1. (30 points) (a) Find the exact value of $\cos^2(-\pi/12)$.

We'll use the double angle/power lower formula for $\cos^2(\theta)$:

$$
\cos^2(-\pi/12) = \frac{1 + \cos(2(-\pi/12))}{2} = \frac{1 + \cos(-\pi/6)}{2} = \frac{1 + \sqrt{3}/2}{2} = \frac{1}{2} + \frac{\sqrt{3}}{4}
$$

(b) (1.6 # 37) Find the exact value of $\log_3 100 - \log_3 18 - \log_3 50$.

We use the laws of logarithms:

$$
\log_3 100 - \log_3 18 - \log_3 50 = \log_3 \left(\frac{100}{18 \cdot 50}\right) = \log_3 \left(\frac{2}{18}\right) = \log_3 \left(\frac{1}{9}\right)
$$

and finally we have

$$
\log_3(1/9) = \log_3(9^{-1}) = \log_3((3^2)^{-1}) = \log_3(3^{-2}) = -2.
$$

where the final equality uses the definition of $\log_3(x)$ as the inverse function to $3^x$. 

(c) (1.1 # 27) Evaluate the following difference quotient, and simplify your answer:

\[ f(x) = \frac{1}{x}, \quad \frac{f(x) - f(a)}{x - a}. \]

We have

\[
\frac{f(x) - f(a)}{x - a} = \frac{1/x - 1/a}{x - a} = \frac{xa(1/x - 1/a)}{xa(x-a)} = \frac{a - x}{xa(x-a)} = -\frac{1}{xa}.
\]

(d) (3.6 # 9) Show that \( \cos(\sin^{-1} x) = \sqrt{1 - x^2} \).

We let \( \theta = \sin^{-1} x \), and we want to find \( \cos(\theta) \). Now, we know \( \sin(\theta) = x \), and \( \sin(\theta) = \text{opp}/\text{hyp} \). So the opposite side has length \( x \) and the hypotenuse has length 1. Therefore the adjacent side has length \( \sqrt{1 - x^2} \). Then we have

\[
\cos(\sin^{-1} x) = \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}
\]
2. (10 points) (1.1 # 57) An open rectangular box with height $h$, width $w$ and length $\ell$ has a volume of 2 cubic meters and has a square base.

(a) Express the surface area of the box as a function of the length of a side of the base. (If you don’t remember a formula for surface area of a box, it might help to draw a picture.)

Say the box has height $h$, width $w$ and length $\ell$. Since the bottom is square, $w = \ell$. We are given that the volume is

$$2m^3 = V = h\ell = h\ell^2 \implies h = \frac{2}{\ell^2}$$

Meanwhile, the open box has surface area

$$A = 2hw + 2\ell h + \ell w = 2h\ell + 2h\ell + \ell^2 = 4h\ell + \ell^2 = \frac{2}{\ell^2} \ell + \ell^2 = 8/\ell + \ell^2,$$

where in the final equality we used our equation for $h$ from above. This gives us $A$ as a function on $\ell$:

$$A = \frac{8}{\ell} + \ell^2$$

(b) What is the domain of the function from part (a)?

From the equation for $A$, we see that $\ell \neq 0$. In fact, since the surface area needs to be positive (otherwise the volume wouldn’t be 2) $\ell > 0$. 
3. (10 points) (1.5 # 35) Show that

\[ f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}} \]

is an odd function. It might help to use the fact that \( e^{-y} = \frac{1}{e^y} \) for any real number \( y \).

An odd function satisfies \( f(-x) = -f(x) \). We calculate

\[
\begin{align*}
  f(-x) &= \frac{1 - e^{-1/x}}{1 + e^{-1/x}} \\
  &= \frac{1 - e^{-1/x}}{1 + e^{-1/x}} \cdot \frac{e^{1/x}}{e^{1/x}} \\
  &= \frac{e^{1/x} - 1}{e^{1/x} + 1} = -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)
\end{align*}
\]

so \( f(x) \) is odd.
4. (15 points) Consider the function \( f: [-4, 7] \to \mathbb{R} \) given by the rule

\[
f(x) = \begin{cases} 
-(x + 4)^2 & \text{if } -4 \leq x \leq -2 \\
2x & \text{if } -2 < x < 1 \\
\ln(x) - 1 & \text{if } 1 \leq x \leq 7 
\end{cases}
\]

(a) Graph \( f(x) \) on the axes provided.
(b) Is \( f(x) \) one-to-one? Explain your answer.

As we can see on the graph above, \( f(x) \) fails the horizontal line test, for example, by taking the horizontal line \( y = 2 \).

(c) Find the \( x \)- and \( y \)-intercepts of \( f(x) \).

The only \( y \)-intercept occurs at \((0, 0)\), which is also an \( x \)-intercept. There are two more \( x \)-intercepts at \((-4, 0)\) and at \((e, 0)\).
5. (10 points) In this problem we’ll study \( F(x) = (\log_7(x^2))^{-1/2} \)

(a) Write this function as the composition of 3 functions, i.e., specify functions \( f, g \) and \( h \) so that \( F(x) = h \circ g \circ f(x) \).

\[
\begin{align*}
  f(x) &= x^2, \\
  g(x) &= \log_7(x), \\
  h(x) &= 1/\sqrt{x}.
\end{align*}
\]

(b) What is the domain of \( F(x) \)?

First, we know that \( x \neq 0 \) since for this value \( \log_7(x^2) \) is undefined. Now, since the domain of \( 1/\sqrt{x} \) is \( x > 0 \), we need to figure out where \( \log_7(x^2) > 0 \). From the graph of \( \log_7(t) \), we know that this happens for \( t > 1 \). Setting \( t = x^2 \), we then get that \( \log_7(x^2) > 0 \) for \( x > 1 \) and \( x < -1 \). So the domain of \( F(x) \) is \(( -\infty, -1 ) \cup ( 1, \infty ) \).