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Parameter Set Estimation of Systems with Uncertain Nonparametric Dynamics and Disturbances

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Abstract A method is presented for parameter set estimation of open-loop stable systems with uncertain parameters and quasi-stationary disturbances. The system model is assumed to contain both parametric and nonparametric uncertainty. Parameter set estimates are obtained from finite data records and they have the property that the parameter set of the true plant is contained in the estimated sets with high probability.

1 Introduction

One difficulty in designing an adaptive control system is that the system which generated the measured data is not in the model set used to develop the parameter estimator, *e.g.*, [12], [1]. The model set involves some unknown parameters where their values are presumed to account for the measured data. An alternate approach is to configure an adaptive control system which specifically accounts for this model uncertainty. Depicted in Figure 1 is one such scheme where the usual parameter estimator is replaced with an estimator that produces a model set. This type of estimator is referred to as an *uncertainty estimator* or a *set estimator*. The robust control design rule then accepts the model set in the form produced by the set estimator. Under these conditions, if the system which generated the measured data is contained in the estimated set, then the adaptive system is not only stable, but if carefully designed may achieve the maximum performance possible given the estimated set of uncertainty.

Proceeding in this way we have transformed the problem of adaptive control design into separate synthesis problems in set estimation and robust control design.

At present, methodologies for the design of set estimators are under development, *e.g.*, [15], [11], [9], [8], [10], and [17]. On the other hand, there is a reasonable maturity of methodologies for robust control design, particularly for plants with uncertain nonparametric linear dynamics, *e.g.*, [14], [4], [7]. Methods for robust control design of plants with parametric uncertainty are described in [2, 3] and the references therein.

In this paper we address the problem of parameter set estimation where the system model contains both parametric and nonparametric uncertainty as well as quasi-stationary disturbances. As a result, parameter set estimates, which are computable from *finite* data records, have the property that the true parameter set is in the estimated sets with high probability. This work follows that described in [16, 17, 10] for the disturbance free case with nonparametric uncertainty. The case with no nonparametric uncertainty but with bounded disturbances has been examined in [5, 6, 13].

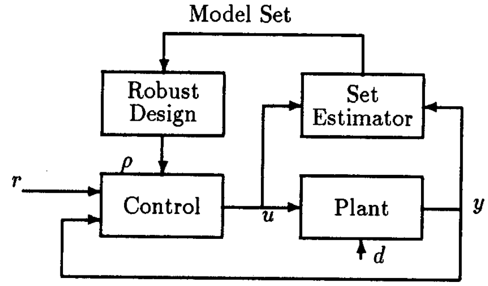

Figure 1: Adaptive control with set estimator.

Notation and Preliminaries We consider sampled-data systems with transfer functions in the complex variable z. A transfer function $G(z)$ is *stable* if all its poles are strictly inside the unit circle $|z| = 1$. The frequency response of $G(z)$ is $\{G(e^{j\omega}) : |\omega| \leq \pi\}$ where ω is the frequency variable normalized with respect to the sampling frequency. For a stable transfer function $G(z)$, the $\mathcal{H}_\infty$ norm is given by

$$\|G\|_{\mathcal{H}_\infty} \triangleq \sup_{|\omega|\leq\pi} |G(e^{j\omega})|$$

A sequence x is evaluated at discrete time points, *i.e.*, $x = \{x(t) : t = 1, 2, \ldots\}$. To reduce notation, we also use z to denote the shift operator, so $z^k x(t) = x(t + k)$, $q^{-k}x(t) = x(t - k)$, and we sometimes write $G(z)x(t)$.

Let $\mathcal{E}(\cdot)$ denote the expectation operator. Following [12], a sequence x is *quasi-stationary* if the following limit

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exists:

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathcal{E}\left[x(t)x(t-\tau)\right], \ \forall\tau$$

The power spectrum of x is defined as

$$S_{xx}(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} r_{xx}(\tau)e^{-j\omega\tau}$$

Similar definitions apply to the cross spectrum $S_{xy}(\omega)$ of the sequences x and y. Note that by the definition of the inverse Fourier transform,

$$\mathcal{E}(x^2) = r_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)d\omega \quad (1)$$

We also use the sample-mean operator $\mathcal{E}_N(\cdot)$,

$$\mathcal{E}_N(x) \triangleq \frac{1}{N}\sum_{t=1}^{N} x(t)$$

2 Problem Formulation

The problem is to use the measured sampled data

$$\mathcal{Z}^N = \{y(t), u(t) : t = 1, \ldots, N\} \quad (2)$$

to identify a model set suitable for robust control design. The system which produced the data is assumed to have the input-output form

$$\begin{aligned} y(t) &= G(z)u(t) + v(t) && (3)\\ v(t) &= H(z)e(t) && (4) \end{aligned}$$

where $u(t)$ is an applied input, $y(t)$ is the measured output, and $v(t)$ is a disturbance. The disturbance $v(t)$ is represented as the output of an linear-time-invariant system with transfer function $H(z)$ whose input is a random sequence e. In general the input $u(t)$ could include a feedback from $y(t)$, but we will assume that no feedback is present and $G(z)$ and $H(z)$ are stable. Unless required for clarity, in the sequel, the arguments z and t will usually be omitted.

The model sets for G, H, and e are as follows:

- G has the structure

  $$G = G_\theta(1 + \Delta_G W_G) \quad (5)$$

  where G_θ is a parametric transfer function with parameters $\theta \in \mathbb{R}^p$ and Δ_G is a stable transfer function satisfying,

  $$\|\Delta_G\|_{\mathcal{H}_\infty} \leq 1 \quad (6)$$

- H has the structure

  $$H = \Delta_H W_H \quad (7)$$

  where Δ_H is a stable transfer function satisfying,

  $$\|\Delta_H\|_{\mathcal{H}_\infty} \leq 1 \quad (8)$$

- e is a sequence of independent random variables with zero means such that,

  $$S_{ee}(\omega) \leq \sigma^2, \quad S_{eu}(\omega) = 0, \quad \forall|\omega| \leq \pi \quad (9)$$

The model set is characterized by the parametric structure of G_θ, W_G, W_H, and σ. Prior knowledge about the set is given as follows:

- G_θ has the standard parametric ARMA form,

  $$\begin{aligned} G_\theta &= B_\theta/A_\theta && (10)\\ B_\theta &= b_1z^{-1} + \cdots + b_mz^{-m} && (11)\\ A_\theta &= 1 + a_1z^{-1} + \cdots + a_nz^{-n} && (12)\\ \theta &= [a_1 \cdots a_n \ \ b_1 \cdots b_m]^T && (13) \end{aligned}$$

- W_G is stable and known.

- W_H is stable, stably invertible, and known.

- σ is known.

The problem we address is to find a set estimate of θ by using the above model set, prior information, and the set of measured data (2).

Before proceeding, we remark that the disturbance set as posed here introduces some limitations. First, since v is quasi-stationary, it follows that a finite data record such as (2) can only produce a "soft"-bounding parameter set, *i.e.*, for large N, the parameter set of the true plant is contained in the set estimate with high probability. A second limitation is that the disturbance set does not depend on the parameters θ. This is adequate when v is a pure output disturbance, *e.g.*, sensor noise. However, if v is the result of a disturbance acting through the plant dynamics, then our representation provides a potentially coarse upper bound.

3 Parameter Set Estimation

From the model structure described in the previous section, the parameter set of the true plant is given by (5) and (6):

True-Plant Parameter Set

$$\Theta^* \triangleq \left\{\theta : \left\| \frac{A_\theta G - B_\theta}{W_G B_\theta} \right\|_{\mathcal{H}_\infty} \leq 1 \right\} \tag{14}$$

The interpretation of the true-plant parameter set is that for every $\theta \in \Theta^*$, there exists a Δ_G satisfying (6) such that $G_\theta(1 + \Delta_G W_G)$ is exactly the true G. Note that computing Θ^* is not possible because it requires G. In addition, Θ^* is the intersection of infinite number of sets from all frequencies.

Our goal is to obtain an estimate of Θ^*. We will start with the plant model set

$$y = G_\theta(1 + \Delta_G W_G)u + \Delta_H W_H e \tag{15}$$

or equivalently,

$$A_\theta y - B_\theta u = \Delta_G W_G B_\theta u + \Delta_H W_H A_\theta e \tag{16}$$

To bring the parameter θ out explicitly, we now define the following:

$$\begin{aligned}
\phi &\triangleq [-D_n y \; D_m u]^T & (17)\\
D_k(z) &\triangleq [z^{-1} \cdots z^{-k}]^T & (18)\\
\theta_A &\triangleq [a_1 \cdots a_n]^T & (19)\\
\theta_B &\triangleq [b_1 \cdots b_m]^T & (20)\\
\psi_u &\triangleq W_G D_m u & (21)\\
\psi_e &\triangleq D_n e & (22)
\end{aligned}$$

Now (16) can be rewritten as

$$W_H^{-1}(y - \theta^T\phi) = \Delta_G W_H^{-1}(\theta_B^T\psi_u) + \Delta_H(e + \theta_A^T\psi_e) \tag{23}$$

After squaring both sides of (23) and taking expectations, we get

$$\begin{aligned}
&\mathcal{E}(W_H^{-1}y)^2 - 2\theta^T\mathcal{E}(W_H^{-1}\phi)(W_H^{-1}y) + \\
&\theta^T\mathcal{E}(W_H^{-1}\phi)(W_H^{-1}\phi)^T\theta = \\
&\qquad \mathcal{E}\left(\Delta_G W_H^{-1}\theta_B^T\psi_u\right)^2 + \mathcal{E}\left(\Delta_H e\right)^2 + \\
&\qquad \mathcal{E}\left(\Delta_H \theta_A^T\psi_e\right)^2
\end{aligned} \tag{24}$$

where the cross terms disappeared because e and u are independent and e is independently distributed. We now use (1), (6), (8), and (9) to obtain

$$\begin{aligned}
\mathcal{E}\left(\Delta_G W_H^{-1}\theta_B^T\psi_u\right)^2 &\leq \mathcal{E}\left(W_H^{-1}\theta_B^T\psi_u\right)^2 & (25)\\
\mathcal{E}\left(\Delta_H e\right)^2 &\leq \sigma^2 & (26)\\
\mathcal{E}\left(\Delta_H \theta_A^T\psi_e\right)^2 &\leq \sigma^2\theta_A^T\theta_A & (27)
\end{aligned}$$

Thus, (24) now becomes

$$\begin{aligned}
&\mathcal{E}(W_H^{-1}y)^2 - 2\theta^T\mathcal{E}(W_H^{-1}\phi)(W_H^{-1}y) + \\
&\theta^T\mathcal{E}(W_H^{-1}\phi)(W_H^{-1}\phi)^T\theta \leq \\
&\qquad \mathcal{E}\left(W_H^{-1}\theta_B^T\psi_u\right)^2 + \sigma^2 + \sigma^2\theta_A^T\theta_A
\end{aligned} \tag{28}$$

which describes a parameter set computable only from an infinite data record. This set is now formally defined as the

Infinite-Data Parameter Set Estimate

$$\Theta_\infty \triangleq \{\theta : \alpha - 2\beta^T\theta + \theta^T\Gamma\theta \leq 0\} \tag{29}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^p$, and $\Gamma \in \mathbb{R}^{p\times p}$ are given by

$$\begin{aligned}
\alpha &= \mathcal{E}(W_H^{-1}y)^2 - \sigma^2\\
\beta &= \mathcal{E}(W_H^{-1}\phi)(W_H^{-1}y)\\
\Gamma &= \mathcal{E}(W_H^{-1}\phi)(W_H^{-1}\phi)^T\\
&\quad - \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & \mathcal{E}(W_H^{-1}\psi_u)(W_H^{-1}\psi_u)^T \end{bmatrix}
\end{aligned}$$

Since our goal is to obtain a parameter set estimate computable from finite data records, we now approximate the expectations in Θ_∞ with sample means and define the

Finite-Data Parameter Set Estimate

$$\Theta_N \triangleq \{\theta : \alpha_N - 2\beta_N^T\theta + \theta^T\Gamma_N\theta \leq 0\} \tag{30}$$

where $\alpha_N \in \mathbb{R}$, $\beta_N \in \mathbb{R}^p$, and $\Gamma_N \in \mathbb{R}^{p\times p}$ are given by

$$\begin{aligned}
\alpha_N &= \mathcal{E}_N(W_H^{-1}y)^2 - \sigma^2\\
\beta_N &= \mathcal{E}_N(W_H^{-1}\phi)(W_H^{-1}y)\\
\Gamma_N &= \mathcal{E}_N(W_H^{-1}\phi)(W_H^{-1}\phi)^T\\
&\quad - \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & \mathcal{E}_N(W_H^{-1}\psi_u)(W_H^{-1}\psi_u)^T \end{bmatrix}
\end{aligned}$$

Provided Γ_N^{-1} exists, Θ_N in (30) can also be expressed as

$$\Theta_N = \left\{\theta : (\theta - \theta_c)^T \Gamma_N (\theta - \theta_c) \leq V\right\}$$

where

$$\begin{aligned}
\theta_c &= \Gamma_N^{-1}\beta_N & (31)\\
V &= \beta_N^T\Gamma_N^{-1}\beta_N - \alpha_N & (32)
\end{aligned}$$

Here $\Gamma_N > 0$ implies that Θ_N is an ellipsoid in $\mathbb{R}^p$ with center at θ_c. However, it is possible for Γ_N to have negative eigenvalues. In that case, Θ_N is a hyperboloid in $\mathbb{R}^p$ (see [10]).

The infinite-data parameter set estimate Θ_∞ given in (29) can be interpreted as a time-domain estimate. We now derive the same set estimate in the frequency domain. Observe that Θ_∞ is the set of θ which satisfies (28). Rewriting (28) so that θ does not appear explicitly, we have

$$\begin{aligned}
\Theta_\infty = \Big\{\theta : &\ \mathcal{E}\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\\
&\leq \mathcal{E}\left(W_H^{-1}W_G B_\theta u\right)^2 + \mathcal{E}\left(A_\theta e\right)^2\Big\}
\end{aligned}$$

Substituting (3) in for y,

$$\begin{aligned}
\Theta_\infty = \Big\{\theta : &\ \mathcal{E}\left[W_H^{-1}(A_\theta G - B_\theta)u + W_H^{-1}A_\theta H e\right]^2\\
&\leq \mathcal{E}\left(W_H^{-1}W_G B_\theta u\right)^2 + \mathcal{E}\left(A_\theta e\right)^2\Big\}
\end{aligned}$$

Since u and e are independent,

$$\Theta_\infty = \left\{\theta : \mathcal{E}\left[W_H^{-1}(A_\theta G - B_\theta)u\right]^2 + \mathcal{E}\left(W_H^{-1}A_\theta He\right)^2 \right.$$
$$\left. \leq \mathcal{E}\left(W_H^{-1}W_G B_\theta u\right)^2 + \mathcal{E}\left(A_\theta e\right)^2\right\}$$

Applying (1) and (9), we have

$$\Theta_\infty = \left\{\theta : \int_{-\pi}^{\pi} |W_H^{-1}|^2 f(\theta,\omega)d\omega \leq 0\right\} \qquad (33)$$

where

$$f(\theta,\omega) = \left(|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2\right) S_{uu}(\omega) + \sigma^2 |A_\theta|^2 \left(|H|^2 - |W_H|^2\right) \qquad (34)$$

We now have all the pieces to state the following:

Main Result Given that the true plant which generated $\mathcal{Z}^N$ has the structure described in (3)-(13), then

$$\Theta_N \to \Theta_\infty \quad \text{w.p. 1 as } N \to \infty \qquad (35)$$

and

$$\Theta^* \subseteq \Theta_\infty \qquad (36)$$

This result is very appealing since although we cannot compute Θ_∞, for large N, Θ_N will contain Θ^* with high probability.

Proof of Main Result Since W_H^{-1} is stable and quasi-stationary disturbances are assumed, from [12], each sample mean in (30) converges as follows:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\left(W_H^{-1}y\right)^2 \quad \to \quad \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\mathcal{E}\left(W_H^{-1}y\right)^2 = \mathcal{E}\left(W_H^{-1}y\right)^2$$

where the convergence is with probability one as N tends to ∞. To show (36), we have from (7) and (8) that $|H(e^{j\omega})|^2 - |W_H(e^{j\omega})|^2 \leq 0$. In addition, from (14), every $\theta \in \Theta^*$ satisfies

$$|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2 \leq 0, \quad \forall|\omega| \leq \pi$$

Thus, $\theta \in \Theta^*$ guarantees the integrand in (33) to be negative and $\Theta^* \subseteq \Theta_\infty$.

4 Simulation Example

In this section, we will generate the data set $\mathcal{Z}^N$ and from them compute or approximate the different parameter sets discussed in the previous section. The system we chose has the following transfer functions:

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s}P(s)\right\}$$
$$H(z) = \frac{0.1}{z - 0.9}$$

where $\mathcal{Z}$ denotes the z-transform and

$$P(s) = \frac{10}{s+1}\frac{10^2}{s^2 + 2(0.005)(10)s + (10)^2}$$

The parametric ARMA model is

$$G_\theta(z) = \frac{bz^{-1}}{1 + az^{-1}}$$

and

$$\theta^T = [a \quad b]$$

The remaining transfer functions needed to set up the problem are

$$W_G(z) = 65\left[(1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s}\frac{s+1}{s+5}\right\}\right]^4$$
$$W_H(z) = \frac{1}{\delta_H}\frac{0.1}{z - 0.9}$$

where δ_H is a constant such that $|\delta_H| \leq 1$. The disturbance v is the output of $H(z)$ driven by e, a sequence of independently distributed Gaussian pseudo random variables with zero mean and σ^2 variance. The sampling frequency is chosen to be 10 Hz.

The magnitude response of G is shown in Figure 2 along with one G_θ, where $\theta \in \Theta^*$. Note that the true system has a resonance at 10 rad/sec and the parametric model shown has a bandwidth of about 1 rad/sec. Figure 3 shows the magnitude response of W_G along with two possible $\Delta_G W_G$, whose corresponding θ are in Θ^*. The plot indicates that $|\Delta_G(e^{j\omega})| \leq 1$. Note that the chosen W_G reflects a low frequency uncertainty of 0.1, and anticipates a rather large resonance at frequencies beyond about 10 rad/sec.

Three series of experiments are carried out to study the effects of noise power, mismatch between H and W_H, and length of data record. In the first two experiments, the input u is a linearly spaced sinesweep from 0.01 to 0.5 rad/sec over 102.3 seconds, giving $N = 1024$ data samples. In the third experiment, N is varied. As mentioned earlier, the true-plant parameter set Θ^* as given by (14) cannot be computed exactly, so the set is approximated by discrete θ's checked at a finite set of 32 frequencies. This set is plotted as asterisks in Figures 4, 5, and 6.

To study the effects of noise power, σ is varied in this experiment. As suggested by (33) and (34), the parameter set estimate should expand as σ increases. This is supported by Figure 4, where Θ_N is plotted for $\sigma = 0.1$, 0.2, and 0.4. Note that in all cases, $\Theta^* \in \Theta_N$.

In Figure 5, the values of δ_H is varied from 0.6 to 1.0. Again, as suggested by (33), as the mismatch between H and W_H becomes larger, *i.e.*, $|\delta_H|$ becomes smaller, Θ_N grows.

The effects of different data record lengths are studied in the last experiment. For the case where $N = 1024$

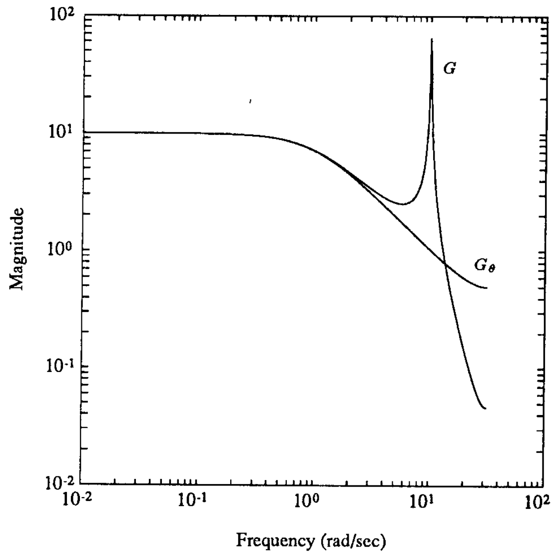

Figure 2: Magnitude responses of the true plant G and one parametric model G_θ.

and 2048 for $\sigma = 0.5$ and $\delta_H = 1.0$, Θ^* is not in Θ_N. This is still in agreement with our results since Θ^* is only guaranteed to be in Θ_N as N tends to infinity. As shown in Figure 6, Θ^* is in Θ_N for $N = 4096$.

5 Conclusion

A method is presented for parameter set estimation of systems with parametric and nonparametric uncertainty. In the presence of quasi-stationary disturbances, the parameter set estimate thus obtained from finite data records is guaranteed to contain the true-plant parameter set as the data length tends to infinity. Some numerical examples are given to support the theoretical results.

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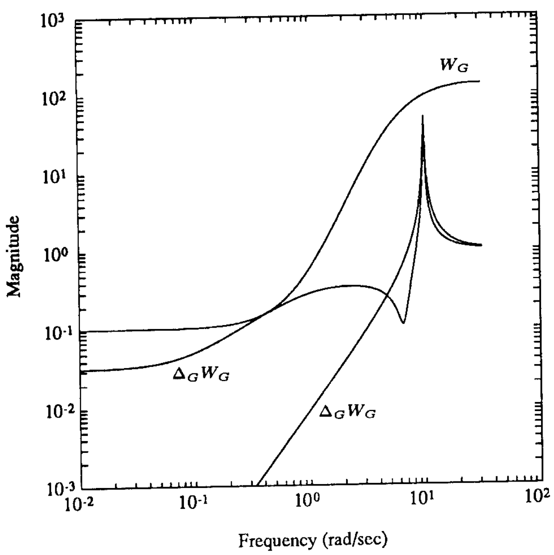

Figure 3: Magnitude responses of W_G and two possible $\Delta_G W_G$.

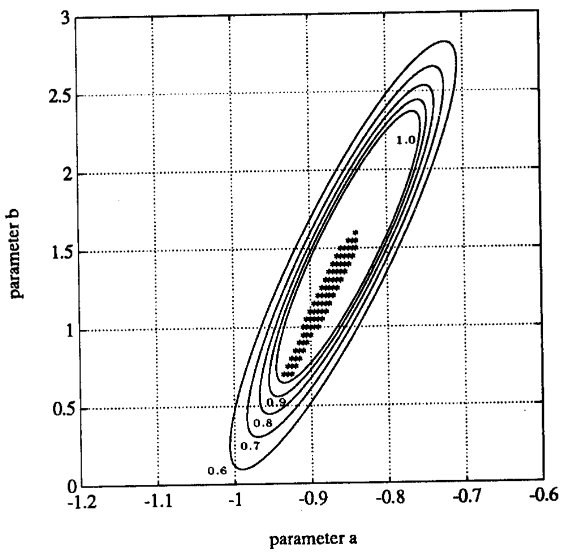

Figure 5: Θ_N for different values of δ_H (σ = 0.2, N = 1024).

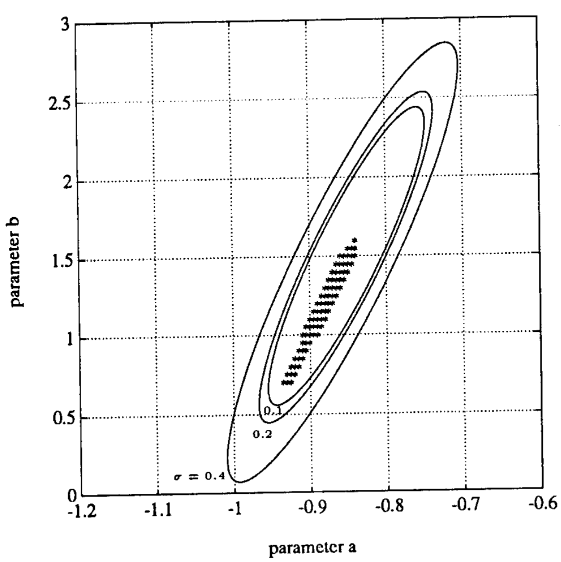

Figure 4: Θ_N for different values of σ, (N = 1024, δ_H = 0.8).

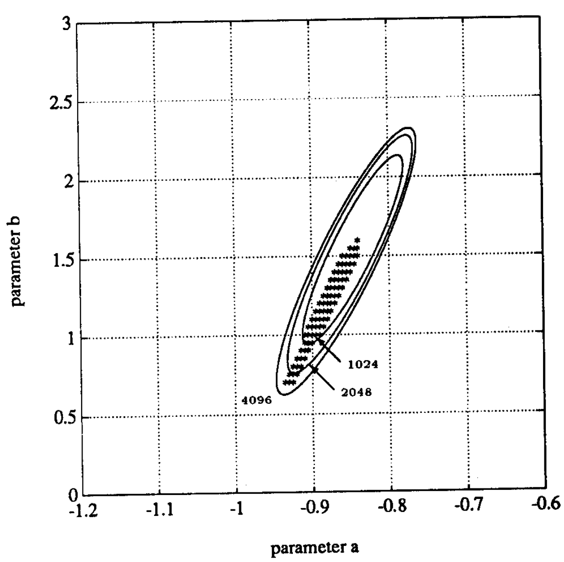

Figure 6: Θ_N for different values of N (σ = 0.5, δ_H = 1.0).