \( B \) and \( B^* \) be two \( d_r \times d_c \) matrices with \( \| B - B^* \|_{\psi_2(\Pi)} \leq 2b^* \), and let \( (X_i)_{i=1}^n \) be a sequence of i.i.d. samples drawn from \( \Pi \). By \( (\varepsilon_i)_{i=1}^n \), we denote a sequence of independent mean-zero \( \sigma^2 \)-sub-Gaussian random variables and we define 

\[
\bar{\sigma}^2 := \frac{1}{n} \sum_{i=1}^n \text{Var}(\varepsilon_i).
\]

Then, for any \( t > 0 \), the inequality 

\[
\left| \frac{1}{n} \| Y - X(B) \|_2^2 - \left( \| B - B^* \|_{L^2(\Pi)}^2 + \bar{\sigma}^2 \right) \right| \leq t,
\]

holds with probability at least 

\[
1 - 6 \exp \left[ -C \min \left( \frac{t^2}{\sigma^4 \vee b^*4}, \frac{t}{\sigma^2 \vee b^*2} \right) n \right],
\]

where \( C > 0 \) is a numerical constant.

**Proof.** Recall from §2 that the vector of all noise values \( \{\varepsilon_i\}_{i=1}^n \) is denoted by \( E \). Note that, 

\[
\| Y - X(B) \|_2^2 = \| E - X(B - B^*) \|_2^2 = \| E \|_2^2 + \| X(B - B^*) \|_2^2 - 2 \langle E, X(B - B^*) \rangle.
\]

Next, using our Lemma A.4 as well as Lemma 5.14 of [29], for each \( i \in [n] \), we have 

\[
\| \varepsilon_i^2 - \mathbb{E} [\varepsilon_i^2] \|_{\psi_1} \leq 2 \| \varepsilon_i^2 \|_{\psi_1} \leq 4 \| \varepsilon_i \|_{\psi_2}^2 \leq 2 \sigma^2.
\]
Also, since \( \| B - B^* \|_{\psi_2(\Pi)} \leq 2b^* \) means that \( \| X_i, B - B^* \|_{\psi_2} \leq 2b^* \) or in other words, \( \langle X_i, B - B^* \rangle^2 \) is sub-exponential. We can now follow similar logic as above and obtain
\[
\begin{align*}
\| X_i, B - B^* \|^2 - \mathbb{E} \left[ \| X_i, B - B^* \|^2 \right] \|_{\psi_2} & \leq 16b^* \cdot \epsilon.
\end{align*}
\]
The same way, \( \epsilon_i \langle X_i, B - B^* \rangle \) that is product two sub-Gaussians becomes sub-exponential with zero mean which gives
\[
\| \epsilon_i \langle X_i, B - B^* \rangle \|_{\psi_1} \leq 8\sigma b^*.
\]
It then follows from Corollary 5.17 in [29] that, defining
\[
E_1(t) := \left\{ \| \epsilon \|_2^2 - \mathbb{E} \left[ \| \epsilon \|_2^2 \right] \geq nt \right\},
\]
\[
E_2(t) := \left\{ \| X(B - B^*) \|_2^2 - \mathbb{E} \left[ \| X(B - B^*) \|_2^2 \right] \geq nt \right\},
\]
\[
E_3(t) := \left\{ \left\| \langle \epsilon, X(B - B^*) \rangle - \mathbb{E} [\langle \epsilon, X(B - B^*) \rangle] \right\| \geq nt \right\},
\]
we have
\[
\mathbb{P}(E_j) \leq 2 \exp \left[ -C \min \left( \frac{t^2}{\sigma^4 \vee b^*}, \frac{t}{\sigma^2 \vee b^*} \right) \frac{n}{n} \right],
\]
for \( j \in \{1, 2, 3\} \) where \( C > 0 \) is a numerical constant. Applying the union bound, we get
\[
\mathbb{P} \left( \left\| \frac{1}{n} \| Y - X(B) \|_2^2 - \| B - B^* \|_{L^2(\Pi)}^2 + \sigma^2 \right\| \geq t \right) \leq 6 \exp \left[ C \min \left( \frac{t^2}{\sigma^4 \vee b^*}, \frac{t}{\sigma^2 \vee b^*} \right) \right],
\]
for some (different) numerical constant \( C > 0 \).

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For all \( k \in [K] \), we define
\[
\bar{\sigma}_k^2 = \frac{1}{n_k} \sum_{i \in I_k} \text{Var}(\epsilon_i).
\]
Moreover, for all \( l \in [L] \) and \( k \in [K] \), define the event \( E_{\ell,k} \) to be
\[
E_{\ell,k} := \left\{ \frac{1}{n_k} \| Y_{\ell,k} - X(\hat{B}_{-k}(\hat{\lambda}_\ell)) \|_2^2 - \left[ \| \hat{B}_{-k}(\hat{\lambda}_\ell) - B^* \|_{L^2(\Pi)}^2 + \bar{\sigma}_k^2 \right] \geq t \right\}.
\]
It follows from Lemma 4.1 and the union bound that the bad event satisfies
\[
\mathbb{P} \left( \bigcup_{k \in [K]} \bigcup_{\ell \in [L]} E_{\ell,k} \right) \leq 6KL \exp \left( -c \min \left( \frac{t^2}{\sigma^4 \vee b^*}, \frac{t}{\sigma^2 \vee b^*} \right) \right),
\]
for some constant \( c \geq 0 \). Note that we used the assumption \( n_k \geq N/(2K) \) for all \( k \in [K] \).

Now, in the complement of the bad event, it follows from the convexity of \( \| \cdot \|_{L^2(\Pi)}^2 \) that
\[
\| \hat{B}_{cv}(\hat{\lambda}) - B^* \|_{L^2(\Pi)}^2 \leq \sum_{k=1}^K \frac{n_k}{n} \cdot \| \hat{B}_{-k}(\hat{\lambda}) - B^* \|_{L^2(\Pi)}^2
\]
\[
\leq \sum_{k=1}^K \frac{n_k}{n} \cdot \left( \frac{1}{n_k} \| Y_{k} - X_k(\hat{B}_{-k}(\hat{\lambda})) \|_2^2 - \bar{\sigma}_k^2 + t \right)
\]
\[
= \hat{E}(\hat{\lambda}) - \bar{\sigma}^2 + t,
\]
which is the desired result. \( \square \)
Before stating the main result of this section, we also define notations \( \Sigma_{R,-k} \) and \( \Sigma_{-k} \) as follows:

\[
\Sigma_{-k} = \sum_{i \in \mathcal{I}_{-k}} \varepsilon_i X_i \quad \text{and} \quad \Sigma_{R,-k} = \sum_{i \in \mathcal{I}_{-k}} \zeta_i X_i,
\]

where \( \{\zeta_i\}_{i \in [n]} \), like in \( \S 3 \), are iid Rademacher random variables.

Now, we are ready to state the main result of this section which is obtained by combining Theorem 4.1 with Corollary 3.2.

**Theorem 4.2.** If Assumptions 3.1-3.3 hold, and let \( \ell_0 \in [L] \) be such that \( \lambda_{\ell_0} \) (in \( \Lambda \)) be larger than \( Cb^* \max_{k \in [K]} \mathbb{E}[\|\Sigma_{R,-k}\|_{op}] \) where \( C \) is an arbitrary (and positive) constant, and \( c \) is defined in \( \S 3 \). Also assume that \( \Lambda, \tilde{\lambda}_{cv}, \) and \( \hat{B}_{cv} \) are defined as above. In addition, assume that \( \{\varepsilon_i\}_{i=1}^n \) are independent mean-zero \( \sigma^2 \)-sub-Gaussian random variables, then for all \( t > 0 \), we have

\[
\|\hat{B}_{cv}(\tilde{\lambda}_{cv}) - B^*\|_{L^2(\Pi)}^2 \leq \frac{C_1 \gamma_{max} \lambda^{2}_{\ell_0} r}{\gamma_{\min}^2} + 2t,
\]

with probability at least

\[
1 - 6KL \exp\left(-C_2 \min\left(\frac{t^2}{\sigma^2 \vee b^2}, \frac{t}{\sigma^2 \vee b^2}^2\right) \cdot \frac{n}{K} \right) - \sum_{k \in [K]} \mathbb{P}\left(\lambda_{\ell_0} \geq 3\|\Sigma_{-k}\|_{op}\right) - K \exp\left(-\frac{C_3 n \lambda^{2}_{\ell_0} r}{c^2 b^2 \gamma_{\min}}\right),
\]

where \( C_1, C_2, \) and \( C_3 \) are positive constants.

**Proof.** The definition of \( \tilde{\lambda}_{cv} \) together with Theorem 4.1, Corollary 3.2, Assumption 3.2, and union bound yields

\[
\|\hat{B}_{cv}(\tilde{\lambda}_{cv}) - B^*\|_{L^2(\Pi)}^2 \leq \hat{E}(\tilde{\lambda}_{cv}) - \sigma^2 + t
\]

\[
\leq \hat{E}(\lambda_{\ell_0}) - \sigma^2 + t
\]

\[
= \sum_{k=1}^K \frac{n_k}{n} \cdot \left[ \frac{1}{n_k} \|Y_k - X_k(\hat{B}_{-k}(\lambda_{\ell_0}))\|_2^2 - \sigma_k^2 + t \right]
\]

\[
\leq \sum_{k=1}^K \frac{n_k}{n} \cdot \left[ \|\hat{B}_{-k}(\lambda_{\ell_0}) - B^*\|_{L^2(\Pi)}^2 + 2t \right]
\]

\[
\leq \sum_{k=1}^K \frac{n_k}{n} \cdot \left[ \frac{C_1 \gamma_{max} \lambda^{2}_{\ell_0} r}{\gamma_{\min}^2} + 2t \right]
\]

\[
= \frac{C_1 \gamma_{max} \lambda^{2}_{\ell_0} r}{\gamma_{\min}^2} + 2t.
\]

with the probability stated in (4.2). Note that we also used the fact that \( |\mathcal{I}_{-k}| \geq n[1 - 1/(2K)] \geq n/2 \) in the last term of (4.2).

**Remark 4.1.** While the tail bound of Theorem 4.2 is stated for \( \|\hat{B}_{cv}(\tilde{\lambda}_{cv}) - B^*\|_{L^2(\Pi)}^2 \), it is straightforward to use Assumption 3.2 and obtain a bound on \( \|\hat{B}_{cv}(\tilde{\lambda}_{cv}) - B^*\|_F^2 \) as well.

### 4.1 Simulations

In this section, we compare the empirical performance of the cross-validated estimator. In order to do so, we generate a \( d \times d \) matrix \( B^* \) of rank \( r \). Following a similar approach as in [13], we first generate \( d \times r \) matrices
Figure 1: Comparison of the relative error (i.e., $\|\hat{B} - B^\star\|^2_F/\|B^\star\|^2_F$) for the proposed estimators when $(d, r) = (50, 2)$, on the left, and when $(d, r) = (100, 3)$, on the right.

$B^\star_L$ and $B^\star_R$ with independent standard normal entries and then, set $B^\star := B^\star_L \cdot B^\star_R^T$. For the distribution of observations, $\Pi$, we consider the matrix completion case. Specifically, for each $i \in [n]$, $r_i$ and $c_i$ are integers in $[d]$, selected independently and uniformly at random. Then, $X_i = e_{r_i} e_{c_i}^T$. This leads to $n$ observations $Y_i = B_{i}^\star e_{r_i} e_{c_i} + \varepsilon_i$ where $\varepsilon_i$ are taken to be i.i.d. standard normal random variables.

Given these observations, we compare the estimation error of the following five different estimators:

1. **Theory-1**, **Theory-2**, and **Theory-3** estimators solve the convex program (1.2) for a given value of $\lambda = \lambda_0$ that is motivated by the theoretical results. Specifically, per Remark 3.3, we need $\lambda_0 \geq 3\|\Sigma\|_{op}$ to hold with high probability, which means we select $\lambda_0$ so that $\lambda_0 \geq 3\|\Sigma\|_{op}$ holds with probability 0.9. For each sample of size $n$, we find $\lambda_0$ by generating 1000 independent datasets of the same size and then, for the estimator **Theory-3**, we choose the 100th biggest value of $3\|\Sigma\|_{op}$. We will see below that this estimator performs very poorly since the constant 3 behind $\|\Sigma\|_{op}$ may be too conservative. Therefore, we also consider two other variants of this estimator where constant 3 is replaced with 1 and 2 respectively and denote these estimators by **Theory-1** and **Theory-2** respectively. Overall, we highlight that these three estimators are not possible to use in practice since they need access to $\|\Sigma\|_{op}$.

2. The **oracle** estimator solves the convex program (1.2) over a set of regularization parameters $\Lambda$. Then, the estimate is obtained by picking the matrix $\hat{B}$ that has the minimum distance to the ground truth matrix $B^\star$ in Frobenius norm. The set $\Lambda$ that is used in this estimator is obtained as follows: let $\lambda_{\text{max}}$ be the minimum real number for which the only minimizer of the convex program is zero. It can be easily shown that $\lambda_{\text{max}} = \|\sum_{i=1}^n Y_i X_i\|_{op}$. Then, we set $\lambda_{\text{min}} := \lambda_{\text{max}}/2$, and then, the sequence of values of $(\lambda_i)_{i=1}^L$ are generated as follows: $\lambda_1 := \lambda_{\text{max}}$ and $\lambda_{L+1} = \lambda_L/2$ such that $L$ is the smallest integer with $\lambda_L \leq \lambda_{\text{min}}$.

3. The **cv** estimator is introduced in the beginning of this section. We a set of regularization parameters $\Lambda' = \{\lambda\}_{i=1}^{\ell} \in [L']$ constructed exactly similar to the ones in oracle estimator, however since **cv** does not have access to $\lambda_0$, $L'$ is the smallest integer with $\lambda_{L'} \leq 0.01\lambda_{\text{max}}$.

Finally, for each of these estimators, we compute the relative error of the estimate $\hat{B}$ from the ground truth $B^\star$ defined as, $\|\hat{B} - B^\star\|^2_F/\|B^\star\|^2_F$ for a range of $n$. The results, averaged over 100 runs with 2SE errorbars, are shown in Figure 1, for two instances $(d, r) = (50, 2)$ and $(d, r) = (100, 3)$. We can see that cv performs close to the oracle and outperforms the theoretical ones.
5 Applications to Noisy Recovery

In this section, we will show the benefit of proving Corollary 3.2 that provides upper bound with a more general norm $\mathcal{R}()$. We will look at four different special cases for the distribution $\Pi$ and in two cases (matrix completion and compressed sensing with Gaussian ensembles) we recover existing results by Negahban and Wainwright [22], Klopp et al. [14] and Candès and Recht [4] respectively. For the other two, multi-task learning and compressed sensing with factored measurements, we obtain (to the best of our knowledge) the first such results (e.g., stated as open problems in Rohde et al. [25] and Recht et al. [24] respectively). Overall, in order to apply Corollary 3.2 in each case, we only need go over the following steps

1. Choose a norm $\mathcal{R}(\cdot)$. In the examples below, we will be using $\|B\|_{\psi_2(\Pi)}$ for an appropriate $p$.
2. Compute $\|B\|_{L^2(\Pi)}$ to find appropriate constant $\gamma_{\min}$ for which Assumption 3.2 holds.
3. Compute $\mathcal{R}(B^*)$ to obtain the constant $b^*$.
4. Choose an appropriate constant $\epsilon$ such that Assumption 3.3 holds.
5. Apply Proposition B.1 (from §B.2) to obtain a bound for $P\left(\lambda < 3\|\Sigma\|_{op}\right)$ as well as calculate $E\left[\|\Sigma R\|_{op}\right]$.

To simplify the notation, we assume $d_c = d_c = d$ throughout this section, however it is easy to see that the arguments hold for $d_r \neq d_c$ as well. We also assume, for simplicity, that $\epsilon_1 \sim \mathcal{N}(0, \sigma^2)$ for all $i \in [n]$.

5.1 Matrix completion

Let $B^*$ be a $d \times d$ matrix and recall that $e_1, e_2, \ldots, e_d$ denotes the standard basis for $\mathbb{R}^d$. Let, also, for each $i \in [n]$, $r_i$ and $c_i$ be integers in $[d]$, selected independently and uniformly at random. Then, let $X_i = \xi_i \cdot e_i, e_i^\top$ where, for each $i$, $\xi_i$ is an independent $4d^2$-sub-Gaussian random variable that is also independent of $r_j$ and $c_j$, $j \in [n]$. If we set $\xi_i := d$ almost surely, then $\|\xi_i\|_{\psi_2} = d/\sqrt{\log 2} \leq 2d$, and so, satisfies our requirement. This corresponds to the problem studied in [22]. Here we show the bounds for the slightly more general case of $\xi_i \sim \mathcal{N}(0, d^2)$.

First, note that,

$$
\|B\|_{L^2(\Pi)} = \|B\|_F,
$$

which means $\gamma_{\min}$ is equal to 1. In order to find a suitable norm $\mathcal{R}(\cdot)$, we next study $\|B\|_{\psi_2(\Pi)}$ to see how heavy-tailed $\langle B, X_i \rangle$ is. We have,

$$
E\left[\exp \left(\frac{|\langle B, X_i \rangle|^2}{4d^2\|B\|_\infty^2}\right)\right] = \frac{1}{d^2} \sum_{j,k=1}^d \mathbb{E}\left[\exp \left(\frac{\xi_i^2 B_{jk}^2}{4d^2\|B\|_\infty^2}\right)\right]
$$

$$
= \frac{1}{d^2} \sum_{j,k=1}^d \left[\frac{1}{\sqrt{1 - \frac{B_{jk}^2}{2\|B\|_\infty^2}}}\right]_{+}
$$

$$
\leq 2
$$

where the second equality uses Lemma A.1 of §A. Therefore,

$$
\|B\|_{\psi_2(\Pi)} \leq 2d\|B\|_\infty,
$$

which guides selection of $\mathcal{R}(\cdot) = d\|\cdot\|_\infty$ and $b^* = 2d\|B^*\|_\infty$. We can now see that $\epsilon = 9$ fulfills Assumption 3.3. The reason is, given $\mathcal{R}(B) = d\|B\|_\infty = 1$, we can condition on $r_i$ and $c_i$ and use Corollaries A.1-A.2 of
§A to obtain
\[
\mathbb{E} \left[ \xi_i^2 B_{r_i c_i}^2 \cdot \mathbb{I} \left( |\xi_i B_{r_i c_i}| \leq 9 \right) \right] \geq \mathbb{E} \left[ \xi_i^2 B_{r_i c_i}^2 \cdot \mathbb{I} \left( |\xi_i B_{r_i c_i}| \leq 5 \sqrt{d|B_{r_i c_i}|} \right) \right] \geq \frac{\mathbb{E} \left[ \xi_i^2 B_{r_i c_i}^2 | r_i, c_i \right]}{2}.
\]

Now, we can take the expectation with respect to \( r_i \) and \( c_i \) and use the tower property to show Assumption 3.3 holds for \( \epsilon = 9 \).

The next step is use Proposition B.1 (from §B.2) for \( Z_i := \xi_i X_i \) to find a tail bound inequality for
\[ P \left( \lambda < 3 \| \Sigma_{op} \| \right) \]
define \( \delta := d\sigma e/(e - 1) \) and let \( G_1 \) and \( G_2 \) be two independent standard normal random variables. Then, it follows that
\[
\mathbb{E} \left[ \exp \left( \frac{\| Z_i \|_{op}}{\delta} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{d\sigma |G_1 G_2|}{\delta} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{d\sigma (G_1^2 + G_2^2)}{2\delta} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{d\sigma G_1^2}{2\delta} \right) \right]^2 \leq \mathbb{E} \left[ \exp \left( \frac{(e - 1)G_1^2}{2e} \right) \right]^2 = \left[ \frac{1}{\sqrt{1 - e^{-1}}} \right]^2 = \epsilon.
\]

Next, notice that
\[
\mathbb{E} \left[ Z_i Z_i^\top \right] = \mathbb{E} \left[ Z_i^\top Z_i \right] = d\sigma^2 I_d.
\]
Therefore, applying Proposition B.1 of §B.2 gives
\[
P \left( \lambda < 3 \| \Sigma_{op} \| \right) \leq 2d \exp \left[ -\frac{Cn\lambda}{d\sigma} \left( \frac{\lambda}{\sigma} \wedge \frac{1}{\log d} \right) \right], \tag{5.1}
\]
for some constant \( C \).

We can follow the same argument for \( \zeta_i X_i \), and use Corollary B.1 of §B.2 to obtain
\[
\mathbb{E} \left[ \| \Sigma_R \|_{op} \right] \leq C_1 \sqrt{\frac{d \log d}{n}}, \tag{5.2}
\]
provided that \( n \geq C_2 d \log^3 d \) for constants \( C_1 \) and \( C_2 \). We can now combine (5.1), (5.2), with Corollary 3.2 to obtain the following result: for any \( \lambda \geq C_3 b^* \sqrt{d \log d/n} \) and \( n \geq C_2 d \log^3 d \), the inequality
\[
\| \hat{B} - B^* \|^2_F \leq C_4 \lambda^2 r
\]
holds with probability at least
\[
1 - 2d \exp \left[ -\frac{Cn\lambda}{\sigma d} \left( \frac{\lambda}{\sigma} \wedge \frac{1}{\log d} \right) \right] - \exp \left( -\frac{C_3 n\lambda^2 r}{b^2} \right).
\]
In particular, setting
\[
\lambda = C_6 (\sigma \vee b^*) \sqrt{\frac{\rho d}{n}}, \tag{5.3}
\]
for some $\rho \geq \log d$, we have that

$$\|\hat{B} - B^*\|_F^2 \leq C_7(\sigma^2 \vee b^* \|B^*\|_\infty^2) \frac{\rho \cdot d}{n}, \quad (5.4)$$

with probability at least $1 - \exp(-C_8 \rho)$ whenever $n \geq C_2 d \log^2 d$. This result resembles Corollary 1 in [22] whenever $\rho = \log d$.

5.2 Multi-task learning

Similar as in §5.1, let $B^*$ be a $d \times d$ matrix and let for each $i \in [n]$, $r_i$ be an integer in $[n]$, selected independently and uniformly at random. Then let $X_i = e_{r_i} \cdot X_i^\top$ where, for each $i$, $X_i$ is an independent $\mathcal{N}(0, d \cdot I_d)$ random vector that is also independent of $\{r_j\}_{j=1}^n$. It then follows that

$$\|B\|_{L^2(\Pi)} = \|B\|_F,$$

which means $\gamma_{\text{min}} = 1$. Also,

$$\|B\|_{\psi_2(\Pi)} \leq 2\sqrt{d} \|B\|_{2,\infty}.$$

To see the latter, for any $X \sim \Pi$ we follow the same steps as in previous section and obtain,

$$\mathbb{E} \left[ \exp \left( \frac{|\langle B, X \rangle|^2}{4d\|B\|_{2,\infty}^2} \right) \right] = \frac{1}{d} \sum_{j=1}^d \mathbb{E}_{X_j} \left[ \exp \left( \frac{|\langle B, e_j \cdot X_j^\top \rangle|^2}{4d\|B\|_{2,\infty}^2} \right) \right]$$

$$\leq \frac{1}{d} \sum_{j=1}^d \left[ \frac{1}{\sqrt{(1 - \|B\|_{2,\infty}^2)}} \right] \leq 2,$$

where $B_j$ denotes the $j^{\text{th}}$ row of $B$ and (5.5) uses Lemma A.1 of §A, since that $\langle B_j, X_j \rangle \sim \mathcal{N}(0, d\|B_j\|_2^2)$. The final step uses $\|B_j\|_2 \leq \|B\|_{2,\infty}$ which follows definition of $\|B\|_{2,\infty}$. Similar to §5.1, we use this to set $\Pi(\cdot) = 2\sqrt{d} \|B^*\|_{2,\infty}$ which means $b^* = 2\sqrt{d} \|B^*\|_{2,\infty}$ works too. Also, similar to §5.1, we can condition on random variable $r_i$ to show that,

$$\mathbb{E} \left[ (B_{r_i}, X_i)^2 \mathbb{I}(\|B_{r_i}X_i\| \leq 9) \mid r_i \right] \geq \frac{1}{2} \mathbb{E} \left[ (B_{r_i}, X_i)^2 \mid r_i \right],$$

which means $\varepsilon = 9$ satisfies requirement of Assumption 3.3.

Let $Z_i$ be as in §5.1, $\delta = d\sigma/e(1 - 1)$, and let $(G_i)_{i=0}^d$ be a sequence of $d + 1$ independent standard normal random variables. We see that

$$\mathbb{E} \left[ \exp \left( \frac{\|Z_i\|_{op}}{\delta} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{\|Z_i\|_F}{\delta} \right) \right]$$

$$= \mathbb{E} \left[ \exp \left( \frac{\sigma d G_0 \sqrt{G_1^2 + \ldots + G_d^2}}{\delta} \right) \right]$$

$$\leq \mathbb{E} \left[ \exp \left( \frac{\sigma d (G_0^2 + G_1^2 + \ldots + G_d^2)}{2\delta} \right) \right]$$

$$= \left( 1 - \frac{\sigma d}{\delta} \right)^{-\frac{1}{2}} \cdot \left( 1 - \frac{\sigma}{\delta} \right)^{-\frac{1}{2}}$$
\[
\left(1 - \frac{e-1}{e}\right)^{-\frac{1}{2}} \cdot \left(1 - \frac{e-1}{ed}\right)^{-\frac{4}{d}} \leq \sqrt{e \cdot \left(1 - \frac{e-1}{e}\right)} \quad \text{(using } 1 + x \leq e^x) \\
\leq e.
\]

Furthermore, we have

\[
\mathbb{E}[Z_t Z_t^\top] = d \sigma^2 I_d \quad \text{and} \quad \mathbb{E}[Z_t^\top Z_t] = \sigma^2 I_d.
\]

This implies that (5.1) and (5.2) hold in this case as well. Since \( \gamma_{\min} \) and \( \epsilon \) are the same as §5.1, we conclude that (5.4) holds when \( n \geq C_2 d \log^3 d \), with the same probability.

### 5.3 Compressed sensing via Gaussian ensembles

Let \( B^* \) be a \( d \times d \) matrix. Let each \( X_i \) be a random matrix with entries filled with i.i.d. samples drawn from \( \mathcal{N}(0, 1) \). It then follows that

\[
\|B\|_{L^2(\Pi)} = \|B\|_F,
\]

which means \( \gamma_{\min} = 1 \), and

\[
\|B\|_{\psi_2(\Pi)} \leq 2\|B\|_F.
\]

Similar as before, to see the latter, since \( (B, X_i) \sim \mathcal{N}(0, \|B\|_F^2) \) via Lemma A.1 of §A,

\[
\mathbb{E}
\left[
\exp\left(\frac{|\langle B, X_i \rangle|^2}{4\|B\|_F^2}\right)
\right]
= \left[ \frac{1}{\sqrt{(1 - \frac{1}{2})_+}} \right] \leq 2.
\]

Therefore, setting \( \mathcal{O}(\cdot) = \|\cdot\|_{\psi_2(\Pi)} \), \( \epsilon = 9 \) works as before. Therefore, similar argument to that of §5.1-5.2, shows that (5.4) holds for this setting as well. This bound resembles the bound of [3].

### 5.4 Compressed sensing via factored measurements

Recht et al. [24] propose factored measurements to alleviate the need to a storage of size \( nd^2 \) for compressed sensing applications with large dimensions. The idea is to use measurement matrices of the form \( X_i = U V^\top \) where \( U \) and \( V \) are random vector of length \( d \). Even though \( U V^\top \) is a \( d \times d \) matrix, we only need a memory of size \( \mathcal{O}(nd) \) to store all the input, which is a significant improvement compared to Gaussian ensembles of §5.3. Now, we study this problem when \( U \) and \( V \) are both \( \mathcal{N}(0, I_d) \) vectors that are independent of each other. In this case we have,

\[
\|B\|_{L^2(\Pi)}^2 = \mathbb{E}\left[\langle B, U V^\top \rangle^2\right]
= \mathbb{E}\left[\langle V^\top B^\top \rangle U (U^\top B V)\right]
= \mathbb{E}\left[V^\top B^\top \cdot \mathbb{E}[U U^\top | V] \cdot B V\right]
= \mathbb{E}\left[V^\top B^\top B V\right]
= \mathbb{E}\left[\text{tr}(V^\top B^\top B V)\right]
= \mathbb{E}\left[\text{tr}(B V V^\top B^\top)\right]
= \text{tr}(B \mathbb{E}[V V^\top] B^\top)
= \text{tr}(B B^\top)
\]
which means $\gamma_{\min} = 1$ works again. Next, let $\mathbf{B} = \mathbf{O}_1 \mathbf{D} \mathbf{O}_2^T$ be the singular value decomposition of $\mathbf{B}$. Then, we get
\[
\langle \mathbf{B}, \mathbf{U} \mathbf{V}^T \rangle = \mathbf{U}^T \mathbf{B} \mathbf{V} \\
= \mathbf{U}^T \mathbf{O}_1 \mathbf{D} \mathbf{O}_2^T \mathbf{V} \\
= (\mathbf{O}_1^T \mathbf{U})^T \mathbf{D} (\mathbf{O}_2^T \mathbf{V}).
\]
As the distribution of $\mathbf{U}$ and $\mathbf{V}$ is invariant under multiplication of unitary matrices, for any $t > 0$, we have
\[
\mathbb{E} \left[ \exp \left( \frac{\langle \mathbf{B}, \mathbf{U} \mathbf{V}^T \rangle}{t} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{\langle \mathbf{D}, \mathbf{U} \mathbf{V}^T \rangle}{t} \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{\mathbf{U}^T \mathbf{D} \mathbf{V}}{t} \right) \right] \\
= \mathbb{E}_U \left[ \mathbb{E}_V \left[ \exp \left( \frac{\mathbf{U}^T \mathbf{D} \mathbf{V}}{t} \right) \right] \mathbf{U} \right] \\
= \mathbb{E} \left[ \exp \left( \frac{\|\mathbf{U}^T \mathbf{D}\|^2}{2t^2} \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{\sum_{i=1}^d U_i^2 \mathbf{D}_i^2}{2t^2} \right) \right] \\
= \prod_{i=1}^d \mathbb{E} \left[ \exp \left( \frac{U_i^2 \mathbf{D}_i^2}{2t^2} \right) \right] \\
= \prod_{i=1}^d \frac{1}{\sqrt{\left( 1 - \frac{\mathbf{D}_i^2}{t^2} \right)_+}}.
\]
Using Lemma A.3 (from §A), we realize that the necessary condition for $\|\langle \mathbf{B}, \mathbf{U} \mathbf{V}^T \rangle\|_{\psi_1} \leq t$ to hold is
\[
\mathbb{E} \left[ \exp \left( \frac{\langle \mathbf{B}, \mathbf{U} \mathbf{V}^T \rangle}{t} \right) \right] \leq 2. \tag{5.6}
\]
This, in particular, implies that
\[
\frac{1}{\sqrt{\left( 1 - \frac{\mathbf{D}_i^2}{t^2} \right)_+}} \leq 2,
\]
or equivalently
\[
\frac{\mathbf{D}_i^2}{t^2} \leq \frac{3}{4},
\]
for all $i \in [d]$. By taking derivatives and concavity of logarithm, we can observe that $-2x \leq \log(1 - x) \leq -x$ for all $x \in [0, \frac{3}{4}]$. This implies that, whenever (5.6) holds, we have
\[
\frac{\mathbf{D}_i^2}{2t^2} \leq \log \left( 1 - \frac{\mathbf{D}_i^2}{t^2} \right)_+ \leq \frac{\mathbf{D}_i^2}{t^2},
\]
and thus
\[
\exp \left( \frac{\sum_{i=1}^d \mathbf{D}_i^2}{2t^2} \right) \leq \mathbb{E} \left[ \exp \left( \frac{\langle \mathbf{B}, \mathbf{U} \mathbf{V}^T \rangle}{t} \right) \right] \leq \exp \left( \frac{\sum_{i=1}^d \mathbf{D}_i^2}{t^2} \right).
\]
Using $\|B\|_F^2 = \sum_{i=1}^d D_{ii}^2$, the above can be simplified to

$$
\exp \left( \frac{\|B\|_F^2}{2t^2} \right) \leq \mathbb{E} \left[ \exp \left( \frac{\langle B, UV^\top \rangle}{t} \right) \right] \leq \exp \left( \frac{\|B\|_F^2}{t^2} \right). \quad (5.7)
$$

Putting all the above together, we may conclude that $\|\langle B, X \rangle\|_{\psi_1} \leq t$ implies

$$
\frac{2D_{11}}{\sqrt{3}} \leq t \quad \text{and} \quad \frac{\|B\|_F}{\sqrt{2 \log 2}} \leq t.
$$

Therefore, we have

$$
\frac{1}{\sqrt{2 \log 2}} \|B\|_F \leq \|B\|_{\psi_1, (II)} \quad (5.8)
$$

Next, define

$$
t := \max \left\{ \frac{2D_{11}}{\sqrt{3}}, \frac{\|B\|_F}{\sqrt{\log 2}} \right\} = \|B\|_F / \sqrt{\log 2}.
$$

Since $D_{11}^2 / t^2 \leq 3/4$, we can use (5.7) which gives

$$
\mathbb{E} \left[ \exp \left( \frac{\langle B, UV^\top \rangle}{t} \right) \right] \leq 2.
$$

Using Lemma A.3 of §A, we can conclude that

$$
\|B\|_{\psi_1, (II)} \leq \frac{8}{\sqrt{\log 2}} \|B\|_F.
$$

Now, setting $\mathcal{H}(\cdot) = \|\cdot\|_{\psi_1, (II)}$, given that the ratio $\|B\|_{\psi_1, (II)}/\|B\|_F$ is at most $8 / \sqrt{\log(2)}$, we can apply Corollary A.1 of §A for $Z = \langle B, X \rangle$ to see that $c = 53$ satisfies Assumption 3.3.

Now, for bounding $\mathbb{P} \left( \lambda < 3 \|\Sigma\|_{\text{op}} \right)$ we need to use a truncation argument for the noise. Specifically, let

$$
\bar{\varepsilon}_i := \varepsilon_i \mathbb{I} \left[ \varepsilon_i \leq C_{\varepsilon} \sigma \sqrt{\log d} \right],
$$

for a large enough constant $C_{\varepsilon}$. Now, defining $\Sigma := \sum_{i=1}^n \bar{\varepsilon}_i X_i$, via union bound we have

$$
\mathbb{P} \left( \lambda < 3 \|\Sigma\|_{\text{op}} \right) \leq \mathbb{P} \left( \lambda < 3 \|\Sigma\|_{\text{op}} \right) + \sum_{i=1}^n \mathbb{P} \left( |\varepsilon_i| > C_{\varepsilon} \sigma \sqrt{\log d} \right)
$$

$$
\leq \mathbb{P} \left( \lambda < 3 \|\Sigma\|_{\text{op}} \right) + 2ne^{-\frac{C_{\varepsilon}^2}{2} \log d}.
$$

Now, defining $\delta := 2C_{\varepsilon} \sigma d \sqrt{\log d}$ and $Z_i = \bar{\varepsilon}_i X_i$, as in §5.1 we aim to use Proposition B.1 again. Let $(G_i)_{i=1}^{2d}$ be a sequence of $2d + 1$ independent standard normal random variables, similar steps as in §5.1-5.2 yields

$$
\mathbb{E} \left[ \exp \left( \frac{\|Z\|_{\text{op}}}{\delta} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{\|\bar{\varepsilon}_i\| \sqrt{\sum_{j=1}^{2d} G_j^2}}{\delta} \sqrt{\sum_{j=d+1}^{2d} G_j^2} \right) \right]
$$

$$
\leq \mathbb{E} \left[ \exp \left( \frac{\|\bar{\varepsilon}_i\| \sum_{j=1}^{2d} G_j^2}{2\delta} \right) \right]
$$

$$
\leq \mathbb{E} \left[ \exp \left( \frac{\|\bar{\varepsilon}_i\| G_i^2}{2\delta} \right) \right]^{2d} \mathbb{E} \left[ \bar{\varepsilon}_i \right]
$$
\[ \leq \mathbb{E} \left[ \exp \left( \frac{C_1^2}{4d} \right) \right]^{2d} \leq \left( \frac{1}{1 - \frac{1}{2d}} \right)^{2d} \leq e. \]

Furthermore, we have
\[ \mathbb{E}[Z_i Z_i^T] \succeq d\sigma^2 I_d \quad \text{and} \quad \mathbb{E}[Z_i^T Z_i] \preceq \sigma^2 I_d. \]

Therefore, the following slight variation of (5.1) holds
\[ \mathbb{P} \left( \lambda < 3\|\Sigma\|_{op} \right) \leq 2d \exp \left[ -\frac{Cn\lambda}{d\sigma} \left( \frac{\lambda}{\sigma} \wedge \frac{1}{\log^{3/2} d} \right) \right] + 2ne^{-\frac{c_2^2}{2} \log d}. \] (5.9)

However, (5.2) stays unchanged since for \( \zeta_i \), unlike \( \varepsilon_i \), we do not need to use any truncation. This means we can define \( \lambda \) as in (5.3) and obtain a bound as in (5.4) with probability at least \( 1 - \exp(-C_8 \rho) \) whenever \( \rho \geq \log d \) and \( n \geq C_d d \log^4 d \). This bound matches Theorem 2.3 of [2], however theirs work for \( n = O(rd) \) which is smaller than ours when \( r < \log^4 d \).

### 6 Applications to Exact Recovery

In this section we study the trace regression problem when there is no noise. It is known that, under certain assumptions, it is possible to recover the true matrix \( B^* \) exactly, with high probability (e.g., [4, 12]). The discussion of Section 3.4 in [22] makes it clear that bounds given in terms of spikiness are not strong enough to obtain exact recovery even in the noiseless setting for the matrix completion problem (studied in §5.1). However, will will show in this section that the methodology from §3 can be used to prove exact recovery for the two cases of compressed sensing studied in previous section (§5.3-5.4). We will conclude this section with a brief discussion on exact recovery for the multi-task learning case (§5.2).

For any arbitrary sampling operator \( \mathfrak{X}(\cdot) \), let \( \mathcal{S} \) be defined as follows:
\[ \mathcal{S} := \{ B \in \mathbb{R}^{d_r \times d_c} : \mathfrak{X}(B) = Y \}. \]

Using \( \sigma_e = 0 \) and the linear model (2.1), one can verify that \( B^* \in \mathcal{S} \) and so, \( \mathcal{S} \) is not empty. The definition of \( \mathcal{S} \) implies that \( \mathcal{S} \) is an affine space and is thus convex. Next, note that, for any \( B \in \mathcal{S} \), the following identity holds:
\[ \mathcal{L}(B) = \frac{1}{n} \|Y - \mathfrak{X}(B)\|_2^2 + \lambda\|B\|_* = \lambda\|B\|_* . \]

Therefore, the minimizers of the optimization problem
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{n} \|Y - \mathfrak{X}(B)\|_2^2 + \lambda\|B\|_* \\
\text{subject to} & \quad \mathfrak{X}(B) = Y,
\end{align*}
\]
are also minimizers of
\[
\begin{align*}
\text{minimize} & \quad \|B\|_* \\
\text{subject to} & \quad \mathfrak{X}(B) = Y.
\end{align*}
\]

Note that, in the above formulation, the convex problem does not depend on \( \lambda \) anymore, and so, \( \lambda \) can be chosen arbitrarily. In the noiseless setting, \( \|\Sigma\|_{op} = 0 \), and so, any \( \lambda > 0 \) satisfies (3.5). Therefore, if RSC condition holds for \( \mathfrak{X}(\cdot) \) with parameters \( \alpha \) and \( \beta \) on the set \( \mathcal{C}(0, \eta) \), Theorem 3.1 leads to
\[ \|\hat{B} - B^*\|_F^2 \leq \frac{8b^2 \beta}{\alpha}. \] (6.1)
Now, defining $\nu_0$ as

$$\nu_0 := \inf_{B \neq 0} \frac{\|B\|_F^2}{\Re(B)^2}, \quad (6.2)$$

one can easily observe that $C(\nu_0, \eta) = C(0, \eta)$. Moreover, assume that $n$ is large enough so that

$$\mathbb{E}[\|\Sigma_R\|_{op}^2] \leq \frac{\gamma_{\min} \nu_0}{800 \epsilon^2 \eta}, \quad (6.3)$$

where $\Sigma_R$ was defined previously to be $(1/n) \sum_{i=1}^n \zeta_i X_i$ with $(\zeta_i)_{i=1}^n$ being a sequence of i.i.d. Rademacher random variables. Using this, together with Corollary 3.1 implies that with probability at least $1 - 2 \exp\left(-\frac{C n \gamma_{\min} \nu_0}{\epsilon^2}\right)$ for all $A \in C(\nu_0, \eta)$, we have

$$\|X(A)\|_2^2 \geq \frac{\gamma_{\min}}{8} \|A\|_{\infty}^2 - \frac{93 \eta \epsilon^2}{\gamma_{\min}} \mathbb{E}[\|\Sigma_R\|_{op}^2].$$

This shows that $X(\cdot)$ satisfies the RSC condition with $\alpha = \gamma_{\min}/8$ and $\beta = 0$. As a result, from Equation (6.1), we can deduce the following proposition:

**Proposition 1.** Let $\nu_0$ and $n$ be as in (6.2) and (6.3). Then, the unique minimizer of the constraint optimization problem

$$\begin{align*}
\text{minimize} & \quad \|B\|_s \\
\text{subject to} & \quad X(B) = Y.
\end{align*}$$

is $B = B^*$, and hence, exact recovery is possible with probability at least $1 - 2 \exp\left(-\frac{C n \gamma_{\min} \nu_0}{\epsilon^2}\right)$.

Now, we can use the above proposition to prove that exact recovery is possible for the two problems of compressed sensing with Gaussian ensembles (§5.3) and compressed sensing with factored measurements (§5.4). Note that in both examples, we have $\gamma_{\min} = 1$, $\nu_0 \geq 0.1$, and $\epsilon \leq 60$.

Therefore, in order to use Proposition 1, all we need to do is to find a lower-bound for $n$ such that (6.3) holds. We study each example separately:

1. **Compressed sensing with Gaussian ensembles.** Here, since entries of $X_i$’s are i.i.d. $\mathcal{N}(0, 1)$ random variables, the entries of $\Sigma_R$ are i.i.d. $\mathcal{N}(0, 1/n)$ random variables. We can, then, use Theorem 5.32 in [29] to get

$$\mathbb{E}[\|\Sigma_R\|_{op}] \leq \frac{2\sqrt{d}}{\sqrt{n}}.$$

Therefore, (6.3) is satisfied if $n \geq C d$ where $C > 0$ is a large enough constant.

2. **Compressed sensing with factored measurements.** Here, the observation matrices are of the form $U_i V_i^\top$ where $U_i$ and $V_i$ are independent vectors distributed according to $\mathcal{N}(0, I_d)$. Note that we have $\|X_i\|_{op} = \|U_i\|_2 \|V_i\|_2 \leq \|U_i\|_2^2 + \|V_i\|_2^2$. Then,

$$\|X_i\|_{\psi_1} \leq \|U_i\|_2^2 + \|V_i\|_2^2$$

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\( \leq 2 \left\| U_i \right\|_{\psi_1}^2 \)
\( = O(d) . \)

An application of Equation (3.9) in [16] gives us
\[
\mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \right] = O \left( \sqrt{\frac{d \log(2d)}{n}} \sqrt{\frac{d \log(2d)^2}{n}} \right).
\]

We can thus infer that (6.3) holds for all \( n \geq Crd \log(d) \) where \( C \) is a large enough constant.

Therefore, Proposition 1 guarantees that, for \( n \) satisfying the conditions stated above, exact recovery is possible in each of the two aforementioned settings, with probability at least \( 1 - \exp(C'n) \) where \( C' > 0 \) is a numerical constant.

**Implications for multi-task learning.** We also apply Proposition 1 to the multi-task learning case (§5.2) as well. We have \( \gamma_{\text{min}} = 1 \) and \( c = 9 \), but, for \( \nu_0 \) we have

\[
\nu_0 = \inf_{B \neq 0} \frac{\left\| B \right\|_{\psi_2}^2}{\mathbb{F}(B)^2} = \inf_{B \neq 0} \frac{\left\| B \right\|_{\psi_2}^2}{4d \left\| B \right\|_{2,\infty}^2} = \frac{1}{4d},
\]

where the infimum is achieved if and only if \( B \) has exactly one non-zero row. Notice that \( \nu_0 \) depends on the dimensions of the matrix in contrast to the previous examples that we had \( \nu_0 \geq 0.1 \).

It is straight-forward to verify that

\[
\left\| X_i \right\|_{\text{op}} \left\| X_i \right\|_{\psi_2} = \left\| X_i \right\|_{2,\psi_2} = O(\sqrt{d}).
\]

Therefore, similar argument as the above shows that

\[
\mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \right] = O \left( \sqrt{\frac{d \log(2d)}{n}} \sqrt{\frac{\sqrt{d \log(2d)^2}}{n}} \right).
\]

This, in turn, implies that, in order for (6.3) to hold, it suffices to have \( n \geq Crd^2 \sqrt{d \log(2d)} \), for a large enough constant \( C \). In this case, Proposition 1 shows that the exact recovery is possible with probability at least \( 1 - 2 \exp(-\frac{C'n}{d}) \). However, this result is trivial, since \( n \geq Crd^2 \sqrt{d \log(2d)} \) means that with high probability, each row is observed at least \( d \) times, and so each row can be reconstructed separately (without using low-rank assumption). This result can not be improved without further assumptions, as it is possible in a rank-2 matrix that all rows are equal to each other except for one row, and that row can be reconstructed only if at least \( d \) observation is made for that row. Since this must hold for all rows, at least \( d^2 \) observations is needed. Nonetheless, one can expect that with stronger assumptions than generalized spikiness, such as incoherence, the number of required observations can be reduced to \( rd \log(d) \).

### A Auxiliary proofs

**Lemma A.1.** Let \( Z \) be a \( \mathcal{N}(0, \sigma^2) \) random variable. Then, for all \( \eta > 0 \),

\[
\mathbb{E} \left[ e^{\eta Z^2} \right] = \frac{1}{\sqrt{(1 - 2\sigma^2 \eta)_+}}.
\]
Proof. Easily follows by using the formula \( \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2a^2})dt = \sqrt{2\pi}a^2. \)

**Lemma A.2.** Let \( Z \) be a non-negative random variable such that \( \|Z\|_{\psi_p} = \nu \) holds for some \( p \geq 1 \), and assume \( c > 0 \) is given. Then, we have

\[
\mathbb{E}[Z^2 \cdot I(Z \geq c)] \leq (2c^2 + 4c\nu + 4\nu^2) \cdot \exp\left(-\frac{c^p}{\nu^p}\right).
\]

Proof. Without loss of generality, we can assume that \( Z \) has a density function \( f(z) \) and \( \nu = 1 \). Moreover, let \( F(z) := \mathbb{P}(Z \leq z) \) be the cumulative distribution function of \( Z \). The assumption that \( \|Z\|_{\psi_p} = 1 \) together with Markov inequality yields

\[
F(z) \geq 1 - 2 \exp(-z^p).
\]

Therefore,

\[
\mathbb{E}[Z^2 \cdot I(Z \geq c)] = \int_{c}^{\infty} z^2 f(z) \, dz
\]

\[
= \int_{c}^{\infty} (-z^2)[-f(z)] \, dz
\]

\[
= c^2 [1 - F(c)] + \int_{c}^{\infty} 2z[1 - F(z)] \, dz
\]

\[
\leq 2c^2 \cdot \exp(-c^p) + \int_{c}^{\infty} 4z \exp(-z^p) \, dz.
\]

Now, note that, the function \( f_c(p) \) defined as

\[
f_c(p) := \int_{c}^{\infty} z \exp(-z^p) \, dz = \frac{\int_{c}^{\infty} z \exp(c^p - z^p) \, dz}{c + 1},
\]

is decreasing in \( p \). So, we have that

\[
\int_{c}^{\infty} z \exp(-z^p) \, dz \leq (c + 1) \exp(-c^p) f_c(1)
\]

\[
= (c + 1) \exp(-c^p),
\]

where we have used \( f_c(1) = 1 \) which can be proved integrals for \( p = 1 \). Therefore, we have

\[
\mathbb{E}[Z^2 \cdot I(Z \geq c)] \leq (2c^2 + 4c + 4) \cdot \exp(-c^p),
\]

which is the desired result.

**Corollary A.1.** Let \( Z \) be a random variable satisfying \( \|Z\|_{\psi_p} = \nu \) holds for some \( p \geq 1 \) and \( \mathbb{E}[Z^2] = \sigma^2 \). Then, for

\[
c_{\sigma,p} := \nu \cdot \max\left\{ 5, \left[ 10 \log\left(\frac{2\nu^2}{\sigma^2}\right)\right]^{\frac{1}{p}} \right\}, \quad (A.1)
\]

we have

\[
\mathbb{E}[Z^2 \cdot I(|Z| \leq c_{\sigma,p})] \geq \frac{\mathbb{E}[Z^2]}{2}.
\]

Proof. Without loss of generality, we can assume that \( \nu = 1 \). Using Lemma A.2 for \( |Z| \) and any \( c \geq 5 \), we have

\[
\mathbb{E}[Z^2 \cdot I(|Z| \leq c)] \geq \mathbb{E}[Z^2] - \mathbb{E}[Z^2 \cdot I(|Z| \geq c)]
\]

\[22\]
\[ \geq \mathbb{E}[Z^2] - 3c^2 \cdot \exp(-c^p). \]

Next, it is easy to show that, for any \( c \geq 5, \)
\[ 3c^2 \cdot \exp\left(-\frac{9c^p}{10}\right) \leq 3c^2 \cdot \exp\left(-\frac{9c}{10}\right) \leq 1. \]

Therefore, letting \( c_\sigma \) be defined as (A.1), we get
\[ 3c^2_{\sigma,p} \cdot \exp\left(-c^p_{\sigma,p}\right) = 3c^2_{\sigma,p} \cdot \exp\left(-\frac{9c^p_{\sigma,p}}{10}\right) \leq \exp\left(-\frac{c^p_{\sigma,p}}{10}\right) \leq \frac{\sigma^2}{2}, \]
which completes the proof of this corollary.

**Corollary A.2.** Let \( Z \) be a \( \mathcal{N}(0,\sigma^2) \) random variable. Then, the constant \( c_{\sigma,2} \) defined in (A.1) satisfies \( c_{\sigma,p} \leq 5\|Z\|_{\psi_2}. \)

**Proof.** Using Lemma A.1, we obtain \( \nu = \|Z\|_{\psi_2} = \sqrt{8\sigma^2/3} \) which means \( \nu^2/\sigma^2 = 8/3. \) The rest follows from Corollary A.1.

The Orlicz norm of a random variable is defined in terms of the absolute value of that random variable, and it is usually easier to work with the random variable rather than its absolute value. The next lemma relates the Orlicz norm to the mgf of a random variable.

**Lemma A.3.** Let \( X \) be a mean zero random variable and
\[ \alpha := \inf \left\{ t > 0 : \max \left\{ \mathbb{E}\left[ \exp\left( \frac{|X|}{t} \right) \right], \mathbb{E}\left[ \exp\left( -\frac{|X|}{t} \right) \right] \right\} \leq 2 \right\}. \]

Then, we have
\[ \alpha \leq \|X\|_{\psi_1} \leq 8\alpha. \]

**Proof.** The first inequality, \( \alpha \leq \|X\|_{\psi_1}, \) follows from monotonicity of the exponential function. For the second one, note that for any \( t > 0, \)
\[ \mathbb{E}\left[ \exp\left( \frac{|X|}{t} \right) \right] = \mathbb{E}\left[ \exp\left( \frac{|X| - \mathbb{E}[|X|]}{t} \right) \right] \cdot \exp\left( \mathbb{E}[|X|]/t \right). \]

Now, union bound and Markov inequality lead to the following tail bound for \( |X|:\)
\[ \mathbb{P}(|X| \geq x) \leq \mathbb{P}(X \geq x) + \mathbb{P}(-X \geq x) \leq 4 \exp\left(-\frac{x}{\alpha}\right). \]

Hence, we have
\[ \mathbb{E}[|X|] = \int_0^\infty \mathbb{P}(|X| \geq x)dx \leq 4 \alpha. \]

Next, assuming that \( X' \) is an independent copy of \( X \) and \( \varepsilon \) is a Rademacher random variable independent of \( X \) and \( X' \), we have
\[ \mathbb{E}\left[ \exp\left( \frac{|X| - \mathbb{E}[|X|]}{t} \right) \right] = \mathbb{E}\left[ \exp\left( \frac{|X| - \mathbb{E}[|X'|]}{t} \right) \right] \]

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\[ \begin{align*}
\leq & \mathbb{E} \left[ \exp \left( \frac{|X| - |X'|}{t} \right) \right] \\
= & \mathbb{E} \left[ \exp \left( \frac{\varepsilon |X| - |X'|}{t} \right) \right] \\
= & \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \frac{\varepsilon |X| - |X'|}{t} \right) \bigg| X, X' \right] \right] \\
\overset{(\ast)}{=} & \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \frac{\varepsilon |X - X'|}{t} \right) \bigg| X, X' \right] \right] \\
= & \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \frac{\varepsilon (X - X')}{t} \right) \bigg| X, X' \right] \right] \\
= & \mathbb{E} \left[ \exp \left( \frac{\varepsilon (X - X')}{t} \right) \right] \\
= & \frac{1}{2} \mathbb{E} \left[ \exp \left( \frac{X - X'}{t} \right) \right] + \frac{1}{2} \mathbb{E} \left[ \exp \left( \frac{X' - X}{t} \right) \right] \\
= & \mathbb{E} \left[ \exp \left( \frac{X - X'}{t} \right) \right] \\
= & \mathbb{E} \left[ \exp \left( \frac{X}{t} \right) \right] \cdot \mathbb{E} \left[ \exp \left( \frac{-X}{t} \right) \right] ,
\end{align*} \]

where (\ast) follows from \( ||a| - |b|| \leq |a - b| \) and the fact that the function \( z \mapsto \frac{1}{2} (\exp(-z) + \exp(z)) \) is increasing for \( z > 0 \).

Therefore, from the above inequalities, we can deduce that

\[ \mathbb{E} \left[ \exp \left( \frac{|X|}{t} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{X}{\alpha} \right) \right] \cdot \mathbb{E} \left[ \exp \left( \frac{-X}{\alpha} \right) \right] \exp \left( \frac{4\alpha}{t} \right) . \]

Now, by setting \( t := 8\alpha \) and using Jensen’s inequality, we get

\[ \mathbb{E} \left[ \exp \left( \frac{|X|}{t} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{X}{\alpha} \right) \right] \cdot \mathbb{E} \left[ \exp \left( \frac{-X}{\alpha} \right) \right] \cdot \exp \left( \frac{4\alpha}{t} \right) \leq \exp \left( \frac{\log 2 + 2}{4} \right) . \]

This implies that \( \|X\|_{\psi_1} \leq t = 8\alpha . \)

**Lemma A.4.** For any sub-exponential random variable \( X \), we have

\[ \| \mathbb{E}[X] \|_{\psi_1} \leq \|X\|_{\psi_1} . \]

**Proof.**

\[ \begin{align*}
\| \mathbb{E}[X] \|_{\psi_1} = & \inf \left\{ t > 0 : \exp \left( \frac{\mathbb{E}[|X|]}{t} \right) \leq 2 \right\} \\
= & \frac{\mathbb{E}[|X|]}{\log 2} \\
\leq & \frac{\mathbb{E}[|X|]}{\log 2} \\
= & \|X\|_{\psi_1} \cdot \log \exp \left( \frac{|X|}{\|X\|_{\psi_1}} \right) \\
= & \frac{\|X\|_{\psi_1}}{\log 2} .
\end{align*} \]
We can alternatively express $P_B$. Then, using the duality between operator norm and trace norm, we get

$$
\|X\|_{\psi_1} \cdot \log \frac{\log 2}{E} \leq \|X\|_{\psi_1} \cdot \log 2 \leq \|X\|_{\psi_1} \cdot \log 2
$$

B.1 Proof of Theorem 3.1

First, it follows from (3.2) that

$$
\frac{1}{n} \|Y - \bar{X}(B)\|_2^2 + \lambda \|\hat{B}\|_* \leq \frac{1}{n} \|Y - \bar{X}(B^*)\|_2^2 + \lambda \|B^*\|_*.
$$

By substituting $Y$ with $\bar{X}(B^*) + E$ and doing some algebra, we have

$$
\frac{1}{n} \|\bar{X}(B^* - \hat{B})\|_2^2 + 2\langle \Sigma, B^* - \hat{B} \rangle + \lambda \|\hat{B}\|_* \leq \lambda \|B^*\|_*.
$$

Then, using the duality between operator norm and trace norm, we get

$$
\frac{1}{n} \|\bar{X}(B^* - \hat{B})\|_2^2 + \lambda \|\hat{B}\|_* \leq 2\|\Sigma\|_{op} \cdot \|B^* - \hat{B}\|_* + \lambda \|B^*\|_*.
$$

(B.1)

For a given set of vectors $S_r$ we denote by $P_S$ the orthogonal projection on the linear subspace spanned by elements of $S$ (i.e., $P_S = \sum_{i=1}^k u_i u_i^T$ if $\{u_1, \ldots, u_k\}$ is an orthonormal basis for $S$). For matrix $B \in \mathbb{R}^{d_r \times d_c}$, let $S_r(B)$ and $S_c(B)$ be the linear subspace spanned by the left and right orthonormal singular vectors of $B$, respectively. Then, for $A \in \mathbb{R}^{d_r \times d_c}$ define

$$
P_B(A) := P_{S_r(B)} A P_{S_c(B)} \quad \text{and} \quad \hat{P}_B(A) := A - P_B(A).
$$

We can alternatively express $P_B(A)$ as

$$
P_B(A) = A - \hat{P}_B(A)
$$

$$
= P_{S_r(B)} A + P_{S_r(B)} A - P_{S_r(B)} A
$$

$$
= P_{S_r(B)} A + P_{S_r(B)} [A - A P_{S_r(B)}]
$$

$$
= P_{S_r(B)} A + P_{S_r(B)} A P_{S_c(B)}
$$

(B.2)

In particular, since $S_r(B)$ and $S_c(B)$ both have dimension rank($B$), it follows from (B.2) that

$$
\text{rank}(P_B(A)) \leq 2 \text{rank}(B).
$$

(B.3)

Moreover, the definition of $P_B(A)$ implies that the left and right singular vectors of $P_B(A)$ are orthogonal to those of $B$. We thus have

$$
\|B + P_B(A)\|_* = \|B\|_* + \|P_B(A)\|_*.
$$

By setting $B := B^*$ and $A := \hat{B} - B^*$, the above equality entails

$$
\|B^* + P_B(B^* - \hat{B})\|_* = \|B^*\|_* + \|P_B(B^* - \hat{B})\|_*.
$$

(B.4)
We can then use the above to get the following inequality:

\[
\|\hat{B}\|_* = \|B' + \hat{B} - B\|_* \\
= \|B' + P_{B'}(\hat{B} - B') + P_{B'}(\hat{B} - B')\|_* \\
\geq \|B' + P_{B'}(\hat{B} - B')\|_* - \|P_{B'}(\hat{B} - B')\|_* \\
= \|B'\|_* + \|P_{B'}(\hat{B} - B')\|_* - \|P_{B'}(\hat{B} - B')\|_*.
\]

Combining (B.1) with (B.5), we get

\[
\frac{1}{n}\|\hat{\chi}(B' - \hat{B})\|_2^2 \leq \frac{2}{\|\Sigma\|_{op}} \|\hat{B} - B\|_* + \lambda\|P_{B'}(\hat{B} - B')\|_* - \lambda\|P_{B'}(\hat{B} - B')\|_* \\
\leq \left(2\|\Sigma\|_{op} + \lambda\right)\|P_{B'}(\hat{B} - B')\|_* + \left(2\|\Sigma\|_{op} - \lambda\right)\|P_{B'}(\hat{B} - B')\|_* \\
\leq \frac{5}{3}\lambda\|P_{B'}(B' - B)\|_* ,
\]

where, in the last inequality, we have used (3.5). Now, using this and the fact that rank\((P_{B'}(\hat{B} - B')) \leq \text{rank}(B')\) from (B.3), we can apply Cauchy-Schwartz to singular values of \(P_{B'}(\hat{B} - B')\) to obtain

\[
\frac{1}{n}\|\hat{\chi}(B' - \hat{B})\|_2^2 \leq \frac{5}{3}\lambda \sqrt{2\text{rank}(B')}\|P_{B'}(B' - B')\|_F \\
\leq \frac{5}{3}\lambda \sqrt{2\text{rank}(B')}\|\hat{B} - B\|_2 ,
\]

The next lemma makes a connection between \(\hat{B}\) and the constraint set \(C(\nu, \eta)\).

**Lemma B.1.** If \(\lambda \geq 3\|\Sigma\|_{op}\), then

\[
\|P_{B'}(\hat{B} - B')\|_* \leq 5\|P_{B'}(B' - B')\|_* .
\]

**Proof.** Note that \(\|\hat{\chi}(\cdot)\|_2^2\) is a convex function. We can then use the convexity at \(B'\) to get

\[
\frac{1}{n}\|\hat{\chi}(\hat{B})\|_2^2 - \frac{1}{n}\|\hat{\chi}(B')\|_2^2 \geq -\frac{2}{n}\sum_{i=1}^{n} (y_i - \langle X_i, B' \rangle) \langle X_i, \hat{B} - B' \rangle \\
= -2\langle \Sigma, \hat{B} - B' \rangle \\
\geq -2\|\Sigma\|_{op} \|\hat{B} - B'\|_* \\
\geq -\frac{2\lambda}{3} \|\hat{B} - B'\|_* .
\]

Combining this with (3.2) and (B.5), we have

\[
\frac{2\lambda}{3} \|\hat{B} - B'\|_* \geq \frac{1}{n}\|\hat{\chi}(B')\|_2^2 - \frac{1}{n}\|\hat{\chi}(\hat{B})\|_2^2 \\
\geq \lambda\|\hat{B}\|_* - \lambda\|B'\|_* \\
\geq \lambda\|P_{B'}(\hat{B} - B')\|_* - \lambda\|P_{B'}(B' - B')\|_* .
\]

Using the triangle inequality, we have

\[
\|P_{B'}(\hat{B} - B')\|_* \leq 5\|P_{B'}(B' - B')\|_* .
\]
Lemma B.1, the triangle inequality, and (B.3) imply that
\[
\|\mathbf{B} - \mathbf{B}^*\|_* \leq 6\|\mathbf{P}_{\mathbf{B}^*}(\mathbf{B} - \mathbf{B}^*)\|_* \\
\leq \sqrt{72} \text{rank}(\mathbf{B}^*)\|\mathbf{P}_{\mathbf{B}^*}(\mathbf{B} - \mathbf{B}^*)\|_F \\
\leq \sqrt{72} \text{rank}(\mathbf{B}^*)\|\mathbf{B} - \mathbf{B}^*\|_F.
\]

Next, define \(b := \mathfrak{N}(\mathbf{B} - \mathbf{B}^*)\) and \(A := \frac{1}{b}(\mathbf{B} - \mathbf{B}^*)\). We then have that
\[
\mathfrak{N}(A) = 1 \quad \text{and} \quad \|A\|_* \leq \sqrt{72} \text{rank}(\mathbf{B}^*)\|A\|_F.
\]

Now, we consider the following two cases:

**Case 1:** If \(\|A\|_F^2 < \nu\), then
\[
\|\mathbf{B} - \mathbf{B}^*\|_F^2 < 4b^2\nu.
\]

**Case 2:** Otherwise, \(A \in C(\nu, \eta)\). We can, now, use the RSC condition, as well as, (B.6) to get
\[
\alpha\|\mathbf{B} - \mathbf{B}^*\|_F^2 - \beta \leq \frac{\|\mathbf{X}(\mathbf{B} - \mathbf{B}^*)\|_2^2}{nb^2}
\]
which leads to
\[
\alpha\|\mathbf{B} - \mathbf{B}^*\|_F^2 - 4b^2\beta \leq \frac{\|\mathbf{X}(\mathbf{B} - \mathbf{B}^*)\|_2^2}{n}
\]
\[
\leq \frac{5\lambda\sqrt{2}\text{rank}(\mathbf{B}^*)}{3}\|\mathbf{B} - \mathbf{B}^*\|_F
\]
\[
\leq \frac{50\lambda^2\text{rank}(\mathbf{B}^*)}{3\alpha} + \frac{\alpha}{2}\|\mathbf{B} - \mathbf{B}^*\|_F^2.
\]

Therefore, we have
\[
\|\mathbf{B} - \mathbf{B}^*\|_F^2 \leq \frac{100\lambda^2\text{rank}(\mathbf{B}^*)}{3\alpha^2} + \frac{8b^2\beta}{\alpha},
\]
which completes the proof of this theorem.

### B.2 Matrix Bernstein inequality

The next Proposition is a variant of the Bernstein inequality (Proposition 11 of [14]).

**Proposition B.1.** Let \((\mathbf{Z}_i)_{i=1}^n\) be a sequence of \(d_r \times d_c\) independent random matrices with zero mean, such that
\[
\mathbb{E}\left[\exp\left(\frac{\|\mathbf{Z}_i\|_{op}}{\delta}\right)\right] \leq e \quad \forall i \in [n],
\]
and
\[
\sigma_{\mathbf{Z}} = \max\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbb{E}[\mathbf{Z}_i\mathbf{Z}_i^T]\right\|_{op}, \left\|\frac{1}{n}\sum_{i=1}^n \mathbb{E}[\mathbf{Z}_i^T\mathbf{Z}_i]\right\|_{op}\right\}^{\frac{1}{2}},
\]
for some positive values \(\delta\) and \(\sigma_{\mathbf{Z}}\). Then, there exists numerical constant \(C > 0\) such that, for all \(t > 0\)
\[
\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\right\|_{op} \leq C\max\left\{\sigma_{\mathbf{Z}}\sqrt{\frac{t + \log(d)}{n}}, \delta\left(\frac{\log(\frac{\delta}{\sigma_{\mathbf{Z}}})}{\sigma_{\mathbf{Z}}}\right)\frac{t + \log(d)}{n}\right\},
\]
with probability at least \(1 - \exp(-t)\) where \(d = d_r + d_c\).
We also state the following corollary of the matrix Bernstein inequality.

**Corollary B.1.** If \((B.7)\) holds and \(n \geq \frac{\delta^2}{c^2 \sigma Z^2} \log d \left( \log \frac{d}{\delta \sigma Z^2} \right)^2\), then

\[
E \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i \right\|_{\text{op}} \right] \leq C' \sigma Z \sqrt{\frac{2e \log d}{n}},
\]

where \(C' > 0\) is a numerical constant.

This corollary has been proved for the case of \(Z = \zeta_i X_i\) in [14]. The proof can be adapted for the general case as well.

**B.3 Proof of Theorem 3.2**

First, we reproduce a slightly modified version of proof Lemma 12 in [14], adapted to our setting. Set

\[
\beta := 93 \gamma_c^2 \frac{2}{3} \min E[\|\Sigma_R\|_{\text{op}}^2].
\]

By \(B\), we denote the bad event defined as

\[
B := \{ \exists A \in C'(\theta, \eta) \text{ such that } \frac{1}{2} \|A\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\mathcal{X}(A)\|_2^2 > \frac{1}{4} \|A\|_{L^2(\Pi)}^2 + \beta \}.
\]

We thus need to bound the probability of this event. Set \(\xi = 6/5\). Then, for \(T > 0\), we define

\[
C'(\theta, \eta, T) := \{ A \in C'(\theta, \eta) \mid T \leq \|A\|_{L^2(\Pi)}^2 < \xi T \}.
\]

Clearly, we have

\[
C'(\theta, \nu) = \bigcup_{l=1}^{\infty} C'(\theta, \eta, \xi^{l-1} \theta).
\]

Now, if the event \(B\) holds for some \(A \in C'(\theta, \eta)\), then \(A \in C'(\theta, \eta, \xi^{l-1} \theta)\) for some \(l \in \mathbb{N}\). In this case, we have

\[
\frac{1}{2} \|A\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\mathcal{X}(A)\|_2^2 > \frac{1}{4} \|A\|_{L^2(\Pi)}^2 + \beta \\
\geq \frac{1}{4} \xi^{l-1} \theta + \beta \\
= \frac{5}{24} \xi^l \theta + \beta.
\]

Next, we define the event \(B_l\) as

\[
B_l := \{ \exists A \in C'(\theta, \eta, \xi^{l-1} \theta) \text{ such that } \frac{1}{2} \|A\|_{L^2(\Pi)}^2 - \frac{1}{n} \|\mathcal{X}(A)\|_2^2 > \frac{5}{24} \xi^l \theta + \beta \}.
\]

It follows that

\[
B \subseteq \bigcup_{l=1}^{\infty} B_l.
\]

The following lemma helps us control the probability that each of these \(B_l\)’s happen.
Lemma B.2. Define
\[ Z_T := \sup_{A \in C'(\theta, \eta, T)} \left\{ \frac{1}{2}\|A\|_{L^2(\Pi)}^2 - \frac{1}{n}\|X(A)\|_2^2 \right\}. \]
Then, assuming that \((X_i)_{i=1}^n\) are i.i.d. samples drawn from \(\Pi\), we get
\[ \mathbb{P}\left( Z_T \geq \frac{5\xi T}{24} + \beta \right) \leq \exp\left( -\frac{Cn\xi T}{\epsilon^2} \right), \tag{B.8} \]
for some numerical constant \(C > 0\).

Proof. We follow the lines of the proof of Lemma 14 in [14]. For a \(d_r \times d_e\) matrix \(A\), define
\[ f(X; A) := \langle X, A \rangle^2 \cdot \mathbb{I}\left( \|X, A\| \leq \epsilon \right). \]
Next, letting
\[ W_T := \sup_{A \in C'(\theta, \eta, T)} \frac{1}{n} \sum_{i=1}^n \left\{ \mathbb{E}[f(X_i; A)] - f(X_i; A) \right\}, \]
\[ \tilde{W}_T := \sup_{A \in C'(\theta, \eta, T)} \left| \frac{1}{n} \sum_{i=1}^n \left[ \mathbb{E}[f(X_i; A)] - f(X_i; A) \right] \right|, \]
it follows from Assumption 3.3 (where \(\epsilon\) is defined) that \(Z_T \leq W_T\), and clearly \(W_T \leq \tilde{W}_T\) hence
\[ \mathbb{P}(Z_T \geq t) \leq \mathbb{P}(\tilde{W}_T \geq t), \]
for all \(t\). Therefore, if we prove (B.8) holds when \(Z_T\) is replaced with \(\tilde{W}_T\), we would be done. In the remaining, we will aim to prove this via Massart’s inequality (e.g. Theorem 3 in [19]). In order to invoke Massart’s inequality, we need bounds for \(\mathbb{E}[\tilde{W}_T]\) and \(\text{Var}(\tilde{W}_T)\).

First, we find an upper bound for \(\mathbb{E}[\tilde{W}_T]\). It follows from the symmetrization argument (e.g. Lemma 6.3 in [18]) that
\[ \mathbb{E}[\tilde{W}_T] \leq 2 \mathbb{E} \left[ \sup_{A \in C'(\theta, \eta, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i f(X_i; A) \right| \right], \tag{B.9} \]
where \((\zeta_i)_{i=1}^n\) is a sequence of i.i.d. Rademacher random variables. Note that Lemma 6.3 of [18] requires the use of a convex function and a norm. Here, the convex function is the identity function and the norm is infinity norm applied to an infinite dimensional vector (indexed by \(A \in C'(\theta, \eta, T)\)).

Next, we will use the contraction inequality (e.g. Theorem 4.4 in [18]). First, we write \(f(X_i; A) = \alpha_i \langle X_i, A \rangle\) where \(\alpha_i = \langle X_i, A \rangle \cdot \mathbb{I}(\|X_i, A\| \leq \epsilon)\). By definition, \(|\alpha_i| \leq \epsilon\). Now, for every realization of the random variables \(X_1, \ldots, X_n\), we can apply Theorem 4.4 in [18] to obtain
\[ \mathbb{E}_\zeta \left[ \sup_{A \in C'(\theta, \eta, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i f(X_i; A) \right| \right] \leq \epsilon \mathbb{E}_\zeta \left[ \sup_{A \in C'(\theta, \eta, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i \langle X_i, A \rangle \right| \right]. \]
Now, taking expectation of both sides with respect to \(X_i\)'s, using the tower property, and combining with (B.9) we obtain
\[ \mathbb{E}[\tilde{W}_T] \leq 8\epsilon \mathbb{E} \left[ \sup_{A \in C'(\theta, \eta, T)} \left| \frac{1}{n} \sum_{i=1}^n \zeta_i \langle X_i, A \rangle \right| \right]. \]
\[ \leq 8c \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \sup_{A \in \mathcal{C}(\theta, \eta, T)} \| A \|_* \right] \]
\[ \leq 8\sqrt{\eta} \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \sup_{A \in \mathcal{C}(\theta, \eta, T)} \| A \|_F \right] \]
\[ \leq 8\sqrt{\eta \gamma_{\min}} \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \sup_{A \in \mathcal{C}(\theta, \eta, T)} \| A \|_{L^2(\Omega)} \right] \]
\[ \leq 8\sqrt{\eta \xi T \gamma_{\min}} \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \right]. \]

In the above, we also used definition of \( \mathcal{C}(\theta, \eta, T) \) as well as Assumption 3.2.

We can, now, use \( 2ab \leq a^2 + b^2 \) to get
\[ \mathbb{E} \left[ \tilde{W}_T \right] \leq \frac{8}{9} \left( \frac{5\xi T}{24} \right) + \frac{87\eta c^2}{\gamma_{\min}} \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \right]^2. \]

Next, we turn to finding an upper bound for the variance of \( \sum_{i=1}^n f(X_i; A) - \mathbb{E} [f(X_i; A)] \):
\[ \text{Var} \left( f(X_i; A) - \mathbb{E} [f(X_i; A)] \right) \leq \mathbb{E} [f(X_i; A)^2] \]
\[ \leq c^2 \mathbb{E} \left[ (X_i, A)^2 \right] \]
\[ \leq c^2 \cdot \| A \|^2_{L^2(\Omega)}. \]

Therefore, we have that
\[ \sup_{A \in \mathcal{C}(\theta, \eta, T)} \frac{1}{n} \text{Var} \left( f(X_i; A) - \mathbb{E} [f(X_i; A)] \right) \leq \frac{c^2}{n} \sup_{A \in \mathcal{C}(\theta, \eta, T)} \| A \|^2_{L^2(\Omega)} \]
\[ \leq \frac{\xi T c^2}{n}. \]

Finally, noting that \( \frac{1}{n} f(X_i; A) \leq \frac{1}{n} c^2 \) almost surely, we can use Massart’s inequality (e.g. Theorem 3 in [19]) to conclude that
\[ \mathbb{P} \left( \tilde{W}_T \geq \frac{5\xi T}{24} + \beta \right) = \mathbb{P} \left( \tilde{W}_T \geq \frac{5\xi T}{24} + \frac{93\eta c^2}{\gamma_{\min}} \mathbb{E} \left[ \left\| \Sigma_R \right\|_{\text{op}} \right]^2 \right) \]
\[ \leq \mathbb{P} \left( \tilde{W}_T \geq \frac{18}{17} \mathbb{E} \left[ \tilde{W}_T \right] + \frac{1}{17} \left( \frac{5\xi T}{24} \right) \right) \]
\[ \leq \exp \left( -C0_n \xi T \right), \]
for some numerical constant \( C_0 > 0 \).

Lemma B.2 entails that
\[ \mathbb{P}(B_l) \leq \exp \left( -C0_n \xi^l \theta \right) \]
\[ \leq \exp \left( -C0_n l \log(\xi) \theta \right). \]

Therefore, by setting the numerical constant \( C > 0 \) appropriately, the union bound implies that
\[ \mathbb{P}(B) \leq \sum_{l=1}^{\infty} \mathbb{P}(B_l) \]

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\[ \sum_{l=1}^{\infty} \exp \left( -\frac{Cnl\theta}{\epsilon^2} \right) = \exp \left( -\frac{Cn\theta}{\epsilon^2} \right) \frac{1}{1 - \exp \left( -\frac{Cn\theta}{\epsilon^2} \right)}. \]

Finally, assuming that \( Cn\theta > \epsilon^2 \), we get that
\[ \mathbb{P}(B) \leq 2 \exp \left( -\frac{Cn\theta}{\epsilon^2} \right), \]
which complete the proof of Theorem 3.2.

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**References**


