High-Rate Analysis of Systematic Lossy Error Protection of a Predictively Encoded Source

Shantanu Rane, David Rebollo-Monedero and Bernd Girod
Information Systems Laboratory,
Stanford University, Stanford, CA 94305
{srane,drebollo,bgirod}@stanford.edu

Abstract

We consider a first-order Markov source, which is predictively encoded using a DPCM-style encoder. The quantized compressed prediction residual is transmitted over an erasure channel. Additionally, a Wyner-Ziv encoded version of the prediction residual is transmitted in order to provide error resilience. When the symbols from the first transmission are erased by the channel, this second description is decoded, and limits the maximum distortion that can occur. Since the quantization step size used in the second description is, in general, larger than that used in the main transmission, error protection is lossy. Using high-rate quantization theory, we derive expressions for the rate and the end-to-end distortion incurred by this system. We show that, compared to conventional lossless forward error correction, this lossy error protection scheme is robust in the sense that it allows the received signal quality to degrade gracefully when the erasure probability increases.

1 Introduction

Consider a communication scheme in which a source, typically an image or video or audio signal, is compressed and transmitted over a lossy channel which drops some of the packets. Without error correction, this would result in an increase in the distortion of the decoded signal. Forward Error Correction (FEC) adds a specified amount of redundant information to protect the compressed bit stream from channel errors. When a systematic channel code is used, the FEC parity data can be considered as a separate bit stream that provides error resilience. Shamai, Verdú and Zamir studied a more general transmission scheme known as systematic *lossy* source-channel coding [1], in which the added redundancy protects the waveform of the transmitted signal, rather than the bit stream itself. In their formulation, an analog signal is transmitted uncoded over an error-prone channel. The received version of the signal is degraded by channel errors. To provide error resilience, a separate bit stream is generated using Wyner-Ziv coding [2] of the input signal. The Wyner-Ziv bit stream is decoded at the receiver using the degraded analog signal as side information. Depending upon the point chosen on the (operational) Wyner-Ziv rate-distortion curve, this scheme provides a trade-off between the amount of error resilience desired and quality of the final decoded signal.

Inspired by [1], a practical scheme for error resilient transmission of compressed video signals, called Systematic Lossy Error Protection (SLEP), was proposed in [3, 4]. By using Wyner-Ziv coding, SLEP has been shown to provide a graceful trade-off between error resilience and decoded picture quality, effectively mitigating the "cliff" effect observed in FEC-based systems.

In this work, we study a SLEP scheme which is simple enough for a closed-form mathematical analysis. We consider robust transmission of a first-order Markov source over an erasure channel. The source is compressed by a first-order DPCM coder. The prediction residual is quantized, entropy-coded and transmitted over an erasure channel. For error resilience, we requantize the prediction residual and use Wyner-Ziv coding to mitigate the effect of erasures on the distortion in the transmitted signal. We derive expressions for the total rate and the end-to-end distortion in the decoded sequence.

The paper is organized as follows: The DPCM source coding scheme is described in Section 2, followed by the Wyner-Ziv coding scheme in Section 3. The high-rate rate-distortion functions for this scheme are derived in Section 4. A comparison of SLEP with traditional FEC is performed in Section 5, and concluding remarks are presented in Section 6.

2 DPCM source coding scheme

We now describe the encoding and decoding scheme for the systematic transmission. In addition, we detail the assumptions on the source data and the coding operations, which will be used to obtain the expressions for rate and distortion:

1. **Source data:** The encoding scheme is shown in Fig. 1. The source data is represented by $(X_n)_{n\in\mathbb{Z}}$, a zero-mean, stationary, first-order Markov process.

2. **Prediction residual:** We consider a simple linear predictor $X_n = \rho\,\widehat{X}_{n-1} + W_n$, where $|\rho| < 1$, and W_n represents the unpredictable component, i.e., the prediction residual. In this example, ρ is the correlation coefficient between X_n and X_{n-1}, and ρX_{n-1} is the best linear unbiased estimate of X_n given X_{n-1}. Note that in the DPCM encoder, we predict X_n from the reconstructed sample $\widehat{X}_{n-1}$ and not from X_{n-1}. At high rates, the quantization of W_n is fine enough so that $X_n \simeq \widehat{X}_n$. Therefore, we immediately become less formal and say that the W_n are i.i.d. and independent of the past values of the source data X_{n-1}, X_{n-2}.... This situation occurs, for example, when the source data are produced by a first-order Gauss-Markov process. Note that whenever a variable, or a difference of variables, is identically distributed, we will drop the time index n.

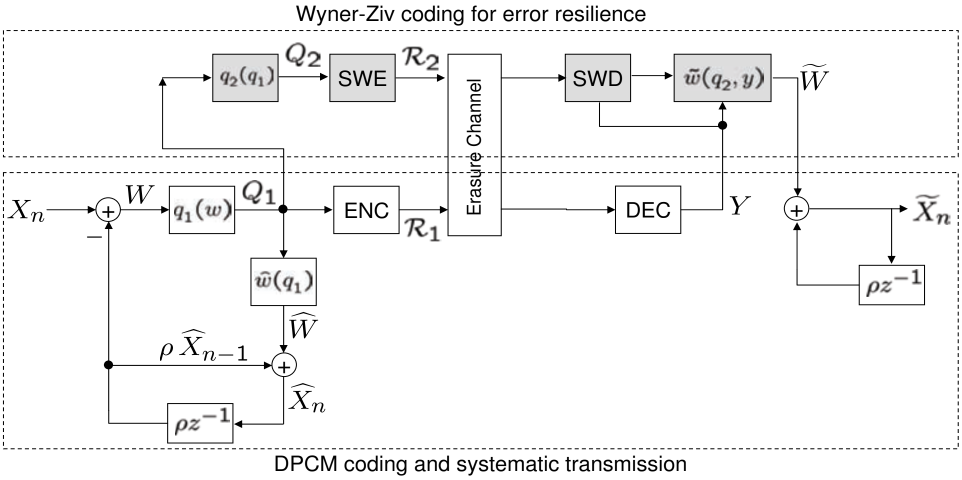


Figure 1: Systematic lossy error protection applied to the prediction residual signal of a DPCM coding scheme.

3. **Quantization of prediction residual:** The quantizer $q_1(w)$ maps the prediction error W into the quantization index Q_1, which is compressed by an ideal entropy coder. Thus, the source coding bit rate is $\mathcal{R}_1 \triangleq \mathrm{H}(Q_1)$. The codewords generated by the entropy coder are transmitted across an error-prone channel. The reconstruction of W corresponding to the index Q_1 is $\widehat{W} = \mathrm{E}[W|Q_1]$. Mean squared error (MSE) is used as the distortion measure, thus the expected source coding distortion in W is $\mathcal{D}_1 \triangleq \mathrm{E}(W - \widehat{W})^2$.

4. **Using local reconstructions as reference samples:** The encoder's local reconstruction of X_n, to be used for predictive encoding of the future samples, is given by $\widehat{X}_n = \rho\widehat{X}_{n-1} + \widehat{W}_n$. Note that, in the absence of channel errors, the receiver would recover the quantization indices and obtain $\widehat{X}_n$ exactly, and there would be no mismatch between encoder and decoder. i.e., $\mathrm{E}(X - \widehat{X})^2 = \mathrm{E}(W - \widehat{W})^2 = \mathcal{D}_1$.

3 Wyner-Ziv coding of the prediction residual

We assume an erasure channel in this work. Specifically, the codewords generated by the entropy coder are erased with probability p. The process causing the erasures is assumed to be independent of the source statistics. At the receiver, reversing the entropy coding operation yields either the quantization index Q_1, or an erasure (denoted by the symbol e). Thus, the side information for the Wyner-Ziv decoder is:

$$Y = \begin{cases} Q_1 & \text{w.p. } 1-p \\ \mathrm{e} & \text{w.p. } p \end{cases} \tag{1}$$

With no error protection in the case of an erasure, the best possible reconstruction of W is $\mathrm{E}[W|\mathrm{e}] = \mathrm{E}\,W = 0$, which would result in a MSE in W of σ_W^2, the variance of

W. Because of the predictive coding structure, this error energy will propagate to the subsequently decoded samples. To mitigate this error propagation, SLEP transmits additional symbols generated by distributed coding of the prediction residual. The Wyner-Ziv coding procedure is as follows:

1. **Quantization:** First, the prediction residual is requantized. Specifically, let the quantizer $q_2(q_1)$ map the quantization index Q_1 from Fig. 1 into the quantization index Q_2. Thus, $q_1(w)$ is embedded inside $q_2(q_1(w))$. The corresponding reconstruction levels for W are given by $\widehat{\widehat{W}} = \mathrm{E}[W|Q_2]$.

2. **Slepian-Wolf coding:** Now, ideal lossless encoding of the quantization indices Q_2 is performed assuming the presence of side information Y at the decoder. Note that the statistics of Y are known to the Slepian-Wolf encoder, but the actual value of Y is unknown. With ideal Slepian-Wolf encoding [5], the bit rate required would be $\mathrm{H}(Q_2|Y)$. However, the Slepian-Wolf code is transmitted over an erasure channel. In order to ensure that the Slepian-Wolf codewords can be recovered in spite of these erasures, the bit rate must be increased to $\mathcal{R}_2 > \mathrm{H}(Q_2|Y)$. $\mathcal{R}_2$ will also be referred to as the error resilience bit rate or the Wyner-Ziv bit rate. At the receiver, Slepian-Wolf decoding returns the quantization index Q_2.

3. **SLEP decoding:** Let $\widetilde{W}$ denote the output of the SLEP decoder. We now define the operation of the SLEP decoder, i.e., its response to erasures that may occur on both the systematic and the Wyner-Ziv transmissions. If there is no erasure on the systematic transmission, it means that the side information $Y = Q_1$ and no error has occurred. In this case, the output is defined to be $\widetilde{W} = \widehat{W} = \mathrm{E}[W|Q_1]$. If there is an erasure on the systematic transmission, Wyner-Ziv decoding must be performed and the output is given by $\widetilde{W} = \widehat{\widehat{W}} = \mathrm{E}[W|Q_2, \mathrm{e}] = \mathrm{E}[W|Q_2]$, because the erasure provides no information about W. To summarize, the output of the SLEP decoder is given by:

$$\widetilde{W} = \mathrm{E}[W|Q_2, Y] = \begin{cases} \widehat{W} & \text{if } Y = Q_1 \\ \widehat{\widehat{W}} & \text{if } Y = \mathrm{e} \end{cases} \tag{2}$$

   We emphasize that, owing to requantization, the Wyner-Ziv representation has lower quality compared to the main transmitted signal, and will only be called upon when the main prediction error signal is lost. Owing to the embedding of the quantizers, $\mathrm{E}[W|Q_1, Q_2] = \mathrm{E}[W|Q_1]$. This justifies the above decoding strategy, since $\widetilde{W}$ is the optimal reconstruction of W in the MSE sense. In this simple setup, SLEP is the same as unequal error protection of the prediction error, in which the higher significant bit-planes in the binary representation of W are protected, while lower significant bit-planes are not. Since the error process of the channel is independent of the prediction and quantization operations, $\widetilde{W}_n$ is i.i.d. and the subscript n has been omitted. The MSE distortion in W, after SLEP decoding, is $\mathcal{D}_2 \triangleq \mathrm{E}(W - \widetilde{W})^2$.

4 Rate-distortion tradeoffs in SLEP

As shown in Fig. 1, the final goal is to reproduce X_n. This final reproduction, denoted by $\widetilde{X}_n$, is obtained by reversing the prediction process at the encoder. Thus, $\widetilde{X}_n = \rho\widetilde{X}_{n-1} + \widetilde{W}_n$. Our goal is to obtain an expression for the total rate, defined as $\mathcal{R} \triangleq \mathcal{R}_1 + \mathcal{R}_2$ and the end-to-end distortion, defined as $\mathcal{D} \triangleq \mathrm{E}(X - \widetilde{X})^2$. Please refer to Lemmas 4 and 5 in the appendix for an explanation of why X_n, $\widetilde{X}_n$ and the difference $X_n - \widetilde{X}_n$ are all identically distributed.

We assume that W is encoded at high rates. The results in this section hold if the Bennett assumptions [6] apply to the probability density function $f_W(w)$. We consider, in turn, the rate-distortion relation for the source coder of W, the rate-distortion relation for the Wyner-Ziv coder of W, and the final expression for end-to-end distortion in X.

Suppose that the statistics of W are such that the differential entropy $\mathrm{h}(W)$ is defined and finite. By a direct application of high rate quantization theory [7], an asymptotically optimal scalar quantization strategy for the prediction residual W is to perform uniform quantization with step-size Δ_1, which satisfies, for large $\mathcal{R}_1$:

$$\mathcal{R}_1 \simeq \mathrm{h}(W) - \log_2 \Delta_1 \ , \ \mathcal{D}_1 \simeq \frac{\Delta_1^2}{12} \ , \ \mathcal{D}_1 \simeq \frac{1}{12}\, 2^{2\,\mathrm{h}(W)}\, 2^{-2\mathcal{R}_1} \tag{3}$$

Note that, since W is encoded at high rates, $\Delta_1 \ll \sigma_W$. We now obtain a rate-distortion relation for the Wyner-Ziv coder.

Proposition 1. *Suppose that the statistics of W are such that the differential entropy $\mathrm{h}(W)$ is defined and finite. Suppose also that asymptotically optimal scalar quantization has been used in the systematic transmission. Then, an asymptotically optimal scalar quantization strategy for the SLEP decoding procedure described in Section 3 is to perform uniform quantization of Q_1 with step-size m, which satisfies, for large $\mathcal{R}_2$, and $\Delta_1 \to 0$:*

$$\mathcal{R}_2 \simeq \frac{p}{1-p}\,(\mathcal{R}_1 - \log_2 m) \ , \ \mathcal{D}_2 \simeq (1 - p + p\,m^2)\,\mathcal{D}_1 \tag{4}$$

$$\mathcal{D}_2 \simeq \frac{(1-p+p\,m^2)}{m^2}\,\frac{1}{12}\,2^{\,2\,\mathrm{h}(W)}\,2^{-2\mathcal{R}_2\frac{1-p}{p}}$$

Proof. Since Q_2 is obtained via requantization of the indices Q_1, knowledge of Q_1 unambiguously determines Q_2. If there were no erasures in the Slepian-Wolf transmission, then the error resilience bit rate would be given by the Slepian-Wolf theorem:

$$\mathrm{H}(Q_2|Y) = (1-p)\ \mathrm{H}(Q_2|Q_1) + p\ \mathrm{H}(Q_2|e) = 0 + p\ \mathrm{H}(Q_2) \tag{5}$$

However, if there are erasures in the Slepian-Wolf transmission, then the Slepian-Wolf theorem cannot be used directly because it assumes error-free transmission of the Slepian-Wolf code[1]. To find $\mathcal{R}_2$, we use the analogy between the Slepian-Wolf code

[1]Intuitively, the error resilience bit rate $\mathcal{R}_2$ should be *higher* than the Slepian-Wolf bit rate because we want the Slepian-Wolf code to provide protection not only against erasures in the DPCM-coded transmission, but also against erasures in the transmission of the Slepian-Wolf codewords themselves.

and the parity portion of a systematic channel code. Consider a systematic channel code in which both the source and the parity symbols are erased with probability p. Let the parity portion of a capacity-achieving channel code be used as a Slepian-Wolf code. Then, the parity bit rate, which equals $\mathcal{R}_2$ in the present problem, is given by:

$$\mathcal{R}_2 = \frac{p}{1-p}\,\mathrm{H}(Q_2) \simeq \frac{p}{1-p}\,(\,\mathrm{h}(W) - \log_2(m\Delta_1)) = \frac{p}{1-p}\,(\mathcal{R}_1 - \log_2 m) \tag{6}$$

Here, requantization to obtain Q_2 is asymptotically equivalent to transcoding W using a uniform quantizer with step-size $\Delta_2 = m\Delta_1,\ m \in \mathbb{Z}^+$. We further claim that there is no loss of optimality if $m \in \mathbb{Z}^+$ (instead of the more general claim that $m \in \mathbb{R}^+$). For a given distortion, since $\Delta_1 \to 0$, the increase in rate due to this introduced gradation is arbitrarily small. Such a gradation gives *points* on the $\mathcal{R}_2(\mathcal{D}_2)$ curve, but these points are arbitrarily close at high rates, so we can take the rate-distortion function to be asymptotically continuous. Further, (6) uses the result that a uniform quantizer of width $\Delta_2 = m\Delta_1$, without index repetition, is asymptotically optimal as shown in [8].

The MSE distortion at the output of the Wyner-Ziv decoder is then given by:

$$\begin{aligned}\mathcal{D}_2 &= \mathrm{E}(W - \widetilde{W})^2 = (1-p)\ \mathrm{E}(W - \widehat{W})^2 + p\ \mathrm{E}(W - \widehat{\widehat{W}})^2 &(7)\\ &\simeq (1-p)\,\frac{\Delta_1^2}{12} + p\,\frac{m^2\Delta_1^2}{12} \simeq (1 - p + p\,m^2)\mathcal{D}_1 &(8)\end{aligned}$$

where (7) is obtained by iterated expectation, and (8) uses the distortions observed at high rates for quantizers with step sizes Δ_1 and $m\Delta_1$. □

For $p = 0$, we have $\mathcal{D}_2 = \mathcal{D}_1, \mathcal{R}_2 = 0$, confirming that no bits need to be spent on error resilience for the error-free case. We now derive an expression for $\mathcal{D}$, the effective distortion in X as a result of the distortion in W, accounting for the effect of error propagation from previously decoded samples.

Theorem 2. *Consider a SLEP system in which the systematic transmission has a rate-distortion relation given by (3) and the Wyner-Ziv transmission has a rate-distortion relation given by Proposition 1. Then, the end-to-end mean squared error distortion in X is given by:*

$$\mathcal{D} \simeq \left(1 + p\,\frac{m^2-1}{1-\rho^2}\right) m^{-2p}\,\frac{1}{12} 2^{\,\mathrm{h}(W)} 2^{-2\mathcal{R}(1-p)} \tag{9}$$

Proof. Consider the error in the reconstruction of X at the decoder:

$$X_n - \widetilde{X}_n = (\rho\,\widehat{X}_{n-1} + W_n) - (\rho\,\widetilde{X}_{n-1} + \widetilde{W}_n) = \rho\,(\widehat{X}_{n-1} - \widetilde{X}_{n-1}) + (W_n - \widetilde{W}_n) \tag{10}$$

From Lemmas 4 and 5 in the appendix, the differences $W_n - \widetilde{W}_n$, $X_n - \widetilde{X}_n$, $\widehat{X}_n - \widetilde{X}_n$ are stationary, and we can drop the time indices while writing the distortions. Moreover, since W is i.i.d., the difference $W_n - \widetilde{W}_n$ is independent of $\widehat{X}_{n-1} - \widetilde{X}_{n-1}$ Then, from (10),

$$\begin{aligned}\mathcal{D} &= \mathrm{E}(X - \widetilde{X})^2 = \rho^2\ \mathrm{E}(\widehat{X} - \widetilde{X})^2 + \mathrm{E}(W - \widetilde{W})^2 + 2\ \mathrm{E}(\widehat{X} - \widetilde{X})\ \mathrm{E}(W - \widetilde{W})\\ &= \rho^2\ \mathrm{E}(\widehat{X} - \widetilde{X})^2 + \mathcal{D}_2 + 0 &(11)\end{aligned}$$

where the last term vanishes because, by iterated expectation, $\mathrm{E}\,\widetilde{W} = \mathrm{E}\,\mathrm{E}[W|Q_2, Y] = \mathrm{E}\,W$. Now consider the difference,

$$V_n = \widehat{X}_n - \widetilde{X}_n = \rho\,(\widehat{X}_{n-1} - \widetilde{X}_{n-1}) + (\widehat{W}_n - \widetilde{W}_n) = \rho\,V_{n-1} + U_n \tag{12}$$

Thus, the new random process V_n is obtained by passing a strict sense stationary zero mean random process U_n through a LTI filter [2]. Then, from Lemma 6, we have,

$$\mathrm{E}(\widehat{X} - \widetilde{X})^2 = \sigma_V^2 = \frac{1}{1-\rho^2}\,\sigma_U^2 = \frac{1}{1-\rho^2}\,\mathrm{E}(\widehat{W} - \widetilde{W})^2 \tag{13}$$

where the MSE in the right hand side can be evaluated as follows:

$$\mathrm{E}(\widehat{W} - \widetilde{W})^2 = (1-p)\,\mathrm{E}(\widehat{W} - \widehat{W})^2 + p\,\mathrm{E}(\widehat{W} - \widehat{\widehat{W}})^2 \simeq 0 + p\,(m^2-1)\mathcal{D}_1 \tag{14}$$

The last term in (14) is the MSE between the reconstruction levels of the source quantizer and Wyner-Ziv quantizer. For any $m \in \mathbb{Z}^+$, this MSE evaluates to $(m^2-1)\mathcal{D}_1$. This calculation is worked out in Proposition 7 in the appendix.

Substituting the expressions of (13) and (14) into (11), the end-to-end MSE distortion in X is given by

$$\mathcal{D} \simeq \frac{\rho^2}{1-\rho^2}\,p\,(m^2-1)\mathcal{D}_1 + \mathcal{D}_2 = \left(1 + p\,\frac{m^2-1}{1-\rho^2}\right)\mathcal{D}_1 \tag{15}$$

This equation may be reduced to the form in the theorem statement by expressing $\mathcal{D}_1$ in terms of $\mathcal{R}_1$, and finally expressing $\mathcal{R}_1$ in terms of the total rate $\mathcal{R}$. For $p = 0$, the familiar high-rate result is obtained, with $\mathcal{D}$ reducing by 6.02 dB/bit. For non-zero p, $\mathcal{D}$ falls at the rate of $6.02(1-p)$ dB/bit. □

5 Observations on lossy versus lossless protection

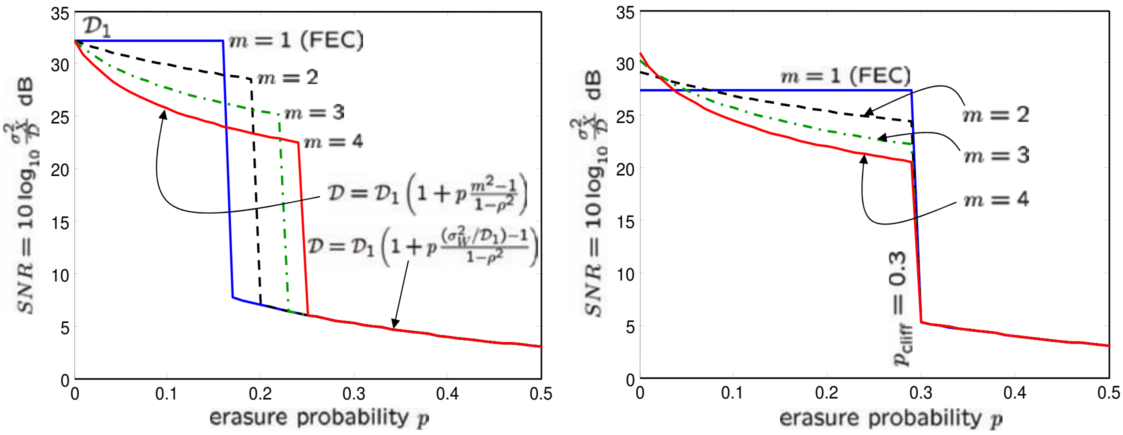
The treatment in the earlier sections assumed that the erasure probability is known. Now consider the case in which $\mathcal{R}_2$ is set to allow error protection for any erasure probability $p \leq p_{\text{cliff}}$. In that case, we can write the overall distortion in X as:

$$\mathcal{D} \simeq \begin{cases} \mathcal{D}_1\left(1 + p\,\frac{m^2-1}{1-\rho^2}\right) & \text{if } p \leq p_{\text{cliff}} \\ \mathcal{D}_1\left(1 + p\,\frac{(\sigma_W^2/\mathcal{D}_1)-1}{1-\rho^2}\right) & \text{if } p > p_{\text{cliff}} \end{cases} \tag{16}$$

where the distortion for $p \leq p_{\text{cliff}}$ is obtained from (15). The distortion for $p > p_{\text{cliff}}$ can be obtained by repeating the steps in the proof of Theorem 2 noting that erasure protection fails for $p > p_{\text{cliff}}$, and so the minimum MSE reconstruction of W is not $\widehat{\widehat{W}}$ but $\mathrm{E}\,W$. From (16), notice that $\mathcal{D}$ has a discontinuity at $p = p_{\text{cliff}}$ because $m^2\mathcal{D}_1 \ll \sigma_W^2$ at high rates. Now we compare SLEP ($m > 1$) against lossless forward error correction ($m = 1$) in two scenarios:

---

[2] The stationarity of $U_n = \widehat{W}_n - \widetilde{W}_n$ arises from the initial assumptions on W and X and is a consequence of Lemma 4 in the appendix



(a) Fix $\mathcal{R}_1 = 5$ bits, $\mathcal{R}_2 = 1$ bit, vary m (b) Fix $\mathcal{R} = 6$ bits, $p_{\text{cliff}} = 0.3$, vary m

Figure 2: The end-to-end distortion $\mathcal{D}$ is evaluated for the case where source data X are generated by a first-order Gauss-Markov process with $\rho = 0.75$ and $\sigma_W^2 = 5$. (a) For a fixed error resilience bit rate, SLEP provides graceful quality degradation over a wider range of erasure probabilities than FEC. (b) If the maximum erasure probability is fixed, then SLEP allocates a larger fraction of the total bit rate $\mathcal{R}$ to source coding, incurring less distortion than FEC in the erasure-free case.

1. The bit rates $\mathcal{R}_1$ and $\mathcal{R}_2$ are fixed. Let $p_{\text{cliff},m}$, indexed by $m = \Delta_2/\Delta_1 \in \mathbb{Z}^+$, be the maximum erasure probability at which the system can provide error protection. Using (4), we have $p_{\text{cliff},m} \geq p_{\text{cliff},1}$. Thus, SLEP provides erasure protection over a wider range of erasure probabilities compared to FEC. As shown in Fig. 2(a), the distortion for FEC is constant for $p \leq p_{\text{cliff},1}$ and increases rapidly for $p > p_{\text{cliff},1}$ owing to the failure of the channel code. This is the familiar "cliff effect". In SLEP, the distortion increases gracefully owing to coarse quantization, as long as $p \leq p_{\text{cliff},m}$. Moreover, the cliff in SLEP is pushed further to the right, as compared to FEC. The larger the value of m, the greater the robustness of the error protection scheme.

2. The total bit rate $\mathcal{R}$ is fixed and the system is designed to tolerate a fixed maximum erasure probability p_{cliff}. Let $\mathcal{R}_{1,m}$ and $\mathcal{R}_{2,m}$ be the optimally chosen source coding bit rate and error resilience bit rate, depending upon the value of m. From (4) and the total bit rate constraint, $\mathcal{R}_{1,m} \geq \mathcal{R}_{1,1}$ and $\mathcal{R}_{2,m} \leq \mathcal{R}_{2,1}$. Thus, SLEP allocates more bits to the source code than FEC. For $p = 0$, the erasure-free case, the SNR with SLEP is higher than that with FEC by $20 p_{\text{cliff}} \log_{10} m$ dB. As shown in Fig. 2(b) for $0 \leq p \leq p_{\text{cliff}}$, FEC incurs constant distortion, while the distortion of SLEP increases with p. The system design ensures that the cliff occurs at probability $p = p_{\text{cliff}}$ for both FEC and SLEP. It can be shown that the distortion plots for FEC and SLEP must cross at:

$$p = \frac{(1-\rho^2)(m^{2p_{\text{cliff}}} - 1)}{m^2 - 1} < p_{\text{cliff}} \text{ for } m > 1$$

6 Conclusion

We have analyzed a simple error-resilient codec in which a first order Markov source is predictively encoded and transmitted over an erasure channel. In addition, a bit stream generated by Wyner-Ziv coding is used to provide lossy error protection. Using high-rate quantization theory, closed form expressions for rate and distortion have been derived for the encoding of the prediction residual and the overall encoding of the Markov source. Using these relations, it is shown that the lossy error protection property can be used to provide graceful degradation over a wider range of erasure probabilities compared to a lossless error correction approach like FEC.

Appendix

The following results are well-known and are provided for the sake of completeness in order to fill in the details in the proofs sketched in the main body of the paper. All references to stationarity will mean stationarity in the strict sense.

Definition 3. *$(U_n)_n$ and $(V_n)_n$ are defined to be jointly stationary processes if and only if the joint process $(U_n, V_n)_n$ is stationary.*

Lemma 4. *If $(U_n, V_n)_n$ is stationary, then $(U_n - V_n)_n$ is stationary.*

Recall, in the DPCM encoder, W_n is i.i.d. Then, by the above definition, W_n, $\widehat{W}_n$, $\widehat{\widehat{W}}_n$, and $\widetilde{W}_n$ are jointly stationary. By Lemma 4, the differences $W_n - \widehat{W}_n$, $W_n - \widehat{\widehat{W}}_n$, $W_n - \widetilde{W}_n$ are all stationary.

Lemma 5. *If U_n is stationary, and $V_n = h * U_n$, where h is the impulse response of a stable Linear Time Invariant (LTI) system, then $(U_n, V_n)_n$ is stationary.*

By the above lemma $(X_n, \widehat{X}_n, \widetilde{X}_n)_n = h * (W, \widehat{W}, \widetilde{W})_n$ is stationary, because $h(n) = \rho^n u(n)$ with $|\rho| < 1$ to ensure stability. By Lemma 4, this implies that $X_n - \widehat{X}_n$ is also stationary. Similarly, it may be shown that the differences, $\widehat{X}_n - \widetilde{X}_n$ and $X_n - \widetilde{X}_n$ are stationary. Therefore, the functionals $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$ may be defined by dropping the time index n.

Lemma 6. *Let $V_n = \rho V_{n-1} + U_n$, where $|\rho| < 1$ and $(U_n)_n$ is a stationary zero mean process with U_n independent of the past values V_{n-k}, $k \in \mathbb{Z}^+$. Then $\mathrm{E}\, V = 0$ and $\sigma_V^2 = \frac{\sigma_U^2}{1-\rho^2}$.*

Proposition 7. *Consider the embedded quantization scheme for quantizing W in which $m = \frac{\Delta_2}{\Delta_1} \in \mathbb{Z}^+$. Then, the MSE between the reconstruction functions of the finer quantizer and the coarser quantizer is given by*

$$\mathrm{E}(\widehat{W} - \widehat{\widehat{W}})^2 = (m^2 - 1)\frac{\Delta_1^2}{12} \simeq (m^2 - 1)\mathcal{D}_1 \tag{17}$$

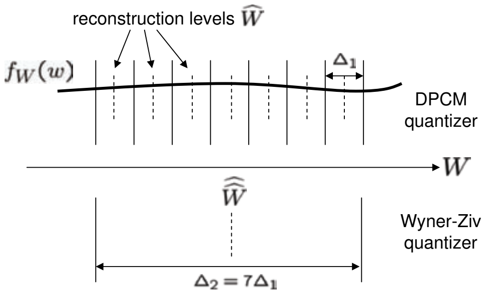


Figure 3: Embedded quantization (successive degradation) of W with $m = \Delta_2/\Delta_1 = 7$. Embedding increases the MSE by a factor of $(m^2 - 1)$

Proof. We prove the result for odd valued m. Note that the proof for even m follows the same method. By the high-rate assumption, W is approximately uniformly distributed over the width of the bins. Fig. 3 shows the embedded quantization scenario for $m = 7$. In this case,

$$\mathrm{E}(\widehat{W} - \widehat{\widehat{W}})^2 = \frac{2}{m} \sum_{i=1}^{\frac{m-1}{2}} i^2 \Delta_1^2 = (m^2 - 1)\frac{\Delta_1^2}{12} \simeq (m^2 - 1)\mathcal{D}_1$$

□

References

[1] S. Shamai, S. Verdú, and R. Zamir, "Systematic lossy source/channel coding," *IEEE Transactions on Information Theory*, vol. 44, no. 2, pp. 564–579, Mar. 1998.

[2] A. D. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," *IEEE Transactions on Information Theory*, vol. IT-22, no. 1, pp. 1–10, Jan. 1976.

[3] S. Rane, P. Baccichet, and B. Girod, "Modeling and Optimization of a Systematic Lossy Error Protection System," in *Picture Coding Symposium (PCS 2006)*, Beijing, China, Apr. 2006.

[4] P. Baccichet, S. Rane, and B. Girod, "Systematic Lossy Error Protection using H.264/AVC Redundant Slices and Flexible Macroblock Ordering," in *Proc. IEEE Packet Video Workshop*, Hangzhou, China, Apr. 2006.

[5] J. D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Transactions on Information Theory*, vol. IT-19, pp. 471–480, July 1973.

[6] W. R. Bennett, "Spectra of quantized signals," Technical Journal 27, Bell Systems, July 1948.

[7] A. Gersho and R. M. Gray, *Vector quantization and signal compression*, Kluwer Academic Publishers, Boston, MA, 1992.

[8] D. Rebollo-Monedero, S. Rane, A. Aaron, and B. Girod, "High-Rate Quantization and Transform Coding with Side Information at the Decoder," *EURASIP Signal Processing journal, Special Issue on Distributed Source Coding*, vol. 86, pp. 3160–3179, Apr. 2006.