The Final
Math 19 2009

• Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.

• This is a closed-book, closed-notes exam. No calculators or other electronic aids permitted.

• In order to receive full credit, you must show all of your work and justify your answers. Your answer should be clearly labeled.

• The following formulas may be of use to you:

\[
\begin{align*}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B. \\
\sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A. \\
\sin(2A) &= 2 \sin A \cos A. \\
\cos(2A) &= \cos^2(A) - 1.
\end{align*}
\]

• Please sign the following:

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Name: ________________________________

Signature: ______________________________

1. _________ (/20 points)

2. _________ (/5 points)

3. _________ (/25 points)

4. _________ (/10 points)

5. _________ (/10 points)

6. _________ (/10 points)

7. _________ (/10 points)

8. _________ (/10 points)

Bonus _________ (/5 points)

Total. _________ (/100 points)
(a) (5 points) State the limit definition of the derivative of a function $f$ at $x$.

(b) (5 points) Using your definition from (a), compute the derivative of $f(x) = \sqrt{x + 2}$ at $x = 2$. No other method will receive credit.
(c) (5 points) Now check your answer from (b) by computing the derivative of \( f(x) = \sqrt{x + 2} \) at \( x = 2 \) using any of the differentiation rules that apply.

(d) (5 points) Find the equation of the tangent line to the curve \( y = f(x) = \sqrt{x + 2} \) at \( x = 2 \).
2 (5 points) Suppose the tangent line to the graph of $y = f(x)$ at $x = -1$ is $y = x + 2$. What are the values of $f(-1)$ and $f'(-1)$? (note: you have all of the information needed to solve this problem. A sketch might help.)
3 (25 points) Compute the limit using any appropriate method. Show your work as much as possible.

(a) \[
\lim_{x \to -1} \frac{x^2 - 2x - 3}{x^2 + 3x + 2}
\]

(b) \[
\lim_{x \to 0^+} \frac{1}{x - x^2}
\]

(c) \[
\lim_{x \to +0} x \ln(x)
\]
(d) \[ \lim_{x \to 2} f(x), \]

where

\[ f(x) = \begin{cases} 
  x^2, & x < 2 \\
  5, & x = 2 \\
  3x - 2, & x > 2 
\end{cases} \]

(e) \[ \lim_{x \to -\infty} \frac{\sqrt{x^2 + 1}}{x - 1}. \]
(10 points) Compute the derivative of each of the functions below using any applicable rules.

(a) (5 points) \( y = \tan(\ln(2x)) \).

(b) (5 points) \( y = e^{\sin(x) + x^2} \).
(10 points) Produce an accurate sketch of the function below. Label all interesting behaviour including intercepts, asymptotes, intervals of increase/decrease, find and classify all critical points, and show concavity as appropriate.

\[ y = xe^{-x}. \]
(10 points) Gravel is being dumped from a conveyor belt at a rate of 30 ft³/min, and its coarseness is such that it always forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing then the pile is 10 feet high?
A farmer has 200 meters of fencing and wants to fence off a rectangular field that borders a straight river. She needs no fence along the river. What are the dimensions that yield the maximum area? Justify that your answer is indeed a maximum.
(a) What is the absolute maximum of \( y = x^2 \) on \((0, 1)\)?

(b) Suppose \( f'(x) > 0 \) on \((0, \infty)\), then what is the relationship between \( f(3) \) and \( f(7) \)? (i.e. choose one of \(<\), \(\geq\), \(\leq\), \(\geq\) or can’t tell?)

(c) True or false: if \( c \) is a critical number of \( f(x) \), then \( c \) is a either a local maximum or local minimum.

(d) True or false: if \( f \) is differentiable at \( x = a \), then \( f \) is continuous at \( x = a \).

(e) Compute \( \cos(\sin^{-1}(3/5)) \).
[Bonus!! up to 5 points] Solve one of the following problems. Circle which you want graded. If you make no choice, the grader will grade the first problem they see.

A. Prove that every polynomial with odd degree has at least one real root.

** OR **

B. Is there a positive solution to $x^3 - 3x + 10 = 0$? Explain why or why not using techniques from class.