Math 19: Fall 2013
Midterm 1

NAME: Solutions
LECTURE:

Time: 75 minutes

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ____________________________

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<tr>
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Problem 1: (24 points)

a) [Section 1.5 # 4] Simplify: \( \frac{x^{2n} \cdot x^{3n-1}}{x^{n+2}} \).

\[
= \frac{x^{5n-1}}{x^{n+2}} \\
= x^{4n-3}
\]

b) Simplify: \( \frac{1 \frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{4}} \).

\[
= \frac{\frac{3}{6} - \frac{2}{6}}{\frac{4}{12} + \frac{3}{12}} \\
= \frac{\frac{1}{6}}{\frac{7}{12}} \\
= \frac{\frac{1}{6} \cdot \frac{12}{7}}{1} \\
= \frac{2}{7}
\]

Alternate Solution:

\[
\frac{(\frac{1}{2} - \frac{1}{3})}{(\frac{1}{3} + \frac{1}{4})} = \frac{6 - 4}{4 + 3} \\
= \frac{2}{7}
\]
c) [HW 1 Trig problem # 4] Find the exact value: \( \sin^2(\frac{\pi}{8}) \).

\[
\sin^2\left(\frac{\pi}{8}\right) = \frac{1 - \cos(\frac{2\pi}{8})}{2} = \frac{1 - \cos(\frac{\pi}{4})}{2} = \frac{1 - \frac{\sqrt{2}}{2}}{2} = 2 - \frac{\sqrt{2}}{2} = 2 - \frac{\sqrt{2}}{4}.
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e) [Section 1.5 # 23] If \( f(x) = 5^x \), simplify the expression \( \frac{f(x + h) - f(x)}{h} \).

\[
= \frac{5^{x+h} - 5^x}{h} = 5^x \left( \frac{5^h - 1}{h} \right)
\]

f) [Section 3.6 # 7] Find the exact value: \( \sin(2 \tan^{-1} \sqrt{2}) \).

Let \( \theta = \tan^{-1} \sqrt{2} \) so \( \tan \theta = \sqrt{2} \)

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
= 2 \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}
\]

\[
= \frac{2\sqrt{2}}{3}
\]

\[
1^2 + (\sqrt{2})^2 = x^2
\]

\[
x = \sqrt{3}
\]
Problem 2: (6 points) [Section 1.1 # 59] A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side \( x \) at each corner and then folding up the sides as in the figure. Express the volume \( V \) of the box as a function of \( x \).

Volume = \( lwh \)

Here, \( h = x \)
\( b = 12 - 2x \)
\( l = 20 - 2x \)

Also, \( 0 \leq x \leq 6 \)
\( \text{cannot have corners wider than } \frac{12}{2} \)
\( \text{cannot have negative length} \)

So \( V: [0, 6] \rightarrow \mathbb{R} \)

\[ V(x) = (20 - 2x)(12 - 2x)x \]
Problem 3: (16 points) [Section 1.3 # 51] Use the given graphs of $f$ and $g$ to evaluate each expression, or explain why it is undefined.

2pts
a) $f(g(2)) = f(5) = 4$

2pts
b) $g(f(0)) = g(0) = 3$

3pts
c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$
3 pts

\[ d) \ (g \circ f)(6) = g(f(6)) = g(6) \text{ is undefined} \]

since \ 6 \ is \ not \ in \ the \ domain \ of \ g

3 pts

e) \ (g \circ g)(-2) = g(g(-2)) = g(1) = 4

3 pts

f) \ (f \circ f)(4) = f(f(4)) = f(2) = -2
Problem 4: (18 points) Consider the function \( f: [-7, \infty) \rightarrow \mathbb{R} \) given by the rule

\[
f(x) = \begin{cases} 
  e^{x+2} & \text{if } -7 \leq x < -2, \\
  (x + 1)^2 + 1 & \text{if } -2 \leq x \leq 0, \\
  2 \cos x & \text{if } 0 < x.
\end{cases}
\]

a) (10 points) Graph \( f \).
b) (2 points) Is \( f \) a one-to-one function?

\[ \text{No} \]

c) (3 points) List all of \( f \)'s \( y \)-intercepts, if any.

\[ f(0) = (0+1)^2 + 1 = 2 \]

\[ (0, 2) \]

d) (3 points) List all of \( f \)'s \( x \)-intercepts, if any.

\[ f(x) = 0: \begin{array}{ll}
\text{between } & -7 \leq -2 & f \text{ has no } x \text{-intercept} \\
\text{between } & -2 \leq 0 & f \text{ has no } x \text{-intercept.}
\end{array} \]

When \( x > 0 \),

\[ f(x) = 2 \cos x = 0 \]

When \( \cos x = 0 \)

\[ x = \frac{\pi}{2} + n\pi, \ n \text{ an integer} \]

or

\[ \left( \frac{\pi}{2} + n\pi, 0 \right), \ n \text{ an integer} \]
Problem 5: (8 points) Let $\theta$ be an angle such that $\tan \theta = \frac{x}{4}$.

a) Write down an expression for $\cos \theta$ in terms of $x$.

\[ y^2 = x^2 + 4^2 \]
\[ y = \sqrt{x^2 + 16} \]

\[ \cos \theta = \frac{4}{\sqrt{x^2 + 16}} \]

b) Write down an expression for $x$ in terms of $\theta$.

\[ x = 4 \tan \theta \]

c) Write down an expression for $x\sqrt{16 + x^2}$ in terms of $\theta$.

we have \[ \sec \theta = \frac{\sqrt{16 + x^2}}{4} \] so \[ \sqrt{16 + x^2} = 4 \sec \theta \]

also, \[ x = 4 \tan \theta \]

so \[ x\sqrt{16 + x^2} = 4 \tan \theta \cdot 4 \sec \theta = 16 \tan \theta \sec \theta \]
\[ = \frac{16 \sin \theta}{\cos^2 \theta} \]
Problem 6: (10 points) Consider the function given by the rule $F(x) = \sqrt{\tan x}$.

3pts a) There exist functions $f$ and $g$ such that $F = g \circ f$. Give the rule for $f$ and for $g$.

\[ f(x) = \tan x \]
\[ g(x) = \sqrt{x} \]

7pts b) Among the values of $x$ that are such that $0 \leq x \leq 4\pi$, which ones are in the domain of $F$?

We need $\tan x > 0$

<table>
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<tr>
<th>$x$</th>
<th>$0$</th>
<th>(\pi)</th>
<th>(\frac{3\pi}{2})</th>
<th>(2\pi)</th>
<th>(\frac{5\pi}{2})</th>
<th>(3\pi)</th>
<th>(4\pi)</th>
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<tr>
<td>$\sin x$</td>
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<tr>
<td>$\cos x$</td>
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undefined undefined undefined

so $\tan x > 0$ when $x$ is in $[0, \frac{\pi}{2}) \cup [\frac{\pi}{2}, \frac{3\pi}{2}) \cup [\frac{3\pi}{2}, \frac{5\pi}{2}) \cup [\frac{5\pi}{2}, 2\pi)$

and also $x = 4\pi$

these are the values of $x$ between $0$ and $4\pi$ that are in the domain of $F$
Problem 7: (18 points) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + 4$.

a) (2 points) Is $f$ one-to-one?

\[ \text{yes} \]

b) (5 points) Find an expression for the rule of the function $f^{-1}$ and state its domain.

\[ y = x + 4 \implies x = y + 4 \]
\[ x - 4 = y \]

\[ \text{domain: } \mathbb{R} \]

c) (5 points) Sketch $f$ and $f^{-1}$ on the grid below. Label which one is which.

[Diagram of a graph showing $y = f(x)$ and $y = f^{-1}(x)$]
d) (3 points) Show that \((f \circ f^{-1})(x) = x\).

\[
\begin{align*}
 f (f^{-1} (x)) & = f (x - 4) \\
 & = (x - 4) + 4 = x  \quad \checkmark
\end{align*}
\]


e) (3 points) Show that \((f^{-1} \circ f)(x) = x\).

\[
\begin{align*}
 f^{-1} (f(x)) & = f^{-1} (x + 4) \\
 & = (x + 4) - 4 \\
 & = x  \quad \checkmark
\end{align*}
\]