Measuring Volterra Kernels

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Abstract —Volterra series have been in the engineering literature for some time now, and yet there have been few attempts to *measure* Volterra kernels. This paper discusses techniques for measuring the Volterra kernels of weakly nonlinear systems. We introduce a new quick method for measuring the second Volterra kernel which is analogous to pseudo-noise testing of a linear device. To illustrate the discussion we present an experimental example, an electro-acoustic transducer. Throughout the paper we emphasize the *practical* aspects of kernel measurement.

I. Introduction: Purpose and Point of View

VOLTERRA SERIES have appeared in the engineering literature for 40 years now. There have been many articles devoted to theoretical issues such as existence of Volterra series (e.g., [1]–[3]) computation of Volterra kernels of special systems (composition, feedback configurations, nonlinear circuits; see [8]–[13]), the formal framework for Volterra series [1], [4]–[6]; we can say that the topic has a firm foundation. However, relatively few attempts have been made, outside the biological areas, to actually measure Volterra kernels.

This paper discusses practical techniques for measuring the Volterra kernels of a weakly nonlinear system (device, plant, network). By a weakly nonlinear system we mean no more than a system which is well described by its first few Volterra kernels; in particular the higher order kernels must fall off rapidly. We assume that the nonlinearities may be subtle (i.e., distortion products 40 dB or more down) and that the measurement noise is low (or that the necessary signal averaging has been done). Examples of such systems are some high quality transformers, electromechanical and electroacoustic transducers, simple communications systems; not included are, e.g., devices with dead zone, hard saturation, or hysteretic nonlinearities (even when these nonlinearities are subtle). While the problems of kernel measurement in biology are quite different, involving stronger nonlinearities and very poor S/N ratios, much of the following is still relevant.

Related work includes that of Narayanan and Meyer *et al.* [22]–[25] who have studied IM distortion in transistor circuits; Weiner and Spina [26] and others have done similar work for simple communications systems. In these studies a model of a transistor or modulator is assumed and expressions derived for the various kernels; then certain distortions such as $2f_1 - f_2$ are measured at a few frequencies and input levels and checked against the model's predictions. Certain recent work by Ewen and Weiner [17] assumes a specific (but important) form for the Volterra kernels and gives methods to solve the resulting parameter identification problem. In contrast to these studies we make no assumption about the form of the kernels. These measurements are thus useful in systems of such complexity that no simple model is obvious, and for model validation when one is.

We have chosen frequency-domain Volterra kernels over time-domain Volterra kernels and Wiener kernels for two reasons. The first is that it is easier to accurately measure frequency-domain kernels than time-domain Volterra kernels when the nonlinearities are subtle. Second and more important, we are usually interested in frequency-domain Volterra kernels precisely because they have an *intuitive interpretation*: for example, $H_2(j\omega_1, -j\omega_2)$ is a measure of the second-order difference intermodulation of ω_1 and ω_2. While a similar interpretation exists for time-domain Volterra kernels, *no such simple interpretation* can be given to the Wiener kernels, whose apparent advantages are types of convergence (L_2 as opposed to local Taylor series; irrelevant to us) and "ease" of measurement with white noise [15], [18]–[20]. Concerning this last "advantage", we feel that in many applications the advent of microcomputers, D/A's, and A/D's has outmoded the use of white noise/correlation techniques. With only a few inexpensive components it is now possible to generate very complicated multitone signals with all distortion products near the noise floor, often 70 dB or more down. Signal processing too has gone far beyond Lee's Laguerre lattice filter [7, p. 91]. These practical considerations allow us to make a more direct attack on the measurement problem than was possible 25 years ago.

The organization of the paper is as follows: Section II contains the preliminaries, Section III covers the two basic methods used to resolve the output into its homogeneous components, Section IV discusses the basic multitone method of measuring the kernels, Section V introduces a new quick method of measuring the second kernel, and Section VI describes a simple experimental example.

With the exception of Section V, much of the material in this paper is known, though perhaps not in the form appearing here. We have tried to keep the exposition practical as opposed to theoretical. Where a statement or method may be true for, or generalizable to, arbitrary n, we

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give it for a specific and practical n, e.g., two or three: most of the following can be formalized.

II. Preliminaries

Under very general conditions a nonlinear causal time invariant operator N has a Volterra series

$$y(t) = Nu(t) = y_0 + y_1 + y_2 + \cdots$$

$$y_n(t) = \int \cdots \int h_n(\tau_1, \tau_2, \cdots, \tau_n) u(t-\tau_1) \cdot u(t-\tau_2) \cdots u(t-\tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n$$

where h_n is a symmetric distribution supported on $(R^+)^n$ and is called the nth Volterra kernel of N. We will be concerned with those systems for which the truncation $y_0 + \cdots + y_n$ is very close to y for the signals of interest; n is some small integer, say five or six.[1] We refer to y_n as the nth degree or order component of the output y and assume for simplicity that $y_0 = 0$. The map $u \to y_n$ is homogeneous of degree n, that is, $\alpha u \to \alpha^n y_n$. Each h_n also determines a symmetric multilinear operator

$$N_n(u_1, u_2, \cdots, u_n) = \int \cdots \int h_n(\tau_1, \tau_2, \cdots, \tau_n) u_1(t-\tau_1) \cdot u_2(t-\tau_2) \cdots u_n(t-\tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n$$

so that $y_n = N_n(u, u, \cdots, u)$. These multilinear operators can also be specified in terms of their Laplace transforms

$$H_n(s_1, s_2, \cdots, s_n) = \int \cdots \int h_n(t_1, t_2, \cdots, t_n) \exp\left(-\sum_{i=1}^{n} s_i t_i\right) dt_1\, dt_2 \cdots dt_n$$

which are called the frequency-domain Volterra kernels.

If u is a multitone, i.e.,

$$u(t) = 1(t) \sum_{i=1}^{K} \alpha_i \exp(s_i t)$$

then as $t \to \infty$, $y(t) \to y_s(t)$, where

$$y_{sn}(t) = \sum_{i_1=1}^{K} \sum_{i_2=1}^{K} \cdots \sum_{i_n=1}^{K} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_n} \cdot \exp\left(\sum_{j=1}^{n} s_{i_j} t\right) H_n(s_{i_1}, s_{i_2}, \cdots, s_{i_n})$$

It will be important later to note that the nth-order component of y_s is a sum of exponentials whose frequencies are sums of n input frequencies, negative frequencies included.

References [1]–[4], [8]–[16], [26], and [27] cover this material in more detail.

III. The Problem of Kernel Separation

In general the output y due to u has components of all degrees, though in the systems we consider their amplitudes fall off quickly. One step in measuring the kernels, in the time or frequency domain, is to estimate the components $y_1, \cdots,$ of y. What we need is a stable method of estimating

$$y_n = \frac{1}{n!} \frac{\partial^n}{\partial \alpha^n} N(\alpha u)\bigg|_{\alpha=0}.$$

While $N(\alpha u)(t)$ is in general an analytic function of α, for the systems we consider it is close to a low-order *polynomial* in α, with coefficients y_i. Thus the problem of estimating the different-order components is *in practice* one of estimating the coefficients of a noisy polynomial. There are many ways to do this, e.g., see [31]–[33] and the references therein. We will first describe the simplest, which we call the interpolation method. Consider the fact that y_n is homogeneous of degree n. Thus if our input is reduced 6 dB, y_1 falls 6 dB, y_2 falls 12 dB and so on; if $-u$ is applied, the odd-degree components change sign while the even ones do not. Suppose we assume that components of degree five and higher are negligible, i.e., buried in the measurement/quantization noise. Let us apply the signals $\alpha_i u(t)$ to the device and call the resulting responses $r_i(t)$, where α_i, $i = 1, \cdots, 4$ are some wisely chosen nonzero distinct constants. Then we have

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 \\ \alpha_2 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4 \\ \alpha_3 & \alpha_3^2 & \alpha_3^3 & \alpha_3^4 \\ \alpha_4 & \alpha_4^2 & \alpha_4^3 & \alpha_4^4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

where the e_i contain measurement noise and terms of degree five and higher. The matrix A above is a Vandermonde matrix, and is invertible since the α_i are distinct and nonzero. Approximating $e = 0$ and solving this equation (for each time or frequency sample point) gives us an estimate of the components y_i in terms of the measurements r_i.[2] This is just a simple polynomial interpolation and is mentioned in Simpson and Power [20, p. 318] and Halme [6, p. 29].

In the frequency domain the α_i may be complex and vary with frequency.[3] Thus the response of a device to a signal passed through various all-pass filters could be used to resolve the output into its homogeneous components. Sometimes we know *a priori* that only certain y_i appear; the other y_i's may then be dropped from the y vector and the corresponding columns from the A matrix. For example, if we know only even-order responses occur, the equations above can be replaced with a two-by-two system involving just y_2 and y_4. This is of course equivalent to interpolating with an even polynomial.

The α_i must be chosen carefully. Choosing the $\alpha_i > 1$ has the advantage of keeping $\|A^{-1}\|$ small, so the error in our resulting estimates is small. The disadvantage is that to estimate the components at some reference level we apply a larger signal, perhaps overloading the device (that is, operating the device where it is not weakly nonlinear in our

[1] We are deliberately vague about what "close" and "signals of interest" mean exactly.

[2] We should point out that the inverse of this matrix can be explicitly found and that there are stable and fast ways to solve these equations: see, e.g., [29].

[3] Some of Victor's (biological) experiments can be interpreted as the interpolation method with complex α_i; see, e.g., [28].

strict sense). The α_i should alternate in sign and not be too close, to keep $\|A^{-1}\|$ small.

But even with careful choice of the α_i, the interpolation method is in general sensitive to measurement error. To see this, consider estimating y_1 and y_2 with $\alpha_1 = 1, \alpha_2 = -1$. We average r_1 and r_2 to get y_2, and since r_1 is very nearly $-r_2$ (y_2 is generally much smaller than y_1) we have committed the cardinal sin of subtracting nearly equal quantities. Of course this example is oversimplified, but it conveys the basic idea. A more formal explanation is that the absolute error in y is bounded by $\|A^{-1}e\|$, but the magnitudes of $y_2, y_3, \cdots$ are generally much smaller than y_1 so the relative error in these entries may be huge. Rescaling the equations, perhaps using $y_1, 10y_2, 100y_3 \cdots$ instead of $y_1, y_2 \cdots$ simply makes A^{-1} blow up.

One improvement is to take additional measurements and use the least squares solution of the resulting overdetermined equations as our estimate of y. This is the method we used, and although it is an improvement over the simplest interpolation method, it still gives poor estimates of the higher order components: estimating the rapidly decreasing coefficients of a noisy polynomial is inherently difficult. What we can say is this: we can get a good estimate of the first coefficient appearing, a poorer estimate of the next, and a very poor estimate of the small high-order coefficients. The frequency separation technique we discuss below is based on this observation. It arranges for the component we need to estimate at some frequency to be the *first* component appearing at that frequency.

For time-domain kernel measurement or when the input u is fixed, we may have no alternative. But in other cases, clever selection of the probing signal u can greatly improve our estimates. The frequency separation technique relies on the fact that the (steady-state) nth-order response to a multitone signal only occurs at specific frequencies, sums of n input frequencies. We assume that the input has the form

$$u(t) = 2\,\mathrm{Re} \sum_{m=1}^{K} b_m \exp(j\omega m t)$$

and that the steady-state output spectrum $\hat{y}(k\omega)$ is measured; for notational convenience we will assume $w = 1$ and drop the qualifier "steady-state" in the following. The simplest and oldest use of frequency separation is as follows: suppose the input frequencies are all odd (i.e., $b_k = 0$ for k even); then the odd- and even-order responses occur at odd- and even-order frequencies, respectively. To isolate a second-order response at some even frequency we need only remove the 4th-, 6th-, etc., order responses, that is, estimate the x^2 coefficient of an *even* polynomial. We could use the interpolation method, modifying the matrix and y, but the estimate will be very accurate since the second-order response we seek is not swamped by a larger first-order response; it is the first large response occurring at that frequency. Moreover, by applying the signal at three levels we can approximately remove the effects of the components through degree six, as opposed to degree three for the general case. This trick is widely known, the requirement is simply that the input signal be odd, i.e., have the inverse repeat property as it is sometimes called. In the next section we'll see very robust methods for measuring even high-order kernels using frequency separation.

It should be mentioned that complete separation of the components of different order by frequency separation is impossible. For whenever ω is an nth-order response frequency, it is also an $n+2, n+4, \cdots$ order response frequency, at least.

IV. The Multitone Method ("Harmonic Probing")

In this section we discuss the actual measurement of the kernels. Suppose we apply a two-tone signal $u(t) = \cos(n_1 t) + \cos(n_2 t)$, $n_1 > n_2 > 0$. Then $\hat{y}(n_1 \pm n_2) = H_2(jn_1, \pm jn_2) +$ terms of order $4, 6, \cdots$ and for certain values of n_1 and n_2, additional terms of order $3, 5, \cdots$. Applying the signal at two or three levels and using the interpolation method to estimate the second-degree component of $\hat{y}(n_1 \pm n_2)$ yields an accurate measurement of $H_2(jn_1, jn_2)$ and $H_2(jn_1, -jn_2)$. At the same time we can measure $H_2(jn_1, jn_1)$ and $H_2(jn_2, jn_2)$ but these are of less interest since they always lie on the line $\omega_1 = \omega_2$. We simply repeat this procedure until a sufficient number of points have been measured.

A variant of this method can be used to measure the third- and higher order kernels. Suppose a three-tone signal is applied. Third-degree responses occur at up to 22 different (positive) frequencies, three of which are the input frequencies n_1, n_2, n_3.[4] If we choose integer triplets such that the full 22 frequencies appear (the "general" triplet has this property), estimation of y_3 yields a good estimate of 19 points of H_3. The four points $H_3(jn_1, \pm jn_2, \pm jn_3)$ are of more importance than the remaining 15 which lie on planes where two frequencies are equal. Note that 12 points of H_2 can be measured from the same experiment.

V. A New Method

Unfortunately, measuring kernels by the multitone method can be quite slow. For example, to measure H_2 at only 100 points (relatively few) requires at least 100 experiments, each experiment consisting of generation of a signal, waiting for steady-state, sampling the output, and then computation (FFT, kernel separation). One may have to wait through half of these before deciding the input level is too low or high or that another frequency range might be more interesting. We've developed a method for getting a quick estimate of the second kernel. We use this method to make decisions about input level, frequency range, etc., before using the slower but more robust multitone method.

It is perhaps surprising that many points of $H_2(j\omega_1, j\omega_2)$ can be simultaneously measured since methods for simultaneously measuring many points of $H(j\omega)$ for a linear

[4] They are:

$$n_1,\ n_2,\ n_3,\ 3n_1,\ 3n_2,\ 3n_3,\ |n_1 \pm n_2 \pm n_3|,$$
$$|2n_1 \pm n_2|,\ |2n_1 \pm n_3|,\ |2n_2 \pm n_1|,\ |2n_2 \pm n_3|,\ |2n_3 \pm n_1|,\ |2n_3 \pm n_2|$$

device (pseudonoise, impulse testing) rely very heavily on linearity. The idea is simple: arrange the second-order IM tones to lie on distinct frequencies which do not include the input frequencies.

We start with two relatively prime integers p and q, q odd. The probing signal will have two parts: one with frequencies $p, 2p, \cdots, p(q-1)/2$ and the other with frequencies $q, 2q, \cdots, q(p-1)$. We claim that the part one–part two intermodulation tones are distinct. These IM tones occur at frequencies $np+mq$, $0<|n|\leqslant (q-1)/2$, $0<m\leqslant p-1$; the input tones are precisely the $n=0$ or $m=0$ cases. Suppose that $\bar{n}p+\bar{m}q=np+mq$, where $0\leqslant \bar{n}$, $n\leqslant (q-1)/2$ and $0\leqslant \bar{m}, m\leqslant p-1$. Taking residues mod q, we have $\bar{n}=n[q]$, and thus $\bar{n}=n$ considering the inequality in $\bar{n}$, n above. Hence $\bar{m}=m$ as well. This shows that the part one–part two IM tones are distinct and do not include any input frequencies. They also do not include any part one(two)–part one(two) intermodulation tones since these are all 0 mod p (mod q); here we use the inequality in $\bar{m}$, m.[5] The conclusion is that at the part one–part two IM frequencies, there is no first-order component and only one second-order contribution. Let us take $p=7, q=5$ as an example. We make a table as follows:

14	19	24	29	34	39	44
7	12	17	22	27	32	37
0	**5**	**10**	**15**	**20**	**25**	**30**
−7	2	3	8	13	18	23
−14	9	4	1	6	11	16

The left column and center row (in bold) are input frequencies; the other entries are the part one–part two IM frequencies and it is easily checked that at these frequencies there is no first-order and just one second-order contribution.

A quick estimate of H_2 is now easy: we apply this multitone signal u at, say, six different levels and use a least squares interpolation to estimate $\hat{y}_2$. Almost every entry of $\hat{y}_2$ gives us a value of H_2: in our example above $\hat{y}_2(8)=H_2(15j,-7j)\alpha\bar{\beta}$ where α and β are the complex amplitudes at 15 and 7 in u. This should be compared to the multitone method where only two or four of the entries of $\hat{y}_2$ are used and in fact the efficiency of using the FFT is questionable.

The distribution of points at which we estimate H_2 is interesting. We measure H_2 at the points in the uniform grid as in the table above, but recall that H_2 has two symmetries: it is symmetric and $\overline{H_2(j\omega_1, j\omega_2)}=H_2(-j\omega_1,-j\omega_2)$. The region $|w_2|<\omega_1$ is a "fundamental region" for H_2, that is a minimal region which determines H_2 everywhere, and in it the distribution is shown in Fig. 1 for $p=13, q=11$.

Several comments are in order concerning this quick method. First, repeated quick-method tests with different p's or q's yield estimates of new points. For example, one test may estimate at 200 points; the next test at 200 new points yielding 400 points altogether; there are no redundant estimates. The second comment concerns the choice of the complex amplitudes of the frequencies in the probing signal. While it is tempting to make them all one, this is the worst choice possible. This results in $\sin(\Theta N/2)/\sin(\Theta/2)$ type signals with very high crest factors; the signals spend most of their time down where the quantization step is significant. For a given peak level (to keep from clipping the device, perhaps) the amplitudes are small, and the second-order distortions we are trying to measure are extremely small (i.e., small squared). To avoid these problems we simply let an optimization routine adjust the phases to minimize the peak (see [30]). The practical result of this is to pack as much probing signal (L_2) as possible into a given peak.[6] For signals with frequencies $f, 2f, \cdots, Kf$ near optimal phases are $\delta_k=(\pi k^2)/(K+1)$; our optimization routine used these as starting points. For the quick tests we used $(7<p,q<19)$, we were able to reduce the peak by more than 10 dB and thus realize a 20-dB gain in measurement sensitivity. This is not far from the bound peak $>\sqrt{K/2}$ for a K tone unit amplitude signal. To illustrate this, Fig. 2 shows two 7–5 quick test signals: the first (darker) with optimized phases and a peak of about 4, the second with all phases zero and a peak of 8. In this case the peak has only been reduced about 6 dB (representing a 12-dB gain in second kernel measurement sensitivity), but in more realistic cases the improvement is greater. We have now arrived at probing signals which at first glance resemble the white noise we complained about in Section I, but we hope the reader will appreciate the difference.

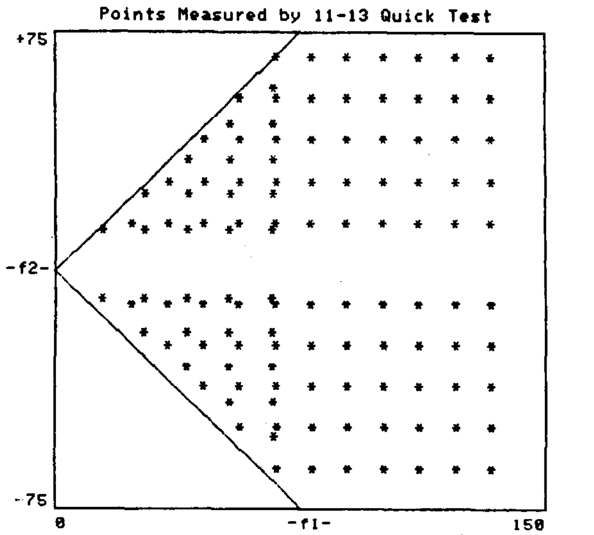

Fig. 1. Distribution of points measured by the 13–11 quick test (described in Section V) in the region $|f_2|<f_1$.

VI. An Example

In this section we briefly describe our test setup and illustrate some of the above with an example. We used a

[5] We could add more tones to the second part and simply ignore every pth column, since these frequencies may have part one–part one contributions.

[6] Incidently we first used the quick method with all the amplitudes one and it really was not that bad.

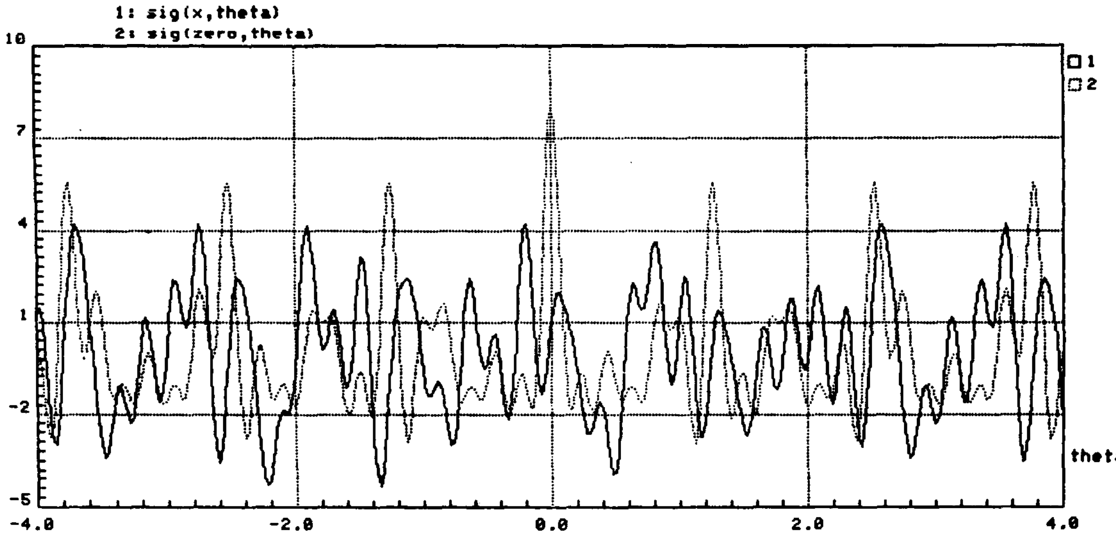

Fig. 2. Two unit amplitude signals for the 7–5 quick test: the first (darker) is with nearly optimal phases, the second is with zero phases. The peak in the optimized signal has been reduced about 6 dB below that of the zero-phases signal, giving a second kernel measurement sensitivity gain of about 12 dB. For more realistic quick tests, e.g., 13–11, the improvement is more drastic.

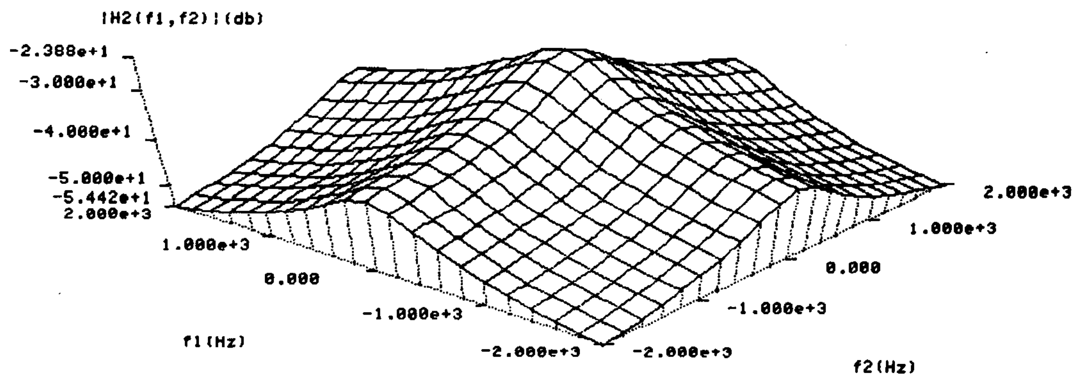

Fig. 3. $|H_2(f_1,f_2)|$ for reference device #1. The reference level is 1 V^{-1}. f_1 and f_2 are actually shifted slightly so that none of $f_1, f_2, f_1+f_2, f_1-f_2$ is zero.

small 8085 based microcomputer to generate the probing and trigger signals and do all computation except the FFT; an HP3582A spectrum analyzer collected and transformed the responses. We built several reference nonlinear devices with known kernels like

$$H_1(s)=\frac{6.4}{1+s/s_0} \qquad H_2(s_1,s_2)=\frac{0.064}{(1+s_1/s_0)(1+s_2/s_0)}$$

$$H_n=0, \quad n>2 \quad s_0=2\pi 350 \quad |V_{in}|<1\text{ V}$$

and used them to check the algorithms above. Note that the distortion is at most 1 percent, i.e., at least 40 dB under y_1. The values of H_2 measured by the multitone and quick methods were within 2 and 7 percent, respectively, of the predicted values. Fig. 3 shows the magnitude of H_2 measured by the multitone method; it is indistinguishable from the graphs based on either the quick test measurement or the expression above.

The example we give is an electro-acoustic transducer, a JBL 2441 compression driver on a Northwest Sound 90 degree radial horn, measured 0.5 m on axis. We chose this example because it has no simple model[7] and as far as we know these measurements have never been made before. To illustrate frequency separation and the fact that $N(\alpha u)$ is indeed close to a low-order (even) polynomial in α, Fig. 4 shows the real part of the output at 800-Hz versus the input amplitude of a 400-Hz signal. The interpolation method correctly estimates a large second-order, small fourth-order, and nearly zero first- and third-order components at 800 Hz. A plot-like Fig. 4 can warn us that a device is not well described by its first few Volterra kernels if it is not close to a low-order polynomial.

Figs. 5 and 6 show typical input and output spectra for this transducer during a 13-11 quick test. In Fig. 6 one can see clearly the large first-order responses at the input frequencies and the smaller higher order responses. The

[7]An accurate model would involve at least: nonlinear flux-coil linkage, nonlinear support compliance, and thermodynamic nonlinearity (called *throat distortion*, distributed!).

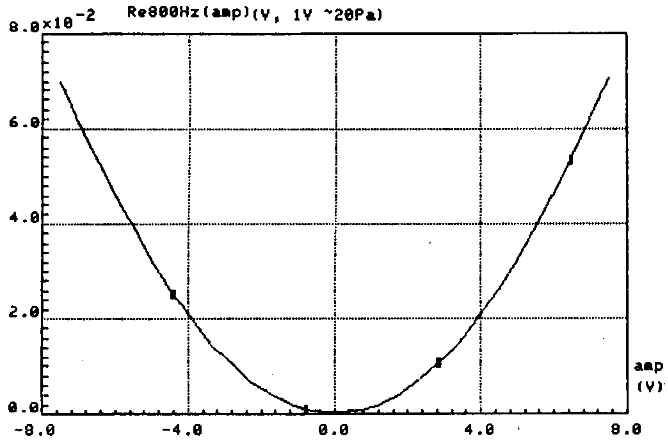

Fig. 4. Real output at 800 Hz versus amplitude of 400-Hz input signal for example of Section VI, JBL 2441 compression driver on Northwest Sound radial horn, 0.5 m on axis.

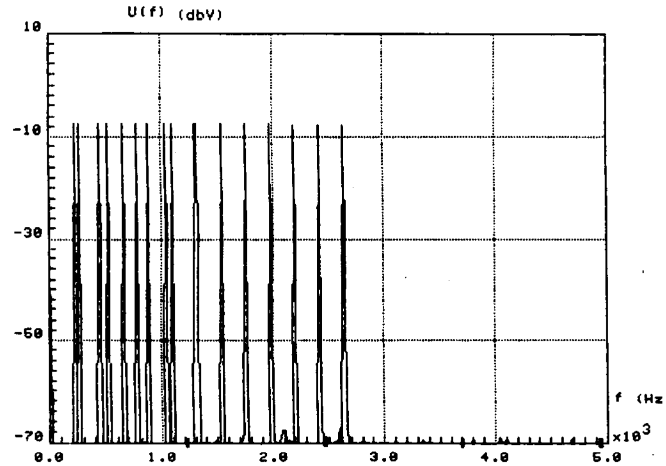

Fig. 5. Typical input spectrum for 13–11 quick test.

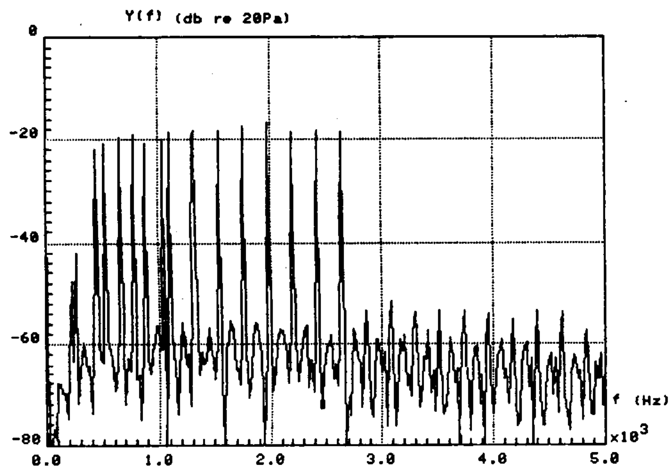

Fig. 6. Typical output spectrum for 13–11 quick test, JBL 2441 driver on Northwest Sound horn, 0.5 m on axis.

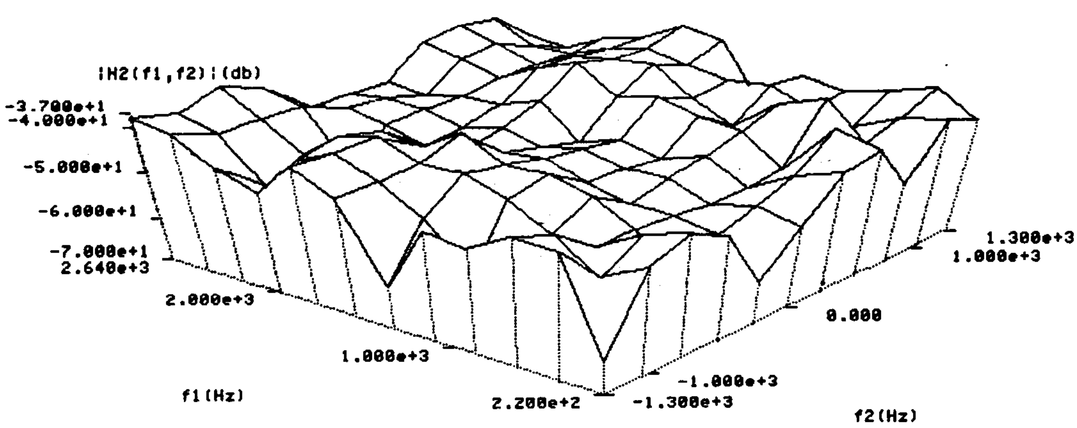

Fig. 7. $|H_2(f_1, f_2)|$ for transducer example in Section VI, JBL 2441 compression driver on Northwest Sound radial horn, 0.5 m on axis. The reference level is $20Pa\ V^{-2}$. $f_2 = 0$ values are not measured by the quick test and are interpolated. The "trough" along $f_1 + f_2 = 0$ represents a linear high-pass filter following a nonlinear operator.

responses on the right which are about 8 dB higher are mostly second-order part II–part II intermodulation. Measurements of the second kernel of the transducer by the quick method and the two-tone method agreed within 5 percent. Fig. 7 shows the magnitude of the second kernel measured by the quick method. The peak distortion here is only 2 percent. Some features are recognizable, for example, the "trough" along the line $f_1 + f_2 = 0$ suggests a linear high-pass filter (horn cutoff) following a nonlinear operator.

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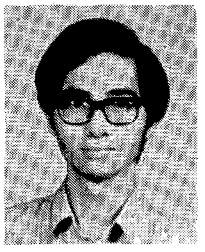

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