Final exam
Math 19
December 12, 2016

Name: ________________________________________________________________

Instructor: __________________________________________ Lecture Time: __________

Time: 3 hours

Do not open this exam until you are told to do so.

To receive full credit, show an appropriate amount of work for each problem. Graders must be able to see not only your answer but how you obtained it. This work could be a sequence of algebraic calculations or a couple sentences, whichever feels most appropriate for the problem.

THIS EXAM IS DOUBLE-SIDED!

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

• You may NOT use a calculator or any notes or book during the exam.

• You may NOT access your cell phone (or any cellular or internet capable device) during the exam for any reason.

• You are required to sit in your assigned seat.

• You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructors/CAs during the exam, or look at anyone else’s solutions.

I understand and accept these instructions.

Signature: ____________________________________________________________
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Do not detach!
1. (4 points) Answer each of the following multiple choice questions by circling one of the given options. You do not need to justify your choices.

(a) Evaluate the limit $\lim_{x \to \infty} \sin \left( \frac{1}{x} \right)$.

(i) $-1$  (ii) $0$  (iii) $1$  (iv) $\infty$
(v) The limit does not exist and it does not diverge to $\pm \infty$.

(b) What is the derivative of $\ln (1 - x)$?

(i) $\frac{1}{1 - x}$  (ii) $-\frac{1}{1 - x}$  (iii) $-\ln (1 - x)$  (iv) $-\ln$

(c) What is the derivative of $2^x$?

(i) $x 2^{x-1}$  (ii) $\frac{2^x}{\ln 2}$  (iii) $2^x$  (iv) $(\ln 2) 2^x$

(d) What is the equation of the line tangent to the graph of $f(x) = x^3 - 3x + 2$ at $(0, 2)$?

(i) $y = -3x + 2$  (ii) $y = 2$  (iii) $y = -3x$  (iv) $y = 3x - 2$
2. (10 points) The different parts of this question are not related to each other.

(a) (5 points) For each of the following, indicate which values of $x$ make the function positive. (You do not need to justify your choices for this part.)

Note: 0 is not considered positive.

i. $-1 - x^2$

ii. $1 + (-x)^2$

iii. $e^{-x}$

iv. $\ln(x)$

v. $|x - 27|$

(b) (5 points) Find the $x$-coordinate of the point where the graphs of $y = e^{-3x}$ and $y = 12 \cdot 6^x$ intersect.
3. (7 points) Write an equation for the tangent line to the curve $xy^2 + 9y^3 - 8x = 2$ at the point $(1, 1)$. 
4. (7 points) Let $a$, $b$, and $c$ be constants and consider the function

$$G(x) = \begin{cases} 
  x^3 - 3x + a & \text{for } x < 0 \\
  5 & \text{for } x = 0 \\
  x^2 + bx + c & \text{for } x > 0 
\end{cases}$$

Each of the parts of this question will ask you to find values of $a$, $b$, and $c$. Fully correct answers will include a statement about the value(s) of each of these numbers as well as justification.

(a) (3 points) For what values of $a$, $b$, and $c$ is $G(x)$ continuous when $x = 0$?

(b) (4 points) For what values of $a$, $b$, and $c$ is $G(x)$ differentiable when $x = 0$?
5. (14 points) Consider the function \( h(x) = \frac{(2 - x)(x + 3)}{(x - 2)(x - 4)}. \)

(a) Does \( h(x) \) have a vertical asymptote at \( x = 2 \)?
Your answer should include a computation of both \( \lim_{x \to 2^-} h(x) \) and \( \lim_{x \to 2^+} h(x) \). That is, indicate if each limit is \( \infty \), \( -\infty \), or some finite number.

(b) Does \( h(x) \) have a vertical asymptote at \( x = 4 \)?
Your answer should include a computation of both \( \lim_{x \to 4^-} h(x) \) and \( \lim_{x \to 4^+} h(x) \). That is, indicate if each limit is \( \infty \), \( -\infty \), or some finite number.
6. (13 points) Consider the function \( f(x) = \frac{1}{\sqrt{x}} \).

(a) (7 points) Find the tangent line to the graph of \( f(x) \) at \( x = 4 \).

(b) (4 points) Use the tangent line you found in part (a) to approximate the value \( \frac{1}{\sqrt{4.04}} \). Your answer should be a single fraction.

(c) (2 points) Is the approximation you found in part (b) an overestimate or an underestimate? Be sure to justify your response.
7. (24 points) Compute the following limits. If they do not exist, explain why. If they diverge to \( \pm \infty \), indicate which one.

*Note: if you use L’Hopital’s rule to simplify a limit computation, please indicate that you are using it and why. For example, write \( \frac{0}{0} \) and then “L’Hopital” or “L’H” near that equals sign. It will help the grader follow your train of thought.*

(a) \[ \lim_{x \to \infty} \frac{x^4 + 2x^2 - 27}{\sqrt{3x^8 + x^2 + 2}} \]

(b) \[ \lim_{x \to -\infty} \frac{|3x - 5|}{x - 1} \]
Repeating the instructions: Compute the following limits. If they do not exist, explain why. If they diverge to ±∞, indicate which one.

*Note: if you use L’Hopital’s rule to simplify a limit computation, please indicate that you are using it and why. For example, write “0/0” and then “L’Hopital” or “L’H” near that equals sign. It will help the grader follow your train of thought.*

(c) \[ \lim_{x \to 0} \frac{(\sin x)^2}{\cos x - \cos(2x)} \]
8. (22 points) Consider the function \( f(x) = x^4e^{-x} \).

(a) (5 points) Compute \( f'(x) \).

(b) (5 points) Determine the intervals of increase and decrease of \( f(x) \).

(c) (4 points) Find all the critical points of \( f(x) \) and classify them as local minima, local maxima, or neither.
(d) (5 points) Recall that we are considering the function \( f(x) = x^4 e^{-x} \). Its second derivative is
\[
f''(x) = x^2 e^{-x} (x - 6)(x - 2).
\]
(You can take our word for it and don’t have to show this.) For which values of \( x \) is \( f(x) \) concave down?

(e) (3 points) For which values of \( x \) is \( f(x) \) increasing and concave down?
9. (10 points) A rectangle is stretched, so that the longer side is always 3 times the length of the shorter side.

(a) If the perimeter of the rectangle is increasing at a rate of 4 cm per second when the shorter side of the rectangle is 3 cm, how quickly is the shorter side increasing at this time?

(b) When the shorter side of the rectangle is 3 cm, how quickly is the area increasing at this time?
10. (24 points) Compute the following derivatives. You do not need to simplify your answers.

(a) \[ \frac{d}{dx} \left( (\ln 3)x^2 + (\ln 4)e^x \right) \]

(b) \[ \frac{d}{dt} \left( \frac{\sin t}{(1 + t)^2} \right) \]
Remember, you do not need to simplify your answers.

(c) \[ \frac{d}{d\theta} \tan \left( (3\theta - \pi)^2 \right) \]

(d) \[ \frac{d}{dx} e^{\sqrt{x}} \ln(x) \]
11. (20 points) The different parts of this question are not related to each other.

(a) (6 points) Let $h(x) = x \ln(x) - 4x$. What is the minimum value of the function $h(x)$ on the interval $(1, e^5)$?

(b) (7 points) Let $g(x) = x^3 + 2x - \sin(x)$. What is the minimum value of the function $g(x)$ on the interval $[0, \pi]$? What is the maximum value of the function $g(x)$ on this interval?
(c) (7 points) Find two positive numbers $x$ and $y$, whose product is 4, so that the sum $2x + 3y$ is as small as possible.
12. (5 points) Each of the following statements is true and is a result of one of the theorems we discussed this quarter: Intermediate Value Theorem, Mean Value Theorem, Extreme Value Theorem, and the Racetrack Principle.

For each statement, identify a theorem that proves the statement. No justification needed; just the name of the theorem is sufficient.

Note: there are 4 theorems and 5 statements, meaning at least one theorem is used more than once.

(a) Let $f(x)$ and $g(x)$ be differentiable functions with $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x < 0$. Then we know that $f(x) \geq g(x)$ for all $x < 0$.

(b) A truck travels 131 miles in two hours. The speed limit for the entire two hours was 65 mph. Therefore, the truck was speeding at some point.

(c) A continuous function on a closed interval obtains both a global maximum and a global minimum.

(d) If $f'(x) < 3$ for all $x$ and $f(2) = 1$ then $f(3) < 4$.

(e) There is a solution to the equation $f(x) = x^4 - 3 + x = 0$ in the interval $(1, 2)$ since $f(x)$ is continuous, $f(1) = -1$ and $f(2) = 15$. 
13. (*BONUS* 6 points) The different parts of this question are not related to each other.

   (a) (*BONUS* 3 points) Compute the derivative of

   \[ g(x) = \frac{(3 - \ln(2))^2(x + 27 - x)^3}{x^{-1}\cos(\arccos x)} \].

   (b) (*BONUS* 3 points) Are there any polynomials \( p(x) \) so that the function \( \sin(x)p(x) \) has no critical points on the interval \([0, \pi]\)? If there exists such polynomials \( p(x) \), explain how you know that they exist, and if there aren’t any such polynomials, explain why.

   \textit{Hint:} You might want to use the Mean Value Theorem.