Practice Final Exam: Winter 2007

1. (40 points) Find $\frac{dy}{dx}$ for each function. Each answer should be a function of $x$ only.

(a) (10 points) $y = \frac{2}{x - 1} - \frac{1}{\sqrt{x}}$.

(b) (10 points) $y = (\sin x)^{\cos x}$.

(c) (10 points) $y = \sqrt{\tan(x^2)}$.

(d) (10 points) $y = \frac{(2x + 1)^4 \sin(x^2)}{(\ln x) \sqrt{3x - 1}}$.

2. (10 points) Find the equation of the tangent line to the curve $e^{x^2} + e^{y^2} = 2e$ at the point $(-1, 1)$.

3. (20 points) Let $f(x) = \ln(x^2 - 1)$.

(a) (10 points) You must show all your work, but please write your final answers in the box.

<table>
<thead>
<tr>
<th>The domain of $f(x)$ is:</th>
<th>______________________</th>
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<tbody>
<tr>
<td>$f(x)$ is increasing on:</td>
<td>______________________</td>
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<tr>
<td>$f(x)$ is decreasing on:</td>
<td>______________________</td>
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<tr>
<td>$f(x)$ has local maxima at:</td>
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<tr>
<td>$f(x)$ has local minima at:</td>
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<tr>
<td>$f(x)$ is concave up on:</td>
<td>______________________</td>
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<tr>
<td>$f(x)$ is concave down on:</td>
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(b) (4 points) Compute the following four limits.

$$\lim_{x \to \infty} \ln(x^2 - 1) = \quad \lim_{x \to -\infty} \ln(x^2 - 1) =$$

$$\lim_{x \to 1^+} \ln(x^2 - 1) = \quad \lim_{x \to -1^-} \ln(x^2 - 1) =$$
(c) (1 points) List all vertical and horizontal asymptotes of \( y = \ln (x^2 - 1) \).

(d) (5 points) Using your answers from parts (a) and (b), sketch a graph of

\[
f(x) = \ln (x^2 - 1).
\]

Even if your answers in parts (a) and (b) are wrong, if your sketch correctly uses those answers, you may earn partial credit.

4. (20 points) A particle is moving along the curve \( x^2 - 4xy - y^2 = -11 \). Given that the \( x \)-coordinate of the particle is changing at 3 units/second, how fast is the distance from the particle to the origin changing when the particle is at the point \((1, 2)\)? Hint: As an intermediate step, you should compute the value of \( \frac{dy}{dt} \) when \( x = 1 \) and \( y = 2 \).

5. (20 points) A balloon is rising at a constant speed of 1 m/sec. A girl is cycling along a straight road at a speed of 2 m/sec. When she passes under the balloon it is 3 m above her. How fast is the distance between the girl and the balloon increasing 2 seconds later?

6. (20 points) A Norman window consists of a rectangle surmounted by a semicircle, as shown. Given that the total area of the window is \( A = 8 + 2\pi \), find the minimum possible perimeter of the window. (Please note the horizontal line between the rectangle and the semicircle does not count as part of the perimeter.) Hint: The total area has been carefully chosen so that the minimum perimeter occurs at a very simple value of \( r \). If your optimal value of \( r \) is complicated, you have done something incorrectly.

7. (20 points) Suppose you have a cone with constant height \( H \) and constant radius \( R \), and you want to put a smaller cone “upside down” inside the larger cone (see figure). If \( h \) is the height of the smaller cone, what should \( h \) be to maximize the volume of the smaller cone? The optimal value of \( h \) will depend on \( H \). Recall that the volume of a cone with base radius \( r \) and height \( h \) is given by the formula \( V = \frac{1}{3} \pi r^2 h \).

8. (10 points) For parts (a) and (b), compute the given limits, if they exist. If you assert that a limit does not exist, you need to justify your answer to get full credit.

(a) (5 points) \( \lim_{x \to \infty} (\sqrt{x^2 - 3x + 1} - \sqrt{x^2 + 2}) \)

(b) (5 points) \( \lim_{x \to 2} e^{\frac{1}{x-2}} \)