Math 19, Midterm 2    February 22nd, 2017

Name: ________________________________

Time: 150 minutes

Do not open this exam until you are told to do so.

To receive full credit, show an appropriate amount of work for each problem. Graders must be able to see not only your answer but how you obtained it. This work could be a sequence of algebraic calculations or a couple of sentences, whichever feels more appropriate for the problem.

THIS EXAM IS DOUBLE-SIDED!

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You may not communicate with anyone other than the instructors/CAs during the exam, or look at anyone else’s solutions.

I understand and accept these instructions.

Signature: ________________________________

Grade table (for instructor use only)

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LIMIT LAWS

Suppose that $a$ and $c$ are any two constants. Then the following limits hold.

1. $\lim_{x \to a} c = c$

2. $\lim_{x \to a} x = a$

Suppose that $f(x)$ and $g(x)$ are any two functions for which $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then the limits on the left hand side of the following equations exist and can be computed by the corresponding formulae on the right hand side.

3. $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} (f(x) + g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$

5. $\lim_{x \to a} (f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$

6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$

Suppose that $h(x)$ is any function which is continuous at the point $\lim_{x \to a} f(x)$. Then the limit of the left hand side of the following equation exists and can be computed by the formula on the right hand side.

7. $\lim_{x \to a} h(f(x)) = h(\lim_{x \to a} f(x))$
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1. (25 points) This problem asks you to calculate the derivatives of various functions. You may perform the calculation in whichever way you wish to, as long as you get the correct answer (partial credit will be given for using adequate techniques).

(a) \[ \frac{d}{dx} \left( \sin(2^x + 3^x) \right) \]

(b) \[ \frac{d}{dx} \left( \frac{2x^5 - x^3 + 1}{1 - x^2} \right) \]
(c) \( \frac{d}{dx} \left( \tan(x) \cdot e^{x^2} \right) \)

(d) \( \frac{d}{dx} \left( \sqrt{x^2 + \sqrt{x^2 + 1}} \right) \)
(e) \( \frac{d}{dx}(x^{2x^2+3}) \)
2. (25 points) This problem focuses on the limit definition of the derivative.

(a) Consider the function \( f(x) = x^2 + 3x + 1 \). Show by explicit computation that \( f'(x) = 2x + 3 \). To be clear, you are not allowed to use the formula for the derivative of a polynomial. Instead, you must compute \( f'(x) \) directly using the limit definition of the derivative

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

At each and every step of your computation you should either be performing a basic algebraic manipulation or using one of the limit laws (which you can find on the second page of this examination booklet). Whenever you do use a limit law, be sure to indicate which one you’re using.
(b) Using the limit definition for the derivative, show that if \( f(x) \) and \( g(x) \) are two differentiable functions, then the function \( f(x) + g(x) \) is differentiable and its derivative is \( f'(x) + g'(x) \). As in part (a), at each and every step of your answer you should be performing a basic algebraic manipulation or using one of the limit laws. Whenever you do use a limit law, be sure to indicate which one you’re using.
3. (25 points) This problem explores the relation between continuity and differentiability. We begin by revisiting the limit definitions of these two notions. Recall that a function \( f(x) \) is said to be continuous at a point \( x = a \) if the limit
\[
\lim_{x \to a} f(x)
\]
exists and equals \( f(a) \). Equivalently, we can define \( f(x) \) to be continuous at \( x = a \) if we have
\[
\lim_{x \to a} (f(x) - f(a)) = 0.
\]
This is the same condition as before because the limit laws imply that
\[
\lim_{x \to a} (f(x) - f(a)) = (\lim_{x \to a} f(x)) - (\lim_{x \to a} f(a)) = (\lim_{x \to a} f(x)) - f(a).
\]
Recall also that \( f(x) \) is said to be differentiable at \( x = a \) if the limit
\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]
exists. When this limit exists, we call it the derivative of \( f(x) \) at \( x = a \) and denote it by the symbol \( f'(a) \). Using the substitution \( x = a + h \), we can equivalently define \( f(x) \) to be differentiable at \( x = a \) if the limit
\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]
exists. Note that this is the same limit \( f'(a) \) as before since demanding that \( h \to 0 \) is equivalent to demanding that \( x \to a \).

(a) (Differentiability implies continuity) Using the limit laws (which you can find on the second page of this examination booklet), prove that if a function \( f(x) \) is differentiable at a point \( x = a \), then it must also be continuous at \( x = a \). (Hint: use the equivalent definitions of continuity and differentiability given above)
(b) (Continuity does not imply differentiability) Write down an explicit example of a function $f(x)$ which is continuous at a point but is not differentiable at that same point. Use the limit definitions of continuity and differentiability to justify that your example works.
4. (25 points) In this problem we will find an approximation for the number $\pi$ using the fact that $\sin(\pi/6) = 1/2$. We take the arcsin function to have domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

(a) Use the chain rule and the fact that $\sin(\arcsin(x)) = x$ to show that
\[
\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}.
\]

(b) Find the linearization of the function $f(x) = 6\arcsin(x)$ at the point $x = 0$. In other words, calculate the tangent line to the curve $y = 6\arcsin(x)$ at the point $(x, y) = (0, 0)$. You may use the formula for the derivative of $\arcsin(x)$ given in part (a) even if you weren’t able to derive it.
(c) Use the linearization that you wrote down in part (b) to estimate the number 
\[ \pi = 6 \arcsin\left(\frac{1}{2}\right). \]

(d) Is your approximation for \( \pi \) an overestimate or an underestimate? Justify your an-
swer by studying the concavity of the function \( f(x) = 6 \arcsin(x) \). A picture helps 
but you must also give a mathematical argument that supports your conclusion.