Multiobjective $H_2/H_\infty$-Optimal Control via Finite Dimensional $Q$-Parametrization and Linear Matrix Inequalities

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In this talk, we focus on $H_\infty$. Same ideas carry through for $H_2/H_\infty$ as well - see paper.
Set of all achievable stable closed loop maps is:

\[
\{G = P_{zr}w_e + P_{zr}uK(I - P_{yu}K)^{-1}P_{yw} | K \text{ stabilizing}\}
\]

- set of stabilizing $K$’s not obvious
- parametrization is linear fractional
- $P$’s and $K$ can be unstable

Can transform into equivalent parametrization:

\[
\{G = H - UQV | Q \text{ stable}\}
\]

- now $H$, $U$, $V$ and $Q$ stable
- affine in $Q \rightarrow$ Good for optimization
General Regulator Problem

\[ z_r = \begin{bmatrix} z \\ u \end{bmatrix} = G w_e = \begin{bmatrix} G_z w_e \\ G_u w_e \end{bmatrix} w_e \]

- Typically want:
  1. Small \( \|G_z w_e\|_\infty \) for “good regulation”
  2. Small \( \|G_u w_e\|_\infty \) for “efficient control”
  3. “Reject” disturbances \( w_e = \begin{bmatrix} w \\ v \end{bmatrix} \)

- Usually **conflicting** requirements:
  “good” regulation requires “large” control
Multiobjective Design Paradigm

- Define:

\[ J_\lambda^{M}(Q) = (1 - \lambda) \|G_{zw e}(Q)\|_\infty^2 + \lambda \|G_{uw e}(Q)\|_\infty^2 \]

- Compute **tradeoff curve**: 

  for \( \lambda = 0 \) to 1 

  solve for \( Q_\lambda: \inf_{Q \in H_\infty} J_\lambda^{M}(Q) \)

  plot \( \|G_{zw e}(Q_\lambda)\|_\infty \) versus \( \|G_{uw e}(Q_\lambda)\|_\infty \)

  end

- **Tradeoff curve gives limits of performance** - very useful in practice!

![Diagram](attachment:diagram.png)
Standard vs Multiobjective $H^\infty$

**Standard $H^\infty$ Problem:** $z_r = \begin{bmatrix} (1 - \lambda)^{\frac{1}{2}} z \\ \lambda^{\frac{1}{2}} u \end{bmatrix}$ minimize

$$J^S_\lambda(Q) = \left\| \begin{bmatrix} (1 - \lambda)^{\frac{1}{2}} G_{zw_e}(Q) \\ \lambda^{\frac{1}{2}} G_{uw_e}(Q) \end{bmatrix} \right\|_\infty^2 = \sup_{w_e \neq 0} \frac{(1 - \lambda) \|z\|_2^2 + \lambda \|u\|_2^2}{\|w_e\|_2^2}$$

**Multiobjective $H^\infty$ Problem:** minimize

$$J^M_\lambda(Q) = (1 - \lambda) \|G_{zw_e}(Q)\|_\infty^2 + \lambda \|G_{uw_e}(Q)\|_\infty^2 = (1 - \lambda) \sup_{w_e \neq 0} \frac{\|z\|_2^2}{\|w_e\|_2^2} + \lambda \sup_{w_e \neq 0} \frac{\|u\|_2^2}{\|w_e\|_2^2}$$

**Comments**

- In multiobjective design maximization of $z$ and $u$ over $w_e$ is done **independently**
- In standard design maximization of $z$ and $u$ over $w_r$ is done **simultaneously** - artificially couples $z$ and $u$
- Why would we care about the gain from $w_e$ to the sum of $z$ and $u$? They might peak at different frequencies.
More Remarks

• Note that since

\[ J^S_\lambda = \text{"sup of sum"} \]

\[ J^M_\lambda = \text{"sum of sups"} \]

we have

\[ J^S_\lambda(Q) \leq J^M_\lambda(Q) \quad \forall Q \in H_\infty \]

\[ \inf_{H_\infty} J^S_\lambda \leq \inf_{H_\infty} J^M_\lambda \]

• Also, since \( G_{zw_e}(Q) \) and \( G_{uw_e}(Q) \) are both affine in \( Q \)

\[ \implies \text{ both problems convex} \]

• Finally, note that the problems are infinite dimensional

• In Standard problem, state space structure provides means for minimizing exactly via bisection applied to Riccati equations or LMI.

• In Multiobjective problem cannot solve exactly in general. Can only minimize conservative upper bound. But no analysis for degree of conservativeness.

• So why not use finite dimensional \( Q \)-based approach which fell out of favor because no analysis was available for degree of approximation?
Previous Research

- State space, upper bound on $H_\infty/H_2$
  - '89: Bernstein & Haddad
  - '91: Khargonekar & Rotea

- Finite dimensional $Q$, convex optimization
  - '88: Boyd, Barratt, Balakrishnan, Kabamba & Meyer
  - '94: Sznaier, Rotstien & Sideris

- Finite dimensional $Q$ and LMIs
  - '95: Chen & Wen
  - '95: Scherer - our method was first proposed

- Lyapunov Shaping, LMIs
  - '95: Scherer, Gahinet & Chilali
  - '95: El-Ghaoui & Folcher

- Solve nonconvex problem
  - '98: Halder, Hassibi & Kailath
\[ \|G\|_\infty \text{ via Bounded Real Lemma} \]

- To avoid truncation errors of QDES, we use state space
- Given closed loop system \( G \) with then

\[
\|G\|_\infty \equiv \|D + C(zI - A)^{-1}B\|_\infty = \gamma^* 
\]

**if and only if** \( \gamma^* \) is optimizer of

\[
\begin{align*}
\text{minimize} & \quad \gamma \\
\text{subject to} & \quad \begin{bmatrix}
A^T X A - X & A^T X B & C^T \\
B^T X A & B^T X B - \gamma I & D^T \\
C & D & -\gamma I
\end{bmatrix} < 0 \\
& \quad X > 0
\end{align*}
\]

- \( A, B, C, D \) are closed loop matrices - contain controller variables
- Due to cross terms between \( A, B, \) and \( X \), have nonlinear matrix inequality
- In ’93 ’94, Gahinet & Apkarian and Iwasaki & Skelton showed that using elimination lemma can reduce to 3 LMI’s

(Similar LMI’s exist for \( H_2 \) norm)
Multiobjective $\mathcal{H}_\infty$ Problem

- We now want to minimize

$$(1 - \lambda) \|G_z\|_\infty + \lambda \|G_u\|_\infty$$

- Apply **bounded real lemma** to $G_z$ and $G_u$ separately

$\rightarrow$ **SDP** in $\gamma_z, \gamma_u, X_z, X_u$, and closed loop matrices of $G_z$ and $G_u$:

$$\min \quad (1 - \lambda) \gamma_z + \lambda \gamma_u$$

$$\begin{bmatrix}
A^T_z X_z A_z - X_z & A^T_z X_z B_z & C^T_z \\
B^T_z X_z A_z & B^T_z X_z B_z - \gamma_z I & D^T_z \\
C_z & D_z & -\gamma_z I
\end{bmatrix} < 0$$

$$X_z > 0$$

$$\begin{bmatrix}
A^T_u X_u A_u - X_u & A^T_u X_u B_u & C^T_u \\
B^T_u X_u A_u & B^T_u X_u B_u - \gamma_u I & D^T_u \\
C_u & D_u & -\gamma_u I
\end{bmatrix} < 0$$

$$X_u > 0$$

- Again **cross terms** between $A$’s, $X$’s and $B$’s.

- But now **elimination lemma fails**

- Note $C$’s and $D$’s appear **linearly**

- If could put all controller variables in $C$’s and $D$’s, get LMI’s - **done!**. This is our goal.
State Space SISO FIR

• Given FIR system $Q$ with **pulse response**

\[
\{ q_0, q_1, q_2, q_3, 0, 0, \ldots \} \]

• We have (control canonical form) **realization**

\[
\begin{bmatrix}
  A_Q & B_Q \\
  C_Q & D_Q
\end{bmatrix} \equiv \begin{bmatrix}
  0 & \begin{bmatrix} 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
  q_1 & q_2 & q_3 & q_0
\end{bmatrix}
\]

• **All variables** $q_i$ are in $C_Q$ and $D_Q$ matrices.
• Matrices $A_Q$ and $B_Q$ are **fixed**.
• Later on, we will **assume** that the component SISO systems $Q_{ij}$ of $Q$ in the $Q$-parametrization are all SISO FIRs.
Pulling Out $Q$

- Recall that in $Q$-parametrization: $H$, $U$, $Q$, and $V$ are just matrices in $\mathcal{H}_\infty$.
- Want to write

$$G(Q) = H - U Q V$$

in terms of **SISO components** of $Q$ explicitly.
- Decompose $Q$ as sum of its **SISO components** $Q_{rs}$ times elementary matrices $E_{rs} = e_r e_s^T$:

$$Q = \sum_{r,s} Q_{rs} e_r e_s^T$$

- Hence:

$$G(Q) = H - U \left( \sum_{r,s} Q_{rs} e_r e_s^T \right) V$$

$$= H - \sum_{r,s} Q_{rs} \left( (U e_r)(e_s^T V) \right)$$

- Therefore:

$$G(Q) = H - \sum_{r,s} Q_{rs} T_{rs}$$

where $T_{rs} = (U e_r)(e_s^T V)$. 

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Kronecker Products

• So we have

\[ G(Q) = H - \sum_{rs} Q_{rs} T_{rs} \]

• Now \( Q_{rs} T_{rs} \) is just \textbf{scalar} (SISO) \( \times \) \textbf{matrix} (MIMO) in \( \mathcal{H}_\infty \). So

\[
Q_{rs} T_{rs} \triangleq \begin{bmatrix}
Q_{rs} T_{rs}^{(11)} & \cdots & Q_{rs} T_{rs}^{(1n)} \\
\vdots & \ddots & \vdots \\
Q_{rs} T_{rs}^{(m1)} & \cdots & Q_{rs} T_{rs}^{(mn)}
\end{bmatrix}
\]

\[ = Q_{rs} \otimes T_{rs} \]

where \( \otimes \) denotes \textbf{Kronecker multiplication}

\[ A \otimes B \triangleq \begin{bmatrix}
a_{11} B & \cdots & a_{1n} B \\
\vdots & \ddots & \vdots \\
a_{m1} B & \cdots & a_{mn} B
\end{bmatrix} \in \mathbb{R}^{mp \times nq} \]

• So to be \textbf{explicit} we write:

\[ G(Q) = H - \sum_{rs} Q_{rs} \otimes T_{rs} \]
State Space Representation of $Q \otimes T$

- **Given:** $Q \in \mathcal{H}^{p \times q}$ and $T \in \mathcal{H}^{m \times n}$

\[
Q \equiv \begin{bmatrix}
A_Q & B_Q \\
C_Q & D_Q
\end{bmatrix} \quad \text{and} \quad T \equiv \begin{bmatrix}
A_T & B_T \\
C_T & D_T
\end{bmatrix}
\]

Then: $Q \otimes T$ has state space

\[
\begin{bmatrix}
A_Q \otimes I_m & B_Q \otimes C_T & B_Q \otimes D_T \\
0 & I_q \otimes A_T & I_q \otimes B_T \\
C_Q \otimes I_m & D_q \otimes C_T & D_Q \otimes D_T
\end{bmatrix}
\]

- If $Q$ has **SISO FIR structure**, then all coefficients $q_i$ of $Q$ (contained in $C_Q$ & $D_Q$) appear only in $C_Q \otimes T$ and $D_Q \otimes T$. 
State Space for Closed Loop System $G$

- **Assume:** that $Q$ is SISO, then there’s just one $Q$ and one $T$. Can then drop $r$ and $s$ indexes:

$$\sum_{r,s} Q_{rs} \otimes T_{rs} = Q \otimes T.$$  

(general case same idea - see paper)

- Then closed loop transfer function

$$G(Q) = H - Q \otimes T$$

- This is just $H$ in parallel with $-(Q \otimes T)$.
- Therefore it’s easy to write down state space for $G$:

$$\begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} = \begin{bmatrix} A_H & A_{Q \otimes T} & B_H \\ C_H & -C_{Q \otimes T} & B_{Q \otimes T} \\ D_H & D_{Q \otimes T} & D_{Q \otimes T} \end{bmatrix}.$$  

- Note that if $Q$ is FIR, then all coefficients of $Q$ are contained in $C_G$ and $D_G$.
State Space for Multiobjective Closed Loop

- Start with

\[ G(Q) = H - \sum_{r,s} Q_{rs} \otimes T_{rs}. \]

- **Partition** \( G, H, T \) according to \( z_r = \begin{bmatrix} z \\ u \end{bmatrix} \):

\[
\begin{bmatrix}
G_z(Q) \\
G_u(Q)
\end{bmatrix} = \begin{bmatrix} H_z \\ H_u \end{bmatrix} + \sum_{r,s} \begin{bmatrix} Q_{rs} \otimes T_{z,rs} \\ Q_{rs} \otimes T_{u,rs} \end{bmatrix}
\]

- Again **assume** just one \( Q \) and \( T \).
- **Now get state space** of \( G_z \) and \( G_u \):

\[
\begin{bmatrix}
A_z & B_z \\
C_z & D_z
\end{bmatrix} = \begin{bmatrix}
A_{Hz} & B_{Hz} \\
A_{Q \otimes T_z} & B_{Q \otimes T_z}
\end{bmatrix} \begin{bmatrix}
C_{Hz} & -C_{Q \otimes T_z} \\
D_{Hz} & D_{Q \otimes T_z}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_u & B_u \\
C_u & D_u
\end{bmatrix} = \begin{bmatrix}
A_{Hu} & B_{Hu} \\
A_{Q \otimes T_u} & B_{Q \otimes T_u}
\end{bmatrix} \begin{bmatrix}
C_{Hu} & -C_{Q \otimes T_u} \\
D_{Hu} & D_{Q \otimes T_u}
\end{bmatrix}
\]

- Note that if \( Q \) is FIR, then all **coefficients** of \( Q \) are contained in \( C_z, D_z \) and \( C_u, D_u \) → **done**!
Numerical Example

The graph illustrates tradeoffs between multiobjective (solid line) and standard (dashed line) approaches.

- **System was:**
  - unstable, second order, $f_0 = 1$, $\zeta = -0.5$.
  - discretized at $T_s \approx 1/6$
  - $0.9T_s$ delay in loop

- Stabilized with LQG to get $H$, $U$, and $V$

- Modified with **12-tap FIR $Q$**

**Result:** 25% reduction in control effort!
Conclusion

Proposed Method

- based on Q-parametrization & finite dimensional convex optimization

- conservative, but can outperform standard $H_\infty$ and Lyapunov shaping

- extends to $H_2/H_\infty$ (and other problems)

- involves more computation than standard methods, but structure can be exploited for speedup