# Closed-loop Convex Analysis of Performance Limits for Linear Control Systems

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## Abstract

In *closed-loop convex analysis and design*, the linear controller design problem is reformulated as a convex optimization problem, which may be more easily solved than the problems resulting from conventional formulations. This reformulation has several unconventional aspects: it synthesizes control laws that have high degree, but nevertheless can be implemented using digital signal processors; the "solution" is rarely expressed in a "closed-form," but nevertheless is readily computable.

This paper describes some of the basic ideas of closed-loop convex analysis and design, in general terms. Full details can be found in the references cited.

## 1 Motivation

Closed-loop convex methods are motivated in part by the following technological developments: high quality integrated sensors and actuators, powerful control processors that can implement complex control algorithms, and powerful computer hardware and software that can be used to design and analyze control systems. We believe that these technological developments have the following ramifications for linear controller design:

- When many high quality sensors and actuators are incorporated into the design of a system, sophisticated control algorithms can outperform the simple control algorithms that have sufficed in the past.
- Current methods of computer-aided control system design underutilize available computing power and need to be rethought.

These technology advances present a number of challenges for controller design:

- *More sensors and actuators.* For only a modest cost, it is possible to incorporate many more sensors, and possibly more actuators, into the design of a system. Clearly the extra information coming from the sensors and the extra degrees of freedom in manipulating the system make better control system performance possible. The challenge for controller design is to take advantage of this extra information and degrees of freedom.
- *Higher quality systems.* As higher quality sensors and actuators are incorporated into the system, the system behavior becomes more repeatable and can be more accurately modeled. The challenge for controller design is to take advantage of this more detailed knowledge of the system.
- *More powerful control processors.* Very complex control laws can be implemented using digital control processors. Clearly a more complex control law could improve control system performance (it could also degrade system performance, if improperly designed). The challenge for controller design is to fully utilize the control processor power to achieve better control system performance.

  In particular, control law specifications should be examined carefully. Historically relevant measures of control law complexity, such as the order of an LTI controller, are now less relevant. For example, the order of the compensator used in a vacuum tube feedback amplifier is the number of inductors and capacitors needed to synthesize the compensation network, and was therefore related to cost, size, and reliability. On a particular digital control processor, however, the order of the controller is essentially unrelated to cost, size, and reliability.
- *Powerful computers to design controllers.* The challenge for controller design is to productively use the enormous computing power available. Many current methods of computer-aided controller design simply automate procedures developed in the 1930's through the 1950's, for example, plotting root loci or Bode

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plots. Even the "modern" state-space and frequency-domain methods (which require the solution of algebraic Riccati equations) greatly underutilize available computing power.

## 2 Summary

Closed-loop convex analysis addresses a restricted but important class of control system problems. The restriction on the systems considered is that they must be linear and time-invariant (LTI). The restriction on the design specifications is that they be *closed-loop convex*, which means that the specifications can be expressed as convex constraints on some closed-loop transfer function or transfer matrix of the closed-loop system. The precise definition, and many examples, can be found in the references. This restricted set of design specifications includes a wide class of performance specifications, and a less complete class of robustness specifications.

Specifications that limit the order or constrain the stucture of a control law are generally *not* closed-loop convex. Our opinion is that many of these specifications are no longer relevant in view of the technology advances described above.

The basic approach involves directly designing a good closed-loop response, as opposed to designing an open-loop controller that yields a good closed-loop response. Given a system that is LTI, and a set of closed-loop convex design specifications, the controller design problem can be cast as a convex optimization problem, and consequently, can be effectively solved. This means that if the specifications are achievable, we can find a controller that meets the specifications; if the specifications are not achievable, this fact can be determined, *i.e.*, we will *know* that the specifications are not achievable. Since we can determine numerically which specifications can be achieved and which cannot, we can determine the *limits of performance* for a given system and control configuration.

In contrast, the designer using a classical controller design scheme is only *likely* to find a controller that meets a given set of specifications that is achievable; and, of course, certain not to find a controller that meets a set of specifications that is not achievable. The problem, however, is to know when to abandon the search for a control law that achieves the specifications, since many controller design techniques do not have any way to determine unambiguously that a set of specifications is not achievable.

No matter which controller design method is used by the engineer, knowledge of the achievable performance is extremely valuable practical information, since it provides an absolute yardstick against which any designed controller can be compared. To know that a certain candidate controller that is easily implemented, or has some other advantage, achieves regulation only 10% worse than the best regulation achievable by *any* LTI controller, is a strong point in favor of the design. In this sense, closed-loop convex analysis is not a particular controller design method or synthesis procedure; rather it is a method of determining what specifications (of a large but restricted class) can be met using any controller design method, for a given system and control configuration.

## 3 An Example

We can demonstrate some of the main ideas with an example. We will consider a specific system that has one actuator and one output that is supposed to track a command input, and is affected by some noises; the system is described in detail in section 2.4 of [4], but the details are not relevant for this example.

Goals for the design of a controller for this system might be:

- *Good RMS regulation*, *i.e.*, the root-mean-square (RMS) value of the output, due to the noises, should be small.
- *Low RMS actuator effort*, *i.e.*, the RMS value of the actuator signal should be small.

It is intuitively clear that by using a larger actuator signal, we may improve the regulation, since we can expend more effort counteracting the effect of the noises. The exact nature of this tradeoff between RMS regulation and RMS actuator effort can be determined; it is shown in figure 1. The shaded region shows every pair of RMS regulation and RMS actuator effort specifications that can be achieved by a controller; the designer must, of course, pick one of these.

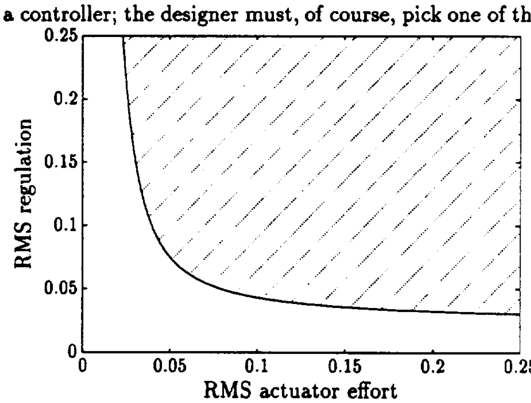


**Figure 1** The shaded region shows specifications on RMS actuator effort and RMS regulation that are achievable. The unshaded region, at the lower left, shows specifications that no controller can achieve: this region shows a fundamental limit of performance for this system.

The *unshaded* region at the lower left is very important: it consists of RMS regulation and RMS actuator effort specifications that cannot be achieved by any controller, no matter which design method is used. This unshaded region therefore describes a *fundamental limit of performance* for this system. It tells us, for example, that if we require an RMS regulation of 0.05, then we *cannot* simultaneously achieve an RMS actuator effort of 0.05.

Each shaded point in figure 1 represents a possible design; we can view many controller design methods as "rummaging around in the shaded region". If the designer knows that a point is shaded, then the designer can find a controller that achieves the corresponding specifications, if the designer is clever enough. On the other hand, each unshaded point represents a limit of performance for our system. Knowing that a point is unshaded is perhaps disappointing, but still very useful information for the designer.

This tradeoff of RMS regulation against RMS actuator effort can be determined using LQG theory. The main point of closed-loop convex analysis is that for a much wider class of specifications, a similar tradeoff curve can be computed. Suppose, for example, that we add the following specification to our goals above:

- *Command to output overshoot limit, i.e.*, the step response overshoot of the closed-loop system, from the command to the output, does not exceed 10%.

Of course, intuition tells us that by adding this specification, we make the design problem "harder": certain RMS regulation and RMS actuator effort specifications that could be achieved without this new specification will no longer be achievable once we impose it.

In this case there is no analytical theory, such as LQG, that shows us the exact tradeoff. Closed-loop convex analysis, however, can be used to determine the exact tradeoff of RMS regulation versus RMS actuator effort with the overshoot limit imposed. This tradeoff is shown in figure 2. The dashed line, below the shaded region of achievable specifications, is the tradeoff boundary when the overshoot limit is not imposed. The "lost ground" represents the cost of imposing the overshoot limit. We can compute this new region because limits on RMS actuator effort, RMS regulation, and step response overshoot are all closed-loop convex specifications.

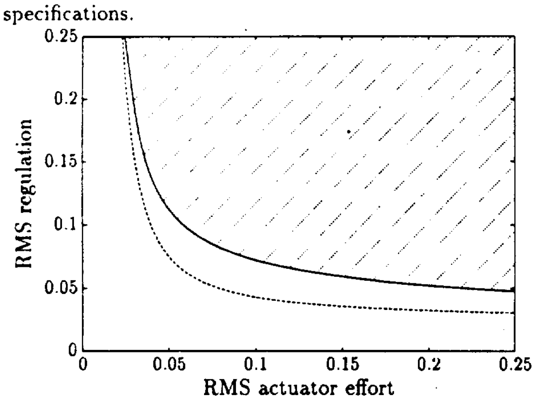


**Figure 2** The shaded region shows specifications on RMS actuator effort and RMS regulation that are achievable when an additional limit of 10% step response overshoot is imposed; it can be computed using closed-loop convex methods. The dashed line shows the tradeoff boundary without the overshoot limit; the gap between this line and the shaded region shows the cost of imposing the overshoot limit.

In contrast, suppose that instead of the overshoot limit, we impose the following control law constraint:

- *The controller is proportional plus derivative (PD), i.e.*, the control law has a specific form.

This constraint might be needed to implement the controller using a specific commercially available control processor. This specification is *not* closed-loop convex, so closed-loop convex methods *cannot* be used to determine the exact tradeoff between RMS actuator effort and RMS regulation. This tradeoff can be computed, however, using a brute force global optimization approach (see for example, [1]) and is shown in figure 3. The dashed line is the tradeoff boundary when the PD controller constraint is not imposed. Specifications on RMS actuator effort and RMS regulation that lie in the region between the dashed line and the shaded region can be achieved by some controller, but no PD controller.

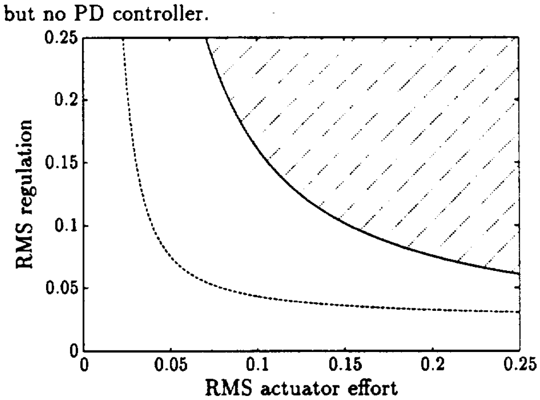


**Figure 3** The shaded region shows specifications on RMS actuator effort and RMS regulation that can be achieved using a PD controller; it cannot be computed using closed-loop convex methods. It can be computed, at much greater cost, using global optimization methods. The dashed line shows the tradeoff boundary when no constraint on the control law is imposed.

An important conclusion is that we can compute tradeoffs among closed-loop convex specifications, such as shown in figure 2, although it requires more computation than determining the tradeoff for a problem that has an analytical solution, such as shown in figure 1; in return, however, a much larger class of problems can be considered. While the computation needed to determine a tradeoff such as shown in figure 2 is more than that required to compute the tradeoff shown in figure 1, it is *much less* than the computation required to compute tradeoffs such as the one shown in figure 3.

The fact that a tradeoff like the one shown in figure 3 is much harder to compute than a tradeoff like the one shown in figure 2 presents a paradox. To produce figure 2 we search over the set of all possible LTI controllers, which has infinite dimension. To produce figure 3, however, we search over the set of all PD controllers, which has dimension two. In fact, convexity makes figure 2 "easier" to produce than figure 3, even though we must search over a far "larger" set of potential controllers.

## 4 Notes

The survey paper[5] and book[4] contain extensive summaries of the closed-loop convex method and large bibliographies. We note here that the method can be traced back at least forty years, to Truxal's 1950 Ph.D. thesis [13] and 1955 book, *Automatic Feedback Control System Synthesis*, in which he credits Guilleman with developing the idea in 1947. The idea comes up again in a series of papers by Fegley and colleagues in the sixties (a summary appears in [7]). In the eighties, the method is considered by Desoer and colleagues (for example, [6, 8]), by Salcudean and Polak [12, 11], and Boyd and colleagues [3].

The article [10] describes the implementation of a controller designed by closed-loop convex methods for a flexible robotic arm. This short and preliminary paper demonstrates, at the least, that it is possible to design and implement on a real physical system a very high order controller which works. (See the discussion above regarding measures of controller complexity—in this case the MacMillan degree of the controller is a traditional but inappropriate measure, while the elapsed real time per sample is more appropriate.)

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