ON MAXIMIZING A ROBUSTNESS MEASURE FOR STRUCTURED NONLINEAR PERTURBATIONS

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Abstract

In this paper, we propose a robustness measure for LTI systems with causal, nonlinear diagonal perturbations with finite $\mathbf{L}_2$-gain. We propose an algorithm to reliably compute this quantity. We show how to find a state-feedback controller that achieves the global maximum of the robustness measure.

1. Definition of the Robustness Measure

We consider the following feedback system:

$$\begin{aligned} \dot{x} &= Ax + Bw, \\ z &= Cx, \qquad (1) \\ w &= \Delta z, \qquad (2) \end{aligned}$$

where $x(t) \in \mathbf{R}^n$, $w(t)$, $z(t) \in \mathbf{R}^m$, and Δ is a causal, possibly nonlinear operator mapping $\mathbf{L}_2^m$ into itself. We assume that Δ has 'diagonal structure': With $z^T = [z_1, \ldots, z_m]$ and $w^T = [w_1, \ldots, w_m]$, we assume that (2) can be expressed as

$$w_i = \Delta_i(z_i), \; i = 1, \ldots, m, \qquad (3)$$

where each Δ_i maps $\mathbf{L}_2$ into itself. We also assume that (A, B, C) is a minimal realization of the transfer matrix $H(s) \triangleq C(sI - A)^{-1}B$.

Denote by $\mathcal{P}$ the set of real diagonal $m \times m$ matrices with positive entries and define

$$\nu(H) \triangleq \inf_{P \in \mathcal{P}} \left\| P^{1/2} H P^{-1/2} \right\|_\infty, \qquad (4)$$

where $\|H\|_\infty$ is the $\mathbf{H}_\infty$-norm of the transfer matrix $H(s)$. The quantity $1/\nu(H)$ is a measure of robustness of $H(s)$ against causal nonlinear diagonal operators $\Delta : \mathbf{L}_2^m \to \mathbf{L}_2^m$. In particular, the system (1,2) is stable for all diagonal Δ with $\mathbf{L}_2$-gain less than $1/\nu(H)$, where the $\mathbf{L}_2$-gain of an operator Δ is defined as

$$\sup_{u \neq 0} \frac{\|\Delta(u)\|_2}{\|u\|_2}.$$

($\|u\|_2$ stands for the usual $\mathbf{L}_2$-norm of u.) The matrices P in (4) are *input-output scalings* that leave the block structure of the system (1,2) invariant. For more details, see [1].

2. Computation of the Robustness Measure

The strict bounded real lemma states the following (see *e. g.* [2]). For a given transfer matrix $H(s) = C(sI - A)^{-1}B$, with (A, B, C) minimal, the following statements are equivalent:

1. $H(s)$ is stable and $\|H\|_\infty < \gamma$.
2. There exists $X = X^T > 0$ such that
$$A^T X + XA + C^T C + \frac{1}{\gamma^2} XBB^T X < 0.$$

Using this lemma we conclude that $\nu(H) < \gamma$ if and only if there exists a $P \in \mathcal{P}$ and $X = X^T > 0$ such that

$$A^T X + XA + C^T PC + \frac{1}{\gamma^2} XBP^{-1}B^T X < 0.$$

Thus

$$\nu(H) = \inf_{\substack{X = X^T > 0 \\ P \in \mathcal{P}}} \{\gamma > 0 \mid \mathcal{R}(\gamma, X, P) > 0\},$$

where $\mathcal{R}(\gamma, X, P) = A^T X + XA + C^T PC + \gamma^{-2} XBP^{-1}B^T X$.

Using a change of variable $W = X^{-1}$ and $Q = \gamma^{-2}P^{-1}$, we obtain the following characterization of $\nu(H)$.

$$\nu(H) = \inf_{\substack{W = W^T > 0 \\ Q \in \mathcal{P}}} \{ \gamma > 0 \mid \mathcal{R}(\gamma, W, Q) > 0 \},$$

where the symmetric matrix function $\mathcal{R}(\gamma, W, Q)$ is defined as

$$\mathcal{R}(\gamma, W, Q) = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{12}^T & \mathcal{R}_{22} \end{bmatrix},$$

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with

$$\begin{aligned}\mathcal{R}_{11} &= -(AW + WA^T + BQB^T),\\ \mathcal{R}_{12} &= WC^T,\\ \mathcal{R}_{22} &= \gamma^2 Q.\end{aligned}$$

Since $\mathcal{R}(\gamma, W, Q)$ is an affine symmetric matrix function in the variables W, Q, we readily recast the problem of computing ν as a constrained *generalized eigenvalue minimization* problem:

$$\begin{array}{c}\text{minimize} \qquad \gamma,\\ (x,\gamma) \in \mathbf{R}^{N+1}\\ \gamma^2\mathcal{B}(x) - \mathcal{A}(x) > 0\\ \mathcal{C}(x) > 0\end{array} \tag{5}$$

where $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$ and $\mathcal{C}(\cdot)$ are affine, symmetric matrix functions in the variable $x \in \mathbf{R}^N$ which contains the independent variables in (W, Q).

Problem (5) is a nondifferentiable, quasi-convex optimization problem [3], so methods such as Kelley's cutting-plane algorithm or the ellipsoid algorithm of Shor, Nemirovksy, and Yudin are guaranteed to minimize it. Interior point algorithms which appear to be very efficient for solving (5) are given in [4, 5]; see also [6] for a brief outline of the algorithm.

3. Minimizing ν via state-feedback

Consider now the following system:

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u,\\ z &= Cx,\\ w &= \Delta z,\end{aligned} \tag{6}$$

where Δ is again diagonal as in (3). We assume that (A, B_1) and (A, B_2) are controllable, and that (C, A) is observable.

We wish to find a (stabilizing) state feedback control law $u = Kx$ which minimizes the robustness measure (4) for the closed-loop transfer matrix from w to z, which we denote by $H_K(s)$. Noting that $H_K(s)$ has a minimal realization $\{A+B_2K, B_1, C\}$, the results of the previous section may now be applied to obtain a characterization for the smallest achievable $\nu(H_K)$ as a minimization problem over an affine matrix matrix inequality.

$$\inf_K \nu(H_K) = \inf_{\substack{X=X^T>0\\ P\in\mathcal{P},\ K}} \{\gamma > 0 \mid \mathcal{R}(\gamma, X, P, K) > 0\}, \tag{7}$$

where $\mathcal{R}(\gamma, X, P, K) = (A+B_2K)^T X + X(A+B_2K) + C^T PC + \gamma^{-2} XB_1P^{-1}B_1^T X$.

Using the change of variables $W = X^{-1}$, $Y = KX^{-1}$ and $Q = \gamma^{-2}P^{-1}$, we obtain the following result.

$$\inf_K \nu(H_K) = \inf_{\substack{W=W^T>0\\ Q\in\mathcal{P},\ Y}} \{\gamma > 0 \mid \mathcal{R}(\gamma, W, Q, Y) > 0\}, \tag{8}$$

where the matrix function $\mathcal{R}(\gamma, W, Q, Y)$ is defined as

$$\mathcal{R}(\gamma, W, Q, Y) = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12}\\ \mathcal{R}_{12}^T & \mathcal{R}_{22}\end{bmatrix},$$

with

$$\begin{aligned}\mathcal{R}_{11} &= -(AW + WA^T + B_2Y + Y^TB_2^T + B_1QB_1^T),\\ \mathcal{R}_{12} &= WC^T,\\ \mathcal{R}_{22} &= \gamma^2 Q.\end{aligned}$$

Problem (8) is thus reducible to (5). The optimal controller is given by $K_{\text{opt}} = Y_{\text{opt}}W_{\text{opt}}^{-1}$, where Y_{opt} and W_{opt} are the optimal Y and W respectively in (8).

4. Conclusions

A measure of robustness of LTI systems against causal nonlinear diagonal perturbations can be computed using generalized eigenvalue minimization. The same algorithm can be used to design the optimal feedback that minimizes this measure. Extensions of these results to more generally structured perturbations (for example block diagonal) are fairly straightforward.

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