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Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization

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Abstract

This paper shows that Lyapunov-based state feedback controller synthesis for piecewise-affine (PWA) slab systems can be cast as an optimization problem subject to a set of linear matrix inequalities (LMIs) analytically parameterized by a vector. Furthermore, it is shown that continuity of the control inputs at the switchings can be guaranteed by adding equality constraints to the problem without affecting its parameterization structure. Finally, it is shown that piecewise-affine state feedback controller synthesis for piecewise-affine slab systems to maximize the decay rate of a quadratic control Lyapunov function can be cast as a set of quasi-concave optimization problems analytically parameterized by a vector. Before casting the synthesis in the format presented in this paper, Lyapunov-based piecewise-affine state feedback controller synthesis could only be formulated as a bi-convex optimization program, which is very expensive to solve globally. Thus, the fundamental importance of the contributions of the paper relies on the fact that, for the first time, the piecewise-affine state feedback synthesis problem has been formulated as a convex problem with a parameterized set of LMIs that can be relaxed to a finite set of LMIs and solved efficiently to a point near the global optimum using available software. Furthermore, it is shown for the first time that, in some situations, the global can be exactly found by solving only one concave problem.

Keywords: Piecewise-affine systems; State feedback; Convex optimization; Ellipsoids; Lyapunov function

1. Introduction

Piecewise-affine systems are multi-model systems that offer a good modeling framework for complex dynamical systems involving nonlinear phenomena. In fact, many nonlinearities that appear frequently in engineering systems are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as piecewise-affine functions. Piecewise-affine systems are also a class of hybrid systems, i.e, systems with a continuous-time state

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and a discrete-event state. For piecewise-affine systems the discrete-event state is associated with discrete modes of operation. The continuous-time state is associated with the affine (linear with offset) dynamics valid within each discrete mode. Piecewise-affine systems pose challenging problems because of its switched structure. In fact, the analysis and control of even some simple piecewise-affine systems have been shown to be either an $\mathcal{NP}$ hard problem or undecidable [4].

State and output feedback control of continuous-time piecewise-affine systems have received increasing interest over the last years [10,13,15,21]. The interesting approach presented in [13,15] relies on computing upper and lower bounds to the optimal cost of the controller obtained as the solution to the Hamilton–Jacobi–Bellman equation. However, the continuous-time controller resulting from the approach in [15] is a patched LQR that cannot be guaranteed to avoid sliding modes at the switching and, therefore, is not provably stabilizing. Previous work of the authors has concentrated on Lyapunov-based controller synthesis methods for continuous-time piecewise-affine (PWA) systems [10,21]. In [21], Lyapunov-based controller synthesis was formulated as a bi-convex optimization problem. The bi-convexity structure arises because of the negativity constraint on the derivative of the piecewise-quadratic Lyapunov function over time. This constraint leads to a bilinear matrix inequality (BMI) [8]. Bi-convex optimization problems are non-convex, $\mathcal{NP}$ hard and, therefore, extremely expensive to solve globally from a computational point of view [8]. Based on this fact, Ref. [21] has adapted three alternative iterative algorithms for solving the non-convex problem to a suboptimal solution. Although the controller synthesis problem for piecewise-affine systems using piecewise-quadratic Lyapunov functions is non-convex, Hassibi and Boyd [10] have shown that for the particular case of piecewise-linear state feedback of slab piecewise-linear systems (without affine terms), globally quadratic stabilization could be cast as a convex optimization problem. Unfortunately, if affine terms are included in the controller, as stated in [10], “it does not seem that the condition for stabilizability can be cast as an LMI”, which apparently destroys the convex structure of the problem, making it hard to solve globally. The current paper shows that piecewise-affine state feedback for piecewise-affine slab systems using a globally quadratic Lyapunov function can indeed be solved to a point near the global optimum in an efficient way by a set of LMIs. Building on the result of [10], this paper formulates piecewise-affine state feedback as an optimization problem involving a set of LMIs analytically parameterized by a vector. Three different algorithms will be suggested to solve relaxations of the optimization problem to a point near the global optimum. One is based on gridding of the domain of the parameterizing vector and yields solutions that approach the global optimum as the density of the grid is increased. The others are based on trace maximization to approximately solve an LMI subject to rank constraints [17], a problem that appears frequently in reduced order controller design. Although yielding solutions approaching the global optimum, the algorithm involving gridding increases the computational cost as the grid becomes denser and can be prohibitive for large systems. However, the gridding approach has already been used in other recent research on analysis [9], LPV control [23], gain-scheduling control [1,2] and some techniques already exist to alleviate the computational cost due to the gridding phase [3,22]. The algorithms for trace maximization are inspired by the work presented in [11,7,6]. One of these algorithms is iterative but typically involves only one or two iterations, thus typically being less computationally expensive than the gridding algorithm. The other proposed trace maximization algorithm is simply a concave program, which is therefore efficient from a computational point of view. It is also shown in the paper that constraints for continuity of the control inputs can be added to the PWA state feedback problem without affecting its parameterization structure. Finally, it is shown that piecewise-affine state feedback controller synthesis for piecewise-affine slab systems to maximize the decay rate of a globally quadratic control Lyapunov function can be cast as a set of quasi-concave optimization problems analytically parameterized by a vector. This problem can also be solved numerically using efficient algorithms.

In this paper, four controller synthesis problems are formulated, relaxed to a finite set of convex optimization problems and solved. The paper starts by presenting the assumptions that are common to all controller design problems, followed by the statements of the four problems. Section 4 formulates the controller synthesis problems as optimization programs. Section 5 presents several algorithms to solve the formulated problems. Finally, after two numerical examples, the paper finishes by presenting the conclusions

2. Problem assumptions

It is assumed that a PWA system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see [20] for generating such a partition). Following [14,18,10], each cell is constructed as the intersection of a finite number (p_i) of half-spaces

$$\mathcal{R}_i = \{x | H_i^{\mathrm{T}} x - \tilde{g}_i < 0\}, \tag{1}$$

where $H_i = [h_{i1} \;\; h_{i2} \ldots h_{ip_i}]$, $\tilde{g}_i = [\tilde{g}_{i1} \;\; \tilde{g}_{i2} \ldots \tilde{g}_{ip_i}]^{\mathrm{T}}$. Moreover, the sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ such that $\bigcup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{x}(t) = A_i x(t) + \tilde{b}_i + B_i u(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. For system (2), we adopt the following definition of trajectories or solutions presented in [13].

Definition 2.1 (*Johansson [13]*). Let $x(t) \in \mathcal{X}$ be an absolutely continuous function. Then $x(t)$ is a trajectory of the system (2) on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$ and Lebesgue measurable $u(t)$, the equation $\dot{x}(t) = A_i x(t) + \tilde{b}_i + B_i u(t)$ holds for $x(t) \in \overline{\mathcal{R}}_i$.

Any two cells sharing a common facet will be called *level*-1 neighboring cells. Let $\mathcal{N}_i = \{$*level*-1 neighboring cells of $\mathcal{R}_i\}$. It is also assumed that vectors $c_{ij} \in \mathbb{R}^n$ and scalars d_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in \mathbb{R}^n \mid c_{ij}^{\mathrm{T}} x - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [10]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{x = \tilde{l}_{ij} + F_{ij} s | s \in \mathbb{R}^{n-1}\} \tag{3}$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in \mathbb{R}^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}^{T}, and $\tilde{l}_{ij} \in \mathbb{R}^n$ is given by $\tilde{l}_{ij} = c_{ij}(c_{ij}^{\mathrm{T}} c_{ij})^{-1} d_{ij}$. For systems whose polytopic cells are slabs, called *piecewise-affine slab systems*, each $\mathcal{R}_i$ can be outer approximated by a degenerate ellipsoid ε_i. This covering will be used to describe the regions instead of the polytopic description. The ellipsoidal description of piecewise-affine systems is useful because it often requires fewer parameters than the polytopic description and it enables to cast the synthesis problem as an optimization program involving a set of LMIs analytically parameterized by a vector. To describe the ellipsoidal covering, it is assumed that matrices E_i and $\tilde{f}_i$ exist such that

$$\mathcal{R}_i \subseteq \varepsilon_i, \tag{4}$$

where

$$\varepsilon_i = \{x | \; \|E_i x + \tilde{f}_i\| \leqslant 1\}. \tag{5}$$

This covering is especially useful in the case where $\mathcal{R}_i$ is a slab because in this case the matrices E_i and $\tilde{f}_i$ are guaranteed to exist and the covering (having one degenerate ellipsoid ε_i) is exact, i.e., $\varepsilon_i \subseteq R_i$ and $R_i \subseteq \varepsilon_i$. More precisely, if $\mathcal{R}_i = \{x \mid d_1 < c_i^{\mathrm{T}} x < d_2\}$, then the degenerate ellipsoid is described by $E_i = 2c_i^{\mathrm{T}}/(d_2 - d_1)$ and $\tilde{f}_i = -(d_2 + d_1)/(d_2 - d_1)$. Finally, it is assumed that the control objective is to stabilize the system to a given point x_{cl}. With the change of coordinates $z = x - x_{\mathrm{cl}}$ the problem is transformed to the stabilization of the origin. In these coordinates, the system dynamics (2) are

$$\dot{z}(t) = A_i z(t) + b_i + B_i u(t), \tag{6}$$

where $b_i = \tilde{b}_i + A_i x_{\mathrm{cl}}$. The parametric description of boundaries (3) is written as

$$\overline{\mathscr{R}_i} \cap \overline{\mathscr{R}_j} \subseteq \{z = l_{ij} + F_{ij}s \mid s \in \mathbb{R}^{n-1}\}, \tag{7}$$

where $l_{ij} = \tilde{l}_{ij} - x_{\mathrm{cl}}$ for $i = 1, \ldots, M,\ j \in \mathscr{N}_i$. The description of the polytopic cells is

$$\mathscr{R}_i = \{z \mid H_i^{\mathrm{T}} z - g_i < 0\}, \tag{8}$$

where $g_i = \tilde{g}_i - H_i^{\mathrm{T}} x_{\mathrm{cl}}$, and the ellipsoidal covering is described by

$$\varepsilon_i = \{z \mid \|E_i z + f_i\| \leqslant 1\}, \tag{9}$$

where $f_i = \tilde{f}_i + E_i x_{\mathrm{cl}}$.

3. Problem statement

There are four Lyapunov-based controller synthesis problems that will be solved in this paper. For the four problems, the piecewise-affine state feedback input signal is parameterized by K_i and m_i in the form

$$u = K_i z + m_i, \quad z \in \mathscr{R}_i \tag{10}$$

with $-l_0 \leqslant m_i \leqslant l_0$ where l_0 is a vector of upper bounds for the entries of m_i, $i = 1, \ldots, M$. The globally quadratic candidate control Lyapunov function is parameterized by $P = P^{\mathrm{T}}$ as

$$V(z) = z^{\mathrm{T}} P z. \tag{11}$$

The four problems are:

1. **Problem 1**. Find a piecewise-affine state feedback controller that exponentially stabilizes the origin of (6) and find a globally quadratic Lyapunov function that proves it.
2. **Problem 2**. The same as problem 1 with continuous input signals at the switching boundaries.
3. **Problem 3**. From the controllers that exponentially stabilize the origin, find the one that maximizes the decay rate of the globally quadratic control Lyapunov function.
4. **Problem 4**. The same as problem 3 with continuous input signals at the switching boundaries.

4. Problem formulation

This section formulates mathematically the stabilization problems 1 and 2 as optimization programs involving a set of LMIs analytically parameterized by a vector. The decay rate maximization problems 3 and 4 are formulated as a set of quasi-concave optimization programs analytically parameterized by the same vector.

4.1. Stabilization—problem 1

The candidate control Lyapunov function (11) becomes a Lyapunov function if for fixed $\alpha \geqslant 0$, $V > 0$ and $\dot{V} < -\alpha V$. Using (6) and (10), sufficient conditions for exponential stability are thus $P = P^{\mathrm{T}} > 0$ and

$$\begin{aligned} z \in \mathscr{R}_i \Rightarrow\ & [(A_i + B_i K_i)z + (b_i + B_i m_i)]^{\mathrm{T}} P z \\ & + z^{\mathrm{T}} P[(A_i + B_i K_i)z + (b_i + B_i m_i)] + \alpha z^{\mathrm{T}} P z < 0. \end{aligned} \tag{12}$$

This expression can be recast as

$$z \in \mathscr{R}_i \Rightarrow \begin{bmatrix} z \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{A}_i^{\mathrm{T}} P + P\bar{A}_i + \alpha P & P\bar{b}_i \\ (P\bar{b}_i)^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < 0, \tag{13}$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{b}_i = b_i + B_i m_i$. If the condition $z \in \mathscr{R}_i$ in (13) is relaxed to $z \in \varepsilon_i$ and if expression (9) is used along with the $\mathscr{S}$*-procedure* [24,5] with multiplier $\lambda_i < 0$ yields the sufficient conditions $P = P^{\mathrm{T}} > 0$ and

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{A}_i^{\mathrm{T}} P + P\bar{A}_i + \alpha P & P\bar{b}_i \\ (P\bar{b}_i)^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < -\lambda_i \begin{bmatrix} z \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{E}_i^{\mathrm{T}} E_i & E_i^{\mathrm{T}} f_i \\ f_i^{\mathrm{T}} E_i & f_i^{\mathrm{T}} f_i - 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}, \tag{14}$$

Rearranging expression (14) the following sufficient conditions for quadratic stabilization are derived:

$$P = P^{\mathrm{T}} > 0, \quad \lambda_i < 0, \quad i = 1, \ldots, M,$$
$$\begin{bmatrix} \bar{A}_i^{\mathrm{T}} P + P\bar{A}_i + \alpha P + \lambda_i E_i^{\mathrm{T}} E_i & P\bar{b}_i + \lambda_i E_i^{\mathrm{T}} f_i \\ (P\bar{b}_i + \lambda_i E_i^{\mathrm{T}} f_i)^{\mathrm{T}} & -\lambda_i (1 - f_i^{\mathrm{T}} f_i) \end{bmatrix} < 0. \tag{15}$$

Using new variables $Q = P^{-1}$, $\mu_i = \lambda_i^{-1}$ this set of conditions are equivalent to

$$Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M,$$
$$\begin{bmatrix} \bar{A}_i^{\mathrm{T}} Q^{-1} + Q^{-1}\bar{A}_i + \alpha Q^{-1} + \mu_i^{-1} E_i^{\mathrm{T}} E_i & Q^{-1}\bar{b}_i + \mu_i^{-1} E_i^{\mathrm{T}} f_i \\ (Q^{-1}\bar{b}_i + \mu_i^{-1} E_i^{\mathrm{T}} f_i)^{\mathrm{T}} & -\mu_i^{-1} (1 - f_i^{\mathrm{T}} f_i) \end{bmatrix} < 0. \tag{16}$$

The following lemma will present a set of equivalent conditions to inequalities (16).

Lemma 4.1. *Conditions* (16) *are equivalent to* $Q = Q^{\mathrm{T}} > 0$, $\mu_i < 0$ *and*

$$1 - f_i^{\mathrm{T}} f_i < 0, \tag{17}$$

$$\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} + (\mu_i \bar{b}_i f_i^{\mathrm{T}} + Q E_i^{\mathrm{T}}) \mu_i^{-1} (I - f_i f_i^{\mathrm{T}})^{-1} (\mu_i \bar{b}_i f_i^{\mathrm{T}} + Q E_i^{\mathrm{T}})^{\mathrm{T}} < 0 \tag{18}$$

for $i = 1, \ldots, M$.

Proof. The conditions $Q = Q^{\mathrm{T}} > 0$, $\mu_i < 0$, $i = 1, \ldots, M$ are the same as in (16). To derive inequalities (17) and (18), using Schur complement, the LMI in (16) is equivalent to $1 - f_i^{\mathrm{T}} f_i < 0$ and

$$Q^{-1}\bar{A}_i + \bar{A}_i^{\mathrm{T}} Q^{-1} + \alpha Q^{-1} + \mu_i^{-1} E_i^{\mathrm{T}} E_i + (Q^{-1}\bar{b}_i + \mu_i^{-1} E_i^{\mathrm{T}} f_i) \mu_i (1 - f_i^{\mathrm{T}} f_i)^{-1} (Q^{-1}\bar{b}_i + \mu_i^{-1} E_i^{\mathrm{T}} f_i)^{\mathrm{T}} < 0. \tag{19}$$

Left multiplying inequality (19) by Q and right multiplying it by $Q^{\mathrm{T}} = Q$ yields the equivalent condition

$$\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i^{-1} Q E_i^{\mathrm{T}} E_i Q^{\mathrm{T}} + (\bar{b}_i + \mu_i^{-1} Q E_i^{\mathrm{T}} f_i) \mu_i (1 - f_i^{\mathrm{T}} f_i)^{-1} (\bar{b}_i + \mu_i^{-1} Q E_i^{\mathrm{T}} f_i)^{\mathrm{T}} < 0. \tag{20}$$

The Matrix Inversion Lemma [16] states that $(1 - f_i^{\mathrm{T}} f_i)^{-1} = 1 + f_i^{\mathrm{T}} (I - f_i f_i^{\mathrm{T}})^{-1} f_i$. Using this result, expression (20) can be rewritten as

$$\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i^{-1} Q E_i^{\mathrm{T}} E_i Q^{\mathrm{T}} + (\bar{b}_i + \mu_i^{-1} Q E_i^{\mathrm{T}} f_i) \mu_i [1 + f_i^{\mathrm{T}} (I - f_i f_i^{\mathrm{T}})^{-1} f_i] (\bar{b}_i + \mu_i^{-1} Q E_i^{\mathrm{T}} f_i)^{\mathrm{T}} < 0. \tag{21}$$

Expression (21) can be expanded as

$$\begin{aligned}&\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i^{-1} Q E_i^{\mathrm{T}} E_i Q^{\mathrm{T}} + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} + \mu_i^{-1}(QE_i^{\mathrm{T}} f_i)(QE_i^{\mathrm{T}} f_i)^{\mathrm{T}} + \bar{b}_i f_i^{\mathrm{T}} E_i Q \\ &\quad + QE_i^{\mathrm{T}} f_i \bar{b}_i^{\mathrm{T}} + (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} f_i f_i^{\mathrm{T}})\mu_i^{-1}(I - f_i f_i^{\mathrm{T}})^{-1}(\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} f_i f_i^{\mathrm{T}})^{\mathrm{T}} < 0. \end{aligned} \tag{22}$$

Expression (22) can be rewritten as

$$\begin{aligned}&\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} + \mu_i^{-1} QE_i^{\mathrm{T}}(I + f_i f_i^{\mathrm{T}})(QE_i^{\mathrm{T}})^{\mathrm{T}} + \bar{b}_i f_i^{\mathrm{T}}(QE_i^{\mathrm{T}})^{\mathrm{T}} \\ &\quad + QE_i^{\mathrm{T}}(\bar{b}_i f_i^{\mathrm{T}})^{\mathrm{T}} + (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} \\ &\quad - QE_i^{\mathrm{T}}(I - f_i f_i^{\mathrm{T}}))\mu_i^{-1}(I - f_i f_i^{\mathrm{T}})^{-1}(\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} - QE_i^{\mathrm{T}}(I - f_i f_i^{\mathrm{T}}))^{\mathrm{T}} < 0. \end{aligned} \tag{23}$$

Inequality (23) can be rearranged in the form

$$\begin{aligned}&\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} + (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})\mu_i^{-1}(I - f_i f_i^{\mathrm{T}})^{-1}(\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})^{\mathrm{T}} \\ &\quad + \mu_i^{-1} QE_i^{\mathrm{T}}(I + f_i f_i^{\mathrm{T}})(QE_i^{\mathrm{T}})^{\mathrm{T}} + \bar{b}_i f_i^{\mathrm{T}}(QE_i^{\mathrm{T}})^{\mathrm{T}} + QE_i^{\mathrm{T}}(\bar{b}_i f_i^{\mathrm{T}})^{\mathrm{T}} + \mu_i^{-1} QE_i^{\mathrm{T}}(I - f_i f_i^{\mathrm{T}})(QE_i^{\mathrm{T}})^{\mathrm{T}} \\ &\quad - (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})\mu_i^{-1}(QE_i^{\mathrm{T}})^{\mathrm{T}} - \mu_i^{-1} QE_i^{\mathrm{T}}(\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})^{\mathrm{T}} < 0. \end{aligned} \tag{24}$$

Finally, the last two rows of expression (24) cancel and we are left with the following conditions:

$$1 - f_i^{\mathrm{T}} f_i < 0, \tag{25}$$

$$\bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} + (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})\mu_i^{-1}(I - f_i f_i^{\mathrm{T}})^{-1}(\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})^{\mathrm{T}} < 0. \quad \square \tag{26}$$

The following result can now be derived.

Corollary 4.1. *For piecewise-affine slab systems conditions* (16) *are equivalent to*

$$Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M,$$

$$\begin{bmatrix} \bar{A}_i Q + Q\bar{A}_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} & \mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} \\ (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})^{\mathrm{T}} & -\mu_i(1 - f_i f_i^{\mathrm{T}}) \end{bmatrix} < 0. \tag{27}$$

Proof. It follows trivially from (25) to (26) applying Schur complement and using the fact that $1 - f_i^{\mathrm{T}} f_i < 0$ and $I - f_i f_i^{\mathrm{T}} < 0$ are equivalent when f_i is a scalar, which is the case for piecewise-affine slab systems. $\square$

Remark 1. For an alternative path that enables to derive similar results in a more general setting please see Duality Lemma 4 in [12].

For the remainder of the paper we will concentrate on piecewise-affine slab systems. Performing the substitution $\bar{A}_i = A_i + B_i K_i$ and introducing new variables $Y_i = K_i Q$ in (27) yields

$$Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M,$$

$$\begin{bmatrix} A_i Q + QA_i^{\mathrm{T}} + B_i Y_i + Y_i^{\mathrm{T}} B_i^{\mathrm{T}} + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^{\mathrm{T}} & \mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}} \\ (\mu_i \bar{b}_i f_i^{\mathrm{T}} + QE_i^{\mathrm{T}})^{\mathrm{T}} & -\mu_i(1 - f_i f_i^{\mathrm{T}}) \end{bmatrix} < 0, \tag{28}$$

where $\bar{b}_i = b_i + B_i m_i$. The piecewise-affine state feedback stabilization problem is now formally defined.

Definition 4.1. The piecewise-affine state feedback synthesis problem (problem 1) is: for fixed $\alpha \geqslant 0$

$$\begin{aligned} \text{find} \quad & Q, Y_i, m_i, \mu_i \\ \text{s.t.} \quad & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (28), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{aligned}$$

where $\succ, \prec$ mean component-wise inequalities and l_0, l_1 are given vector bounds.

Section 5 presents three algorithms to (approximately) solve this problem. The feasibility problem 1 can be transformed into an optimization problem if the Q with minimum condition number is sought as follows:

Definition 4.2. The minimum condition number piecewise-affine state feedback problem is: for fixed $\alpha \geqslant 0$, $\varepsilon > 0$

$$\begin{array}{ll} \min & \eta \\ \text{s.t.} & \eta > 0, \quad \varepsilon I < Q < \eta\varepsilon I, \\ & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (28), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{array}$$

where $\succ$, $\prec$ mean component-wise inequalities and l_0, l_1 are given vector bounds.

Usually ε is selected to be unitary.

4.2. Stabilization—problem 2

To formulate problem 2, similarly to what was done in [21], the boundary description (7) is used to yield the following constraints for continuity of the control signals:

$$(K_i - K_j)F_{ij} = 0, \tag{29}$$

$$(K_i - K_j)l_{ij} + (m_i - m_j) = 0, \quad \forall_j \in \mathcal{N}_i. \tag{30}$$

These constraints for continuity cannot be directly used in the problem from Definition 4.1 because K_i, $i = 1, \ldots, M$, are not variables in that problem since the change of variables $Y_i = K_i Q$ has been used. To be able to express constraints (29)–(30) on the variables Y_i, $i = 1, \ldots, M$, define the matrix

$$X_{ij} = [F_{ij} \;\; l_{ij}]. \tag{31}$$

Note that X_{ij} is invertible because F_{ij} is full rank and l_{ij} does not belong to the column space of F_{ij} by construction. Using (31), (29)–(30) can be rewritten as

$$(K_i - K_j)X_{ij} = [0_{m\times(n-1)} \;\; m_j - m_i], \quad \forall_j \in \mathcal{N}_i. \tag{32}$$

Then, using (32), the change of variables $Y_i = K_i Q$ and inverting X_{ij}, we can write the constraints on Y_i, Y_j as

$$Y_i = Y_j + [0_{m\times(n-1)} \;\; m_j - m_i]X_{ij}^{-1}Q, \quad \forall_j \in \mathcal{N}_i. \tag{33}$$

The stabilization problem 2 is now formally defined.

Definition 4.3. The stabilization problem 2 is: for fixed $\alpha \geqslant 0$

$$\begin{array}{ll} \text{find} & Q, Y_i, m_i, \mu_i \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (28), (33), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{array}$$

where $\succ$, $\prec$ mean component-wise inequalities and l_0, l_1 are vector bounds.

In summary, constraints (33) must be included in the optimization problem to guarantee that the control signals are continuous at the switching boundaries.

4.3. Decay rate maximization—problems 3 and 4

In the problems of Sections 4.1 and 4.2 the parameter α was fixed. Let now α, the desired decay rate for the globally quadratic control Lyapunov function, be a variable. Then, we define the performance criterion

$$\mathscr{J} = \alpha. \tag{34}$$

The controller design problem is now to find from the class of control signals parameterized in the form $u = K_i z + m_i$ in each region $\mathscr{R}_i$, the one that maximizes the performance $\mathscr{J}$. This is formally defined next.

Definition 4.4. The decay rate optimization problem 3 is:

$$\begin{array}{rl} \max & \alpha \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad \alpha > 0, \quad (28), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{array}$$

where $\succ, \prec$ mean component-wise inequalities and l_0, l_1 are vector bounds.

Finally, to formulate problem 4, it suffices to include the continuity constraints (33) in the optimization 4.4 yielding the following new optimization problem.

Definition 4.5. The decay rate optimization problem 4 is

$$\begin{array}{rl} \max & \alpha \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad \alpha > 0, \quad (28), (33), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{array}$$

where $\succ, \prec$ mean component-wise inequalities and l_0, l_1 are vector bounds.

5. Solution algorithms

Previous work [10] stated that piecewise-affine state feedback controller synthesis using a quadratic control Lyapunov function does not seem to be convex. In fact, it is clear from (28) that this synthesis problem cannot be formulated as one convex program because (28) is not an LMI if the parameters $m_i, i = 1, \ldots, M$ are unknown but rather an infinite set of parameterized LMIs. However, this section shows how the piecewise-affine state feedback synthesis problem for piecewise-affine slab systems using a globally quadratic Lyapunov function can be relaxed and solved to a point near the global optimum in an efficient way by a finite set of LMIs. Three different algorithms will be presented next.

5.1. Solution Algorithm #1: sampling method

The interesting observation here is that for fixed m_i, $i = 1, \ldots, M$, expression (28) is indeed an LMI and the problem is convex. Therefore, although the problem formulated in (28) cannot be cast as one convex program, it is an infinite set of convex problems involving an LMI or, equivalently, an infinite number of LMIs analytically parameterized by the vector $\gamma = [m_1^{\mathrm{T}} m_2^{\mathrm{T}} \ldots m_M^{\mathrm{T}}]^{\mathrm{T}}$. Since each element $m_i, i = 1, \ldots, M$ has bounded components, γ belongs to an hypercube. Effective meshing techniques can then be used to sample the hypercube and solve a finite number of LMIs. The following algorithm is suggested to solve the state feedback problems 1 and 2:

Algorithm #1: sampling method.

1. Define a grid for the domain of the vector γ to sample it at N points,

2. For fixed $\alpha \geqslant 0$, solve the corresponding feasibility Definition 4.1 for each of the points in the grid until a feasible point is found.
3. If step 2 is successful or if the maximum number of iterations was reached, stop. Otherwise, increase the grid density and go back to Step 2.

Remark 2. Algorithm #1 can be changed to solve the problem stated in Definition 4.2 and store for all grid points the one that yields the minimum value of η. The algorithm can be further improved if the derivative of the cost with respect to γ is computed at each point. Then, for each selected sample point, the next sample point should be chosen in the direction opposite to the vector derivative. This reduces the number of points from the grid that need to be used, thus reducing the computational burden of the algorithm.

Algorithm #1 increases the computational cost as the grid becomes denser and can be prohibitive for large systems. However, the gridding approach has already been used in other recent research on analysis [9], LPV control [23], gain-scheduling control [23,1,2] and some techniques already exist to alleviate the computational cost due to the gridding phase [3,22].

5.2. Solution Algorithms #2 and #3: trace maximization methods

An alternative algorithm can be developed to solve the state feedback problem stated in Definition 4.1 when the ellipsoidal cover for each region is formed by only one ellipsoid, which is the case for *piecewise-affine slab systems*. To develop alternative algorithms for such case, let us return to inequality (28) and perform the change of variables

$$Z_i = \mu_i m_i, \tag{35}$$

$$W_i = \mu_i m_i m_i^{\mathrm{T}} = \mu_i^{-1} Z_i Z_i^{\mathrm{T}}. \tag{36}$$

Then, inequality (28) can be rewritten as

$$\begin{bmatrix} A_i Q + Q A_i^{\mathrm{T}} + B_i Y_i + Y_i^{\mathrm{T}} B_i^{\mathrm{T}} + \alpha Q + \mu_i b_i b_i^{\mathrm{T}} + b_i Z_i^{\mathrm{T}} B_i^{\mathrm{T}} + B_i Z_i b_i^{\mathrm{T}} + B_i W_i B_i^{\mathrm{T}} & (\cdot) \\ ((\mu_i b_i + B_i Z_i) f_i^{\mathrm{T}} + Q E_i^{\mathrm{T}})^{\mathrm{T}} & -\mu_i (1 - f_i f_i^{\mathrm{T}}) \end{bmatrix} < 0. \tag{37}$$

This inequality is an LMI. Note that (35) is just a change of variables because m_i is not involved in (37). After knowing μ_i and Z_i, m_i can be obtained as $m_i = \mu_i^{-1} Z_i$. Therefore, constraint (35) can be handled after the solution to the controller design optimization problem is obtained. However, constraint (36) must be included in the controller design optimization problem because it involves variables that appear in (37). Therefore, the following problem can be defined, which is equivalent to problem 1:

Definition 5.1. The piecewise-affine state feedback synthesis problem (problem 1b) is: for fixed $\alpha \geqslant 0$

$$\begin{aligned} \text{find} \quad & Q, W_i, Z_i, Y_i, m_i, \mu_i \\ \text{s.t.} \quad & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (36), (37), \\ & -l_1 \prec Y_i \prec l_1, -l_0 \prec m_i \prec l_0, \quad i = 1, \ldots, M, \end{aligned}$$

where $\succ, \prec$ mean component-wise inequalities and l_0, l_1 are given vector bounds.

The main obstacle here is that constraint (36) is not convex. Therefore the approach to be followed next is to replace this constraint by a set of convex constraints and formulate a convex optimization problem whose solution will also be a solution to problem 1b whenever the optimal value of the convex problem is zero. To do this, observe that because $\mu_i < 0$, if constraint (36) is verified then rank $(W_i) = 1$ and by a Schur complement argument the following constraints are verified:

$$\tilde{W}_i = \begin{bmatrix} W_i & Z_i \\ Z_i^{\mathrm{T}} & \mu_i \end{bmatrix} \leqslant 0, \quad i = 1, \ldots, M. \tag{38}$$

Let us now define the functional

$$\mathscr{J}_s \triangleq \sum_i [\mathrm{trace}(W_i) - \mu_i^{-1} Z_i^{\mathrm{T}} Z_i]. \tag{39}$$

If constraint (38) is verified, we observe that $\mathscr{J}_s \leqslant 0$ and $\mathscr{J}_s = 0$ if and only if $W_i = \mu_i^{-1} Z_i Z_i^{\mathrm{T}}$, $i = 1, \ldots, M$, which agrees with (36). This discussion motivates the definition of the following maximization problem:

Definition 5.2. The trace maximization problem (problem 1c) is: for fixed $\alpha \geqslant 0$

$$\begin{array}{rl} \max & \mathscr{J}_s \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (37), (38), \\ & -l_1 \prec Y_i \prec l_1, -\tilde{l}_0 \prec Z_i \prec \tilde{l}_0, \quad i = 1, \ldots, M, \end{array}$$

where $\succ, \prec$ mean component-wise inequalities, $\tilde{l}_0, l_1$ are given vector bounds and Q, Y_i, Z_i, W_i, μ_i, $i = 1, \ldots, M$ are the optimization variables.

From the previous discussion it is clear that if the optimal value of problem 1c in Definition 5.2 is zero, then the solution to problem 1c automatically yields a stabilizing piecewise-affine state feedback controller with guaranteed Lyapunov decay rate α, i.e, it yields a solution to problem 1b and to the original problem 1 in Definition 4.1. Since the functional $\mathscr{J}_s$ is not convex, following the ideas expressed in [11,7], an approximation to the solution of problem 1c is obtained by the algorithm that is presented next. Let nominal values of the matrices Z_i at each iteration $k \geqslant 1$ be given by $Z_{i,k-1}$, $i = 1, \ldots, M$. Let also nominal values for the $\mathscr{S}$-procedure multiplier parameters be given as $\mu_{i,0}$ and let $\Delta\mu_i = \mu_i - \mu_{i,0}$, $i = 1, \ldots, M$. The description of the algorithm now follows.

Algorithm #2: Trace maximization.

1.
$$\begin{array}{rl} \text{find} & Q, Y_i, Z_i, W_i, \mu_i \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (37), (38), \\ & -l_1 \prec Y_i \prec l_1, -\tilde{l}_0 \prec Z_i \prec \tilde{l}_0, \quad i = 1, \ldots, M. \end{array}$$
   If the problem is not feasible stop. Otherwise, set $\mu_{i,0} = \mu_i$, $Z_{i,0} = Z_i$, $i = 1, \ldots, M$, $k = 1$ and select a tolerance parameter $\delta > 0$.
2. Solve
$$\begin{array}{rl} \max & \sum_i [\mathrm{trace}(W_{i,k}) - 2\mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k} + \mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k-1}] \\ \text{s.t.} & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad \Delta\mu_i \geqslant 0, \quad (37), (38), \\ & -l_1 \prec Y_i \prec l_1, -\tilde{l}_0 \prec Z_i \prec \tilde{l}_0, \quad i = 1, \ldots, M. \end{array}$$
3. If the absolute value of $\mathscr{J}_s$ obtained for the solution parameters of step 2 is less than δ, or if the maximum number of iterations has been reached, stop. Otherwise, set $Z_{i,k-1} = Z_{i,k}$, $i = 1, \ldots, M$, $k = k + 1$, and go to step 2.

Remark 3. Note that an equivalent functional to be maximized is $\sum_i [\mathrm{trace}(W_{i,k}) - 2\mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k}]$ because the term $\mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k-1}$ is constant at iteration k.

Theorem 5.1. *The sequence* $\tilde{J}_k \overset{\triangle}{=} \sum_i [\mathrm{trace}(W_{i,k}) - 2\mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k} + \mu_{i,0}^{-1} Z_{i,k-1}^{\mathrm{T}} Z_{i,k-1}]$ *is upper bounded by zero and is nondecreasing. Therefore, if there exists a feasible point for problem* 1*c, Algorithm #* 2 *converges to some objective value* $\rho \leqslant 0$. *If* $\rho = 0$ *then Algorithm #* 2 *yields the solution to problem* 1*b in Definition* 5.1 *and therefore also the solution to problem* 1 *in Definition* 4.1

Proof. The proof that the algorithm converges is divided into two parts:

1. $\tilde{J}_k$ *is upper bounded by zero*. As noticed previously, because of constraints (38) we have $\sum_i [\mathrm{trace}(W_{i,k}) - \mu_{i,k}^{-1} Z_{i,k}^{\mathrm{T}} Z_{i,k}] \leqslant 0$. Since $\mu_{i,0} < 0$, $\mu_{i,0} + \Delta\mu_i < 0$, $\Delta\mu_i \geqslant 0$, $i = 1, \ldots, M$, setting $\Delta Z_i = Z_{i,k} - Z_{i,k-1}$ yields

$$
\begin{aligned}
0 \geqslant J_s &= \sum_i [\mathrm{trace}(W_{i,k}) - \mu_{i,k}^{-1} Z_{i,k}^{\mathrm{T}} Z_{i,k}] \\
&= \sum_i [\mathrm{trace}(W_{i,k}) - (\mu_{i,0} + \Delta\mu_i)^{-1} (Z_{i,k-1}^{\mathrm{T}} Z_{i,k-1} + 2 Z_{i,k-1}^{\mathrm{T}} \Delta Z_i + \Delta Z_i^{\mathrm{T}} \Delta Z_i)] \\
&\geqslant \sum_i [\mathrm{trace}(W_{i,k}) - \mu_{i,0}^{-1} (2 Z_{i,k-1}^{\mathrm{T}} Z_{i,k} - Z_{i,k-1}^{\mathrm{T}} Z_{i,k-1})] \\
&= \tilde{J}_k.
\end{aligned} \tag{40}
$$

2. $\tilde{J}_k$ *is non-decreasing*. For any $Z_{i,k-1}$, since $Z_{i,k-2}^{\mathrm{T}} Z_{i,k-1} = Z_{i,k-1}^{\mathrm{T}} Z_{i,k-2}$ and $\mu_{i,0} < 0$, $i = 1, \ldots, M$, then

$$
\begin{aligned}
\tilde{J}_{k-1}(W_{i,k-1}, Z_{i,k-1}) &= \tilde{J}_k(W_{i,k-1}, Z_{i,k-1}) + \sum_i \mu_{i,0}^{-1} [Z_{i,k-1} - Z_{i,k-2}]^{\mathrm{T}} [Z_{i,k-1} - Z_{i,k-2}] \\
&\leqslant \tilde{J}_k(W_{i,k-1}, Z_{i,k-1}) \\
&\leqslant \tilde{J}_k(W_{i,k}, Z_{i,k})
\end{aligned} \tag{41}
$$

because $(W_{i,k-1}, Z_{i,k-1})$ is a feasible solution and $(W_{i,k}, Z_{i,k})$ corresponds to the optimal solution.

When $\rho = 0$ notice from (40) that $0 = \rho = \tilde{J}_\infty \leqslant \mathscr{J}_s \leqslant 0$, so this implies that $\mathscr{J}_s = 0$ when $\rho = 0$. But, as mentioned before, $\mathscr{J}_s = 0$ if and only if $W_i = \mu_i^{-1} Z_i^{\mathrm{T}} Z_i$, $i = 1, \ldots, M$, as required. □

As will be shown in the examples, this algorithm works very well in practice and it has yielded a solution close to the optimal value $\mathscr{J}_s = 0$ in only one iteration (or two if finding a feasible point is counted as an iteration) for the examples analyzed in Section 6.

Algorithm #3. An alternative way to find the solution to problem 1c is by solving the following concave problem:

Definition 5.3. The modified trace maximization problem (problem 1d) is: for fixed $\alpha \geqslant 0$

$$
\begin{aligned}
\max \quad & \sum_i [\mathrm{trace}(W_i)] \\
\text{s.t.} \quad & Q = Q^{\mathrm{T}} > 0, \quad \mu_i < 0, \quad (37), (38), \\
& -l_1 \prec Y_i \prec l_1, -\bar{l}_0 \prec Z_i \prec \bar{l}_0, \quad i = 1, \ldots, M.
\end{aligned}
$$

We have seen that when the optimal value of the solution to problem 1c is zero, the solution of problem 1c is also a solution of problem 1. The following result shows that, in this case, a more efficient way to get this solution is to solve the concave problem 1d.

Theorem 5.2. *If the optimal value of the solution to problem* 1*c is zero then problem* 1*d has the same solution and this solution is also the solution to problem* 1.

Proof. It follows from the following three facts:

1. the solution of problem 1c is a feasible point for problem 1d,
2. if the solution to problem 1c yields an optimal value of zero then $W_i = \mu_i^{-1} Z_i Z_i^{\mathrm{T}}$, $i = 1, \ldots, M$ implying that $\sum_i [\mathrm{trace}(W_i)] = \sum_i [\mu_i^{-1} Z_i^{\mathrm{T}} Z_i]$, and
3. no other feasible point yields a higher value of the objective function of problem 1d because $\sum_i [\mathrm{trace}(W_i)] \leqslant \sum_i [\mu_i^{-1} Z_i^{\mathrm{T}} Z_i]$ given constraints (38). □

Remark 4. An approach to solve problem 1 is thus to solve problem 1d (a concave problem) and then check if condition (36) is verified.

5.3. *Solution algorithm for maximizing the decay rate*

Regarding problems 3 and 4, notice that if there is only one region in the partition of the state space, then $M = 1$, $m_1 = 0$, the system is linear and the decay rate maximization problem is a quasi-concave problem because of the product of variables αQ (see [5] for details about an equivalent quasi-convex problem). Following the same reasoning as the one used at the beginning of Section 5.1, for the general case of piecewise-affine systems, the decay rate maximization problem is an infinite set of quasi-concave programs analytically parameterized by the vector γ.

To solve the problems stated in Definitions 4.4 and 4.5, note that if γ is again sampled using Algorithm # 1, for each fixed value of γ there is one quasi-concave optimization problem to be solved. For each of these quasi-concave optimization problems, a lower bound to the corresponding maximum value of α can then be found, as tight as desired, using the familiar bisection algorithm.

Algorithm #4: Bisection.

1. Set $\alpha = 0$, and solve the corresponding convex stabilization problem 1 (or problem 2). If the problem is infeasible, stop because there is no piecewise-affine state feedback controller that can quadratically stabilize the system. If the problem is feasible, set $\alpha_{\mathrm{lower}} = 0$ and $\alpha = \delta$ for small δ and go to step 2.
2. Solve stabilization problem 1 (or problem 2) with $\alpha \leftarrow 10\alpha$ until an infeasible solution is reported.
3. Set $\alpha_{\mathrm{upper}} = \alpha$, where α is the one that made problem 1 (or problem 2) infeasible in step 2. Given the desired degree of ε tightness of the lower bound, choose the tolerance $\mathrm{tol} = \varepsilon$.
4. While $\alpha_{\mathrm{upper}} - \alpha_{\mathrm{lower}} < tol$ solve the convex stabilization problem 1 (or problem 2) with $\alpha \leftarrow 0.5\alpha_{\mathrm{lower}} + 0.5\alpha_{\mathrm{upper}}$. If the problem is feasible set $\alpha_{\mathrm{lower}} = \alpha$, otherwise set $\alpha_{\mathrm{upper}} = \alpha$
5. The ε-tight lower bound is α_{lower} and the ε-optimal controller and control Lyapunov function parameters are the ones that are provided as the solution to problem 1 (or problem 2) using $\alpha = \alpha_{\mathrm{lower}}$.

This procedure should be done for each point in the grid. Finally, the point in the grid that has the highest value of α is selected.

6. Examples

The purpose of this section is to show that the formulation for controller synthesis presented in this paper is applicable to systems in many different areas. Two examples will be shown: one in the area of circuit control and another in the area of vehicle control. For both examples, the controller synthesis can now be obtained by solving globally only one concave program as opposed to previously existing techniques that could only solve locally a bi-convex optimization problem in a set of iterations (each involving one or two convex programs).

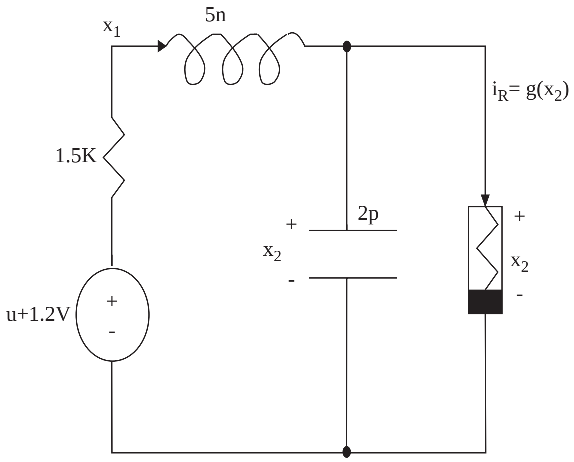

Fig. 1. Circuit with nonlinear resistor.

6.1. *Example 1*

This example considers a circuit with a nonlinear resistor shown in Fig. 1 [10,19]. This nonlinear resistor model is sometimes used to approximate the behavior of a tunnel diode. With time expressed in 10^{-10} seconds, the inductor current in milliAmps and the capacitor voltage in Volts, the dynamics are written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 24 \\ -50g(x_2) \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u.$$

Following [10], the characteristic of the nonlinear resistor $g(x_2)$ is defined to be the piecewise-affine function shown in Fig. 2 which generates the polytopic regions

$$\mathscr{R}_1 = \{x \in \mathbb{R}^2 \mid -L < x_2 < 0.2\}, \quad \mathscr{R}_2 = \{x \in \mathbb{R}^2 \mid 0.2 < x_2 < 0.6\}, \quad \mathscr{R}_3 = \{x \in \mathbb{R}^2 \mid 0.6 < x_2 < L\},$$

where L was selected to be $L = 2 \times 10^4$. The (exact) ellipsoidal covering is

$$E_1 = \frac{2}{0.2 + L} e_1, \quad E_2 = \frac{2}{0.6 - 0.2} e_2, \quad E_3 = \frac{2}{L - 0.6} e_3,$$

$$\tilde{f}_1 = \frac{L - 0.2}{L + 0.2}, \quad \tilde{f}_2 = -\frac{0.6 + 0.2}{0.6 - 0.2}, \quad \tilde{f}_3 = -\frac{L + 0.6}{L - 0.6},$$

where $e_1 = e_2 = e_3 = [0 \;\; 1]$. Assume that the affine terms of the control law have magnitude bounded by 0.2 so that $l_0 = [0.2 \;\; 0.2 \;\; 0.2]^{\mathrm{T}}$. The objective is to design a piecewise-affine state feedback controller to stabilize the open-loop equilibrium point of region $\mathscr{R}_3$ so

$$x_{\mathrm{cl}} = x_{\mathrm{ol}}^3 = \begin{bmatrix} 0.3714 \\ 0.6429 \end{bmatrix}.$$

For region $\mathscr{R}_3$ we then must have $m_3 = 0$ so that x_{cl} is the closed-loop equilibrium point of the dynamics that are valid within this region. We start by applying Algorithm #2 with $\alpha = 1 \times 10^{-9}$, $l_1 = 1 \times 10^{-9}$ and $\tilde{l}_0 = 1 \times 10^{-10}$. After only one iteration, the algorithm found a solution with $|\mathscr{J}_s| = 6.07 \times 10^{-11}$. Since Algorithm #2 found a

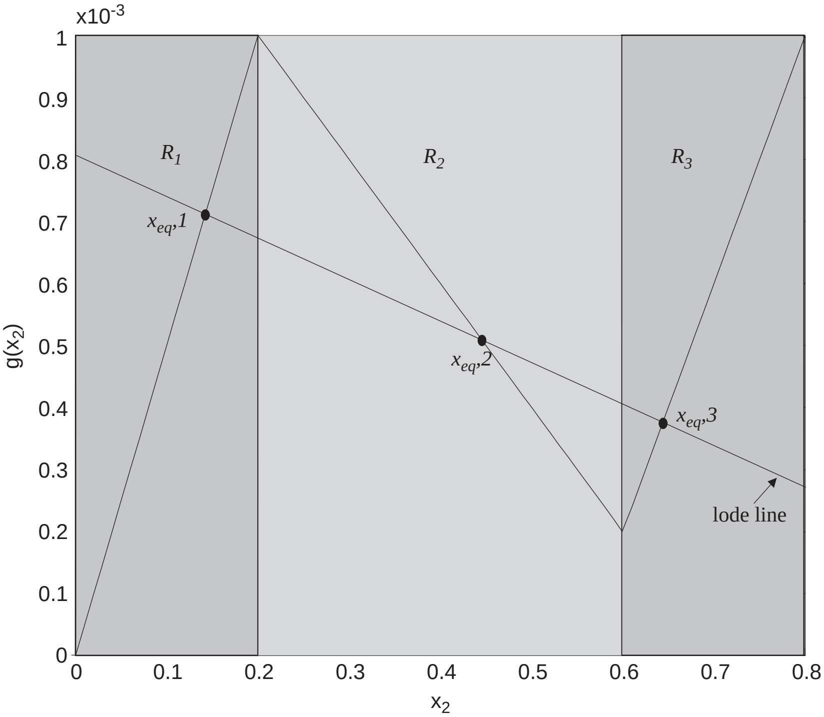

Fig. 2. Nonlinear resistor characteristic.

solution whose optimal value is very close to zero, based on the result of theorem 5.2, Algorithm #3 was then used and yielded the following controller (for which $|\mathcal{J}_s| = 1.16 \times 10^{-12}$):

$$\begin{aligned} K_1 &= [1.390 \quad -0.329], \quad m_1 = 0.000, \\ K_2 &= [1.432 \quad +0.372], \quad m_2 = 0.000. \\ K_3 &= [1.379 \quad -0.484], \quad m_3 = 0.000. \end{aligned}$$

Finally, if each of the affine terms m_1 and m_2 are now sampled in the interval $[-0.2, 0.2]$ with increments of 0.1, a mesh is obtained for the domain of $\gamma = [m_1 m_2]^{\mathrm{T}}$ with 25 points. A loop with Algorithm #4 inside Algorithm #1 was then used to maximize the decay rate of the Lyapunov function subject to the constraint $\alpha < 1$ (to impose a maximum value for alpha and thus reduce the execution time of the algorithm). The resulting controller yielded $\alpha = 0.993$ and is described by

$$\begin{aligned} K_1 &= [1.339 \quad -7.067], \quad m_1 = 0.200, \\ K_2 &= [1.261 \quad -8.527], \quad m_2 = -0.200, \\ K_3 &= [1.310 \quad -10.547], \quad m_3 = 0.000. \end{aligned}$$

The simulation results for the initial conditions $x_1^0 = 0.5$, $x_2^0 = 0.1$ (inside region $\mathcal{R}_1$) are overplotted in Fig. 3 for the controller that maximizes the decay rate and the controller obtained from Algorithm # 3. It can be clearly seen that the controller that maximizes the decay rate is significantly faster.

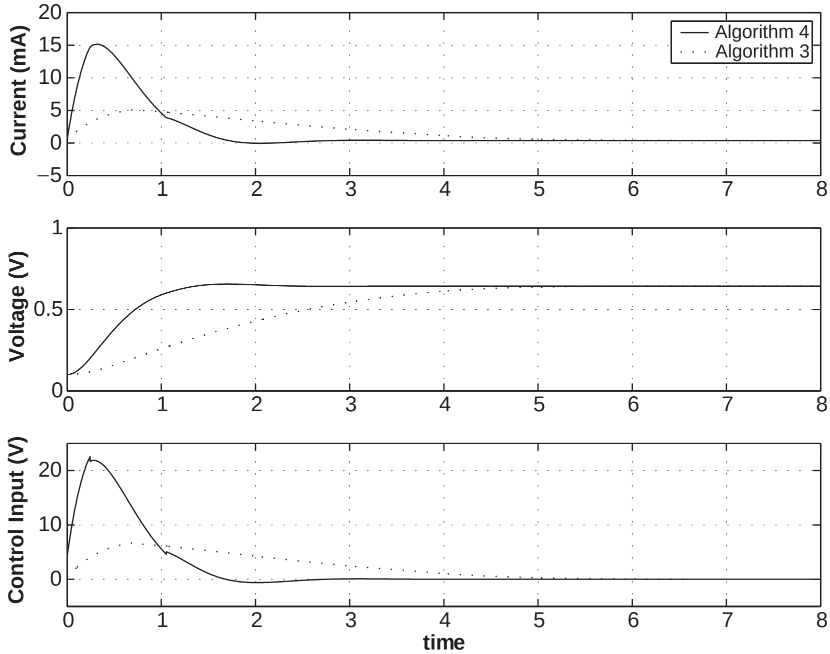

Fig. 3. Comparison between the controllers from Algorithms #3 (dashed) and #4 (solid).

6.2. Example 2

The objective of this example is to design a controller that forces a cart on the x–y plane to follow the straight line $y=0$ with a constant velocity $u_0=1$ m/s. It is assumed that a controller has already been designed to maintain a constant forward velocity. The cart's path is then controlled by the torque T about the z-axis according to the following dynamics:

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{k}{I} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_0 \sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \\ 0 \end{bmatrix} T, \tag{42}$$

where ψ is the heading angle with time derivative r, $I=1\,\mathrm{kg\,m^2}$ is the moment of inertia of the cart with respect to the center of mass, $k=0.01$ Nms is the damping coefficient, and T is the control torque. The state of the system is $(x_1,x_2,x_3)=(\psi,r,y)$. Assume the trajectories can start from any possible initial angle in the range $\psi_0 \in [-3\pi/5, 3\pi/5]$ and any initial distance from the line. The function $\sin(\psi)$ is approximated by a piecewise-affine function (see [20]) yielding a piecewise-affine slab system with 5 regions as follows:

$$\mathscr{R}_1 = \left\{ x \in \mathbb{R}^3 \mid x_1 \in \left(-\frac{3\pi}{5}, -\frac{\pi}{5}\right)\right\},$$
$$\mathscr{R}_2 = \left\{ x \in \mathbb{R}^3 \mid x_1 \in \left(-\frac{\pi}{5}, -\frac{\pi}{15}\right)\right\},$$
$$\mathscr{R}_3 = \left\{ x \in \mathbb{R}^3 \mid x_1 \in \left(-\frac{\pi}{15}, \frac{\pi}{15}\right)\right\},$$

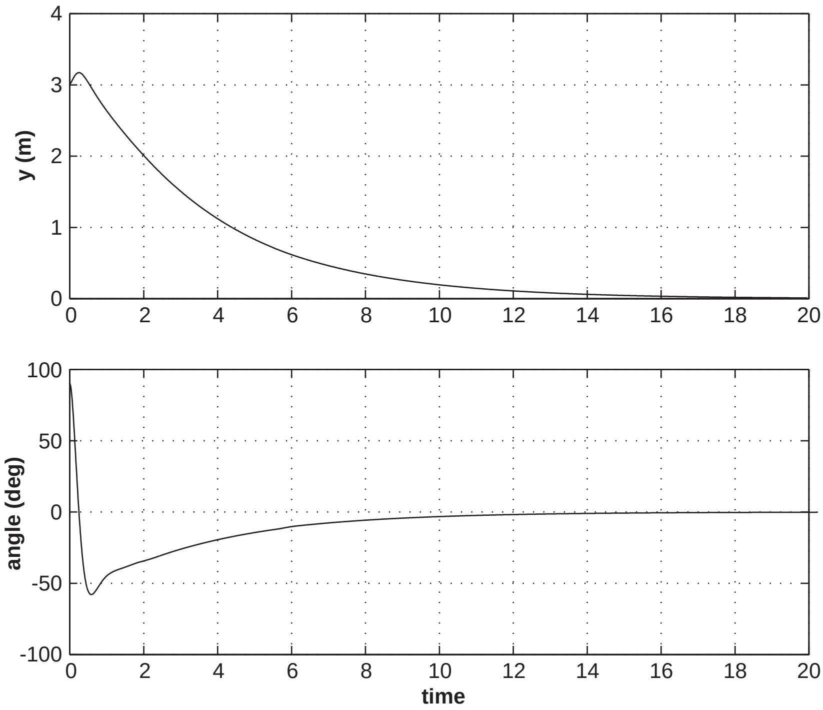

Fig. 4. Time response for $\psi_0 = \pi/2$, $r_0 = 0\,\text{rad/s}$, $y_0 = 3\,\text{m}$.

and $\mathscr{R}_4$ is symmetric to $\mathscr{R}_2$ and $\mathscr{R}_5$ is symmetric to $\mathscr{R}_1$, all with respect to the origin. For this system, a controller was designed to stabilize the origin (which lies inside region $\mathscr{R}_3$) using Algorithm #2. After only one iteration, the algorithm found a solution with $|\mathscr{J}_s| = 9.58 \times 10^{-11}$. Since Algorithm #2 found a solution whose optimal value is very close to zero, based on the result of Theorem 5.2, Algorithm #3 was then used and yielded the following controller (for which $|\mathscr{J}_s| = 1.84 \times 10^{-12}$):

$$\begin{aligned}
K_1 &= [-49.907 \quad -9.468 \quad -13.925], \quad m_1 = 0.00,\\
K_2 &= [-48.315 \quad -9.330 \quad -13.812], \quad m_2 = 0.00,\\
K_3 &= [-50.147 \quad -9.468 \quad -13.742], \quad m_3 = 0.00,\\
K_4 &= [-48.316 \quad -9.330 \quad -13.812], \quad m_4 = 0.00,\\
K_5 &= [-49.907 \quad -9.468 \quad -13.925], \quad m_5 = 0.00.
\end{aligned}$$

The simulations results for this controller are shown in Figs. 4 and 5 The trajectory on the *x*–*y* plane is shown in Fig. 6 where it is clear that the controller makes the cart trajectory converge to the desired straight line.

7. Conclusions

This paper has presented four piecewise-affine state feedback controller synthesis problems. The two stabilization problems were formulated as an infinite set of convex feasibility problems analytically parameterized by a vector. The two decay rate maximization problems were formulated as an infinite set of quasi-concave optimization problems analytically parameterized by the same vector. The feasibility problems can be solved by a finite set of LMIs either

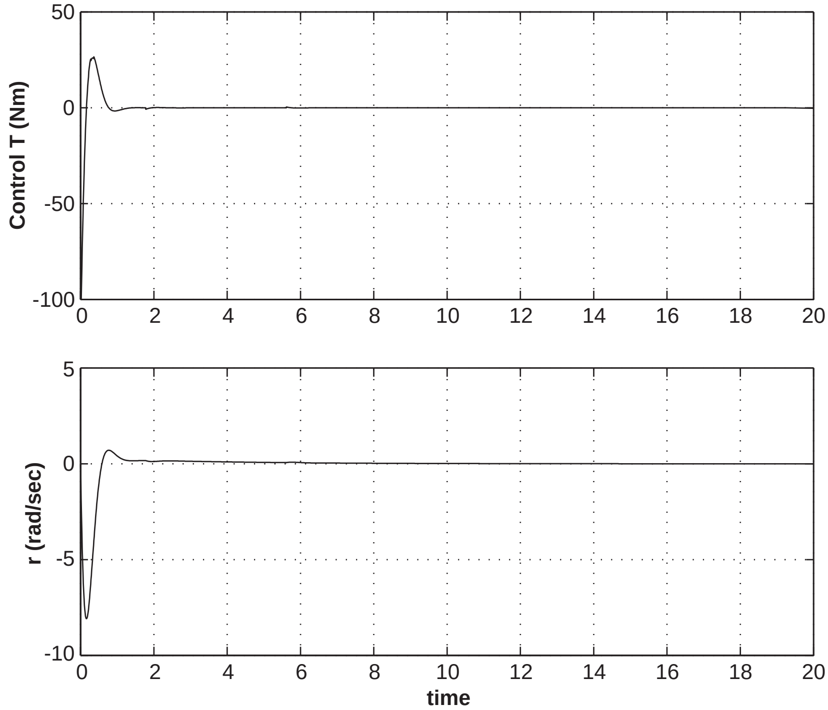

Fig. 5. Time response for $\psi_0 = \pi/2$, $r_0 = 0\,\text{rad/s}$, $y_0 = 3\,\text{m}$.

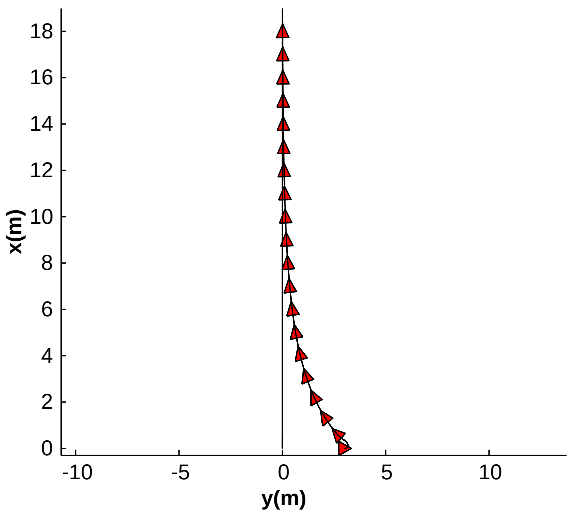

Fig. 6. x–y trajectory for $\psi_0 = \pi/2$, $y_0 = 3\,\text{m}$.

by discretizing the domain of the parameterizing vector or by using the trace maximization algorithms proposed in the paper. It was shown that these algorithms converges and if the functional value of the solution is zero then a feasible point of the original problem was found. The decay rate maximization problems can also be solved to a given desired approximation level to the global optimum by discretizing the domain of the parameterizing vector. If each optimization problem takes approximately 10 s to be solved in a standard pentium 4 personal computer, the problems formulated in this paper can be approximately solved in a matter of few hours of computation time for a number of discretizing points up to 5000. There are more effective methods of discretization that can save a lot of computation time. For example, the gradient with respect to γ can be used to decide which points of the mesh should be selected at each iteration. Alternative techniques, using for example directional convexity concepts [3], have been recently developed to alleviate the computational effort of gridding. Whichever method presented in the current paper is used, the fundamental conclusion is that it is now possible to solve piecewise-affine state feedback synthesis to suboptimal solutions that can be proved to be close to the global optimum. The existing methods from previous research were either not able to give this guarantee or were not able to include affine terms in the control law. On one hand, if affine terms are not included in the control law then the synthesis problem is restricted to systems for which the closed-loop equilibrium points of all regions are placed at the origin. This might not be possible for certain systems and, even if it is possible, it will limit performance. In Example 6.1 shown in this paper, the closed-loop equilibrium points of all regions are different and only the equilibrium point of region 3 is at the translated origin. On the other hand, bi-convex optimization problems are computationally expensive to solve globally. If local solution algorithms are used to perform the design then there are no guarantees that the suboptimal solution found is close to the global optimum. The biggest disadvantage of local algorithms to solve the bi-convex optimization problem is that the solution heavily depends on the initial point and it is not usually clear how to get good initial points. It should be mentioned however, that another advantage of the new algorithms proposed in the paper is that they can provide good initial points for the bi-convex optimization. The importance of this fact is that a piecewise-quadratic Lyapunov function can then be searched for to see if the performance of the resulting controller can be improved over the performance of the controller obtained for a globally quadratic Lyapunov function.

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References

[1] P. Apkarian, R. Adams, Advanced gain-scheduling techniques for uncertain systems, in: Proceedings of the American Control Conference, 1997.
[2] P. Apkarian, R. Adams, Advanced gain-scheduling techniques for uncertain systems, Advances in Linear Matrix Inequality Methods in Control, Advances in Design and Control, SIAM, Philadelphia, 2000.
[3] P. Apkarian, H.D. Tuan, Parameterized LMIs in control theory, SIAM J. Control Optim. 38 (4) (2000) 1241–1264.
[4] V.D. Blondel, J.N. Tsitsiklis, Complexity of stability and controllability of elementary hybrid systems, Automatica 35 (1999) 479–489.
[5] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Studies in Applied Mathematics, vol. 15, SIAM, Philadelphia, June 1994.
[6] S. Boyd, L. Vandenberghe, Semidefinite programming relaxations of non-convex problems in control and combinatorial optimization, in: A. Paulraj, V. Roychowdhuri, C. Schaper (Eds.), Communications, Computation, Control and Signal Processing: a tribute to Thomas Kailath, Kluwer, Dordrecht, 1997, pp. 279–288, (Chapter 15).
[7] F.O.L. El Ghaoui, M. Aitrami, A cone complementarity linearization algorithm for static output-feedback and related problems, IEEE Trans. Automat. Control 42 (8) (1997) 1171–1176.
[8] K.C. Goh, J.H. Ly, L. Turand, M.G. Safonov, Biaffine matrix inequality properties and computational methods, in: Proceedings of the American Control Conference, 1994, pp. 850–855.
[9] J.M. Gonçalves, A. Megretski, M. Dahleh, Global stability of relay feedback systems, IEEE Trans. Automat. Control 46 (4) (2001) 550–562.

[10] A. Hassibi, S.P. Boyd, Quadratic stabilization and control of piecewise-linear systems, in: Proceedings of the American Control Conference, 1998, pp. 3659–3664.
[11] S. Ibaraki, M. Tomizuka, Rank minimization approach for solving bmi problems with random search, in: Proceedings of the American Control Conference, 2001, pp. 1870–1875.
[12] T. Iwasaki, S. Hara, Well-posedness of feedback systems: insights into exact robustness analysis and approximate computations, IEEE Trans. Automat. Control 43 (5) (1998) 619–630.
[13] M. Johansson, Piecewise linear control systems, Ph.D. Thesis, Lund Institute of Technology, 1999.
[14] M. Johansson, A. Rantzer, Computation of piecewise quadratic lyapunov functions for hybrid systems, IEEE Trans. Automat. Control 43 (1998) 555–559.
[15] M. Johansson, A. Rantzer, Piecewise linear quadratic optimal control, IEEE Trans. Automat. Control 45 (4) (2000) 629–637.
[16] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[17] M. Mesbahi, On the rank minimization problem and its control applications, System Control Lett. 33 (1998) 31–36.
[18] S. Pettersson, Analysis and design of hybrid systems. Ph.D. Thesis, Chalmers University of Technology, 1999.
[19] L. Rodrigues, A. Hassibi, J. How, Output feedback controller synthesis for piecewise-affine systems with multiple equilibria, in: Proceedings of the American Control Conference, 2000, pp. 1784–1789.
[20] L. Rodrigues, J. How, Automated control design for a piecewise-affine approximation of a class of nonlinear systems, in: Proceedings of the American Control Conference, 2001, pp. 3189–3194.
[21] L. Rodrigues, J. How, Observer-based control of piecewise-affine systems, Internat. J. Control 76 (2003) 459–477.
[22] H.D. Tuan, P. Apkarian, Relaxations of parameterized LMIs with control applications, Internat. J. Robust Nonlinear Control 9 (1999) 59–84.
[23] A.P.F. Wu, X. Yang, G. Becker, Induced l_2-norm control for lpv system with bounded parameter variations rates, in: Proceedings of the American Control Conference, 1995, pp. 2379–2383.
[24] V.A. Yakubovich, The S procedure in nonlinear control theory, Vestnik Leningradskogo Universiteta, vol. 1, 1971, pp. 62–77 (English translation in Vestnik Leningrad Univ. Math., vol. 4, 1977, pp. 73–93).