Practice Final

1. Determine whether each statement is true or false. Unless otherwise stated, any function below is arbitrary. If the statement is true, very briefly cite your reasoning. If it is false, provide an example showing the statement to be false.

(a) True or False. If $f'(x) < 0$ for $1 < x < 6$, then $f(x)$ is decreasing on $(1,6)$.

(b) True or False. If $f(x)$ has an absolute minimum value at $x = c$, the $f'(c) = 0$.

(c) True or False. $f'(x)$ has the same domain as $f(x)$.

(d) True or False. If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

(e) True or False. A function can have two different horizontal asymptotes.

(f) True or False. $\lim_{x \to 4} \left( \frac{2x}{x - 4} - \frac{8}{x - 4} \right) = \lim_{x \to 4} \frac{2x}{x - 4} - \lim_{x \to 4} \frac{8}{x - 4}$. 

1
2. Complete the following sentence:
The function \( f(x) \) is continuous on the interval \([a, b]\) if

3. Compute the following limits. Justify your results. If the limit is positive or negative infinity, it should be clearly indicated instead of just saying the limit does not exist

   (a) \( \lim_{x \to \pi} \cos(x + \sin(x)) \)

   (b) \( \lim_{x \to 1} \frac{2 - x}{(x - 1)^2} \)

   (c) \( \lim_{x \to -4} \frac{1}{4} + \frac{1}{x} \)
4. Use the limit definition of the derivative to compute the derivative of \( f(x) = \sqrt{1 + 2x} \).
5. Compute the following derivatives. You do not have to use the definition of the derivative. If you can “do them in your head” instead of showing every step that is up to you (though if you get it wrong we cannot give you partial credit.)

(a) Let \( f(x) = 3^x \ln(x) \). Find \( f'(x) \).
(d) Find \( \frac{d}{dx} \left( \frac{(4x - 1)^3}{(2x^2 - 1)^{3/2}(x + 1)^2} \right) \).

(e) Find \( \frac{d}{dx}(x^2 \cos(x)e^{3x} \sin(\pi/2)) \).
6. Find the equation of the line tangent to

\[ y = (2 + x)e^{-x} \]

at the point \((0, 2)\)

7. Show that there exists a solution to the equation \(\ln(x) = \sin\left(\frac{\pi}{2}x\right)\) on the interval \((0, \infty)\).
8. Consider the function \( f(x) = \frac{1}{1-x^2} \).

(a) Write the domain of \( f(x) \) in interval notation and find the coordinates of all points, if any, where the graph of \( f \) crosses the \( x \)-axis.

(b) Find the equations for all of the vertical and horizontal asymptotes of this function, or state why there are none. Justify asymptotes with limit calculations.
(c) Compute $f'(x)$ and $f''(x)$. You do not have to use the limit definition of the derivative.

(d) Determine the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.
(e) Determine intervals where \( f(x) \) is concave up and the intervals where \( f(x) \) is concave down.

(f) Find all the critical points of \( f(x) \) and determine which are actually local maxima and which are actually local minima (for each, be sure to justify why it is a max/min or neither.)
(g) Using your answers from parts (a) through (f), Sketch a graph of this function. Be sure to label your axes appropriately.
9. For each of the following conditions, provide an example of a function $f(x)$ which satisfies the given condition. Unless otherwise indicated, you may express you function either with an explicit formula or by a graph.

(a) $f(x)$ has neither an absolute minima nor an absolute maxima on its domain.

(b) $\lim_{h \to 0} f(h) = f(0)$ yet $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

(c) $f(x)$ satisfies $f(x) < 0$, $f'(x) < 0$, and $f''(x) < 0$ for all $x$. 
10. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2 \text{ cm}^3/\text{sec}$, how fast is the water level rising when the water is 5 cm deep? (Note: the volume of a cone with height $h$ and radius $r$ is given by $V = \frac{1}{3} \pi r^2 h$.)
11. Find the equation of the line through the point (3,5) that cuts off the least area from the first quadrant. Justify your answer completely.