NAME:

LECTURE:

Time: 3 hours

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ________________________________

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Some derivatives

- \[ \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}} \]
- \[ \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}} \]
- \[ \frac{d}{dx} \arctan x = \frac{1}{1 - x^2} \]
Problem 1 : (10 points)

a) (4 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable everywhere. Write the definition of the linearization of $f$ at the point $a$.

b) (3 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by the rule $f(x) = 2x + 3$. Find the linearization of $f$ at $a = 4$.

c) (3 points) Now suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by the rule $f(x) = mx + b$, for some real numbers $m$ and $b$. Find the linearization of $f$ at $a$. Simplify your answer.
Problem 2: (8 points) Compute or simplify the following quantities, as appropriate.

a) (2 points) \( \frac{3^{\sqrt[3]{b}}a^2}{b^{\sqrt[3]{a}}} \)

b) (2 points) \( \frac{x^{-s+1}(x^2)^{s+t}}{x^{-4t-2}} \)

c) (2 points) \( e^{5\ln x + \ln(x-1) - \ln(x-2)} \)

d) (2 points) \( \sin \left( \arctan \frac{2}{3} \right) \)
Problem 3 : (12 points) Compute the following limits:

a) (3 points) For this limit only, graph the basic function and use it to determine what the limit is: \( \lim_{x \to \infty} \ln x \)

b) (3 points) \( \lim_{x \to 2} \frac{x^2 - 2x - 3}{x^2 + x - 6} \)
c) (3 points) \[ \lim_{\theta \to 0} \frac{\sin 6\theta}{\theta} \]

d) (3 points) Let \( f : [0, 2] \to \mathbb{R} \). Assume that \( 3x \leq f(x) \leq x^3 + 2 \) for all \( x \) in the domain of \( f \). Calculate \( \lim_{x \to 1} f(x) \).
Problem 4: (12 points) In each of the following problems, let $y$ be a function of $x$. Compute the rule for $\frac{dy}{dx}$.

a) (3 points) $y: \mathbb{R}\setminus\{0\} \to \mathbb{R}, \ y = \frac{e^x}{x}$

b) (3 points) $y: [-1, 1] \to \mathbb{R}, \ y = (\arccos x)^{1/2}$
c) (3 points) $y: \mathbb{R} \to \mathbb{R}$, $xy^2 + e^y = x^3$

d) (3 points) $y: (5/3, \infty) \to \mathbb{R}$, $y = \frac{(5x - 3)^{11}(2x^3 + 1)^{1/2}}{(-3x + 5)^3}$
Problem 5: (13 points) Consider the function \( f: [-10, 3\pi] \to \mathbb{R} \) given by the rule

\[
    f(x) = \begin{cases} 
        2x + 8 & \text{if } -10 \leq x \leq -2, \\ 
        x^2 & \text{if } -2 < x \leq \pi/2, \\ 
        \cos x & \text{if } \pi/2 < x \leq 3\pi. 
    \end{cases}
\]

a) (3 points) Sketch the graph of \( f \).

b) (2 points) Is \( f \) continuous at \( x = \pi/2 \)? Justify.
Recall that $f$ is given by the rule

$$f(x) = \begin{cases} 2x + 8 & \text{if } -10 \leq x \leq -2, \\ x^2 & \text{if } -2 < x \leq \pi/2, \\ \cos x & \text{if } \pi/2 < x \leq 3\pi. \end{cases}$$

c) (4 points) Is $f$ continuous at $x = -2$? Justify.

d) (4 points) Is $f$ differentiable at $x = -2$? Justify.
Problem 6 : (10 points) Consider the function $f$ whose graph is given below. Sketch a plausible graph for the function $f'$. Be as accurate as you can.
Problem 7: (10 points) The cost, in dollars, of producing $x$ calculus textbooks is given by the cost function $C: [0, \infty) \rightarrow \mathbb{R}$ with rule

$$C(x) = x^3 - 6x^2 + 13x.$$ 

a) (2 points) Find the rule for marginal cost function.

b) (4 points) Find the $x$-coordinate of the inflection point of $C$. 

Recall that the cost function is given by the rule

\[ C(x) = x^3 - 6x^2 + 13x. \]

This is the graph of \( C \):

c) (2 points) What is the significance of the inflection point you found in b)?
Problem 8 : (10 points)

a) (5 points) Use implicit differentiation and the fact that $\frac{d}{dx}e^x = e^x$ to show that 

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$
b) (5 points) Let $D$ be a subset of the real numbers and suppose that $f : D \to \mathbb{R}$ is a one-to-one differentiable function such that its inverse $f^{-1}$ is also differentiable. Use implicit differentiation to show that

\[
(f^{-1})' (x) = \frac{1}{f'(f^{-1}(x))}
\]

for all $x$ such that $f'(f^{-1}(x))$ is not zero.
Problem 9: (15 points) In this problem, parts a) and c) go together, and you can do part b) independently.

a) (3 points) State the Intermediate Value Theorem.

b) Consider the function $f : [0, 1] \to \mathbb{R}$,

$$f(x) = x(x - 1).$$

(a) (3 points) $f$ has an extremum (either a maximum or a minimum) between 0 and 1. Find its location (both $x$ and $y$ coordinates).

(b) (2 points) Is the extremum a maximum or a minimum?
c) Now consider the function $g: [0, 1] \rightarrow \mathbb{R}$,

$$g(x) = x(x - 1)(x - 2)(x - 3).$$

(a) (3 points) Show that $g$ has an extremum between 0 and 1. You do not have to give its location.

(b) (2 points) Is the extremum a maximum or a minimum?

(c) (2 points) Is the extremum to the left or to the right of $x = 1/2$?