Solutions to Math 19 Practice Final B, Winter 2011

1. (10 points) Find each of the following limits, with justification (show steps). If there is an infinite limit, then explain whether it is $\infty$ or $-\infty$.

(a) (5 points) $\lim_{x \to 1} \left( \frac{x^2 - 3x + 2}{3x^2 - 21x + 18} - 5 \right)$

If we plug in $x = 1$, we get $0 - 5$, so we need to manipulate:

$$\lim_{x \to 1} \left( \frac{x^2 - 3x + 2}{3x^2 - 21x + 18} - 5 \right) = \lim_{x \to 1} \left( \frac{(x-1)(x-2)}{3(x-1)(x-6)} - 5 \right) = \lim_{x \to 1} \left( \frac{x-2}{3(x-6)} - 5 \right) = \frac{-1}{-15} - 5 = \frac{-74}{5}$$

(b) (5 points) $\lim_{x \to \infty} (\sqrt{4x^2 - x} - 2x)$

This is an indetermination of the form $\infty - \infty$. We deal with it by multiplying and dividing by the conjugate.

$$\lim_{x \to \infty} (\sqrt{4x^2 - x} - 2x) = \lim_{x \to \infty} \frac{(\sqrt{4x^2 - x} - 2x)(\sqrt{4x^2 - x} + 2x)}{\sqrt{4x^2 - x} + 2x} = \lim_{x \to \infty} \frac{4x^2 - x - 4x^2}{\sqrt{4x^2 - x} + 2x} = \lim_{x \to \infty} \frac{-x}{\sqrt{4x^2 - x} + 2x} = \lim_{x \to \infty} \frac{-1}{\sqrt{4 - \frac{1}{x}} + 2} = \frac{-1}{4}$$
2. (10 points)  (a) (3 points) What are the critical numbers of a function $f(x)$?

The critical numbers of a function $f(x)$ are the numbers $a$ in the domain such that $f'(a) = 0$ or such that $f'(a)$ does not exist.

(b) (8 points) Find the global maximum and the global minimum of the function $f(x) = (x - 2)e^{-x}$ on the domain $[0, 10]$.

The domain is given, it is $[0, 10]$. The function is continuous on $[0, 10]$, so we can use the closed interval method.

Now we find the critical numbers.

$f'(x) = e^{-x} + (x - 2)(-e^{-x}) = e^{-x}(1 - x + 2) = e^{-x}(3 - x)$

$f'(x) = 0$ when $x = 3$ (because $e^{-x}$ can not be zero)

There are no more critical numbers because the function is differentiable on $[0, 10]$

$f(0) = -2$

$f(10) = 8e^{-10} = \frac{8}{e^{10}} < \frac{8}{2^{10}} = \frac{2^3}{2^{10}} = \frac{1}{2^7} = \frac{1}{128}$

$f(3) = e^{-3} = \frac{1}{e^3} > \frac{1}{3\pi} = \frac{1}{27}$

The maximum is $e^{-3}$ and the minimum is $-2$

Alternative solution without estimating for the last part:

Using logic and the stuff we use to graph, we see that the derivative is negative if $x > 3$, so the function is decreasing on the interval $(3, 10)$, and so $f(10)$ is smaller than $f(3)$. 

3. (8 points) Mark each statement below as true or false by circling T or F. No justification is necessary.

**T**  **F**  The function \( f(x) = 2x + \ln(x) \) is one-to-one

*True*

The functions \( y = 2x \) and \( y = \ln(x) \) are increasing, so \( f(x) \) is also increasing and thus one-to-one

**T**  **F**  If \( \lim_{x \to 0} f(x) = \infty \) and \( \lim_{x \to 0} g(x) = \infty \), then \( \lim_{x \to 0} [f(x) - g(x)] = 0 \)

*False*

It is an indetermination. It may be zero, but it may be anything really. For example, if \( f(x) = x \) and \( g(x) = x^2 \), then the third limit is \(-\infty\)

**T**  **F**  The function \( \ln(x) \) is even

*False*

\( \ln(x) \) is not defined for negative numbers.

**T**  **F**  The function \( |x - 7| \) is differentiable everywhere

*False*

It is not differentiable at \( x = 7 \). There is a corner there. In fact, if \( f(x) = |x - 7| \), then

\[
\lim_{h \to 0^-} \frac{f(7 + h) - f(7)}{h} = -1
\]

\[
\lim_{h \to 0^+} \frac{f(7 + h) - f(7)}{h} = 1
\]

Try to do those!

**T**  **F**  If \( f''(1) > 0 \), then \( f'(x) \) is increasing at \( x = 1 \)

*True*

The second derivative is the derivative of \( f'(x) \), and positive derivative implies increasing function.

**T**  **F**  If \( f'(2) = 0 \), then \( f(x) \) has a local maximum or a local minimum at \( x = 2 \)

*False*
For example, the function \( f(x) = (x - 2)^3 \) satisfies \( f'(2) = 0 \), but it does not have a local maximum or minimum at \( x = 2 \).

**T  F**  If two functions have the same derivative, then they must be equal

False

For example, the functions \( f(x) = x \) and \( g(x) = x + 1 \) have the same derivative, but they are not equal.

**T  F**  The derivative of \( \ln(x^3) \) is \( \frac{3}{x} \)

True

\[
[\ln(x^3)]' = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}
\]
4. (11 points) For the function \( f(x) = \begin{cases} \frac{x+2}{|x+2|} & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ 3x^2 + 5x - 3 & \text{if } x > -2 \end{cases} \)

(a) (6 points) Find \( \lim_{x \to -2} f(x) \) (if it exists).

We need to do the side limits.

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x + 2}{|x + 2|} = 1
\]

Because \( x \to -2^- \), that means \( x < -2 \implies x + 2 < 0 \implies |x + 2| = -(x + 2) \)

\[
= \lim_{x \to -2^-} \frac{x + 2}{-(x + 2)} = -1
\]

\[
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (3x^2 + 5x - 3) = 12 - 10 - 3 = -1
\]

Therefore \( \lim_{x \to -2} f(x) = -1 \)

(b) (3 points) Is \( f(x) \) continuous at \( x = -2 \)? Justify your answer.

\( f(-2) = 3 \) is not equal to \( \lim_{x \to -2} f(x) = -1 \), so the function is not continuous at \( x = -2 \)
5. (25 points) Consider the function \( f(x) = \frac{(x + 1)^2}{x^2 + 1} \)

(a) (2 points) Find the domain of \( f(x) \).

The denominator cannot be zero, so the domain is \((-\infty, \infty)\)

(b) (5 points) Where is \( f(x) \) increasing? And decreasing? Find the \( x \)-coordinates of all the local maxima and minima of \( f(x) \).

\[
f'(x) = \frac{2(x + 1)(x^2 + 1) - 2x(x + 1)^2}{(x^2 + 1)^2} = \frac{2x^3 + 2x + 2x^2 + 2 - 2x^3 - 2x - 4x^2}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}
\]

\( f'(x) = 0 \) when \( 2 - 2x^2 = 0 \) \( \Rightarrow x = \pm 1 \)

Our chart now would have two points, 1 and -1

\[
f'(-2) = \frac{-6}{25} < 0
\]

\[
f'(0) = \frac{2}{1} > 0
\]

\[
f'(2) = \frac{-6}{25} < 0
\]

So \( f(x) \) is increasing on \((-1, 1)\) and is decreasing on \((-\infty, -1) \cup (1, \infty)\)

Therefore there is a local minimum at \( x = -1 \) and a local maximum at \( x = 1 \)

For convenience later, we find \( f(-1) = 0 \) and \( f(1) = 2 \)

(c) (5 points) It is known that \( f''(x) = \frac{4x^3 - 12x}{(x^2 + 1)^3} \). Where is \( f(x) \) concave up? And concave down? Find the \( x \)-coordinates of all the inflection points of \( f(x) \).

\[
f''(x) = 0 \text{ when } 4x^3 - 12x = 0 \Rightarrow x(4x^2 - 12) = 0 \Rightarrow x = 0 \text{ or } x = \pm \sqrt{3}
\]

So now our chart would have three points, \(-\sqrt{3}\), 0 and \( \sqrt{3} \)

\[
f''(-2) = \frac{-32 + 24}{125} < 0
\]

\[
f''(-1) = \frac{-4 + 12}{8} > 0
\]

\[
f''(1) = \frac{4 - 12}{8} < 0
\]

\[
f''(2) = \frac{32 - 24}{125} > 0
\]
So \( f(x) \) is concave up on \((-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)\) and concave down on \((-\infty, -\sqrt{3}) \cup (0, \sqrt{3})\).

There are inflection points at \( x = -\sqrt{3} \), \( x = \sqrt{3} \) and \( x = 0 \).

For convenience later, we find \( f(0) = 1 \), \( f(-\sqrt{3}) = \frac{(1 - \sqrt{3})^2}{4} \) and \( f(\sqrt{3}) = \frac{(1 + \sqrt{3})^2}{4} \). (Well, actually these last two are not very convenient.)
(d) (5 points) Find all the asymptotes of \( f(x) \).

\[
\lim_{x \to \infty} \frac{(x+1)^2}{x^2+1} = \lim_{x \to \infty} \frac{x^2 + 2x + 1}{x^2 + 1} = \\
\lim_{x \to \infty} \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 1
\]

So \( y = 1 \) is a horizontal asymptote.

\[
\lim_{x \to -\infty} \frac{(x+1)^2}{x^2+1} = \lim_{x \to -\infty} \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 1
\]

This asymptote is also present at \(-\infty\).

There are no vertical asymptotes because the function is continuous everywhere.

(e) (2 points) Find the \( y \)-intercept of \( f(x) \) and the \( x \)-intercepts (if there are any).

\( f(0) = 1 \), so the \( y \)-intercept is the point \((0, 1)\)

\( f(x) = 0 \) when \((x+1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1\)

So there is only one \( x \)-intercept, \((-1, 0)\)
(f) (6 points) Sketch the graph of $f(x)$ in the axes provided below.
6. (14 points) Find the derivatives of the following functions using any method you like (that works). You do not need to simplify your answers.

(a) (4 points) \( f(x) = \ln(x^4 - 5) \cdot e^{(e^x)} \)

\[
f'(x) = \frac{4x^3}{x^4 - 5} \cdot e^{(e^x)} + \ln(x^4 - 5) \cdot e^{(e^x)} \cdot e^x
\]

(b) (6 points) \( g(x) = [\arcsin(\sqrt{x})]^x \)

We use logarithmic differentiation

\[
\ln(g(x)) = \ln([\arcsin(\sqrt{x})]^x) = x \ln(\arcsin(\sqrt{x}))
\]

\[
g'(x) = \frac{g(x)}{g(x)} \left[ \ln(\arcsin(\sqrt{x})) + x \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{\arcsin(\sqrt{x})} \right]
\]

\[
g'(x) = \frac{g(x)}{g(x)} \left[ \ln(\arcsin(\sqrt{x})) + x \cdot \frac{1}{2 \sqrt{x} \sqrt{1-x}} \right]
\]

\[
g'(x) = [\arcsin(\sqrt{x})]^x \left[ \ln(\arcsin(\sqrt{x})) + \frac{x}{2 \sqrt{x} \sqrt{1-x} \arcsin(\sqrt{x})} \right]
\]

(c) (4 points) \( Z(x) = \frac{e^{1/x}}{x^2} \)

\[
Z'(x) = \frac{e^{1/x} \cdot \frac{1}{x^2} \cdot x^2 - e^{1/x} \cdot 2x}{x^4}
\]
7. (8 points) We know that \( g'(3) = 2 \) and \( g(3) = 4 \). Estimate the value of \( \frac{1}{g(3.1)} \) using linear approximation. Please simplify your answer.

We know \( \frac{1}{g(3)} = \frac{1}{4} \)

So we are going to use \( f(x) = \frac{1}{g(x)} \), \( a = 3 \) and \( x = 3.1 \)

\[
    f'(x) = \frac{-g'(x)}{g(x)^2}
\]

\[
    f'(3) = \frac{-g'(3)}{g(3)^2} = \frac{-2}{16} = \frac{-1}{8}
\]

Now linear approximation says \( f(3.1) \approx f(3) + f'(3)(3.1 - 3) \)

\[
    \frac{1}{g(3.1)} \approx \frac{1}{4} + \frac{-1}{8} \cdot \frac{1}{10} = \frac{19}{80}
\]
8. (14 points) A student can not use a calculator for the final exam, so every time he has to do a calculation with $\pi$, he uses 3 instead. Assume that the maximum error is 0.15 in this quantity then.

(a) (6 points) Using differentials, estimate the maximum error in his computation of the area of a circle of radius 4 cm. Your answer should include the units.

The area is $A = \pi r^2$

Now $r$ is known to be 4, so $A = 4\pi$. Since we are changing $\pi$ by 3, we need to use the function $A(x) = 4x$.

In this case, $dx = 0.15$

$dA = A'(3)dx = 4 \cdot 0.15 = 0.6$ cm$^2$

The maximum error in computing the area is 0.6 cm$^2$

(b) (8 points) He needs to know which number is bigger, $\frac{\pi^2}{2}$ or $\pi + 1.6$, so he changes $\pi$ by 3, and gets to the conclusion that the second number must be bigger. Using differentials, find the possible error in each of these approximations and explain why his conclusion may not be true. (Hint: The real number will be in the interval centered at the value computed using 3 and with length given by twice the propagated error.

We use the function $y = \frac{x^2}{2}$ for $\frac{\pi^2}{2}$.

$y' = x$

$dy = y'(3)dx = 3 \cdot 0.15 = 0.45$

So the possible error when computing $\frac{\pi^2}{2}$ is 0.45

Using 3 instead of $\pi$, we would obtain 4.5

This means that $\frac{\pi^2}{2}$ must be in the interval $[4.5 - 0.45, 4.5 + 0.45] = [4.05, 4.95]$

Now for $\pi + 1.6$ we use $z = x + 1.6$

$z' = 1$

$dz = z'(3)dx = 1 \cdot 0.15 = 0.15$

So the possible error when computing $\pi + 1.6$ is 0.15

Using 3 instead of $\pi$, we would obtain 4.6

This means that $\pi + 1.6$ must be in the interval $[4.6 - 0.15, 4.6 + 0.15] = [4.45, 4.75]$. 
These two intervals intersect, which means it could be possible for example that $\pi + 1.6$ was 4.45 and $\frac{\pi^2}{2}$ was 4.8. That is why his conclusion may not be true.
9. (10 points) Find the coordinates \((x, y)\) of the two points in the curve \(x^2 - xy + y^2 = 1\) where the tangent line is horizontal.

We use implicit differentiation

\[
2x - y - xy' + 2yy' = 0
\]

\[
(2y - x)y' = y - 2x
\]

\[
y' = \frac{y - 2x}{2y - x}
\]

The tangent line is horizontal where \(y' = 0\) \(\Rightarrow y = 2x\)

Since the points are on the curve, they must also satisfy the initial equation

\[
x^2 - x \cdot 2x + (2x)^2 = 1
\]

\[
x^2 - 2x^2 + 4x^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}}
\]

So the points are \(\left( \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right)\) and \(\left( \frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}} \right)\)
10. (15 points) A company has three processing plants near a city. One of them is located 1 km north of the city, another 1 km south of the city and the third one is located 3 km east of the city. The company plans to construct a distribution center at some point in between the city and the third factory (or even at the city or at the third factory). Where should the center be located in order to minimize the sum of the distances to the three processing plants?

We want to minimize the sum of the distances $D = a + b + c$

$$D(x) = 2\sqrt{1 + x^2} + 3 - x$$

The domain of this function is $[0, 3]$

This function is continuous on $[0, 3]$, so we can use the closed interval method.

$$D'(x) = \frac{4x}{2\sqrt{1 + x^2}} - 1$$

$D'(x) = 0$ when $\frac{4x}{2\sqrt{1 + x^2}} - 1 = 0 \Rightarrow 4x = 2\sqrt{1 + x^2}$
\[2x = \sqrt{1 + x^2} \Rightarrow 4x^2 = 1 + x^2 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}}\]

The negative solution is not valid because it is not in the domain.

There are no more critical numbers because the function is differentiable on \([0, 3]\)

\[D(0) = 5\]

\[D(3) = 2\sqrt{10} > 6\]

\[D\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{1 + \frac{1}{3}} + 3 - \frac{1}{\sqrt{3}} = 2\sqrt{\frac{4}{3}} + 3 - \frac{1}{\sqrt{3}} =\]

\[= \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} + 3 = \frac{3}{\sqrt{3}} + 3 = \sqrt{3} + 3 < 5\]

So the center should be located \(\frac{1}{\sqrt{3}}\) km east of the city.
11. (12 points). A certain population of rodents numbers

\[ P = 100 + 30 \cdot 3^t - 4 \cdot 3^{2t} \]

after \( t \) years.

(a) (1 point) What was the original population?

\[ P(0) = 100 + 30 - 4 = 126 \text{ rodents.} \]

(b) (6 points) What is the rate of growth of the population after 1 year? And after 2 years? Include the units in your answers. Interpret the sign of your answers.

\[ P'(t) = 30 \cdot 3^t \cdot \ln(3) - 4 \cdot 3^{2t} \cdot 2 \cdot \ln(3) = \ln(3)(30 \cdot 3^t - 8 \cdot 3^{2t}) \]

\[ P'(1) = \ln(3)(90 - 72) = 18 \ln(3) \]

The rate of growth of the population after 1 year is 18 \( \text{ln}(3) \) rodents/year. So the population is increasing.

\[ P'(2) = \ln(3)(270 - 648) = -378 \ln(3) \]

The rate of growth of the population after 1 year is \(-378 \text{ln}(3) \) rodents/year. So the population is decreasing.

(c) (5 points) The rodents are placed in a controlled environment where the only animals are them and rodent-eating eagles. It is known that the population of rodents still follows the same formula and the total number of animals is always constant. Is the population of eagles growing or decaying after 1 year? Justify your answer.

Let us call \( Q(t) \) the population of rodent-eating eagles. Then, \( P(t) + Q(t) = a \), where \( a \) is just a number.

\[ P'(t) + Q'(t) = 0 \Rightarrow Q'(t) = -P'(t) \]

So \( Q'(1) = -P'(1) = -18 \text{ln}(3) < 0 \), so the population of eagles is decreasing.