1. (20 points) (HW 4, problem 2) Compute the following limits. At every step, justify your work with a limit rule or a theorem.

(a) \( \lim_{x \to 3} \frac{x^2 - 9}{x + 2} \)

The function \( \frac{x^2 - 9}{x + 2} \) is continuous at 3, so the limit equals the value of the function,

\[
\lim_{x \to 3} \frac{x^2 - 9}{x + 2} = \frac{3^2 - 9}{3 + 2} = 0.
\]

(b) \( \lim_{t \to 0} \left( \frac{1}{t\sqrt{1 + t}} - \frac{1}{t} \right) \)

We do the following algebraic manipulation:

\[
\lim_{t \to 0} \left( \frac{1}{t\sqrt{1 + t}} - \frac{1}{t\sqrt{1 + t}} \right) = \lim_{t \to 0} \frac{1 - \sqrt{1 + t}}{t\sqrt{1 + t}} = \lim_{t \to 0} \frac{(1 - \sqrt{1 + t})(1 + \sqrt{1 + t})}{(t\sqrt{1 + t})(1 + \sqrt{1 + t})}.
\]

Expanding the numerator we find the limit equals

\[
= \lim_{t \to 0} \frac{1 - (1 + t)}{t\sqrt{1 + t}(1 + \sqrt{1 + t})} = \lim_{t \to 0} \frac{-t}{t\sqrt{1 + t}(1 + \sqrt{1 + t})} = \lim_{t \to 0} -\frac{1}{\sqrt{1 + t}(1 + \sqrt{1 + t})}.
\]

The final expression is continuous at \( t = 0 \), so we may evaluate, finding

\[
\lim_{t \to 0} \left( \frac{1}{t\sqrt{1 + t}} - \frac{1}{t\sqrt{1 + t}} \right) = \frac{-1}{\sqrt{1 + 0}(1 + \sqrt{1 + 0})} = -\frac{1}{2}.
\]
(c) $\lim_{x \to 0} \frac{|x|}{x}$

We have that

$$|x| = \begin{cases} 
    x & x > 0 \\
    -x & x \leq 0
\end{cases}$$

and so

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

and

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

Since the limit from the left and the limit from the right do not agree, the limit does not exist.

(d) $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$.

We have the inequalities,

$$-1 \leq \cos\left(\frac{2}{x}\right) \leq +1 \implies -x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4,$$

and since

$$\lim_{x \to 0} -x^4 = \lim_{x \to 0} x^4 = 0,$$

by the squeeze theorem we find

$$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0.$$
2. (15 points) Consider the function
\[ f(x) = \begin{cases} \frac{-5x^2 - 20}{(x + 2)(x + 1)} & x \neq -2 \\ 12 & x = -2 \end{cases} \]

(a) Locate and determine the type of discontinuities (e.g., jump, removable, infinite) of the function \( f(x) \).

We have
\[
\frac{5x^2 - 20}{(x + 2)(x + 1)} = 5 \cdot \frac{x^2 - 4}{(x + 2)(x + 1)} = \frac{5(x - 2)(x + 2)}{(x + 2)(x + 1)},
\]
so canceling the factors of \( x + 2 \) we find
\[
\lim_{x \to -2} -\frac{5x^2 - 20}{(x + 2)(x + 1)} = -\frac{5 \cdot (-2) - 2}{-2 + 1} = -20 \neq f(-2) = 12.
\]

Hence, at \( x = -2 \) there is a removable discontinuity. We also have
\[
\lim_{x \to -1^+} \frac{5x^2 - 20}{(x + 2)(x + 1)} = \lim_{x \to -1^+} -\frac{5x - 2}{x + 1} = +\infty, \quad \lim_{x \to -1^-} \frac{5x^2 - 20}{(x + 2)(x + 1)} = -\infty
\]
so at \( x = -1 \) there is an infinite discontinuity.

(b) Calculate the limit \( \lim_{x \to \infty} f(x) \).

We have
\[
\lim_{x \to \infty} \frac{5x^2 - 20}{(x + 2)(x + 1)} = \lim_{x \to \infty} -\frac{5x - 2}{x + 1} = \lim_{x \to \infty} -\frac{1 - 2/x}{1 + 1/x} = -5.
\]
(c) In part (a) we saw that $f(x)$ is continuous on $[0, 2]$ (you can assume this even if you haven’t finished part (a)).

Apply the Intermediate Value Theorem to show that $f(x) = e^x$ has a solution for some $x$. (HINT: Look at the function $g(x) = e^x - f(x)$ and consider the $x$-values 0 and 2.)

Using the hint, we calculate

$$g(0) = e^0 - \left( -\frac{5 \cdot 0 - 20}{(0 + 2)(0 + 1)} \right) = -9$$

and

$$g(2) = e^2 - \left( -\frac{5 \cdot 4 - 20}{4 \cdot 3} \right) = e^2.$$

Since $g(2) = e^2 > 0$ and $g(0) = -9 < 0$ and $g(x)$ is a continuous function on $[0, 2]$, by the Intermediate Value Theorem there exists some $c$ with $0 < c < 2$ such that $g(c) = 0$. Hence, there is a solution to the equation $f(x) = e^x$. 
3. (15 points) Compute the following derivatives. You do not need to simplify your answer.

(a) Find the derivative of \( f(t) = (t + e^t)(3 - \sqrt{t}) \).

By the product rule, we have

\[
\frac{df}{dt}(t) = \frac{d}{dt}(t + e^t)(3 - \sqrt{t}) + (t + e^t)\frac{d}{dt}(3 - \sqrt{t}) = (1 + e^t)(3 - \sqrt{t}) + (t + e^t)(-\frac{1}{2}t^{-1/2}).
\]

(b) Differentiate \( e^{\cos(2\theta)} \).

By the chain rule, we have

\[
\frac{d}{d\theta} e^{\cos(2\theta)} = e^{\cos(2\theta)} \frac{d}{d\theta}(\cos(2\theta)) = e^{\cos(2\theta)}(-\sin(2\theta))2.
\]
(c) Using the limit definition, find the derivative of $f(x) = 5^x$. You might find the formula $\lim_{h \to 0} \frac{5^h - 1}{h} = \ln(5)$ useful.

\[
\begin{align*}
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \lim_{h \to 0} \frac{5^{x+h} - 5^x}{h} \\
&= \lim_{h \to 0} \frac{5^x(5^h - 1)}{h} \\
&= 5^x \lim_{h \to 0} \frac{5^h - 1}{h} \\
&= 5^x \ln(5).
\end{align*}
\]
4. (10 points) (HW 6, problem 4) Suppose that a particle travels along the curve \( C \) given by the parametric equations \( x(t) = t^2, y(t) = t^3 - 3t \), where \( t \) is any real number.

(a) At what time(s) is the particle at the point \((3, 0)\)?

If \( x(t) = 3 \), then \( t^2 = 3 \) and \( t = \pm \sqrt{3} \). If \( y(t) = 0 \), then \( t^3 - 3t^2 = 0 \), so \( t = 0 \) or \( t = \pm \sqrt{3} \). Taken together, we find that for the values \( t = \pm \sqrt{3} \), \((x(t), y(t)) = (3, 0)\).

(b) Find the equation of the tangent line(s) to the curve at the point \((3, 0)\).

We use the formula \[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \]
and calculate \[ \frac{dy}{dt} = 3t^2 - 3, \quad \frac{dx}{dt} = 2t, \]
and so \[ \frac{dy}{dx} = \frac{3t^2 - 3}{2t} \]
so at \( t = +\sqrt{3} \), we have \( dy/dx = \sqrt{3} \) and at \( t = -\sqrt{3} \) we get \( dy/dx = -\sqrt{3} \). Then we use point slope form to get the pair of lines \[ y = \sqrt{3}(x - 3), \quad y = -\sqrt{3}(x - 3). \]
5. (20 points) In this problem we will determine the graph of the function \( f(x) = xe^{-x^2} \).

(a) Find the intervals on which \( f(x) \) is increasing and decreasing, and the values of \( x \) for which \( f(x) \) has a horizontal tangent line.

Using the chain and product rule we calculate
\[
f'(x) = xe^{-x^2} + x(-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2).
\]
Since \( e^{-x^2} \) is always positive, \( f'(x) \) is positive/negative/zero when \( 1 - 2x^2 \) is positive/negative/zero. We find that \( 1 - 2x^2 = 0 \) for \( x = \pm \frac{1}{\sqrt{2}} \), and these are the \( x \)-values at which \( f(x) \) has a horizontal tangent line. For \(-1/\sqrt{2} < x < 1/\sqrt{2}\), we have that \( 1 - 2x^2 > 0 \), so \( f(x) \) is increasing. For \( x > 1/\sqrt{2} \) and \( x < 1/\sqrt{2} \), we get that \( 1 - 2x^2 < 0 \), so \( f(x) \) is decreasing.

(b) Find the intervals on which \( f(x) \) is concave down and concave up.

We have
\[
f''(x) = (f'(x))' = -2xe^{-x^2}(1 - 2x^2) + e^{-x^2}(-4x) = e^{-x^2}(4x^3 - 6x).
\]
By the same argument as in (a), \( f''(x) = 0 \) when \( 4x^3 - 6x = 0 \). Factoring, \( 4x^3 - 6x = 2x(x^2 - 3) = 2x(x + \sqrt{3})(x - \sqrt{3}) \), so that \( f''(x) = 0 \) for \( x = -\sqrt{3}, 0, +\sqrt{3} \). So we have \( f''(x) < 0 \) (and \( f(x) \) is concave down) for \( x < -\sqrt{3} \) and \( 0 < x < \sqrt{3} \). We have \( f''(x) > 0 \) (and \( f(x) \) is concave up) for \( x > +\sqrt{3} \) and \(-\sqrt{3} < x < 0 \).
(c) Using parts (a) and (b), graph \( f(x) \) on the axes provided.