Problem 1. Consider \( f(x) = \sin(x) - x^2 + x \). We want to show that there is some \( x \) between 1 and 2 such that \( f(x) = 0 \).

Notice that \( f(x) \) is continuous on \([1,2]\) and
\[
\begin{align*}
  f(1) &= \sin(1) - 1 + 1 = \sin(1) > 0 \\
  f(2) &= \sin(2) - 4 + 2 = \sin(2) - 2 < 0.
\end{align*}
\]
The result then follows from intermediate value theorem.

Problem 2.

(1) By inspection, 1 is a root of \( f(x) = 0 \). By doing long division, we can then find that \( f(x) = (x - 1)^2(x + 2) \). So \( f(x) = 0 \) has two roots, -2 and 1, and we want to look at the sign of \( f(x) \) in three intervals: \((-\infty, -2), (-2, 1), (1, \infty)\).

Since \((x - 1)^2\) is always non-negative, the sign of \( f(x) \) depends only on the sign of \( x + 2 \), which is positive if \( x > -2 \) and negative if \( x < -2 \). So
\[
\text{\( f(x) \) is negative on \((-\infty, -2)\), and positive on \((-2, 1)\) and \((1, \infty)\).}
\]

(2) Remember that \( f(x) \) increasing in an interval is the same as \( f'(x) \geq 0 \) in an interval.

\[ f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1). \] So \( f'(x) = 0 \) has two roots, -1, 1, and to study the sign of \( f'(x) \), we look at three intervals: \((-\infty, -1), (-1, 1), (1, \infty)\). Considering them case by case, \( f'(x) > 0 \) on \((-\infty, -1) \) and \((1, \infty)\), and \( f'(x) < 0 \) on \((-1, 1)\). Therefore,
\[
\text{\( f(x) \) is increasing on \((-\infty, 1)\) and \((1, \infty)\), and \( f(x) \) is decreasing on \([-1, 1]\).}
\]

(3) Concave up means \( f''(x) > 0 \), and concave down means \( f''(x) < 0 \). Now \( f''(x) = 6x \), so
\[
\text{\( f(x) \) is concave up when \( x > 0 \) and is concave down when \( x < 0 \).}
\]

Problem 3.

(1) \[
\begin{align*}
\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} &= \lim_{h \to 0} \frac{-h}{x(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2}.
\end{align*}
\]

(2) Notice that \(-e^{-x} \leq e^{-x} \sin x \leq e^{-x}\), and that \( \lim_{x \to \infty} -e^{-x} = \lim_{x \to \infty} e^{-x} = 0 \). Therefore, by sandwich rule we see that \( \lim_{x \to \infty} e^{-x} \sin(x) = 0 \). This tells us that
\[
\lim_{x \to \infty} (4 + 3e^{-x} \sin(x)) = 4 + 3 \lim_{x \to \infty} (e^{-x} \sin(x)) = 4.
\]
Problem 5.

For the denominator, \( \lim_{x \to -\infty} (e^{-x} - 4) = -4 \). Therefore,

\[
\lim_{x \to -\infty} \frac{4 + 3e^{-x} \sin(x)}{e^{-x} - 4} = \lim_{x \to -\infty}(4 + 3e^{-x} \sin(x)) = \frac{4}{-4} = -1.
\]

(3)

\[
\lim_{x \to -\infty} \left( \sqrt{x^3} - \sqrt{x^3 - x} \right) = \lim_{x \to -\infty} \frac{(\sqrt{x^3} - \sqrt{x^3 - x})}{\sqrt{x^3} + \sqrt{x^3 - x}}
\]

\[
= \lim_{x \to -\infty} \frac{(x^3) - (x^3 - x)}{\sqrt{x^3} + \sqrt{x^3 - x}}
\]

\[
= \lim_{x \to -\infty} \frac{x}{\sqrt{x^3} + \sqrt{x^3 - x}}
\]

\[
= \lim_{x \to -\infty} \frac{1}{\sqrt{x} + \sqrt{x - \frac{x}{x}}}
\]

\[
= 0.
\]

Problem 4.

(1)

\[
f'(x) = \frac{d}{dx} ((x^2 - 1) e^x)^{90}
\]

\[
= 90 ((x^2 - 1) e^x)^{89} \frac{d}{dx} ((x^2 - 1) e^x)
\]

\[
= 90 ((x^2 - 1) e^x)^{89} ((\frac{d}{dx} (x^2 - 1)) e^x + (x^2 - 1) \frac{d}{dx} e^x)
\]

\[
= 90 ((x^2 - 1) e^x)^{89} (2x e^x + (x^2 - 1) e^x)
\]

\[
= 90 (x^2 - 1)^{89} (e^x)^{89} (x^2 + 2x - 1) e^x
\]

\[
= 90 (x^2 + 2x - 1)(x^2 - 1)^{89} (e^x)^{90}
\]

(2)

\[
g'(x) = \frac{d}{dx} \sin(\sqrt{1 + x^3})
\]

\[
= \cos(\sqrt{1 + x^3}) \frac{d}{dx} \sqrt{1 + x^3}
\]

\[
= \cos(\sqrt{1 + x^3}) \frac{1}{2 \sqrt{1 + x^3}} \frac{d}{dx} (1 + x^3)
\]

\[
= \cos(\sqrt{1 + x^3}) \frac{1}{2 \sqrt{1 + x^3}} 3x^2
\]

\[
= \frac{3x^2}{2 \sqrt{1 + x^3}} \cos(\sqrt{1 + x^3}).
\]

Problem 5.

(1) \( f(-x) = \frac{-x}{1+(-x)^2} = \frac{-x}{1+x^2} = -f(x) \), so \( f \) is odd.

(2)

\[
\lim_{x \to -\infty} \frac{x}{1 + x^2} = \lim_{x \to -\infty} \frac{1}{x^2 + 1} = \lim_{x \to -\infty} \frac{1}{x^2 + 1} = 0 = 0.
\]

\[
\lim_{x \to -\infty} \frac{x}{1 + x^2} = \lim_{x \to -\infty} \frac{1}{x^2 + 1} = \lim_{x \to -\infty} \frac{1}{x^2 + 1} = 0 = 0.
\]

(3) We first compute \( \frac{d}{dx} f(x) \).

\[
\frac{d}{dx} f(x) = \frac{(1 + x^2) \frac{d}{dx} x - x \frac{d}{dx} (1 + x^2)}{(1 + x^2)^2} = \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}.
\]

Then \( \frac{d}{dx} f(x) = 0 \Leftrightarrow \frac{1 - x^2}{(1 + x^2)^2} = 0 \Leftrightarrow 1 - x^2 = 0 \Leftrightarrow x = \pm 1. \)
(4) Slope of the tangent line is \( f'(3) = \frac{1 - 3^2}{(1 + 3^2)^2} = \frac{-8}{100} = -0.08 \). The tangent line also passes through \((3, 0.3)\). Therefore the equation for the tangent line is \( \frac{y - 0.3}{x - 3} = -0.08 \). After simplification, we get \( y = -0.08x + 0.54 \).

(5) From part (3) one can invoke that \( f \) is increasing on \([-1,1]\) and decreases otherwise. Part (1) says the function is odd, which means that the graph of \( f \) on the left of \( y \)-axis and on the right of \( y \)-axis should be the same, up to a flip of the horizontal axis. This suggests us to first plot the graph for \( x \geq 0 \), then do flipping. The end result should look like this.

Problem 6.

(1)

\[
F'(-2) = f'(1 + g(-2)) \left( \sqrt{1 + g(x)} \right)'(-2) \\
= f'(1 + g(-2)) \left( \frac{1}{2\sqrt{1 + g(-2)}} \right) g'(-2) \\
= f'(1 + 0) \left( \frac{1}{2\sqrt{1 + 0}} \right) (3) \\
= (4) \left( \frac{1}{2} \right) (3) \\
= 6.
\]

(2)

\[
G'(-2) = \sin(f(-2)g(-2))(f(x)g(x))'(-2) \\
= \sin(f(-2)g(-2))(f'(-2)g(-2) + f(-2)g'(-2)) \\
= \sin(0)(f'(-2)g(-2) + f(-2)g'(-2)) \\
= 0.
\]